Contents

Chapter 1

Double Integrals over Rectangular Regions

In this chapter, we extend this powerful idea into higher dimensions using the tools of multiple integration. While single integration enables us to calculate the area under a curve or the volume under a surface, multiple integration allows us to calculate volumes in three dimensions, and even hypervolumes in higher dimensions.

First, we will discuss double integration, which allows us to find the volume under a surface in three dimensions. This method involves slicing the solid into infinitesimally small columns, and summing the volumes of these columns.

In a subsequent chapter, we will cover triple integration, a tool that lets us find the volume of more complicated solids in three-dimensional space. The idea is similar to double integration.

To properly implement these techniques, we will also discuss the different coordinate systems that can be used in multiple integration, such as rectangular, cylindrical, and spherical coordinates, and when it's advantageous to use one system over another.

By the end of this chapter, you will have a deeper understanding of the techniques of multiple integration and how to apply them to find the volumes of various types of solids. The methods we study here will serve as a foundation for many topics in higher mathematics and physics, including electromagnetism, fluid dynamics, and quantum mechanics.

1.1 Double Integrals

Double integrals extend single-variable integration to functions of two variables, allowing us to calculate quantities like area, volume, and mass over a two-dimensional region. By integrating a function across a specified domain in the xy-plane, they help analyze how a quantity changes in both dimensions. Common in physics, engineering, and economics, double integrals involve setting up limits for the region and performing two successive integrations, often tailored to the region's geometry. We begin by discussing double integrals over rectangular regions, then extending that discussion to regions of any general shape. Finally, we discuss applications of double integrals.

1.1.1 Over Rectangular Regions

Suppose there is some function, $z = f(x, y)$, that is defined over the rectangular region, R, defined by $R = [a, b] \times [c, d] = \{(x, y) | a \le x \le b, c \le y \le d\}$, and f is such that $f \ge 0$ for all $(x, y) \in \mathbb{R}$. So, the graph of f is a surface that lies above the rectangular region, R (see figure 1.1).

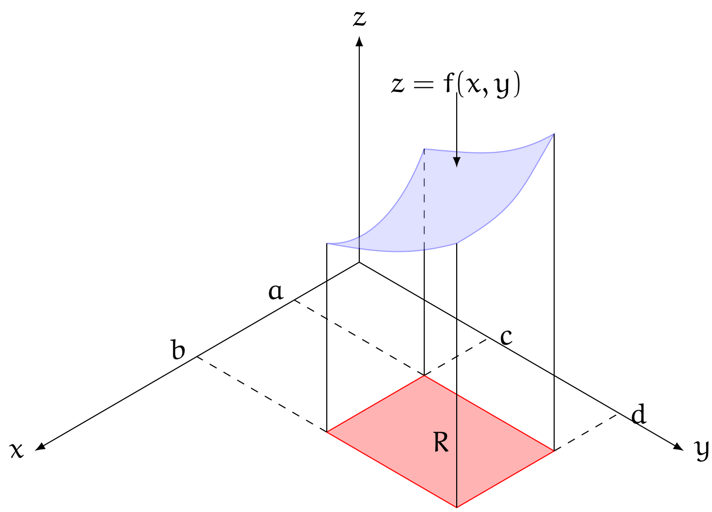

Figure 1.1: The graph of f over the region R

Let's call the solid that fills the space between the xy-plane and the surface $z = f(x, y)$ S. Formally, this is written as

$$S = \{(x, y, z) \in \mathbb{R}^3 | 0 \le z \le f(x, y), (x, y) \in \mathbb{R}\}$$

How can we find the volume of the solid, S? We will apply what we learned about Riemann sums and definite integrals in two dimensions to this three-dimensional problem.

First, we divide R into rectangular subregions. We do this by dividing the interval $[a, b]$ into m subintervals with width $\Delta x = (b - a)/m$ and the interval $[c, d]$ into n subintervals with width $\Delta y = (d - c)/n$. Drawing lines through these divisions parallel to the x- and y-axes, we create a field of subrectangles, each with area $\Delta A = \Delta x \Delta y$ (see figure 1.2). Each subrectangle is defined by:

$$R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] - \{(x, y) | x_{i-1} \le x \le x_i, y_{j-1} \le y \le y_j\}$$

Since $f(x, y)$ in continuous over the R, there is some point, (x^*_{ij}, y^*_{ij}), equal to the average value of $f(x, y)$ over the subrectangle. We can approximate the volume between the xy-plane and $z = f(x, y)$ over the subrectangle as a column with base area ΔA and height

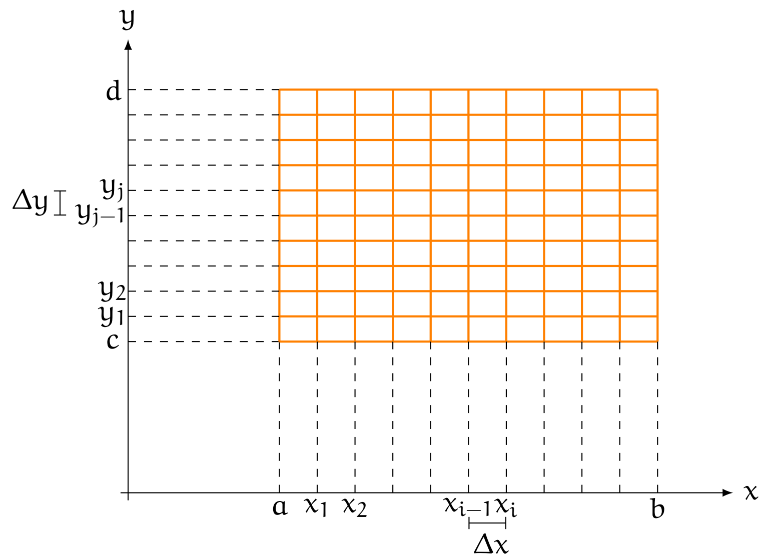

Figure 1.2: The region, R, on the xy-plane divided into subrectangles

$f(x_{ij}^*, y_{ij}^*)$ (seefigure 1.3) and the volume of the column is given by:

$$V_{ij} = f(x_{ij}^*, y_{ij}^*)\Delta A$$

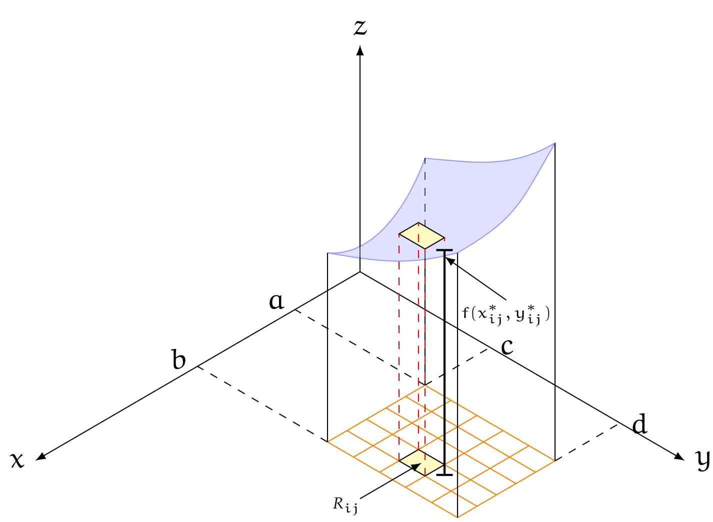

Figure 1.3: A single column with base ΔA and height $f(x_{ij}^*, y_{ij}^*)$

Therefore, the approximate volume of the solid, S, that lies between the region, R, and $z = f(x, y)$ is the sum of all the columns over i and j:

$$V_S \approx \sum_{i=1}^{n} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A$$

Just like with the area under a curve, we get the true volume by taking the limit as $n \to \infty$, which becomes a **double integral**:

Volume of a Solid over a Region

$$V_S = \lim_{n\to\infty} \sum_{i=1}^{n} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A = \iint_R f(x,y)\, dA$$

1.2 Iterated Integrals

To be able to evaluate the double integral as outlined above, we must first discuss iterated integrals. Iterated integrals happen when you evaluate two single integrals, one inside the other. Consider some function, $g(x,y)$. We could integrate that function from $x = q$ to $x = r$ thusly:

$$\int_q^r g(x,y)\, dx$$

Notice that we are integrating with respect to x, so y terms will be treated as constants (recall partial differentiation: this is the opposite process). Let's call the result of this first integral $A(y)$:

$$A(y) = \int_q^r g(x,y)\, dx$$

We can then integrate the resulting function, $A(y)$, from $y = s$ to $y = t$:

$$\int_s^t A(y)\, dy = \int_s^t \left[\int_q^r g(x,y)\, dx\right] dy$$

This is called an **iterated integral**. When evaluating iterated integrals, we work from the inside out. You can also write it without the brackets:

$$\int_s^t \int_q^r g(x,y)\, dx\, dy$$

Example: Evaluate the iterated integral $\int_0^3 \int_1^2 xy^2\, dy\, dx$.

Solution: We can re-write this to more explicitly show the inner and outer integrals:

$$\int_0^3 \left[\int_1^2 xy^2\, dy\right] dx$$

As you can see, the inner integral is with respect to y. Let's isolate and evaluate the inner integral:

$$\int_1^2 xy^2\,dy = x\int_1^2 y^2\,dy = x\left[\frac{1}{3}y^2\right]_{y=1}^{y=2}$$
$$= \frac{x}{3}\left[2^3 - 1^3\right] = \frac{x}{3}\,[8-1] = \frac{7x}{3}$$

We were able to move x outside the integral because when we are integrating with respect to a specific variable (in this case, y), other variables are treated as constants. Now, we can substitute $\int_1^2 xy^2\,dy = \frac{7x}{3}$ into the iterated integral:

$$\int_0^3 \left[\int_1^2 xy^2\,dy\right]\,dx = \int_0^3 \left[\frac{7x}{3}\right]\,dx$$
$$= \frac{7}{3}\left[\frac{1}{2}x^2\right]_{x=0}^{x=3} = \frac{7}{6}\left[3^2 - 0^2\right] = \frac{7\cdot 9}{6} = \frac{21}{2}$$

Exercise 1 Order of Evaluating Iterated Integrals

Show that $\int_0^3\int_1^2 xy^2\,dy\,dx = \int_1^2\int_0^3 xy^2\,dx\,dy$.

Working Space

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Exercise 2 Evaluating Iterated Integrals

Evaluate the following iterated integrals.

Working Space

1. $\int_0^1 \int_1^2 (x + e^{-y})\, dx\, dy$
2. $\int_{-3}^3 \int_0^{\pi/2} \left(2y + y^2 \cos x\right)\, dx\, dy$
3. $\int_0^3 \int_0^{\pi/2} t^2 \sin^3 \theta\, d\theta\, dt$

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1.3 Fubini's Theorem for Double Integrals

Fubini's theorem states that for a function, f, that is continuous over the rectangular region, R, the double integral of f over the region $R = \{(x, y) | a \leq x \leq b, c \leq y \leq d\}$ is equal to the iterated integral of f with respect to x and y. This is expressed mathematically below:

Fubini's Theorem

If f is continuous on the rectangle $R = \{(x, y) \,|\, a \leq x \leq b, c \leq y \leq d\}$, then

$$\iint_R f(x, y)\ dA = \int_a^b \int_c^d f(x, y)\, dy\, dx = \int_c^d \int_a^b f(x, y)\, dx\, dy$$

Exercise 3 Applying Fubini's Theorem

Rewrite the following double integrals as iterated integrals.

1. $\iint_R \frac{xy^2}{x^2+1}\ dA,\ R = \{(x, y) | 0 \leq x \leq 1, -3 \leq y \leq 3\}$
2. $\iint_R \frac{\sec\theta}{\sqrt{1+t^2}}\ dA,\ R = \{(\theta, t) | 0 \leq \theta \leq \frac{\pi}{4}, 0 \leq t \leq 1\}$

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Exercise 4

Evaluate the double integral.

1. $\iint_R \frac{xy^2}{x^2+1}\, dA$, $R = \{(x, y) \mid 0 \le x \le 2, \ -3 \le y \le 3\}$
2. $\iint_R \frac{\tan\theta}{\sqrt{1-t^2}}\, dA$, $R = \{(\theta, t) \mid 0 \le \theta \le \pi/3, \ 0 \le t \le \frac{1}{2}\}$
3. $\iint_R x \sin(x+y)\, dA$, $R = [0, \pi/6] \times [0, \pi/3]$

Working Space

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Chapter 2

Double Integrals Over Non-Rectangular Regions

Now that we've seen how to evaluate double integrals over rectangular regions, let's consider non-rectangular regions. Suppose we are interested in the integral of a function, $f(x, y)$, over a region, D, which exists such that it can be bounded by inside a rectangular region, R (see figure 2.1). We can then define a new function:

$$F(x, y) = \begin{cases} f(x, y) & \text{if } (x, y) \text{ is in } D \\ 0 & \text{if } (x, y) \text{ is in } R \text{ but not in } D \end{cases}$$

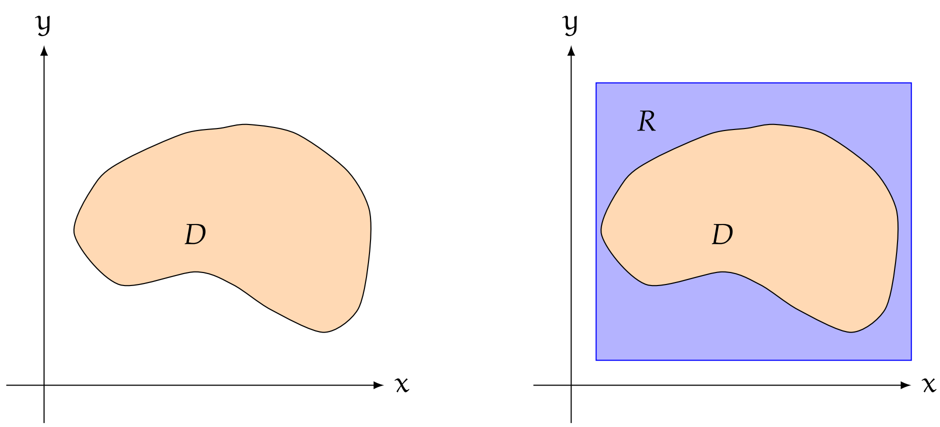


Figure 2.1: We can find a rectangular region, R, that completely encloses D

Then, we can see that:

$$\iint_D f(x, y)\, dA = \iint_R F(x, y)\, dA$$

This makes sense intuitively, since integrating over F outside of D doesn't contribute anything to the integral, and the integral of F inside D is equal to the integral of f inside D. In general, there are two types of regions for D. A region is **type I** if it lies between two continuous functions of x and can be defined thusly:

$$D = \{(x, y) \mid a \le x \le b, g_1(x) \le y \le g_2(x)\}$$

Some type I regions are shown in figure 2.2. To evaluate $\iint_D f(x, y)\, dA$, we begin by choosing a rectangle $R = [a, b] \times [c, d]$ such that D is completely contained in R. We again

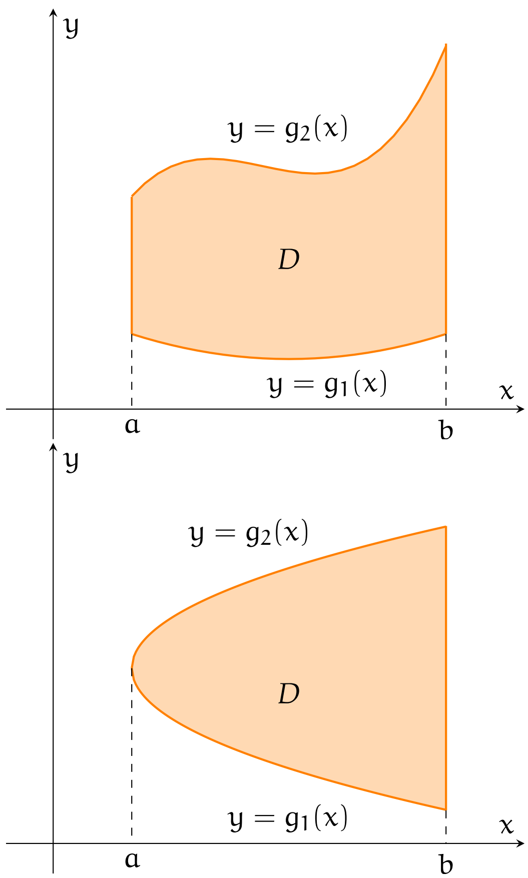


Figure 2.2: Two examples of type I domains

define $F(x, y)$ such that $F(x, y) = f(x, y)$ on D and $F = 0$ outside of D. Then, by Fubini's theorem:

$$\iint_D f(x, y)\, dA = \iint_R F(x, y)\, dA = \int_a^b \int_c^d F(x, y)\, dy\, dx$$

Since $F(x, y) = 0$ when $y \leq g_1(x)$ or $y \geq g_2(x)$, we know that:

$$\int_c^d F(x, y)\, dy = \int_{g_1(x)}^{g_2(x)} F(x, y)\, dy = \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy$$

Substituting this into the iterated integral above, we see that for a type I region $D = \{(x, y) \mid a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}$,

$$\iint_D f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx$$

Another way to visualize the double integral over a type I region is shown in figure 2.3. For any value of $x \in [a, b]$, we know that $g_1(x) \leq y \leq g_2(x)$. The inner integral represents moving along one blue line from $y = g_1(x)$ to $y = g_2(x)$ and integrating with respect to y. Next, for the outer integral, we integrate with respect to x, which is represented by moving the line from $x = a$ to $x = b$.

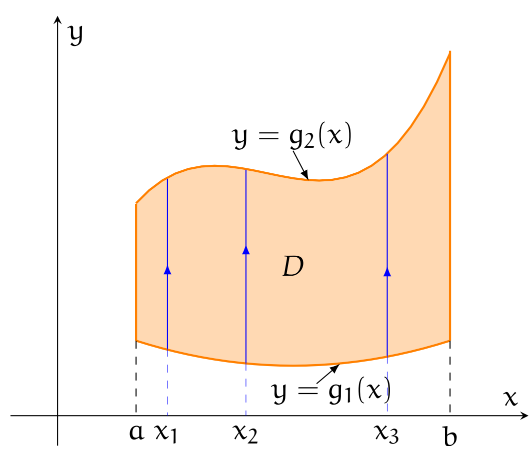


Figure 2.3: On type I domains, for a given value of x, $g_1(x) \leq y \leq g_2(x)$

A **type II** region is a region such that we can define the limits of x in terms of y (see figure 2.4). In other words, a type II region can be defined as:

$$D = \{(x, y) \mid c \leq y \leq d, h_1(y0 \leq x \leq h_2(y)\}$$

In a similar manner to above, we can show that:

$$\iint_D f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$

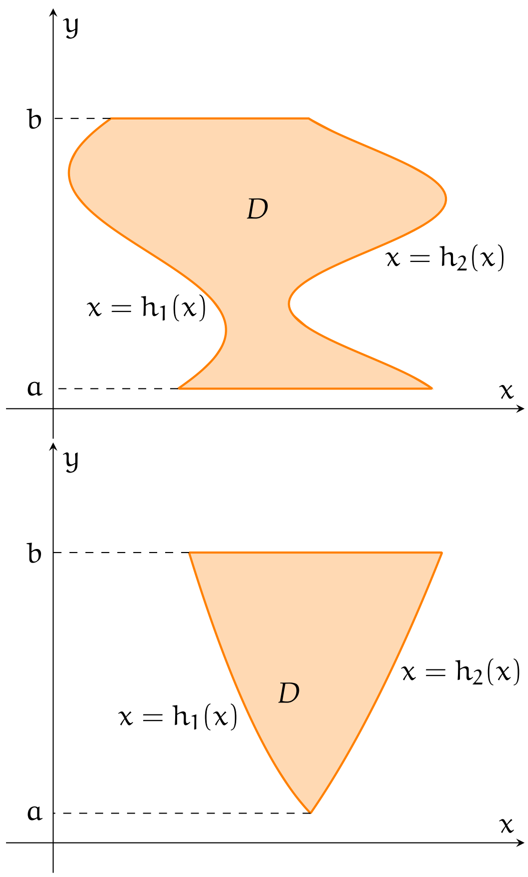


Figure 2.4: Two examples of type II domains

You can annotate type II regions with horizontal lines to show that, for a given y values, all x values in the region are contained in $h_1(y) \leq x \leq h_2(y)$ (see figure 2.5).

2.1 Determining Region Type

Many regions can be described as either type I or type II. Consider the region between the curves $y = \frac{3}{2}(x-1)$ and $y = \frac{1}{2}(x-1)^2$ (see figure 2.6).

We could classify this as a type I (see figure 2.7) or a type II domain (see figure 2.8).

However, not all domains can be classified as both type I or type II. A region can be classified as type I if you can take a vertical line ($x = c$, where c is some number in the domain of the region) and move it across the region without any gaps. Consider the

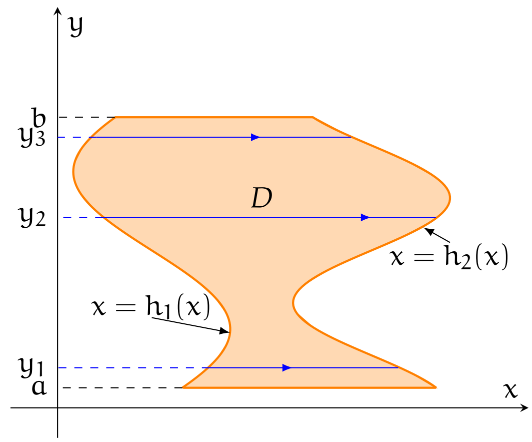


Figure 2.5: On type II domains, for a given value of y, $h_1(y) \leq x \leq h_2(y)$

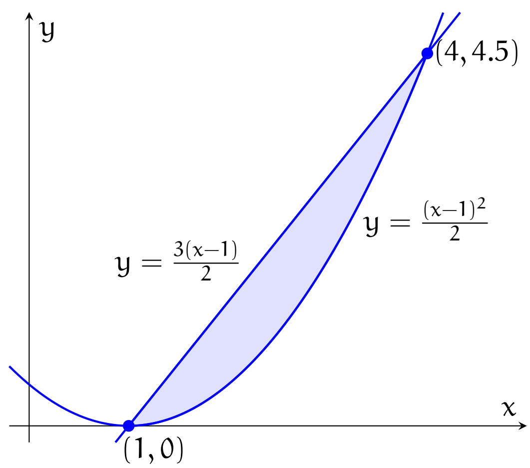


Figure 2.6: The region that lies between $y = \frac{(x-1)^2}{2}$ and $y = \frac{3(x-1)}{2}$ can be classified as type I or type II

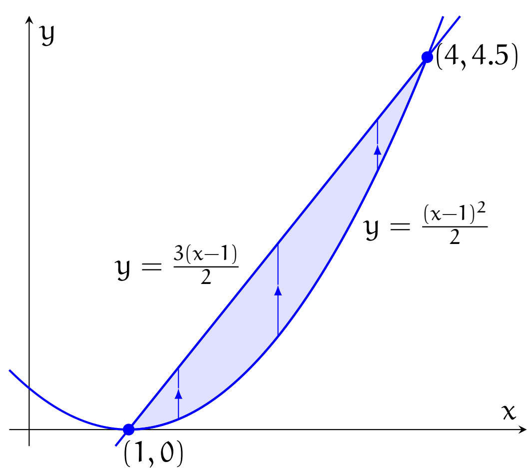


Figure 2.7: The region could be classified as type I, with $\frac{(x-1)^2}{2} \leq y \leq \frac{3(x-1)}{2}$

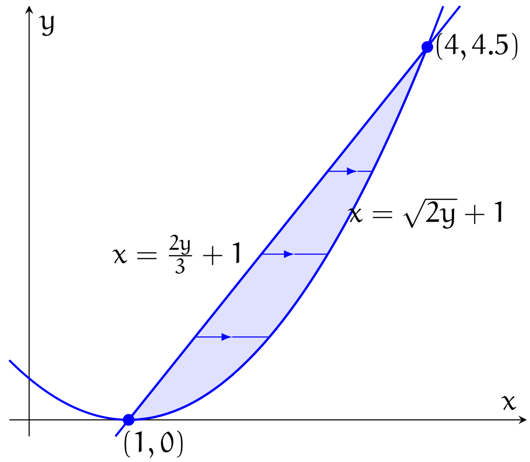


Figure 2.8: The region could be classified as type II, with $\frac{2y}{3} + 1 \leq x \leq \sqrt{2y} + 1$

two regions shown in figure 2.9. The top is type I, and if you put vertical lines, the line traverses the entire region without leaving it. On the other hand, the lower region is not type I. There are vertical lines where the line exits the lower region before traversing the entire region.

To determine if a region can be classified as type II, you can use a horizontal line. If you can move a horizontal line ($y = c$, where c is in the domain of the region) across the region and the line always traverses the entire region without leaving and re-entering it, then the region is type II (see figure 2.10).

Example: Evaluate $\iint_D (2x + y)\, dA$, where D is the region bounded by the parabolas $y = 3x^2$ and $y = 2 + x^2$. Region D is shown in figure 2.11.

Solution:This is a type I region, since for a given x, $y \in \left[3x^2, 2 + x^2\right]$. We can define region D as $D = \{(x, y) \mid -1 \leq x \leq 1, 3x^2 \leq y \leq 2 + x^2\}$. Therefore,

$$\iint_D (2x + y)\, dA = \int_{-1}^{1} \int_{3x^2}^{2+x^2} (2x + y)\ dy\, dx$$

$$= \int_{-1}^{1} \left[\int_{3x^2}^{2+x^2} 2x\, dy + \int_{3x^2}^{2+x^2} y\, dy \right] dx$$

$$= \int_{-1}^{1} \left[2xy|_{y=3x^2}^{y=2+x^2} + \frac{1}{2} y^2|_{y=3x^2}^{y=2+x^2} \right] dx$$

$$= \int_{-1}^{1} \left[2x \left(2 + x^2 - 3x^2\right) + \frac{1}{2}\left((2 + x^2)^2 - (3x^2)^2\right) \right] dx$$

$$= \int_{-1}^{1} \left[2 + 4x + 2x^2 - 4x^3 - 4x^4\right] dx$$

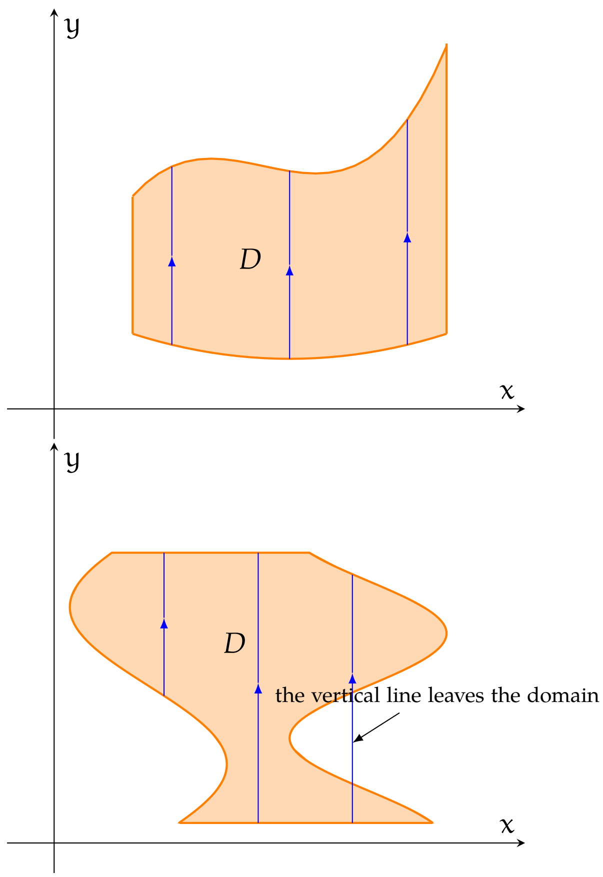


Figure 2.9: The top domain can be classified as type I; the bottom cannot.

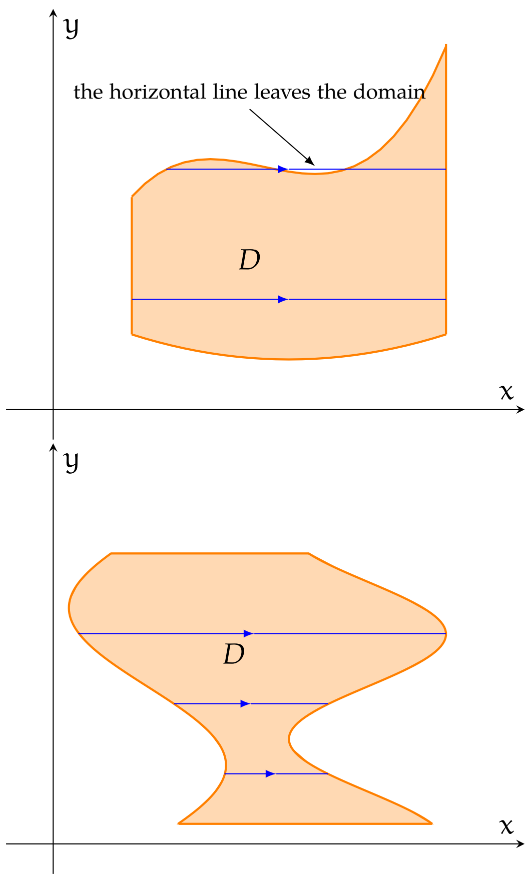


Figure 2.10: The bottom domain can be classified as type II; the top cannot.

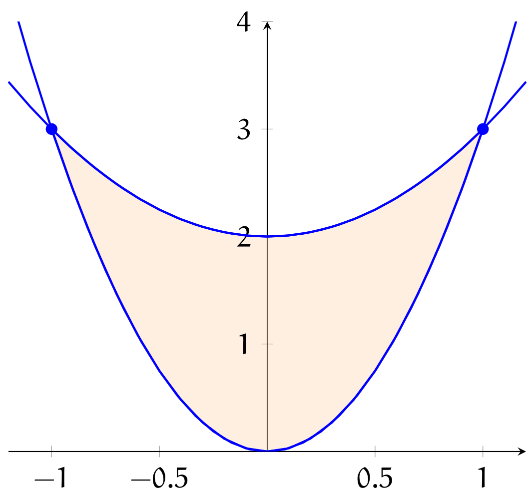


Figure 2.11: Region D is bounded above by $y = 2 + x^2$ and below by $y = 3x^2$

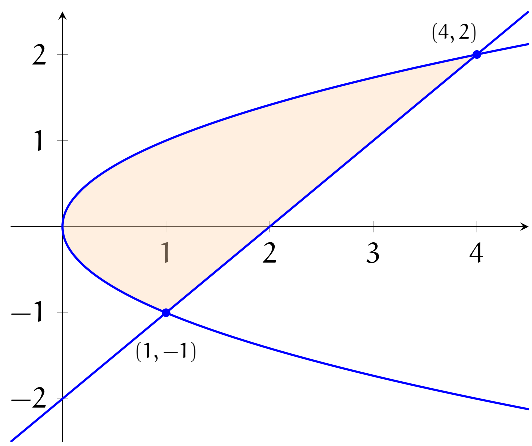
$$= \left[2x + 2x^2 + \frac{2}{3}x^3 - x^4 - \frac{4}{5}x^5\right]_{x=-1}^{x=1}$$

$$= \left(2 + 2 + \frac{2}{3} - 1 - \frac{4}{5}\right) - \left(-2 + 2 - \frac{2}{3} - 1 + \frac{4}{5}\right)$$

$$= 4 + \frac{4}{3} - \frac{8}{5} = \frac{56}{15}$$

Example: Set up integrals to evaluate $\iint_D xy\, dA$ if D is the region bounded by $y = x - 2$ and $x = y^2$ as both a type I and type II region. Which method will be easier to evaluate? Evaluate the easier double integral.

Solution: Let's begin by visualizing D:



This region could be classified as type I and type II. To set up an integral as if D were a type I region, we need to describe y in terms of x. However, we run into a little problem. For $0 \leq x \leq 1$, the region is bounded by the parabola $x = y^2$, and for $1 \leq x \leq 4$, the region is bounded by both the parabola and the line. So, we will have to split the integral into to parts: $0 \leq x \leq 1$ and $1 \leq x \leq 4$:

$$\iint_D xy\, dA = \int_0^1 \int_{-\sqrt{x}}^{\sqrt{x}} xy\, dy\, dx + \int_1^4 \int_{x-2}^{\sqrt{x}} xy\, dy\, dx$$

To set up an integral as if D were a type II region, we need to describe x in terms of y. This time, we will not need to split the integral, because $y + 2 \leq x \leq y^2$ over the entire region:

$$\iint_D xy\, dA = \int_{-1}^{2} \int_{y+2}^{y^2} xy\, dx\, dy$$

It is easier to evaluate the integral with D as a type II region, since we do not need to split the integral. Evaluating:

$$\int_{-1}^{2}\int_{y+2}^{y^2} xy\,dx\,dy = \int_{-1}^{2}\frac{y}{2}\left[x^2\right]_{x=y+2}^{x=y^2}\,dy$$

$$= \int_{-1}^{2}\frac{y}{2}\left[\left(y^2\right)^2-(y+2)^2\right]\,dy = \frac{1}{2}\int_{-1}^{2} y\left[y^4-\left(y^2+4y+4\right)\right]\,dy$$

$$= \frac{1}{2}\int_{-1}^{2} y^5-y^3-4y^2-4y\,dy = \frac{1}{2}\left[\frac{1}{6}y^6-\frac{1}{4}y^4-\frac{4}{3}y^3-\frac{4}{2}y^2\right]_{y=-1}^{y=2}$$

$$= \frac{1}{2}\left[\frac{1}{6}(2)^6-\frac{1}{4}(2)^4-\frac{4}{3}(2)^3-2(2)^2-\left(\frac{1}{6}(-1)^6-\frac{1}{4}(-1)^4-\frac{4}{3}(-1)^3-2(-1)^2\right)\right]$$

$$= \frac{1}{2}\left[\frac{32}{3}-4-\frac{32}{3}-8-\frac{1}{6}+\frac{1}{4}-\frac{4}{3}+2\right] = -\frac{45}{8}$$

Exercise 5 Double Integrals over Non-Rectangular Regions

Evaluate the double integral.

1. $\iint_D e^{-y^2}\, dA$, $D = \{(x, y) \mid 0 \leq y \leq 3, 0 \leq x \leq 2y\}$.
2. $\iint_D x \sin y\, dA$, D is bounded by $y = 0$, $y = x^2$, $x = 2$.
3. $\iint_D (2y - x)\, dA$, D is bounded by the circle with center at the origin and radius 3.

Working Space

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2.2 Double Integrals in Other Coordinate Systems

Consider a region composed of a semi-circular ring (see figure **??**). Describing the region in Cartesian coordinates is complicated; you would have to split it into three regions (see figure ...). However, in polar coordinates, we can describe the whole region in one statement:

$$D = \{(r, \theta) \mid 1 \leq r \leq 4,\ 0 \leq \theta \leq \pi\}$$

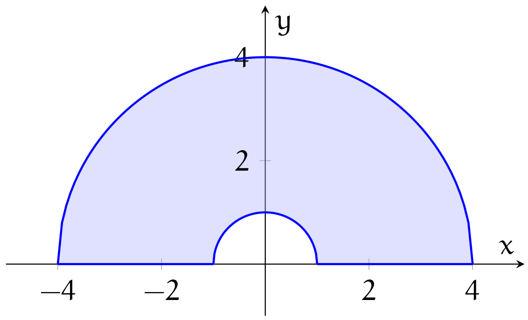


Figure 2.12: A semi-circular ring

There are many instances where a region is simpler to describe in polar coordinates, so how do we take double integrals in polar coordinates? Suppose we want to integrate some function, $f(x, y)$, over a polar rectangle described by $D = \{(r, \theta) \mid a \leq r \leq b,\ \alpha \leq \theta \leq \beta\}$ (see figure 2.13). Similar to Cartesian coordinates, we can divide this region into many smaller polar rectangles, with each subrectangle defined by $D_{ij} = \{(r, \theta) \mid r_{i-1} \leq r \leq r_i,\ \theta_{i-1} \leq \theta \leq \theta_i\}$. The center of each subrectangle has polar coordinates (r_i^*, θ_j^*), where:

$$r_i^* - \frac{1}{2}\left(r_{i-1} + r_i\right)$$

$$\theta_j^* = \frac{1}{2}\left(\theta_{j-1} + \theta_j\right)$$

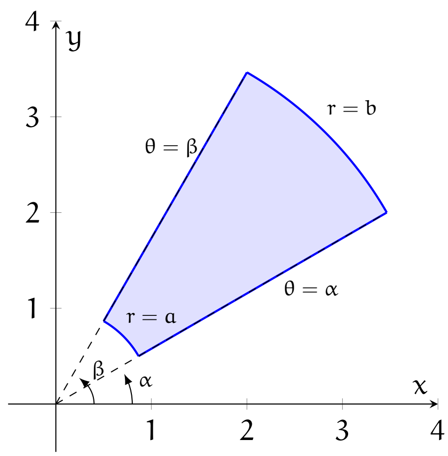


Figure 2.13: A polar rectangle described by $D = \{(r, \theta) \mid a \leq r \leq b,\ \alpha \leq \theta \leq \beta\}$

Each subrectangle is a larger radius sector minus a smaller radius sector, each with the same central angle, $\Delta\theta = \theta_j - \theta_{j-1}$. The total area of each subrectangle is given by:

$$\Delta A_i = \frac{1}{2}\left(r_i\right)^2 \delta\theta - \frac{1}{2}\left(r_{i-1}\right)^2 \Delta\theta = \frac{1}{2}\left(r_i^2 - r_{i-1}^2\right) \Delta\theta$$

Substituting $\left(r_i^2 - r_{i-1}^2\right) = (r_i + r_{i-1})(r_i - r_{i-1})$, we see that:

$$\Delta A_i = \frac{1}{2}(r_i + r_{i-1})(r_i - r_{i-1})\Delta\theta$$

Recall that we have defined $r_i^* = \frac{1}{2}(r_{i-1} + r_i)$. Additionally, $\Delta r = r_i - r_{i-1}$. Substituting this, we find a simplified expression for the area of each subrectangle:

$$\Delta A_i = r_i^* \Delta r \Delta\theta$$

Therefore, the Riemann sum of $f(x, y)$ over the region is:

$$\sum_{i=1}^{n}\sum_{j=1}^{n} f(r_i^* \cos\theta_j^*, r_i^* \sin\theta_j^*)\Delta A_i$$

(Recall that to convert from Cartesian to polar coordinates, we use $x = r\cos\theta$ and $y = r\sin\theta$). Substituting for δA_i:

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} f(r_i^* \cos\theta_j^*, r_i^* \sin\theta_j^*) r_i^* \Delta r \Delta\theta$$

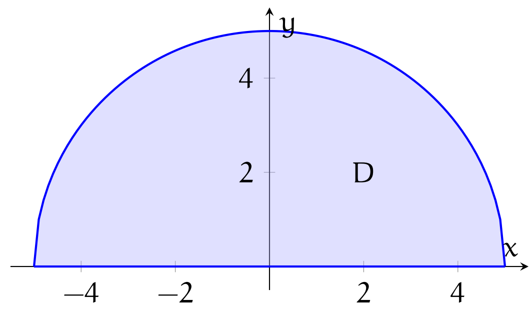
Taking the limit as $n \to \infty$, the Riemann sum becomes the double integral:

$$\int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta) r \, dr \, d\theta$$

Therefore, if f is continuous on the polar rectangle $a \leq r \leq b$, $\alpha \leq \theta \leq \beta$, then:

$$\iint_D f(x, y)\, dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta) r \, dr \, d\theta$$

Example: Evaluate $\iint_D x^2 y \, dA$, where D is the semi-circle shown below.



Solution: Since the region is a semi-circle with radius 5, we can describe D as $D = \{(r, \theta) \mid 0 \le r \le 5,\ 0 \le \theta \le \pi\}$. Therefore,

$$\iint_D x^2 y \, dA = \int_0^{\pi} \int_0^{5} (r\cos\theta)^2 (r\sin\theta)\, r \, d\theta \, dr$$

$$= \int_0^{\pi} \int_0^{5} r^4 \cos^2\theta \sin\theta \, dr \, d\theta$$

$$= \int_0^{\pi} \cos^2\theta \sin\theta \left[\frac{1}{5} r^5\right]_{r=0}^{r=5} d\theta$$

$$= \int_0^{\pi} \cos^2\theta \sin\theta \frac{5^5}{5} \, d\theta = 625 \int_0^{\pi} \cos^2\theta \sin\theta \, d\theta$$

Using u-substitution, let $u = \cos\theta$. Then $-du = \sin\theta d\theta$, therefore:

$$625 \int_0^{\pi} \cos^2\theta \sin\theta \, d\theta = 625 \int_{\theta=0}^{\theta=\pi} -u^2 \, du$$

$$= -625 \frac{1}{3} u^3 |_{\theta=0}^{\theta=\pi} = -625 \frac{1}{3} \left(\cos^3\theta\right) |_{\theta=0}^{\theta=\pi}$$

$$= -\frac{625}{3} \left[(-1)^3 - (1)^3\right] = -\frac{625}{3} (-2) = \frac{1250}{3}$$

Exercise 6 Changing to Polar Coordinates

Evaluate the following iterated integrals by converting to polar coordinates:

1. $\int_0^2 \int_0^{\sqrt{4-x^2}} e^{-x^2-y^2}\, dy\, dx$
2. $\int_0^{1/2} \int_{\sqrt{3}y}^{\sqrt{1-y^2}} xy^2\, dx\, dy$
3. $\int_0^2 \int_0^{\sqrt{2x-x^2}} \sqrt{x^2+y^2}\, dy\, dx$

Working Space

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Exercise 7 Using Polar Coordinates in Multiple Integration

Working Space

Find the volume of the solid that lies under the surface $z = 4 - x^2 - y^2$ and above the xy-plane.

Answer on Page 54

Exercise 8 The volume of a pool

A circular swimming pool has a 40-ft diameter. The depth of the pool is constant along the north-south axis and increases from 3 feet at the west end to 10 feet at the east end. What is the total volume of water in the pool?

Working Space

Answer on Page 55

Chapter 3

Applications of Double Integrals

3.1 Total Mass and Charge

Suppose there is a generic, thin layer (called a *lamina*) with a variable density that occupies an area B (see figure 3.1). Further, let the density of the lamina be described by a function, $\rho(x, y)$, which is continuous over B. For some small rectangle centered at (x, y), the density is given by:

$$\rho(x, y) = \frac{\Delta m}{\Delta A}$$

where Δm is the mass of the small rectangle and ΔA is the area. Then the mass of the rectangle is given by:

$$\Delta m = \rho(x, y)\Delta A$$

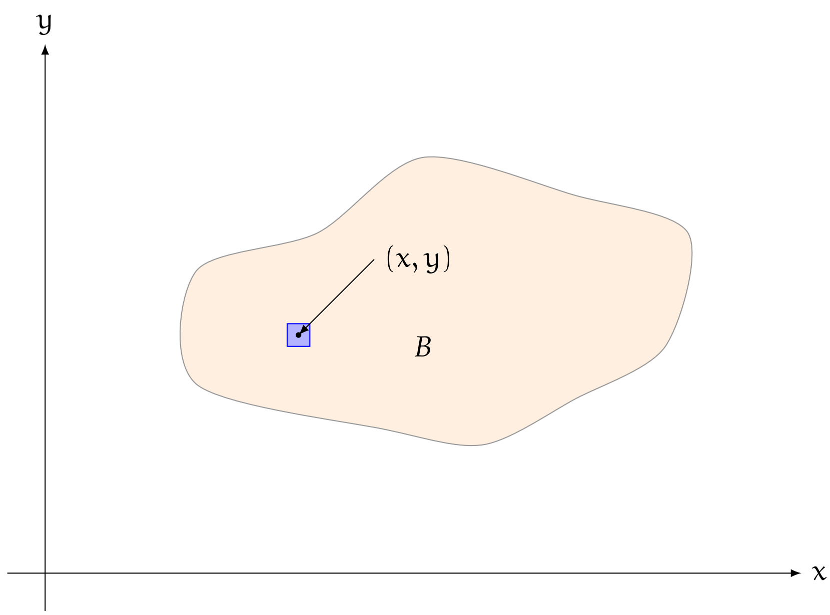


Figure 3.1: A generic lamina that occupies the region B

We can find the mass of the entire lamina by dividing it into many of these small rectangles and adding the masses of all the rectangles (see 3.2). Just like in previous examples, there is some point (x^*_{ij}, y^*_{ij}) in each rectangle, R_{ij}, such that the mass of the part of the lamina that occupies R_{ij} is $\rho(x_{ij}^*, y_{ij}^*)\Delta A$. Adding all these masses yields:

$$m_{total} \approx \sum_{i=1}^{m} \sum_{j=1}^{n} \rho(x_{ij}^*, y_{ij}^*)\Delta A$$

Taking the limit as $m, n \to \infty$ increases the number of rectangles to yield the true total mass:

$$m_{total} = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} \rho(x_{ij}^*, y_{ij}^*)\Delta A = \iint_B \rho(x, y)\, dA$$

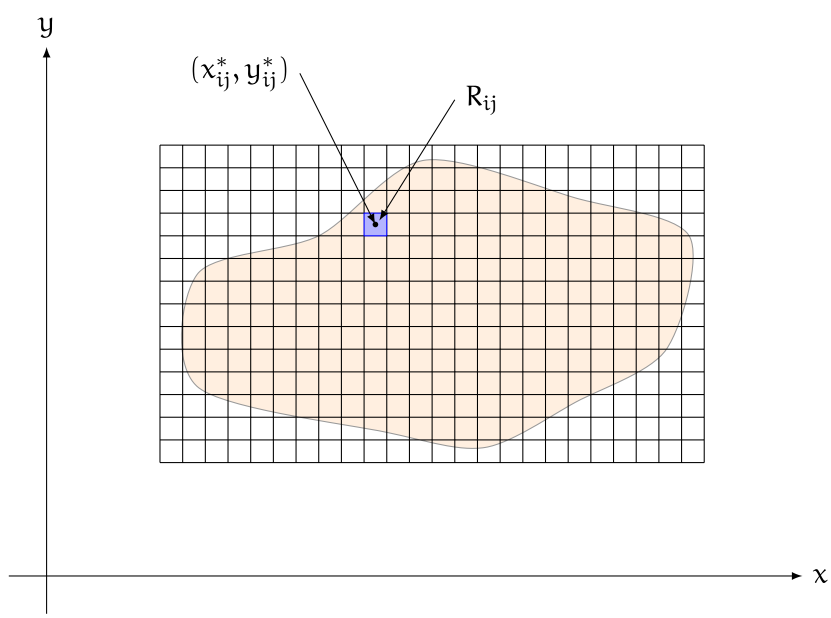


Figure 3.2: A generic lamina divided into many rectangles

Example: Find the total mass of a lamina that occupies the region $D = \{(x, y) \mid 1 \le x \le 3, 1 \le y \le 4\}$ with a density function $\rho(x, y) = 3y^2$.

Solution: We know that the total mass is given by:

$$\iint_D 3y^2\, dA$$

Applying Fubini's theorem, we see that:

$$\iint_D 3y^2\, dA = \int_1^3 \int_1^4 3y^2\, dy\, dx$$

$$= \int_1^3 \left[y^3\right]_{y=1}^{y=4} dx = \int_1^3 \left[4^3 - 1^3\right] dx$$

$$= \int_1^3 63\,dx = 63x\Big|_{x=1}^{x=3} = 126$$

Exercise 9 Finding Total Mass

Find the mass of the lamina that occupies the region, D, and has the given density function, ρ.

1. $D = \{(x, y) \mid 0 \leq x \leq 4, 0 \leq y \leq 3\}$; $\rho(x, y) = 1 + x^2 + y^2$
2. D is the triangular region with vertices $(0, 0)$, $(2, 1)$, $(0, 3)$; $\rho(x, y) = x + y$

Working Space

Answer on Page 56

This method applies not only to mass density, but any other type of density. Some examples could include animals per acre of forest, cells per square centimeter of petri dish, or people per city block. A density physicists are often interested in is charge density (that is, the amount of charge, Q, per unit area). Charge is measured in coulombs (C). Often, charge density is given by a function, $\sigma(x, y)$, in units of coulombs per area (such as cm^2 or m^2). If there is some region, D, with charge distributed across it such that the charge density can be described by a continuous function, $\sigma(x, y)$, then the total charge, Q, is given by:

$$Q = \iint_D \sigma(x, y)\,dA$$

Example: Charge is distributed over the region B shown in figure 3.3 such that the charge density is given by $\sigma(x, y) = xy$, measured in C/m^2. Find the total charge.

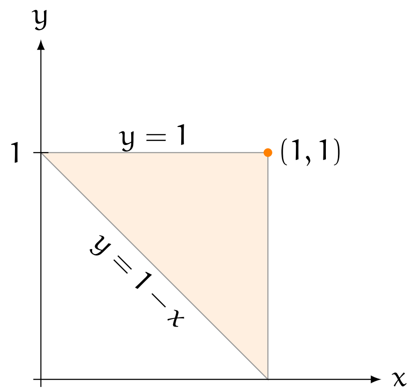


Figure 3.3: A triangular region over which charge is distributed such that $\sigma(x, y) = xy$

Solution: We know that total charge is given by:

$$Q = \iint_B xy \, dA$$

Examining figure 3.3, we see that:

$$\begin{aligned}
\iint_B xy \, dA &= \int_0^1 \int_{1-x}^1 xy \, dy \, dx \\
&= \int_0^1 \frac{x}{2} \left[y^2\right]_{y=1-x}^{y=1} dx = \int_0^1 \frac{x}{2} \left[1^2 - (1-x)^2\right] dx \\
&= \frac{1}{2} \int_0^1 x \left(1 - 1 + 2x - x^2\right) dx = \frac{1}{2} \int_0^1 x \left(2x - x^2\right) dx \\
&= \frac{1}{2} \int_0^1 2x^2 - x^3 \, dx = \frac{1}{2} \left[\frac{2}{3}x^3 - \frac{1}{4}x^4\right]_{x=0}^{x=1} = \frac{1}{2}\left(\frac{2}{3} - \frac{1}{4}\right) = \frac{5}{24} C
\end{aligned}$$

3.2 Center of Mass

For a thin disk (lamina) of variable density in the xy-plane, the coordinates of the center of mass, $(\overline{x}, \overline{y})$, are given by:

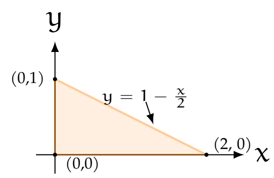
$$\overline{x} = \frac{1}{m} \iint_D x\rho(x, y) \, dA$$

$$\overline{y} = \frac{1}{m} \iint_D y\rho(x, y) \, dA$$

where m is the total mass and ρ is the density of the lamina as a function of x and y.

Example: Find the center of mass of a triangular lamina with vertices at $(0,0)$, $(2,0)$, and $(0,1)$ and a density function $\rho(x,y) = 2 + x + 3y$.

Solution: We begin by visualizing the region so we can determine if it is type I or type II:



Recall that the total mass is given by $m = \iint_D \rho(x,y)\, dA$. As shown above, we can define $D = \{(x,y)\,|0 \le x \le 2, 0 \le y \le 1 - \frac{x}{2}\}$:

$$m = \int_0^2 \int_0^{1-x/2} (2 + x + 3y)\; dy\, dx = \int_0^2 \left[2y + xy + \frac{3}{2}y^2\right]_{y=0}^{y=1-x/2} dx$$

$$= \int_0^2 \left[2(1 - \frac{x}{2}) + x(1 - \frac{x}{2}) + \frac{3}{2}(1 - \frac{x}{2})^2\right] dx = \int_0^2 \left[\frac{7}{2} - \frac{3x}{2} - \frac{x^2}{8}\right] dx$$

$$= \left[\frac{7x}{2} - \frac{3x^2}{4} - \frac{x^3}{24}\right]_{x=0}^{x=2} = \frac{7(2)}{2} - \frac{3(4)}{4} - \frac{8}{24} = 7 - 3 - \frac{1}{3} = \frac{11}{3}$$

Finding $\overline{x}$:

$$\overline{x} = \frac{1}{m} \iint_D x\,(2 + x + 3y)\; dA$$

$$\overline{x} = \frac{3}{11} \int_0^2 \int_0^{1-x/2} \left[2x + x^2 + 3xy\right] dy\, dx$$

$$\overline{x} = \frac{3}{11} \int_0^2 \left[2xy + x^2y + \frac{3}{2}xy^2\right]_{y=0}^{y=1-x/2} dx$$

$$\overline{x} = \frac{3}{11} \int_0^2 \left[\frac{7x}{2} - \frac{3x^2}{2} - \frac{x^3}{8}\right] dx$$

$$\overline{x} = \frac{3}{11} \left[\frac{7x^2}{4} - \frac{x^3}{2} - \frac{x^4}{32}\right]_{x=0}^{x=2}$$

$$\overline{x} = \frac{3}{11} \left[\frac{7(4)}{4} - \frac{8}{2} - \frac{16}{32}\right] = \frac{3}{11}\left(7 - 4 - \frac{1}{2}\right) = \frac{3}{11}\left(\frac{5}{2}\right) = \frac{15}{22}$$

We can similarly find $\overline{y}$:

$$\overline{y} = \frac{1}{m} \iint_D y\,(2 + x + 3y)\; dA$$

$$\overline{y} = \frac{3}{11} \int_0^2 \int_0^{1-x/2} \left[2y + xy + 3y^2\right] dy\, dx$$

$$\overline{y} = \frac{3}{11}\int_0^2 \left[y^2 + \frac{x}{2}y^2 + y^3\right]_{y=0}^{y=1-x/2} dx$$

$$\overline{y} = \frac{3}{11}\int_0^2 \left[\left(1-\frac{x}{2}\right)^2 + \frac{x}{2}\left(1-\frac{x}{2}\right)^2 + \left(1-\frac{x}{2}\right)^3\right] dx$$

$$\overline{y} = \frac{3}{11}\int_0^2 \left[2 - 2x + \frac{x^2}{2}\right] dx = \frac{3}{11}\left[2x - x^2 + \frac{x^3}{6}\right]_{x=0}^{x=2}$$

$$\overline{y} = \frac{3}{11}\left[2(2) - 2(2) + \frac{8}{6}\right] = \frac{3}{11}\left(\frac{4}{3}\right) = \frac{4}{11}$$

Therefore, the center of mass $(\overline{x}, \overline{y})$ is $(\frac{15}{22}, \frac{4}{11})$.

Exercise 10 Center of Mass

Find the center of mass of:

1. A lamina that occupies the area enclosed by the curves $y = 0$ and $y = 2\sin x$ from $0 \leq x \leq \pi$ if its density is given by $\rho(x, y) = x$.
2. The region D if $D = \{(x, y) \mid 0 \leq x \leq 4, 0 \leq y \leq 3\}$; $\rho(x, y) = 1 + x^2 + y^2$
3. The triangular region D with vertices $(0, 0)$, $(2, 1)$, $(0, 3)$; $\rho(x, y) = x + y$

Working Space

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3.3 Moment of Inertia

We can also use double integrals to find the **moment of inertia** of a lamina about a particular axis (we will extend this to three-dimensional objects in the next chapter on triple integrals). Recall that the moment of inertia for a particle with mass m a distance r from the axis of rotation is mr^2. Dividing a lamina into small pieces, we see that the moment of inertia of each piece about the x-axis is:

$$\left(y_{ij}^*\right)^2 \rho\left(x_{ij}^*, y_{ij}^*\right) \Delta A$$

Where x_{ij}^* and y_{ij}^* are the x- and y-coordinates of the small piece. The moment of inertia of the entire lamina about the x-axis is then the sum of all the individual moments:

$$I_x = \sum_{i=1}^{n} \sum_{j=1}^{n} \left(y_{ij}^*\right)^2 \rho\left(x_{ij}^*, y_{ij}^*\right) \Delta A = \iint_D y^2 \rho(x, y)\ dA$$

Similarly, the moment of inertia of a lamina about the y-axis is:

$$I_y = \sum_{i=1}^{n} \sum_{j=1}^{n} \left(x_{ij}^*\right)^2 \rho\left(x_{ij}^*, y_{ij}^*\right) \Delta A = \iint_D x^2 \rho(x, y)\ dA$$

Example: Find the moment of inertia of a square centered at the origin with side length r and constant density ρ about the x-axis.

Solution: We can describe the square as the region bounded by $D = \{(x, y)| - r/2 \leq x \leq r/2, -r/2 \leq y \leq r/2\}$ with density function $\rho(x, y) = \rho$. Therefore, the moment of inertia about the x-axis is given by:

$$\begin{aligned}
I_x &= \int_{-r/2}^{r/2} \int_{-r/2}^{r/2} \rho y^2\, dy\, dx \\
&= \rho \int_{-r/2}^{r/2} \left[\frac{1}{3} y^3\right]_{y=-r/2}^{y=r/2} dx \\
&= \frac{\rho}{3} \int_{-r/2}^{r/2} \left[\frac{r^3}{8} - \left(-\frac{r^3}{8}\right)\right] dx \\
&= \frac{\rho}{3} \int_{-r/2}^{r/2} \left[\frac{r^3}{4}\right] dx = \frac{r^3 \rho}{12} \int_{-r/2}^{r/2} 1\, dx \\
&= \frac{r^3 \rho}{12} [x]_{x=-r/2}^{x=r/2} = \frac{r^3 \rho}{12} \cdot r = \frac{r^4 \rho}{12}
\end{aligned}$$

We can also find the moment of inertia about the origin, I_o. This is the moment of inertia for an object rotating in the xy-plane about the origin. The moment of inertia about the origin is the sum of the moments of inertia about the x- and y-axes:

$$I_o = I_x + I_y = \iint_D \left(x^2 + y^2\right) \rho(x, y)\, dA$$

Example: Find the moment of inertia about the origin of a disk with density $\rho(x, y) = b$, centered at the origin, with a radius of a. Show that this is equal to the expected moment of inertia, $\frac{1}{2}MR^2$, where M is the total mass of the disk and R is the radius of the disk.

Solution: Since we are examining a circle about the origin, the region can be described in polar coordinates as $D = \{(r, \theta) \mid 0 \le r \le a,\ 0 \le \theta \le 2\pi\}$. Converting from Cartesian coordinates to polar coordinates:

$$\begin{aligned} I_o &= \iint_D \left(x^2 + y^2\right) b\, dA = \int_0^a \int_0^{2\pi} r \left(r^2\right) b\, d\theta\, dr \\ &= \int_0^a r^3\, dr \cdot \int_0^{2\pi} b\, d\theta = \frac{a^4}{4} \cdot (2\pi b) = \frac{\pi a^4 b}{2} \end{aligned}$$

The total mass of this disk is the density, b, multiplied by the area, πa^2. Therefore,

$$R = a$$

$$M = \pi a^2 b$$

Substituting into the result of our double integral, we see that:

$$\frac{\pi a^4 b}{2} = \left(\pi a^2 b\right) \cdot \left(\frac{a^2}{2}\right) = M \cdot \frac{R^2}{2} = \frac{1}{2} M R^2$$

3.3.1 Radius of Gyration

When modeling rotating objects, it can be helpful to have a simplified model. A spinning, continuous object can be modeled as a point mass by using the lamina's *radius of gyration*. The radius of gyration of a lamina about the origin is a radius, R, such that:

$$mR^2 = I_o$$

where m is the mass of the lamina and I is the moment of inertia of the lamina. Essentially, we are finding a radius such that if the lamina were shrunk down to a point mass and rotated about the axis at that radius, the moment of inertia would be the same.

We can also find radii of gyration about the x- and y-axes:

$$m\bar{\bar{y}}^2 = I_x$$

$$m\bar{\bar{x}}^2 = I_y$$

About the origin, $R = \sqrt{\bar{\bar{x}}^2 + \bar{\bar{y}}^2}$.

Example: Find the radius of gyration about the y-axis for a disk with density $\rho(x, y) = y$ if the disk has radius 2 and is centered at $(0, 2)$.

Solution: We are ultimately looking for a radius such that $m\bar{\bar{x}}^2 = I_y$, so we need to know the mass, m, and the moment of inertia about the y-axis, I_y. First, let's find the total mass, m, of the disk. We can describe the disk in polar coordinates as $D = \{(r, \theta) \mid 0 \leq r \leq 4\sin\theta, 0 \leq \theta \leq \pi\}$, and therefore the mass is given by:

$$m = \iint_D y\, dA = \int_0^\pi \int_0^{4\sin\theta} r\,(r\sin\theta)\ dr\, d\theta$$

$$= \int_0^\pi \sin\theta \int_0^{4\sin\theta} r^2\, dr\, d\theta = \frac{1}{3}\int_0^\pi \sin\theta\,[4\sin\theta]^3\ d\theta$$

$$= \frac{64}{3}\int_0^\pi \sin^4\theta\, d\theta = \frac{64}{3}\int_0^\pi \left(\frac{1-\cos 2\theta}{2}\right)^2 d\theta = \frac{64}{3}\left(\frac{1}{2}\right)^2 \int_0^\pi \left(1 - 2\cos 2\theta + \cos^2 2\theta\right) d\theta$$

$$= \frac{16}{3}\left[(\theta - \sin 2\theta)_{\theta=0}^{\theta=\pi} + \int_0^\pi \frac{1+\cos 4\theta}{2}\, d\theta\right] = \frac{16}{3}\left[\pi + \frac{1}{2}\left(\theta + \frac{1}{4}\sin 4\theta\right)_{\theta=0}^{\theta=\pi}\right]$$

$$= \frac{16}{3}\left[\pi + \frac{\pi}{2}\right] = \frac{16}{3}\left(\frac{3\pi}{2}\right) = 8\pi$$

Now that we have found the mass, let's find the moment of inertia, I_y:

$$I_y = \iint_D x^2 y\, dA = \int_0^\pi \int_0^{4\sin\theta} r\,(r\cos\theta)^2\,(r\sin\theta)\ dr\, d\theta$$

$$= \int_0^\pi \left[\cos^2\theta\sin\theta \int_0^{4\sin\theta} r^4\, dr\right] d\theta = \int_0^\pi \cos^2\theta\sin\theta\left[\frac{1}{5}r^5\right]_{\theta=0}^{\theta=4\sin\theta} d\theta$$

$$= \frac{1024}{5}\int_0^\pi \cos^2\theta\sin^6\theta\, d\theta = \frac{1024}{5}\int_0^\pi \left(\frac{1+\cos 2\theta}{2}\right)\left(\frac{1-\cos 2\theta}{2}\right)^3 d\theta$$

$$= \frac{1024}{5}\left(\frac{1}{2}\right)^4 \int_0^\pi (1+\cos 2\theta)(1-\cos 2\theta)(1-\cos 2\theta)^2\ d\theta$$

$$= \frac{64}{5}\int_0^\pi \left(1 - \cos^2 2\theta\right)\left(1 - 2\cos 2\theta + \cos^2 2\theta\right) d\theta$$

$$
\begin{aligned}
&= \frac{64}{5}\int_0^{\pi} 1 - 2\cos 2\theta + \cos^2 2\theta - \cos^2 2\theta + 2\cos^3 2\theta - \cos^4 2\theta \, d\theta \\
&= \frac{64}{5}\int_0^{\pi} 1 - 2\cos 2\theta + 2\cos 2\theta\left(1 - \sin^2 2\theta\right) - \left(\frac{1+\cos 4\theta}{2}\right)^2 \, d\theta \\
&= \frac{64}{5}\int_0^{\pi} 1 + 2\cos 2\theta \sin^2 2\theta - \frac{1}{4}\left(1 + 2\cos 4\theta + \cos^2 4\theta\right) \, d\theta \\
&= \frac{64}{5}\int_0^{\pi} 1 + 2\cos 2\theta \sin^2 2\theta - \frac{1}{4} - \frac{\cos 4\theta}{2} - \frac{1}{4}\left(\frac{1+\cos 8\theta}{2}\right) \, d\theta \\
&= \frac{64}{5}\left[\frac{5\theta}{8} + \frac{1}{3}\sin^3\theta - \frac{\sin 4\theta}{8} - \frac{\sin 8\theta}{64}\right]_{\theta=0}^{\theta=\pi} = \frac{64}{5}\cdot\frac{5\pi}{8} = 8\pi
\end{aligned}
$$

We have found that $m = 8\pi$ and $I_y = 8\pi$. Substituting to find the radius of gyration:

$$
\begin{aligned}
m\bar{\bar{x}}^2 &= I_y \\
(8\pi)\bar{\bar{x}}^2 &= 8\pi \\
\bar{\bar{x}} &= 1
\end{aligned}
$$

Therefore, the radius of gyration about the y-axis is $\bar{\bar{x}} = 1$

Exercise 11 Moments of Inertia and Radii of Gyration

Find the requested moment of inertia and radius of gyration of the lamina with the given density function.

1. about the x-axis, $D = \{(x, y) \mid 1 \leq x \leq 4,\ 0 \leq y \leq 3\}$, $\rho(x, y) = xy$.
2. about the y-axis, D is enclosed by the curves $y = 0$ and $y = 2\cos x$ for $-\pi/2 \leq x \leq \pi/2$, $\rho(x, y) = x$.
3. about the origin, $D = \{(r, \theta) \mid 1 \leq r \leq 2,\ 0 \leq \theta \leq \pi\}$, $\rho(r, \theta) = r$.

Working Space

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3.4 Surface Area

We have already seen how to find the areas of surfaces of revolution using single-variable calculus. Now, we will use multivariable calculus to find the surface area of a generic, two-variable function, $z = f(x, y)$. Suppose a surface, S, is defined by the continuous, partially differentiable function, $f(x, y)$, over a rectangular region, R (see figure 3.4).

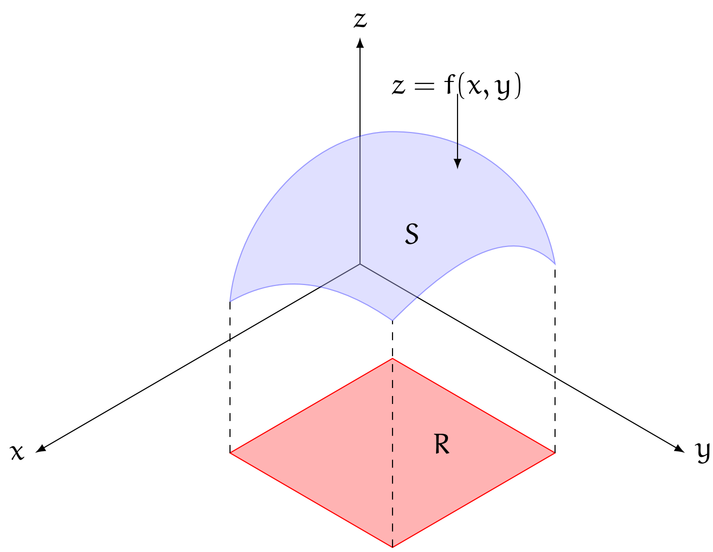


Figure 3.4: The graph of f over the region R creates a surface, S

We begin by dividing the region, R, into sub-rectangles, R_{ij}, each with area $\Delta A = \Delta x \Delta y$. Then, projecting upwards from the point closest to the origin, $(x_i, y_j, 0)$, we find a point on the surface, $P_{ij} = (x_i, y_j, f(x_i, y_j))$. Next, there is a small plane, ΔT_{ij}, tangent to the surface at P_{ij}, and the area of the tangent plane is approximately the same as the area of the surface over the sub-rectangle R_{ij} (see figure 3.5).

It follows that the total surface area of the surface, S, is the sum of all these little tangent surfaces as the number of tangent surfaces approaches infinity:

$$A(S) = \lim_{m,n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} \Delta T_{ij}$$

How can we find an expression for ΔT_{ij}? We will define two vectors, $\mathbf{a}$ and $\mathbf{b}$, that are equal to the sides of ΔT_{ij} (see figure 3.6). Geometrically, the area of ΔT_{ij} is the absolute value of the cross product of the two vectors. Mathematically,

$$\Delta T_{ij} = |\mathbf{a} \times \mathbf{b}|$$

Recall the three unit vectors: $\mathbf{i}$ in the x-direction, $\mathbf{j}$ in the y-direction, and $\mathbf{k}$ in the z-

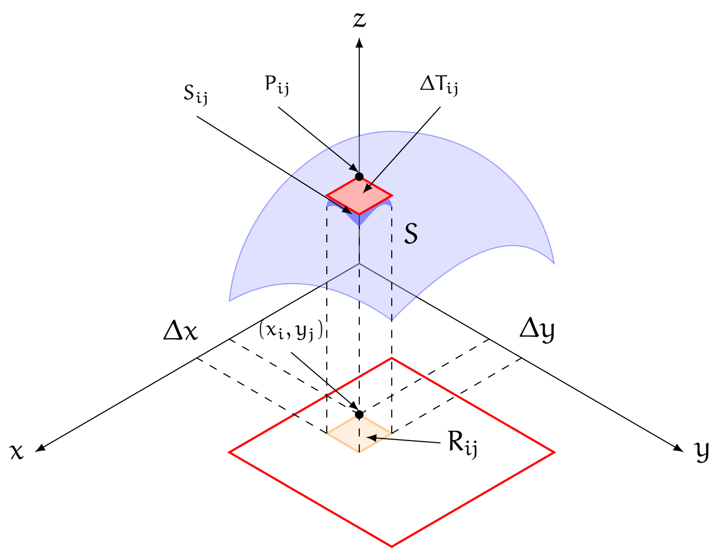


Figure 3.5: The tangent surface, ΔT_{ij}, is approximately the same surface area as the surface, S_{ij}, over the sub-rectangle, R_{ij}

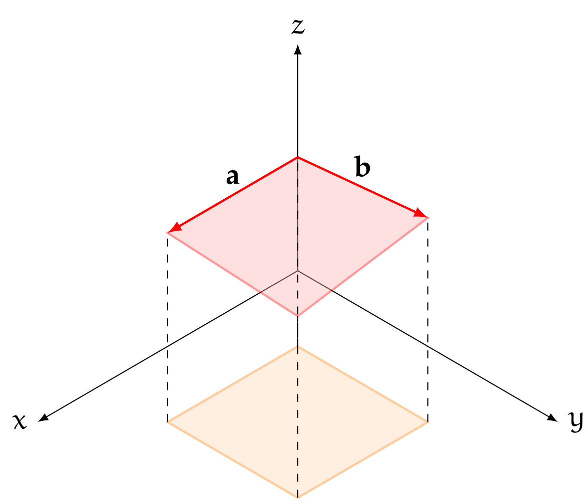


Figure 3.6: The vectors **a** and **b** define the sides of the tangent surface ΔT_{ij}

direction. We can then describe **a** and **b** in terms of **i**, **j**, and **k**:

$$\mathbf{a} = \Delta x\mathbf{i} + f_x(x_i, y_j)\Delta x\mathbf{k}$$

$$\mathbf{b} = \Delta y\mathbf{j} + f_y(x_i, y_j)\Delta y\mathbf{k}$$

(Recall that f_x is the partial derivative of $f(x, y)$ with respect to x, and f_y is the partial derivative with respect to y.) This is true because the partial derivative of f_x gives the slope of a tangent line parallel to the x-axis, and f_y parallel to the y-axis. We then find an expression for $|\mathbf{a} \times \mathbf{b}|$ (we've omitted some details here):

$$\mathbf{a} \times \mathbf{b} = -f_x(x_i, y_j)\Delta x\Delta y\mathbf{i} - f_y(x_i, y_j)\Delta x\Delta y\mathbf{j} + \Delta x\Delta y\mathbf{k}$$

Substituting $\Delta A = \Delta x\Delta y$:

$$\mathbf{a} \times \mathbf{b} = [-f_x(x_i, y_j)\mathbf{i} - f_y(x_i, y_j)\mathbf{j} + \mathbf{k}]\,\Delta A$$

To find the area of ΔT_{ij}, we need to find the length of $\mathbf{a} \times \mathbf{b}$. Recall that we can use the Pythagorean theorem to find the length of a vector. For a 3-dimensional vector $\mathbf{v} = r\mathbf{i} + s\mathbf{j} + t\mathbf{k}$, it's length is given by:

$$|\mathbf{v}| = \sqrt{r^2 + s^2 + t^2}$$

Applying this, we find the length of $\mathbf{a} \times \mathbf{b}$ (which is the same as the area of ΔT_{ij}) is:

$$\Delta T_{ij} = |\,[-f_x(x_i, y_j)\mathbf{i} - f_y(x_i, y_j)\mathbf{j} + \mathbf{k}]\,\Delta A|$$

$$= \sqrt{(-f_x(x_i, y_j)\Delta A)^2 + (-f_y(x_i, y_j)\Delta A)^2 + (\Delta A)^2}$$

$$= \sqrt{[f_x(x, y)]^2 + [f_y(x, y)]^2 + 1}\Delta A$$

So, the area of the entire surface over region R is:

$$A(S) = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} \sqrt{[f_x(x, y)]^2 + [f_y(x, y)]^2 + 1}\Delta A$$

This is the definition of a double integral; therefore, the surface area of a two-variable function, $f(x, y)$ over a region, R, where f_x and f_y are continuous, is:

$$A(S) = \iint_R \sqrt{[f_x(x, y)]^2 + [f_y(x, y)]^2 + 1}\, dA$$

Using the notation of partial derivatives, this is also expressed as:

$$A(S) = \iint_R \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA$$

Example: Find the surface area of the part of the surface $z = 2 - y^2$ that lies over the triangle whose vertices are at $(0,0)$, $(0,4)$, and $(3,4)$.

Solution: We can define $R = \{(x,y) \mid 0 \leq x \leq 34y,\ 0 \leq y \leq 4\}$. Additionally,

$$\frac{\partial z}{\partial x} = 0$$

$$\frac{\partial z}{\partial y} = -2y$$

Therefore, the area of the surface that lies above R is:

$$A(S) = \iint_R \sqrt{1 + 0^2 + (-2y)^2}\, dA = \int_0^4 \int_0^{\frac{3}{4}y} \sqrt{1 + 4y^2}\, dx\, dy$$

$$= \int_0^4 \sqrt{1 + 4y^2}\, [x]_{x=0}^{x=\frac{3}{4}y}\ dy = \frac{3}{4} \int_0^4 y\sqrt{1 + 4y^2}\, dy$$

Let $u = 1 + 4y^2$, then $du = (8y)dy$ and $(y)dy = \frac{du}{8}$. Substituting:

$$A(S) = \frac{3}{4} \int_{y=0}^{y=4} \frac{1}{8}\sqrt{u}\, du = \frac{3}{32} \left[\frac{2}{3}u^{3/2}\right]_{y=0}^{y=4}$$

$$= \frac{1}{16} \left[\left(1 + 4y^2\right)^{3/2}\right]_{y=0}^{y=4} = \frac{1}{16} \left[(65)^{3/2} - 1\right] \approx 32.69$$

Exercise 12 Surface Area of Two-Variable Functions

Find the area of the surface.

1. The part of the plane $9x+6y-3z+6=0$ that lies above the rectangle $[2,6]\times[1,4]$.
2. The part of the paraboloid in the circle $z=2x^2+2y^2$ that lies under the plane $z=32$.
3. The part of the surface $z=3xy$ that lies in the cylinder $x^2+y^2=4$.

Working Space

Answer on Page 60

3.5 Average Value

Recall that the average value of a one-variable function over the interval $x\in[a,b]$ is given by:

$$f_{ave}=\frac{1}{a-b}\int_a^b f(x)\,dx$$

For a two-variable function, the average value over a region, R, is given by:

$$f_{ave}=\frac{1}{A(R)}\iint_R f(x,y)\,dA$$

Where $A(R)$ is the area of the two-dimensional region.

Example: Find the average value of $f(x,y)=xy^2$ over the rectangle with vertices at $(-2,0)$, $(-2,4)$, $(2,4)$, and $(2,0)$.

Solution: The rectangular region has an area of $(2-(-2))\cdot(4-0)=4\cdot4=16$. Therefore, the average value is given by:

$$f_{ave}=\frac{1}{16}\iint_R xy^2\,dA=\frac{1}{16}\int_{-2}^{2}\int_0^4 xy^2\,dy\,dx$$

$$= \frac{1}{16}\int_{-2}^{2}\frac{x}{3}y^3|_{y=0}^{y=4}\,dx = \frac{1}{16}\int_{-2}^{2}\frac{x}{3}\left(4^3\right)\,dx = \frac{4}{3}\int_{-2}^{2}x\,dx$$
$$= \frac{4}{3}\left(\frac{1}{2}\right)x^2|_{x=-2}^{x=2} = 0$$

Exercise 13 Average Value

Find the average value of the function over the region D:

1. $f(x,y) = x\sin y$, $D = [0,2]\times[-\pi/2,\pi/2]$
2. $f(x,y) = x+y$, D is the circle with radius 1 centered at $(1,0)$
3. $f(x,y) = xy$, D is the triangle with vertices at $(0,0)$, $(2,0)$, $(2,2)$

Working Space

Answer on Page 62

Chapter 4

Multivariate Distributions

The world of probability and statistics doesn't limit itself to the study of single variables. Often, we are interested in the interconnections, relationships, and associations among several variables. In such a scenario, the univariate distributions that we have studied so far become inadequate. To comprehend the joint behavior of these variables and to uncover the underlying patterns of dependency, we must turn to the realm of multivariate distributions.

This chapter aims to introduce the reader to the concept of multivariate probability distributions. These are probability distributions that take into account and describe the behavior of more than one random variable. We will start our exploration with a discussion on the joint probability mass and density functions. These functions extend the concepts of probability mass and density functions for one variable to the situation where we have multiple variables.

Next, we will explore important properties of joint distributions, including the concept of marginal distribution and conditional distribution, which allow us to explore the probability of a subset of variables while conditioning on, or ignoring, other variables. We will also introduce the idea of independence of random variables in the multivariate context.

Subsequently, we will discuss some commonly used multivariate distributions such as the multivariate normal distribution, and the multivariate Bernoulli and binomial distributions. These specific distributions will provide us with practical tools for modelling multivariate data.

Finally, we will delve into covariance and correlation, two key measures that give us a sense of how two variables change together. Understanding these measures is critical for capturing the relationships in multivariate data.

Appendix A

Answers to Exercises

Answer to Exercise ?? (on page 7)

We have already shown that $\int_0^3 \int_1^2 xy^2\,dy\,dx = \frac{21}{2}$. We will evaluate $\int_1^2 \int_0^3 xy^2\,dx\,dy$ and see if we get the same result.

$$\int_0^3 xy^2\,dx = y^2 \int_0^3 x\,dx = y^2 \left[\frac{1}{2}x^2\right]_{x=0}^{x=3}$$

$$= \frac{y^2}{2}\left[3^2 - 0^2\right] = \frac{9y^2}{2}$$

Substituting this back into the iterated integral:

$$\int_1^2 \int_0^3 xy^2\,dx\,dy = \int_1^2 \frac{9y^2}{2}\,dy = \frac{9}{2}\int_1^2 y^2\,dy$$

$$= \frac{9}{2}\left[\frac{1}{3}y^3\right]_{y=1}^{y=2} = \frac{9}{2}\cdot\frac{1}{3}\left[2^3 - 1^3\right]$$

$$= \frac{3}{2}(8-1) = \frac{21}{2}$$

Answer to Exercise 2 (on page 8)

1. Answer: $\frac{5}{2}-\frac{1}{e}$. Solution: $\int_0^1 \int_1^2 \left(x + e^{-y}\right)\,dx\,dy = \int_0^1 \left(\frac{1}{2}x^2 + xe^{-y}\right)\big|_{x=1}^{x=2}\,dy = \int_0^1 \left(2 - \frac{1}{2} + 2e^{-y} - e^{-y}\right)\,d$ $\int_0^1 \left(\frac{3}{2} + e^{-y}\right)\,dy = \left[\frac{3}{2}y - e^{-y}\right]_{y=0}^{y=1} = \left(\frac{3}{2}(1) - e^{-1}\right) - \left(\frac{3}{2}(0) - e^0\right) = \frac{5}{2} - \frac{1}{e}$

2. Answer: 18. Solution: $\int_{-3}^3 \int_0^{\pi/2} \left(2y + y^2\cos x\right)\,dx\,dy = \int_{-3}^3 \left[2xy + y^2\sin x\right]_{x=0}^{x=\pi/2}\,dy =$ $\int_{-3}^3 \left[\left(\pi y + y^2\right) - (0+0)\right]\,dy = \int_{-3}^3 \left(\pi y + y^2\right)\,dy = \left[\frac{\pi}{2}y^2 + \frac{1}{3}y^3\right]_{y=-3}^{y=3} = \left(\frac{\pi}{2}(9) + \frac{1}{3}(27)\right) -$ $\left(\frac{\pi}{2}(9) + \frac{1}{3}(-27)\right) = 9 - (-9) = 18$

3. Answer: 6. Solution: $\int_0^3 \int_0^{\pi/2} t^2 \sin^3\theta\,d\theta\,dt = \left(\int_0^3 t^2\,dt\right)\times\left(\int_0^{\pi/2}\sin^3\theta\,d\theta\right) = \left[\frac{1}{3}t^3\right]_{t=0}^{t=3}\times$ $\left(\int_0^{\pi/2}\sin\theta\sin^2\theta\,d\theta\right) = 9\int_0^{\pi/2}\sin\theta\left(1-\cos^2\theta\right)\,d\theta = 9\left[\int_0^{\pi/2}\sin\theta\,d\theta - \int_0^{\pi/2}\sin\theta\cos^2\theta\,d\theta\right] =$ $9\left[\left(-\cos\theta\right)\big|_{\theta=0}^{\theta=\pi/2} + \left(\frac{1}{3}\cos^3\theta\right)\big|_{\theta=0}^{\theta=\pi/2}\right] = 9\left[-(-\cos 0) + (-\frac{1}{3}\cos^3 0)\right] = 9\left(1 - \frac{1}{3}\right) =$

$9\left(\frac{2}{3}\right) = 6$

Answer to Exercise 3 (on page 9)

1. $\int_0^1 \int_{-3}^3 \frac{xy^2}{x^2+1}\, dy\, dx$ OR $\int_{-3}^3 \int_0^1 \frac{xy^2}{x^2+1}\, dx\, dy$
2. $\int_0^{\pi/4} \int_0^1 \frac{\sec\theta}{\sqrt{1+t^2}}\, dt\, d\theta$ OR $\int_0^1 \int_0^{\pi/4} \frac{\sec\theta}{\sqrt{1+t^2}}\, d\theta\, dt$

Answer to Exercise 4 (on page 10)

1. $\iint_R \frac{xy^2}{x^2+1}\, dA$, $R = \{(x,y)\,|0 \le x \le 2, -3 \le y \le 3\} = \int_0^2 \int_{-3}^3 \frac{xy^2}{x^2+1}\, dy\, dx = \int_0^2 \frac{x}{x^2+1}\, dx \cdot \int_{-3}^3 y^2\, dy$. To evaluate the integral with respect to x, we use the u-substitution $u = x^2+1$, $(x)dx = \frac{1}{2}du$: $\int_0^2 \frac{x}{x^2+1}\, dx \cdot \int_{-3}^3 y^2\, dy = \int_{x=0}^{x=2} \frac{1}{2}\frac{1}{u}\, du \cdot \int_{-3}^3 y^2\, dy = \frac{1}{2}\ln|u||_{x=0}^{x=2} \cdot \frac{1}{3}\left[y^3\right]_{y=-3}^{y=3} = \frac{1}{2}\left[\ln\left(2^2+1\right) - \ln\left(0^2+1\right)\right] \cdot \frac{1}{3}\left[3^3 - (-3)^3\right] = \frac{1}{2}\ln 5 \cdot \frac{1}{3}\left(27 - (-27)\right) = \frac{\ln 5}{2}\frac{54}{3} = 9\ln 5$
2. $\iint_R \frac{\tan\theta}{\sqrt{1-t^2}}\, dA$, $R = \{(\theta,t)\,|\, 0 \le \theta \le \pi/3,\ 0 \le t \le \frac{1}{2}\} = \int_0^{\pi/3}\int_0^{1/2} \frac{\tan\theta}{\sqrt{1-t^2}}\, dt\, d\theta = \left[\int_0^{\pi/3} \tan\theta\, d\theta\right] \cdot \left[\int_0^{1/2} \frac{1}{\sqrt{1-t^2}}\, dt\right]$. Recall that $\frac{d}{dt}\arcsin t = \frac{1}{\sqrt{1-t^2}}$. Applying FTC, then $\left[\int_0^{\pi/3} \tan\theta\, d\theta\right] \cdot \left[\int_0^{1/2} \frac{1}{\sqrt{1-t^2}}\, dt\right] = \left[\int_0^{\pi/3} \tan\theta\, d\theta\right] \cdot \left[\arcsin t\right]_{t=0}^{t=1/2} = \left[\int_0^{\pi/3} \tan\theta\, d\theta\right] \cdot \left[\arcsin\frac{1}{2} - \arcsin 0\right] = \left[\int_0^{\pi/3} \tan\theta\, d\theta\right] \cdot \left[\frac{\pi}{6}\right] = \frac{\pi}{6}\int_0^{\pi/3} \frac{\sin\theta}{\cos\theta}\, d\theta$. To evaluate this final integral, we use the u-substitution $u = \cos\theta$ and $-du = \sin\theta d\theta$: $\frac{\pi}{6}\int_0^{\pi/3} \frac{\sin\theta}{\cos\theta}\, d\theta = -\frac{\pi}{6}\int_{\theta=0}^{\theta=\pi/3} \frac{1}{u}\, du = -\frac{\pi}{6}\ln u|_{\theta=0}^{\theta=\pi/3} = -\frac{\pi}{6}\left[\ln(\cos\theta)\right]_{\theta=0}^{\theta=\pi/3} = \frac{\pi}{6}\left[\ln(\cos 0) - \ln\left(\cos\frac{\pi}{3}\right)\right] = \frac{\pi}{6}\left[\ln 1 - \ln\frac{1}{2}\right] = \frac{\pi}{6}\ln\frac{1}{1/2} = \frac{\pi}{6}\ln 2$
3. $\iint_R x\sin(x+y)\, dA$, $R = [0,\pi/6] \times [0,\pi/3] = \int_0^{\pi/6}\int_0^{\pi/3} x\sin(x+y)\, dy\, dx$. Recall the sum formula for sine:

$$\sin(x+y) = \sin x\cos y + \cos x\sin y$$

We can substitute this into our iterated integral:

$$\int_0^{\pi/6}\int_0^{\pi/3} x\sin(x+y)\, dy\, dx = \int_0^{\pi/6}\int_0^{\pi/3} x\left[\sin x\cos y + \cos x\sin y\right]\, dy\, dx$$

$$= \int_0^{\pi/6}\left[\int_0^{\pi/3} x\sin x\cos y\, dy + \int_0^{\pi/3} x\cos x\sin y\, dy\right]\, dx$$

Let's designate $\int_0^{\pi/3} x \sin x \cos y \, dy$ as integral **A**, and $\int_0^{\pi/3} x \cos x \sin y \, dy$ as integral **B**. First, we will evaluate integral **A**:

$$\int_0^{\pi/3} x \sin x \cos y \, dy = x \sin x \int_0^{\pi/3} \cos y \, dy$$
$$= x \sin x \, [\sin y]_{y=0}^{y=\pi/3} = x \sin x \left[\sin \frac{\pi}{3} - \sin 0\right]$$
$$= x \sin x \left(\frac{\sqrt{3}}{2}\right) = \frac{x\sqrt{3}}{2} \sin x$$

Next, we evaluate integral **B**:

$$\int_0^{\pi/3} x \cos x \sin y \, dy = x \cos x \int_0^{\pi/3} \sin y \, dy$$
$$= x \cos x \, [-\cos y]_{y=0}^{y=\pi/3} = x \cos x \left[-\cos \frac{\pi}{3} - (-\cos 0)\right]$$
$$= x \cos x \left[-\frac{1}{2} - (-1)\right] = \frac{x}{2} \cos x$$

Substituting back in for integrals **A** and **B**:

$$\int_0^{\pi/6} \left[\int_0^{\pi/3} x \sin x \cos y \, dy + \int_0^{\pi/3} x \cos x \sin y \, dy\right] dx = \int_0^{\pi/6} \left[\frac{x\sqrt{3}}{2} \sin x + \frac{x}{2} \cos x\right] dx$$
$$= \frac{\sqrt{3}}{2} \int_0^{\pi/6} x \sin x \, dx + \frac{1}{2} \int_0^{\pi/6} x \cos x \, dx$$

Again, we will designate $\int_0^{\pi/6} x \sin x \, dx$ as integral **C** and $\int_0^{\pi/6} x \cos x \, dx$ as integral **D**. We start by using integration by parts to evaluate integral **C**:

Let $u = x$ and $dv = \sin x dx$. Then, $v = -cos x$ and $du = dx$. Therefore:

$$\int_0^{\pi/6} x \sin x \, dx = [x\,(-\cos x)]_{x=0}^{x=\pi/6} - \int_0^{\pi/6} (-\cos x) \, dx$$
$$= \left[\frac{\pi}{6}\left(-\cos \frac{\pi}{6}\right)\right] - [0\,(-\cos 0)] + \sin x|_{x=0}^{x=\pi/6}$$
$$= -\frac{\pi}{6} \cdot \frac{\sqrt{3}}{2} - 0 + \sin \frac{\pi}{6} - \sin 0 = \frac{1}{2} - \frac{\pi\sqrt{3}}{12} = \frac{6 - \pi\sqrt{3}}{12}$$

Next, we will use integration by parts to evaluate integral **D**. Let $u = x$ and $dv = \cos x dx$. Then $du = dx$ and $v = \sin x$ and therefore:

$$\int_0^{\pi/6} x \cos x \, dx = [x \sin x]_{x=0}^{x=\pi/6} - \int_0^{\pi/6} \sin x \, dx$$

$$= \left[\frac{\pi}{6}\sin\frac{\pi}{6} - 0\sin 0\right] - (-\cos x)\,\Big|_{x=0}^{x=\pi/6} = \frac{\pi}{6}\cdot\frac{1}{2} + \cos\frac{\pi}{6} - \cos 0$$
$$= \frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1 = \frac{\pi + 6\sqrt{3} - 12}{12}$$

Substituting back in for integrals **C** and **D**:

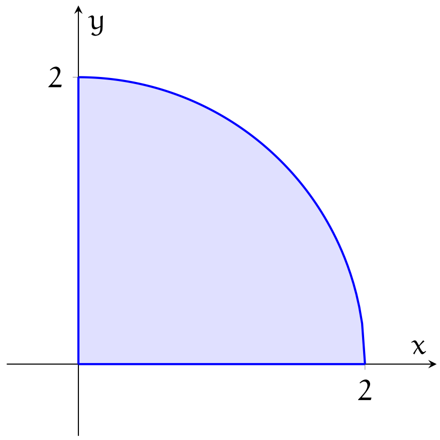
$$\frac{\sqrt{3}}{2}\int_0^{\pi/6} x\sin x\,dx + \frac{1}{2}\int_0^{\pi/6} x\cos x\,dx = \frac{\sqrt{3}}{2}\left(\frac{6-\pi\sqrt{3}}{12}\right) + \frac{1}{2}\left(\frac{\pi + 6\sqrt{3} - 12}{12}\right)$$
$$= \frac{6\sqrt{3} - 3\pi + \pi + 6\sqrt{3} - 12}{24} = \frac{6\sqrt{3} - 6 - \pi}{12}$$

Answer to Exercise 5 (on page 21)

1. $\iint_D e^{-y^2}\,dA = \int_0^3\int_0^{2y} e^{-y^2}\,dx\,dy = \int_0^3 \left[e^{-y^2}x|_{x=0}^{x=2y}\right]\,dy = \int_0^3 2ye^{-y^2}\,dy = -e^{-y^2}|_{y=0}^{y=3} = 1 - e^{-9} \approx 0.9999$
2. $\iint_D x\sin y\,dA = \int_0^2\int_0^{x^2} x\sin y\,dy\,dx = \int_0^2 x\int_0^{x^2}\sin y\,dy\,dx = \int_0^2 x\,[-\cos y]_{y=0}^{y=x^2}$
$= \int_0^2 x\left(\cos 0 - \cos x^2\right)\,dx = \int_0^2 \left(x - x\cos x^2\right)\,dx = \left[\frac{1}{2}x^2 - \frac{1}{2}\sin x^2\right]_{x=0}^{x=2} = \frac{1}{2}(2)^2 - \frac{1}{2}\left(\sin 2^2 - \sin 0\right)$
$= 2 - \frac{1}{2}(\sin 4 - 0) = 2 - \frac{\sin 4}{2} \approx 2.378$
3. We can describe the region as $D = \{(x,y) \mid -3 \le x \le -3, -\sqrt{9-x^2} \le y \le \sqrt{9-x^2}\}$. Therefore, $\iint_D (2y - x)\,dA = \int_{-3}^3\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} (2x - y)\,dy\,dx = \int_{-3}^3 \left[2xy - \frac{1}{2}y^2\right]_{y=-\sqrt{9-x^2}}^{y=\sqrt{9-x^2}}\,dx$
$= \int_{-3}^3 \left[2x\left(\sqrt{9-x^2} + \sqrt{9-x^2}\right) - \frac{1}{2}\left(9 - x^2 - (9-x^2)\right)\right]\,dx = \int_{-3}^3 4x\sqrt{9-x^2}\,dx$. Let $u = 9 - x^2$, then $du = -2x$ and $4x = -2du$. Substituting, $\int_{-3}^3 4x\sqrt{9-x^2}\,dx = \int_{x=-3}^{x=3} -2\sqrt{u}\,du = -2\cdot\frac{2}{3}u^{3/2}|_{x=-3}^{x=3} = -\frac{4}{3}\left[(9-x^2)\right]_{x=-3}^{x=3} = 0$

Answer to Exercise 6 (on page 25)

1. Let's visualize the region in the xy-plane:



The region is a quarter-circle that can be described with $D = \{(r,\theta) \mid 0 \le r \le 2,\ 0 \le \theta \le \pi/2\}$. We can then rewrite the integral in polar coordinates:

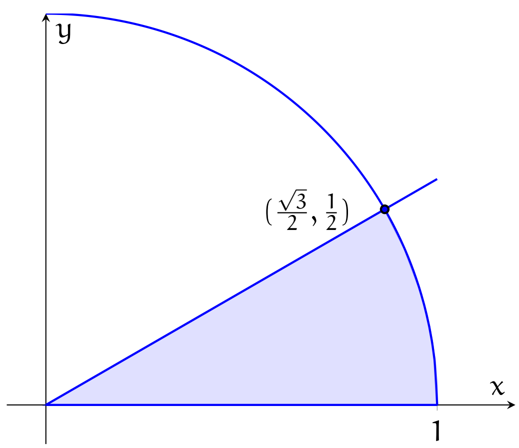
$$\int_0^2 \int_0^{\sqrt{4-x^2}} e^{-x^2-y^2}\, dy\, dx = \int_0^{\pi/2} \int_0^2 re^{-r^2}\, dr\, d\theta$$

$$= \int_0^{\pi/2} \left[-\frac{1}{2}e^{-r^2}\right]_{r=0}^{r=2} d\theta = \int_0^{\pi/2} \left(-\frac{1}{2}\right)\left[e^{-4}-1\right]\, d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} 1 - e^{-4}\, d\theta = \frac{1}{2}\left(1-\frac{1}{e^4}\right)\int_0^{\pi/2} 1\, d\theta$$

$$= \frac{1}{2}\left(1-\frac{1}{e^4}\right)\theta\Big|_{\theta=0}^{\theta=\pi/2} = \frac{\pi}{4}\left(1-\frac{1}{e^4}\right)$$

2. The region is bounded by the x-axis, the line $y = x/\sqrt{3}$, and the circle $x^2 + y^2 = 1$:

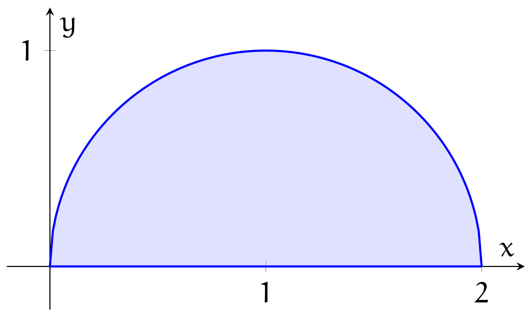


We see that the region defined in polar coordinates is $D = \{(r,\theta) \mid 0 \le r \le 1,\ 0 \le \theta \le \pi/6\}$. And therefore:

$$\int_0^{1/2} \int_{\sqrt{3}y}^{\sqrt{1-y^2}} xy^2\, dx\, dy = \int_0^{\pi/6} \int_0^1 r\,(r\cos\theta)\,(r\sin\theta)^2\, dr\, d\theta$$

$$
\begin{aligned}
&= \int_0^{\pi/6} \left[\cos\theta \sin^2\theta\right] \, d\theta \cdot \int_0^1 r^4 \, dr \\
&= \left(\frac{1}{3}\sin^3\theta|_{\theta=0}^{\theta=\pi/6}\right) \cdot \left(\frac{1}{5}r^5|_{r=0}^{r=1}\right) \\
&= \frac{1}{15} \cdot \left(\frac{1}{2}\right)^3 = \frac{1}{120}
\end{aligned}
$$

3. Visualizing the region:

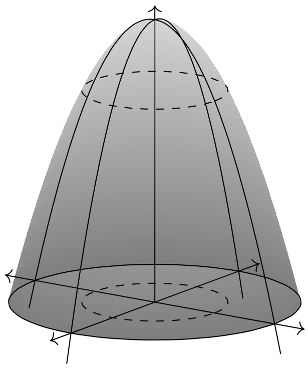


We see that the region is the top half of a circle of radius 1 centered at $(1, 0)$. In polar coordinates, this region is $D = \{(r, \theta) \mid 0 \le r \le 2\cos\theta, 0 \le \theta \le \pi/2\}$. And therefore:

$$
\begin{aligned}
\int_0^2 \int_0^{\sqrt{2x-x^2}} \sqrt{x^2+y^2} \, dy \, dx &= \int_0^{\pi/2} \int_0^{2\cos\theta} r\sqrt{r^2} \, dr \, d\theta \\
&= \int_0^{\pi/2} \int_0^{2\cos\theta} r^2 \, dr \, d\theta = \int_0^{\pi/2} \frac{1}{3}\left[r^3\right]_{r=0}^{r=2\cos\theta} \, d\theta \\
&= \frac{8}{3}\int_0^{\pi/2} \cos^3\theta \, d\theta = \frac{8}{3}\int_0^{\pi/2} \cos\theta \left(1-\sin^2\theta\right) \, d\theta \\
&= \frac{8}{3}\left[\int_0^{\pi/2} \cos\theta \, d\theta - \int_0^{\pi/2} \cos\theta \sin^2\theta \, d\theta\right] \\
&= \frac{8}{3}\left[(\sin\theta)_{\theta=0}^{\theta=\pi/2} - \left(\frac{1}{3}\sin^3\theta\right)_{\theta=0}^{\theta=\pi/2}\right] \\
&= \frac{8}{3}\left[(1-0) - \frac{1}{3}\left(1^3 - 0^3\right)\right] = \frac{8}{3} \cdot \frac{2}{3} = \frac{16}{9}
\end{aligned}
$$

Answer to Exercise 7 (on page 26)

We are finding the volume of the solid that lies under the surface $z = 4 - x^2 - y^2$ and above the xy-plane.



We can use polar coordinates to simplify the double integral. In polar coordinates, $x = r\cos(\theta)$ and $y = r\sin(\theta)$, so $x^2 + y^2 = r^2$. The volume under the surface and above the xy-plane is given by

$$V = \int\int (4 - r^2) r \, dr \, d\theta, \tag{2.1}$$

where r ranges from 0 to 2 (since $4 - r^2 \geq 0$ if $0 \leq r \leq 2$) and θ ranges from 0 to 2π.

Hence,

$$\begin{aligned} V &= \int_0^{2\pi} \int_0^2 (4r - r^3) \, dr \, d\theta \\ &= \int_0^{2\pi} \left[2r^2 - \frac{1}{4} r^4 \right]_0^2 d\theta \\ &= \int_0^{2\pi} (8 - 4) \, d\theta \\ &= \int_0^{2\pi} 4 \, d\theta \\ &= [4\theta]_0^{2\pi} \\ &= 8\pi. \end{aligned}$$

So, the volume of the solid is 8π cubic units.

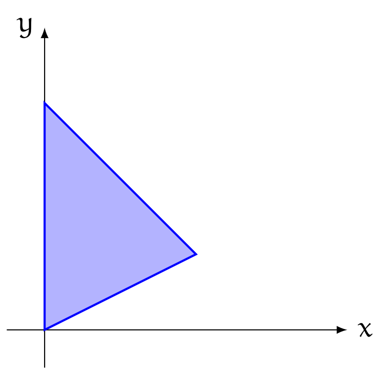
Answer to Exercise 8 (on page 27)

Let's describe the footprint of the pool as a 20-foot radius circle centered at the origin (that is, a region $D = \{(r, \theta) \mid 0 \leq r \leq 20, \ 0 \leq \theta \leq 2\pi\}$). Further, let's take north-south as parallel to the y-axis and east-west as parallel to the x-axis. So, the depth of water is then given by $z = f(x, y) = \frac{7}{40}x + \frac{13}{2}$ over the footprint of the pool. The total volume of water is given by:

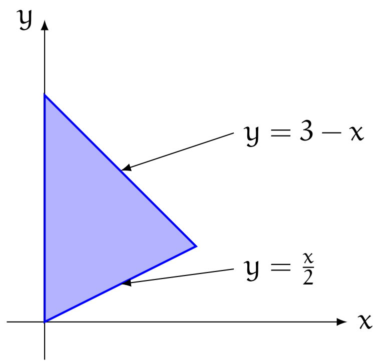
$$\begin{aligned}
&\int_0^{2\pi}\int_0^{20} r\left(\frac{7}{40}r\cos\theta + \frac{13}{2}\right)\,dr\,d\theta \\
&= \int_0^{2\pi}\int_0^{20}\left[\frac{7}{40}r^2\cos\theta + \frac{13}{2}r\right]\,dr\,d\theta \\
&= \int_0^{2\pi}\left[\frac{7\cos\theta}{40}\int_0^{20} r^2\,dr + \frac{13}{2}\int_0^{20} r\,dr\right]\,d\theta \\
&= \int_0^{2\pi}\left[\frac{7\cos\theta}{40}\left(\frac{1}{3}r^3\right)_{r=0}^{r=20} + \frac{13}{2}\left(\frac{1}{2}r^2\right)_{r=0}^{r=20}\right]\,d\theta \\
&= \int_0^{2\pi}\left[\frac{1400}{3}\cos\theta + 1300\right]\,d\theta = \left[\frac{1400}{3}\sin\theta + 1300\theta\right]_{\theta=0}^{\theta=2\pi} \\
&= 2600\pi \text{ cubic feet}
\end{aligned}$$

Answer to Exercise 9 (on page 31)

1. $\iint_D \left(1 + x^2 + y^2\right)\,dA = \int_0^4\int_0^3 \left(1 + x^2 + y^2\right)\,dy\,dx = \int_0^4 \left[y + x^2y + \frac{1}{3}y^3\right]_{y=0}^{y=3}\,dx = \int_0^4 \left[3 + 3x^2 + \frac{1}{3}(3)^3\right]\,dx$
   $= \int_0^4 \left(12 + 3x^2\right)\,dx = \left[12x + x^3\right]_{x=0}^{x=4} = 12(4) + 4^3 = 112$
2. First, let's visualize this region, since it isn't a rectangle:

   Let's divide the triangle horizontally and write equations for each of the sides that do not lie on the y-axis.



We see that we can describe region D as $D = \{(x, y) \mid 0 \le x \le 2, \frac{x}{2} \le y \le 3 - x\}$. Therefore $\iint_D (x+y)\ dA = \int_0^3 \int_{x/2}^{3-x} (x+y)\ dy\, dx = \int_0^3 \left[xy + \frac{1}{2}y^2\right]_{y=x/2}^{y=3-x}\ dx = \int_0^3 \left[(x(3-x)) - (x(x/2)) + \frac{1}{2}\left((3-x)^2 - (x/2)^2\right)\right]\ dx$

$= \int_0^3 \left[\left(3x - x^2 - \frac{x^2}{2}\right) + \frac{1}{2}\left(9 - 6x + x^2 - \frac{x^2}{4}\right)\right]\ dx = \int_0^3 \left[-x^2 - \frac{x^2}{2} + \frac{x^2}{2} - \frac{x^2}{8} + 3x - 3x + \frac{9}{2}\right]\ dx$

$= \int_0^3 \left(-\frac{9x^2}{8} + \frac{9}{2}\right)\ dx = \left[\frac{9x}{2} - \frac{3x^3}{8}\right]_{x=0}^{x=2} = \frac{9(2)}{2} - \frac{3(8)}{8} = 9 - 3 = 6$

Answer to Exercise 10 (on page 35)

1. First, we find the total mass: $m = \int_0^\pi \int_0^{2\sin x} x\, dy\, dx = \int_0^\pi [xy]_{y=0}^{y=2\sin x}\ dx = \int_0^\pi 2x \sin x\, dx.$

   We apply integration by parts to evaluate the integral: $\int_0^\pi 2x\sin x\, dx = (-2x\cos x)\,|_{x=0}^{x=\pi} + \int_0^\pi 2\cos x\, dx = [-2\pi(-1)] - (0) + \sin x|_{x=0}^{x=\pi} = 2\pi + \sin\pi - \sin 0 = 2\pi$

   Now that we know $m = 2\pi$, we can find $\bar{x}$ and $\bar{y}$: $\bar{x} = \frac{1}{2\pi}\int_0^\pi \int_0^{2\sin x} x \cdot x\, dy\, dx = \frac{1}{2\pi}\int_0^\pi x^2 y|_{y=0}^{y=2\sin x}\ dx = \frac{1}{2\pi}\int_0^\pi x^2\,(2\sin x)\ dx = \frac{1}{\pi}\int_0^\pi x^2 \sin x\, dx.$

   Applying integration by parts: $\frac{1}{\pi}\int_0^\pi x^2\sin x\, dx = \frac{1}{\pi}\left[x^2\,(-\cos x)\,|_{x=0}^{x=\pi} - \int_0^\pi 2x\,(-\cos x)\ dx\right]$
   $= \frac{1}{\pi}\left[\left(-\pi^2\cos\pi\right) + 2\int_0^\pi x\cos x\, dx\right] = \frac{1}{\pi}\left[\pi^2 + 2\int_0^\pi x\cos x\, dx\right] = \pi + \frac{2}{\pi}\int_0^\pi x\cos x\, dx.$

   Applying integration by parts again: $\pi + \frac{2}{\pi}\int_0^\pi x\cos x\, dx = \pi + \frac{2x\sin x}{\pi}|_{x=0}^{x=\pi} - \frac{2}{\pi}\int_0^\pi \sin x\, dx$
   $= \pi - \frac{2}{\pi}\int_0^\pi \sin x\, dx = \pi + \frac{2}{\pi}[\cos x]_{x=0}^{x=\pi} = \pi + \frac{2}{\pi}[\cos\pi - \cos 0] = \pi + \frac{2}{\pi}(-1-1) = \pi - \frac{4}{\pi} = \bar{x}$

   And finding $\bar{y}$: $\bar{y} = \frac{1}{2\pi}\int_0^\pi \int_0^{2\sin x} y \cdot x\, dy\, dx = \frac{1}{2\pi}\int_0^\pi \left[\frac{1}{2}xy^2\right]_{y=0}^{y=2\sin x}\ dx = \frac{1}{4\pi}\int_0^\pi x\,[2\sin x]^2\ dx$
   $= \frac{1}{4\pi}\int_0^\pi 4x\sin^2 x\, dx = \frac{1}{\pi}\int_0^\pi x\sin^2 x\, dx = \frac{1}{\pi}\int_0^\pi x\frac{1-\cos(2x)}{2}\ dx = \frac{1}{\pi}\int_0^\pi \frac{x}{2}\ dx - \frac{1}{\pi}\int_0^\pi x\cos(2x)\ dx$
   $= \frac{1}{2\pi}\left[\frac{x^2}{2}\right]_{x=0}^{x=\pi} - \frac{1}{\pi}\left[\frac{1}{2}x\sin(2x)|_{x=0}^{x=\pi} - \frac{1}{2}\int_0^\pi \sin(2x)\ dx\right] = \frac{1}{2\pi}\left(\frac{\pi^2}{2}\right) - \frac{1}{2\pi}\left[\pi\sin(2\pi) - 0 + \frac{1}{2}\cos(2x)|_{x=0}^{x=\pi}\right]$
   $= \frac{\pi}{4} - \frac{1}{2\pi}\left[\frac{1}{2}(\cos 2\pi - \cos 0)\right] = \frac{\pi}{4}$

   Therefore, the center of mass is found at $(\bar{x}, \bar{y}) = \left(\pi - \frac{4}{\pi}, \frac{\pi}{4}\right)$

2. We know from a previous question that the total mass of this lamina is 112 (see *Finding Total Mass*).

   Finding $\bar{x}$: $\bar{x} = \frac{1}{112}\int_0^4 \int_0^3 x\left(1 + x^2 + y^2\right)\ dy\, dx = \frac{1}{112}\int_0^4 \int_0^3 \left(x + x^3 + xy^2\right)\ dy\, dx$

$= \frac{1}{112}\int_0^4 \left[xy + x^3y + \frac{x}{3}y^3\right]_{y=0}^{y=3} dx = \frac{1}{112}\int_0^4 \left[3x + 3x^3 + 9x\right] dx = \frac{3}{112}\int_0^4 \left[4x + x^3\right] dx = \frac{3}{112}\left[2x^2 + \frac{x^4}{4}\right]_{x=0}^{x=4} = \frac{3}{112}\left[2(4)^2 - 2(0)^2 + \frac{4^4}{4} - \frac{0^4}{4}\right] = \frac{3}{112}\left[32 + 64\right] = \frac{3\cdot 96}{112} = \frac{3\cdot 2\cdot 2\cdot 2\cdot 2\cdot 2\cdot 3}{2\cdot 2\cdot 7\cdot 2\cdot 2} = \frac{18}{7}$

Finding $\overline{y}$: $\overline{y} = \frac{1}{112}\int_0^4\int_0^3 y\left(1 + x^2 + y^2\right) dy\, dx = \frac{1}{112}\int_0^4\int_0^3 \left[y + x^2y + y^3\right] y\, dx$

$= \frac{1}{112}\int_0^4 \left[\frac{y^2}{2} + \frac{x^2y^2}{2} + \frac{y^4}{4}\right]_{y=0}^{y=3} dx = \frac{1}{112}\int_0^4 \left[\frac{3^2}{2} + \frac{3^2x^2}{2} + \frac{3^4}{4}\right] dx = \frac{1}{112}\int_0^4 \left[\frac{99}{4} + \frac{9}{2}x^2\right] dx$

$= \frac{1}{112}\left[\frac{99}{4}x + \frac{3}{2}x^3\right]_{x=0}^{x=4} = \frac{3}{224}\left[\frac{33}{2}(4) + 4^3\right] = \frac{3}{224}(66 + 64) = \frac{3\cdot 130}{224} = \frac{3\cdot 65}{112} = \frac{195}{112}$

Therefore, the center of mass of the rectangular region D is $\left(\frac{18}{7}, \frac{195}{112}\right)$

3. We know from a previous question (see *Finding Total Mass*) that the total mass of D is 6, and it can be described as $D = \{(x, y) \mid 0 \le x \le 2, \frac{x}{2} \le y \le 3 - x\}$

Finding $\overline{x}$:

$\overline{x} = \frac{1}{6}\int_0^2\int_{x/2}^{3-x} x(x + y)\, dy\, dx = \frac{1}{6}\int_0^2\int_{x/2}^{3-x} \left(x^2 + xy\right) dy\, dx = \frac{1}{6}\int_0^2 \left[x^2y + \frac{x}{2}y^2\right]_{y=x/2}^{y=3-x} dx$

$= \frac{1}{6}\int_0^2 \left[x^2\left(3 - x - \frac{x}{2}\right) + \frac{x}{2}\left((3-x)^2 - \left(\frac{x}{2}\right)^2\right)\right] dx = \frac{1}{6}\int_0^2 \left[3x^2 - x^3 - \frac{x^3}{2} + \frac{x}{2}\left(9 - 6x + x^2 - \frac{x^2}{4}\right)\right] dx$

$= \frac{1}{6}\int_0^2 \left[3x^2 - \frac{3}{2}x^3 + \frac{x}{2}\left(9 - 6x + \frac{3}{4}x^2\right)\right] dx = \frac{1}{6}\int_0^2 \left[3x^2 - \frac{3}{2}x^3 + \frac{9}{2}x - 3x^2 + \frac{3}{8}x^3\right] dx = \frac{1}{6}\int_0^2 \left[\frac{9}{2}x - \frac{9}{8}x^3\right] dx = \frac{1}{6}\left[\frac{9}{4}x^2 - \frac{9}{32}x^4\right]_{x=0}^{x=2} = \frac{1}{6}\left[\frac{9\cdot 4}{4} - \frac{9\cdot 16}{32}\right] = \frac{1}{6}\left[9 - \frac{9}{2}\right] = \frac{1}{6}\cdot\frac{9}{2} = \frac{9}{12} = \frac{3}{4}$

And finding $\overline{y}$:

$\overline{y} = \frac{1}{6}\int_0^2\int_{x/2}^{3-x} y(x + y)\, dy\, dx = \frac{1}{6}\int_0^2\int_{x/2}^{3-x} \left(xy + y^2\right) dy\, dx = \frac{1}{6}\int_0^2 \left[\frac{x}{2}y^2 + \frac{1}{3}y^3\right]_{y=x/2}^{y=3-x} dx$

$= \frac{1}{6}\int_0^2 \left[\frac{x}{2}\left((3-x)^2 - \left(\frac{x}{2}\right)^2\right) + \frac{1}{3}\left((3-x)^3 - \left(\frac{x}{2}\right)^3\right)\right] dx$

$= \frac{1}{6}\int_0^2 \left[\frac{x}{2}\left(9 - 6x + x^2 - \frac{x^2}{4}\right) + \frac{1}{3}\left(27 - 27x + 9x^2 - x^3 - \frac{x^3}{8}\right)\right] dx$

$= \frac{1}{6}\int_0^2 \left[\frac{x}{2}\left(9 - 6x + \frac{3x^2}{4}\right) + \frac{1}{3}\left(27 - 27x + 9x^2 - \frac{9x^3}{8}\right)\right] dx$

$= \frac{1}{6}\int_0^2 \left[\frac{9}{2}x - 3x^2 + \frac{3}{8}x^3 + 9 - 9x + 3x^2 - \frac{3}{8}x^3\right] dx = \frac{1}{6}\int_0^2 \left[9 - \frac{9}{2}x\right] dx = \frac{1}{6}\left[9x - \frac{9}{4}x^2\right]_{x=0}^{x=2}$
$= \frac{3}{6}\left[3(2) - \frac{3}{4}(2)^2\right] = \frac{1}{2}(6 - 3) = \frac{1}{2}\cdot 3 = \frac{3}{2}$

Therefore, the center of mass is $(\overline{x}, \overline{y}) = \left(\frac{3}{4}, \frac{3}{2}\right)$

Answer to Exercise 11 (on page 40)

1.
$$I_x = \iint_D y^2\rho(x, y)\, dA = \int_1^4\int_0^3 y^2(xy)\, dy\, dx$$
$$= \int_1^4\int_0^3 xy^3\, dy\, dx = \int_1^4 x\left[\frac{1}{4}y^4\right]_{y=0}^{y=3} dx = \frac{1}{4}\int_1^4 81x\, dx$$
$$= \frac{81}{4}\left[\frac{1}{2}x^2\right]_{x=1}^{x=4} = \frac{81}{2}\left(4^2 - 1^2\right) = \frac{81}{2}\cdot 15 = \frac{1215}{2}$$

To find the radius of gyration, first we need to find the total mass:

$$m = \iint_D \rho(x,y)\, dA = \int_1^4 \int_0^3 xy\, dy\, dx$$

$$= \int_1^4 \frac{x}{2}\left[y^2\right]_{y=0}^{y=3} dx = \frac{9}{2}\int_1^4 x\, dx = \frac{9}{2}\cdot\left(\frac{1}{2}\right)\cdot\left[x^2\right]_{x=1}^{x=4} = \frac{9}{4}\left[16-1\right] = \frac{135}{2}$$

Finding the radius of gyration about the x-axis:

$$I_x = m\overline{\overline{y}}^2$$

$$\frac{1215}{2} = \left(\frac{135}{2}\right)\overline{\overline{y}}^2$$

$$9 = \overline{\overline{y}}^2$$

$$\overline{\overline{y}} = 3$$

2.

$$I_y = \iint_D x^2\rho(x,y)\, dA = \int_{-\pi/2}^{\pi/2}\int_0^{2\cos x} x^2 y\, dy\, dx$$

$$= \int_{-pi/2}^{\pi/2} x^2\left[\frac{1}{2}y^2\right]_{y=0}^{y=2\cos x} dx = \frac{1}{2}\int_{-\pi/2}^{\pi/2} x^2\left(2\cos x\right)^2\, dx$$

$$= 2\int_{-\pi/2}^{\pi/2} x^2\cos^2 x\, dx = 2\int_{-\pi/2}^{\pi/2} x^2\left(\frac{1+\cos 2x}{2}\right)\, dx = \int_{-\pi/2}^{\pi/2} x^2\, dx + \int_{-\pi/2}^{\pi/2} x^2\cos 2x\, dx$$

$$= \frac{1}{3}\left[x^3\right]_{x=-\pi/2}^{x=\pi/2} + \frac{1}{2}x^2\sin 2x\Big|_{x=-\pi/2}^{x=\pi/2} - \int_{-\pi/2}^{\pi/2}\frac{1}{2}\sin 2x\,(2x)\, dx$$

$$= \frac{1}{3}\left[\left(\frac{\pi}{2}\right)^3 - \left(\frac{-\pi}{2}\right)^3\right] - \int_{-\pi/2}^{\pi/2} x\sin 2x\, dx$$

$$= \frac{1}{3}\left(\frac{2\pi^3}{8}\right) - \left(\left[-\frac{1}{2}x\cos 2x\right]_{x=-\pi/2}^{\pi/2} - \int_{-\pi/2}^{\pi/2}(-\frac{1}{2}\cos 2x)\, dx\right)$$

$$= \frac{\pi^3}{12} + \left(\frac{1}{2}\left(\frac{\pi}{2}\right)\cos\left(\pi\right) - \frac{1}{2}\left(\frac{-\pi}{2}\right)\cos\left(-\pi\right)\right) - \left[\frac{1}{4}\sin 2x\right]_{x=-\pi/2}^{\pi/2}$$

$$= \frac{\pi^3}{12} + \frac{\pi}{4}(-1) + \frac{\pi}{4}(-1) = \frac{\pi^3}{12} - \frac{\pi}{2}$$

In order to find the radius of gyration, we need to first know the total mass:

$$m = \iint_D \rho(x,y)\, dA = \int_{-\pi/2}^{\pi/2}\int_0^{2\cos x} y\, dy\, dx$$

$$= \int_{-\pi/2}^{\pi/2} \frac{1}{2} y^2 \Big|_{y=0}^{y=2\cos x} dx = \frac{1}{2} \int_{-\pi/2}^{\pi/2} 4\cos^2 x \, dx = \int_{-\pi/2}^{\pi/2} 1 + \cos 2x \, dx$$

$$= \left[x + \frac{1}{2} \sin 2x \right]_{x=-\pi/2}^{\pi/2} = \pi$$

We can then find the radius of gyration about the y-axis:

$$m\bar{\bar{x}}^2 = I_y$$

$$\pi\bar{\bar{x}}^2 = \frac{\pi^3}{12} - \frac{\pi}{2}$$

$$\bar{\bar{x}} = \sqrt{\frac{\pi^2}{12} - \frac{1}{2}}$$

3.

$$I_o = \iint_D \left(x^2 + y^2 \right) \rho(x, y) = \int_1^2 \int_0^\pi r(r^2) r \, d\theta \, dr$$

$$= \int_1^2 r^4 \theta \Big|_{\theta=0}^{\theta=\pi} dr = \pi \int_1^2 r^4 \, dr = \frac{\pi}{5} r^5 \Big|_{r=1}^{r=2} = \frac{\pi}{5} \left(2^5 - 1 \right) = \frac{\pi}{5}(31) = \frac{31\pi}{5}$$

We find the total mass:

$$m = \iint_D \rho(x, y) \, dA = \int_1^2 \int_0^\pi r^2 \, d\theta \, dr = \int_1^2 r^2 \theta \Big|_{\theta=0}^{\theta=\pi} dr$$

$$= \pi \int_1^2 r^2 \, dr = \frac{\pi}{3} r^3 \Big|_{r=1}^{r=2} = \frac{\pi}{3} \left(2^3 - 1 \right) = \frac{\pi}{3}(7) = \frac{7\pi}{3}$$

To find the radius of gyration about the origin:

$$mR^2 = I_o$$

$$\left(\frac{7\pi}{3} \right) R^2 = \frac{31\pi}{5}$$

$$R^2 = \frac{31}{5} \cdot \frac{3}{7} = \frac{93}{35}$$

$$R = \sqrt{\bar{\bar{x}}^2 + \bar{\bar{y}}^2} = \sqrt{\frac{93}{35}}$$

Answer to Exercise 12 (on page 45)

1. Rearranging the formula for the plane, we find that $z = 3x + 2y + 2$. Therefore,

$\partial z/\partial x = 3$ and $\partial z/\partial y = 2$. Then the surface area is given by:

$$A(S) = \int_2^6 \int_1^4 \sqrt{1 + 3^2 + 2^2}\, dy\, dx = \int_2^6 \sqrt{14} y|_{y=1}^{y=4}\, dx$$

$$= \int_2^6 3\sqrt{14}\, dx = 3\sqrt{14} x|_{x=2}^{x=6} = 12\sqrt{14}$$

2. The paraboloid intersects the plane when $2x^2 + 2y^2 = 32$, which is the circle of radius 4 centered at the origin. So, we are looking for the area of the surface $z = 2x^2 + 2y^2$ that lies above the region $R - \{(r, \theta) \mid 0 \le r \le 4,\ 0 \le \theta \le 2\pi\}$. The surface area is:

$$A(S) = \iint_R \sqrt{1 + (4x)^2 + (4y)^2}\, dA = \int_0^4 \int_0^{2\pi} r\sqrt{1 + (4r\cos\theta)^2 + (4r\sin\theta)^2}\, d\theta\, dr$$

$$= \int_0^4 \int_0^{2\pi} r\sqrt{1 + 16r^2}\, d\theta\, dr = \int_0^4 r\sqrt{1 + 16r^2}\, [\theta]_{\theta=0}^{\theta=2\pi}\ dr$$

$$= \int_0^4 2\pi r\sqrt{1 + 16r^2}\, dr$$

Let $u = 1 + 16r^2$, then $du = 32r(dr)$ and $r(dr) = du/32$. Substituting:

$$A(S) = \frac{2\pi}{32} \int_{r=0}^{r=4} \sqrt{u}\, du = \frac{\pi}{16} \left(\frac{2}{3}\right) \left[u^{3/2}\right]_{r=0}^{r=4}$$

$$= \frac{\pi}{24} \left[\left(1 + 16r^2\right)^{3/2}\right]_{r=0}^{r=4} = \frac{\pi}{24} \left[(257)^{3/2} - 1\right] \approx 6470.15$$

3. The region, R, we are interested in is the circle of radius 2 centered at the origin of the xy-plane, described by $R = \{(r, \theta) \mid 0 \le r \le 2,\ 0 \le \theta \le 2\pi\}$. Noting that $\partial z/\partial x = 3y$ and $\partial z/\partial y = 3x$, we see that the surface area is given by:

$$A(S) = \iint_R \sqrt{1 + (3y)^2 + (3x)^2}\, dA = \int_0^2 \int_0^{2\pi} r\sqrt{1 + 9r^2\sin^2\theta + 9r^2\cos^2\theta}\, d\theta\, dr$$

$$= \int_0^2 \int_0^{2\pi} r\sqrt{1 + 9r^2}\, d\theta\, dr = 2\pi \int_0^2 r\sqrt{1 + 9r^2}\, dr$$

Let $u = 1 + 9r^2$, then $du = 18r(dr)$, which means that $r(dr) = du/18$. Substituting:

$$A(S) = \frac{2\pi}{18} \int_{r=0}^{r=2} \sqrt{u}\, du = \frac{\pi}{9} \left(\frac{2}{3}\right) \left[u^{3/2}\right]_{r=0}^{r=2}$$

$$= \frac{2\pi}{27} \left[(1 + 9(4))^{3/2} - 1\right] = \frac{2\pi}{27} \left[(37)^{3/2} - 1\right] \approx 52.14$$

Answer to Exercise 13 (on page 46)

1. The area of D is 2π. Therefore, the average value is:

$$\frac{1}{2\pi}\iint_D x\sin y\,dA = \frac{1}{2\pi}\int_0^2\int_0^\pi x\sin y\,dy\,dx$$

$$= \frac{1}{2\pi}\int_0^2 -x\cos y|_{y=0}^{y=\pi}\,dx = \frac{1}{2\pi}\int_0^2 -x\left(\cos\pi-\cos 0\right)\,dx$$

$$= \frac{1}{2\pi}\int_0^2 (-x)(-1-1)\,dx = \frac{1}{2\pi}\int_0^2 2x\,dx$$

$$= \frac{1}{2\pi}x^2|_{x=0}^{x=2} = \frac{2}{\pi}$$

2. Since D is a circle of radius $r = 1$, the area is $A = \pi r^2 = \pi$. D can be described with $D = \{(r,\theta) \mid 0 \le r \le 2\cos\theta,\ -\pi/2 \le \theta \le \pi/2\}$. Therefore, the average value of $f(x,y) = x + y$ over D is:

$$f_{ave} = \frac{1}{\pi}\iint_D (x+y)\,dA = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}\int_0^{2\cos\theta} r\cdot\left(r\cos\theta + r\sin\theta\right)\,dr\,d\theta$$

$$= \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}(\cos\theta+\sin\theta)\left[\int_0^{2\cos\theta} r^2\,dr\right]\,d\theta$$

$$= \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}(\cos\theta+\sin\theta)\cdot\left[\frac{1}{3}r^3\right]_{r=0}^{r=2\cos\theta}\,d\theta$$

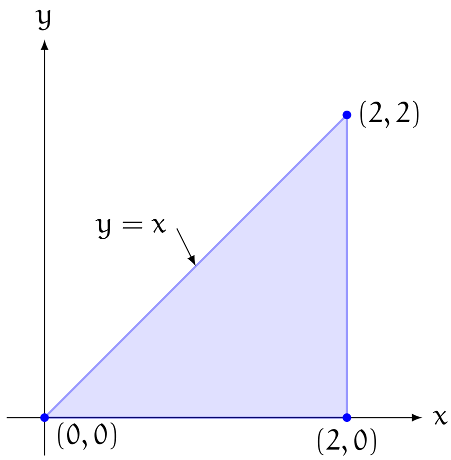
$$= \frac{8}{3\pi}\int_{-\pi/2}^{\pi/2}(\cos\theta+\sin\theta)\cos^3\theta\,d\theta = \frac{8}{3\pi}\int_{\pi/2}^{\pi/2}\left(\cos^4\theta + \sin\theta\cos^3\theta\right)\,d\theta$$

$$= \frac{8}{3\pi}\left[\int_{-\pi/2}^{\pi/2}\left(\frac{1+\cos 2\theta}{2}\right)^2\,d\theta - \left[\frac{1}{4}\cos^4\theta\right]_{\theta=-\pi/2}^{\theta=\pi/2}\right]$$

$$= \frac{2}{3\pi}\int_{-\pi/2}^{\pi/2}\left(1+2\cos 2\theta+\cos^2 2\theta\right)\,d\theta = \frac{2}{3\pi}\left[(\theta+\sin 2\theta)_{\theta=-\pi/2}^{\theta=\pi/2} + \int_{-\pi/2}^{\pi/2}\frac{1+\cos 4\theta}{2}\,d\theta\right]$$

$$= \frac{2}{3\pi}\left[\pi + \frac{1}{2}\left(\theta+\frac{1}{4}\sin 4\theta\right)_{\theta=-\pi/2}^{\theta=\pi/2}\right] = \frac{2}{3\pi}\left[\pi+\frac{1}{2}(\pi)\right] = \frac{2}{3\pi}\left(\frac{3\pi}{2}\right) = 1$$

3. Let's visualize D:



So, D can be described $D = \{(x, y) \mid 0 \leq x \leq 2,\ 0 \leq y \leq x\}$. Additionally, D has area $A = \frac{1}{2}\left(2^2\right) = 2$. Therefore, the average value of $f(x, y) = xy$ over D is:

$$f_{ave} = \frac{1}{2}\iint_D (xy)\ dA = \frac{1}{2}\int_0^2\int_0^x (xy)\ dy\,dx$$

$$= \frac{1}{2}\int_0^2 x\left[\frac{1}{2}y^2\right]_{y=0}^{y=x} dx = \frac{1}{4}\int_0^2 x^3\,dx = \frac{1}{4}\left[\frac{1}{4}x^4\right]_{x=0}^{x=2}$$

$$= \frac{1}{16}\left(2^4\right) = 1$$

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