CONTENTS

Chapter 1

Power Series

Consider the function $f(x) = \frac{1}{1-x}$. This looks similar to the value of convergent geometric series, $\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$. If we let $a = 1$ and $r = x$, then we see that $\sum_{n=1}^{\infty} x^{n-1} = \frac{1}{1-x}$. We can reindex to begin at $n = 0$ and see that:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

This is a **power series**. We use power series in place of functions for many applications, such as integrals where an explicit antiderivative don't exist, solving differential equations, and computer scientists representing functions on computers. Consider $f(x) = \frac{1}{1-x^2}$. What is $\int f(x)\,dx$? We can't directly use u-substitution, and this is not a derivative of any inverse trigonometric function. [You may have realized we could integrate this explicitly by using partial fractions, but this is not true for other functions, and we are using this as a demonstration anyway.] One way to evaluate this integral would be to represent $f(x)$ as a power series, then integrate the series. This is easier, since we know how to take the integral of any polynomial ($\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C$). First, we discuss what power series are further.

1.1 Power Series

Power series are series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 = c_1 x + c_2 x^2 + c_3 x^3 + \cdots + c_n x^n$$

for some fixed x. Depending on x, the series may converge or diverge. For example, the power series $\sum_{n=0}^{\infty} x^n$ converges for $-1 < x < 1$ and diverges for all other values of x. This is because $\sum_{n=0}^{\infty} x^n$ is essentially a geometric series with $r = x$, which we already know converges for $|r| < 1$.

The form given above is for a power series centered on 0, but a power series can be centered on any value, a. In that case, it looks like this:

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n$$

Which we say is a *power series in* $(x - a)$, or a *power series centered at* a, or a *power series about* a.

Example: Find a power series representation for $f(x) = \frac{2x-4}{x^2-4x+3}$.

Solution: Since we know $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, we will use partial fractions to decompose the function into two fractions. (The process is left as an exercise for the student.) We find that:

$$\frac{2x-4}{x^2-4x+3} = \frac{1}{x-1} + \frac{1}{x-3}$$

Noting that $\frac{1}{x-1} = (-1) \cdot \frac{1}{1-x}$, we can say that:

$$\frac{1}{x-1} = (-1) \cdot \sum_{n=0}^{\infty} x^n$$

Now, let's look at $\frac{1}{x-3}$. We can show that:

$$\frac{1}{x-3} = \frac{\frac{1}{3}}{\frac{x}{3}-1} = \frac{1}{3}\frac{1}{\frac{x}{3}-1} = \frac{-1}{3}\frac{1}{1-\frac{x}{3}}$$

Substituting $\frac{x}{3}$ for x into $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ we see that:

$$\frac{1}{1-\frac{x}{3}} = \sum_{n=0}^{\infty} \left(\frac{x}{3}\right)^n$$

Therefore:

$$\frac{1}{x-3} = \left(\frac{-1}{3}\right) \sum_{n=0}^{\infty} \left(\frac{x}{3}\right)^n$$

Adding the terms, we see that:

$$\frac{1}{x-1} + \frac{1}{x-3} = (-1)\sum_{n=0}^{\infty} x^n + \left(\frac{-1}{3}\right) \sum_{n=0}^{\infty} \left(\frac{x}{3}\right)^n$$

1.2 Power Series Convergence

Sometimes, you will be asked to find one or more values of x for which a power series converges. To do this, choose a test to apply, then find x such that the test is passed.

Example: For what values of x is the series $\sum_{n=0}^{\infty} n!x^n$ convergent?

Solution: We will apply the Ratio Test (since there is a factorial in the series) and find x

such that $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!x^{n+1}}{n!x^n}\right|$$
$$= \lim_{n\to\infty}\frac{(n+1)n!x\cdot x^n}{n!x^n} = \lim_{n\to\infty}\frac{(n+1)\cdot x}{1}$$

which converges to 0 when $x = 0$ and diverges for all other values of x. Therefore, $\sum_{n=0}^{\infty} n!x^n$ converges if $x = 0$.

Example: For what values of x does the series $\sum_{n=1}^{\infty}\frac{(x-4)^n}{2n}$ converge?

Solution: We will use the Ratio Test again. We are looking for an x such that:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$$
$$\lim_{n\to\infty}\left|\frac{\frac{(x-4)^{n+1}}{2(n+1)}}{\frac{(x-4)^n}{2n}}\right| < 1$$
$$\lim_{n\to\infty}\left|\frac{(x-4)(x-4)^n}{2n+2}\cdot\frac{2n}{(x-4)^n}\right| < 1$$
$$\lim_{n\to\infty}\left|\frac{(x-4)(x-4)^n(2n)}{(x-4)^n(2n+2)}\right| < 1$$
$$\lim_{n\to\infty}\left|\frac{(x-4)(2n)}{2n+2}\right| < 1$$
$$\lim_{n\to\infty}\left|\frac{2(x-4)}{2+\frac{2}{n}}\right| < 1$$
$$2\cdot\lim_{n\to\infty}\left|\frac{x-4}{1+\frac{1}{n}}\right| < 1$$
$$2\cdot|x-4| < 1$$
$$|x-4| < \frac{1}{2}$$

Which is true when

$$-\frac{1}{2} < x-4 < \frac{1}{2}$$
$$3.5 < x < 4.5$$

We are not done yet, though! We know the series converges for $3.5 < x < 4.5$ and diverges for $x < 3.5$ and $x > 4.5$. What about when $x = 3.5$ and $x = 4.5$? (These are the cases where the Ratio Test is indeterminate, because $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$.) We need to test each case. Substituting $x = 3.5$ into the series yields:

$$\sum_{n=1}^{\infty}\frac{(3.5-4)^n}{2n} = \sum_{n=1}^{\infty}\frac{\left(\frac{-1}{2}\right)^n}{2n}$$

$$= \sum_{n=1}^{\infty}\frac{(-1)^n}{n\cdot 2^{n+1}}$$

This is an alternating series, so we apply the alternating series test. First, we check that $|a_{n+1}| < |a_n|$:

$$\frac{1}{(n+1)\cdot 2^{n+2}} < \frac{1}{n\cdot 2^{n+1}}$$

Which is true for all $n > 0$. Next, we check if $\lim_{n\to\infty}|a_n| = 0$:

$$\lim_{n\to\infty}\frac{1}{n\cdot 2^{n+1}} = \frac{1}{\infty} = 0$$

Therefore, $\sum_{n=1}^{\infty}\frac{(-1)^n}{n\cdot 2^{n+1}}$ is convergent and $\sum_{n=1}^{\infty}\frac{(x-4)^n}{2n}$ is convergent for $x = 3.5$. Next, we test $x = 4.5$ for convergence:

$$\sum_{n=1}^{\infty}\frac{(4.5-4)^n}{2n} = \sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n\cdot\frac{1}{2n} = \sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^{n+1}\frac{1}{n}$$

This series is less than the harmonic series $\sum_{n=1}^{\infty}\frac{1}{n}$ for all n. We know the harmonic series diverges, therefore, by the direct comparison test, $\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^{n+1}\frac{1}{n}$ must also diverge. So, our final answer to the original question is that the series is convergent fort $3.5 \leq x < 4.5$.

1.2.1 Radius of Convergence

There are three possible outcomes when testing a power series $\sum_{n=0}^{\infty}c_n(x-a)^n$ for convergence:

1. The series only converges for $x = a$
2. The series converges for all x
3. The series converges if $|x-a| < R$ and diverges for $|x-a| > R$, where R is some positive number

We call R the **radius of convergence** . If we rearrange $|x-a| < R$, we can see why this is called a radius (see figure 1.1):

$$a - R < x < a + R$$

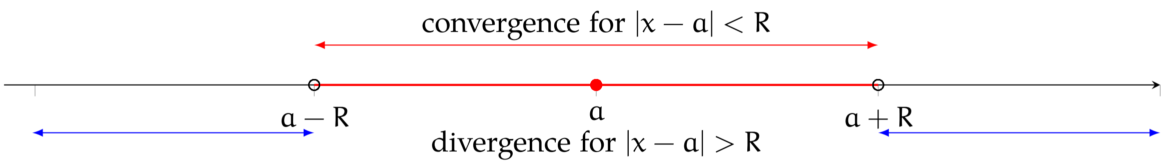

Figure 1.1: R is called the radius of convergence because it is half the width of the window of convergence

When $x = a \pm R$, the series could be convergent or divergent. You will need to test the endpoints of the windown of convergence to determine if the interval is open or closed. Thus, there are four possiblities for the interval of convergence:

1. $(a - R, a + R)$
2. $[a - R, a + R)$
3. $(a - R, a + R]$
4. $[a - R, a + r]$

In the example of $\sum_{n=1}^{\infty} \frac{(x-4)^n}{2n}$ (shown above), $a = 4$ and $R = 0.5$, and we found that the power series is convergent for $x \in [3.5, 4.5)$.

Example: For what values of x is the Bessel function $J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$ convergent?

Solution: Because there is a factorial, we will apply the Ratio Test:

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$$

$$\lim_{n\to\infty} \left| \frac{\frac{(-1)^{n+1} x^{2(n+1)}}{2^{2(n+1)}((n+1)!)^2}}{frac(-1)^n x^{2n} 2^{2n}(n!)^2} \right| < 1$$

$$\lim_{n\to\infty} \frac{x^{2n} x^2 2^{2n} n! n!}{2^{2n} 2^2 (n = 1)!(n+1)! x^{2n}} < 1$$

$$\lim_{n\to\infty} \frac{x^2 n! n!}{2^2 (n+1) n! (n+1) n!} < 1$$

$$\lim_{n\to\infty} \frac{x^2}{4(n+1)^2} = 0 < 1$$

Because $lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = 0$ for all x, the Bessel function $J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$ is convergent for all real values of x, and the interval of convergence is $(-\infty, \infty)$.

Example: Find the radius and interval of convergence for the series $\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$.

Solution: Again, we apply the ratio test to find values of x such that $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$:

$$\lim_{n\to\infty}\left|\frac{(-3)^{n+1}x^{n+1}}{\sqrt{n+1+1}}\cdot\frac{\sqrt{n+1}}{(-3)^n x^n}\right| < 1$$

$$\lim_{n\to\infty}\left|\frac{(-3)x}{\sqrt{n+2}}\cdot\frac{\sqrt{n+1}}{1}\right| < 1$$

$$\lim_{n\to\infty}\left|(-3)x\sqrt{\frac{n+2}{n+1}}\right| < 1$$

$$3|x|\lim_{n\to\infty}\sqrt{\frac{n+2}{n+1}} < 1$$

$$3|x|\lim_{n\to\infty}\sqrt{\frac{1+2/n}{1+1/n}} = 3|x|(1) < 1$$

$$3|x| < 1$$

$$|x| < \frac{1}{3}$$

Therefore, the radius of convergence is $\frac{1}{3}$. We need to test the endpoints, $x = \frac{-1}{3}$ and $x = \frac{1}{3}$, to determine the interval of convergence. First, we will test if $\sum_{n=0}^{\infty}\frac{(-3)^n x^n}{\sqrt{n+1}}$ when $x = \frac{-1}{3}$:

$$\sum_{n=0}^{\infty}\frac{(-3)^n\left(\frac{-1}{3}\right)^n}{\sqrt{n+1}} = \sum_{n=0}^{\infty}\frac{1^n}{\sqrt{n+1}} = \sum_{n=0}^{\infty}\frac{1}{\sqrt{n+1}}$$

This is a p-series such that $p < 1$, so it is divergent, and our original series does not converge for $x = \frac{-1}{3}$. Next, we test $x = \frac{1}{3}$:

$$\sum_{n=0}^{\infty}\frac{(-3)^n\left(\frac{1}{3}\right)^n}{\sqrt{n+1}} = \sum_{n=0}^{\infty}\frac{(-1)^n}{\sqrt{n+1}}$$

which is an alternating series that converges by the alternating series test. Therefore, the interval of convergence for $\sum_{n=0}^{\infty}\frac{(-3)^n x^n}{\sqrt{n+1}}$ is $x \in \left(\frac{-1}{3}, \frac{1}{3}\right]$.

Exercise 1

[This problem was originally presented as a no-calculator, multiple-choice question on the 2012 AP Calculus BC exam.] What is the radius of convergence of the series $\sum_{n=0}^{\infty} \frac{(x-4)^{2n}}{3^n}$?

Working Space

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Exercise 2

[This problem was originally presented as a no-calculator, multiple-choice question on the 2012 AP Calculus BC exam.] A power series is given by $\frac{x}{3} - \frac{x^3}{5} + \frac{x^5}{7} - \frac{x^7}{9} + \cdots$. Write the series in sigma notation and use the Ratio Test to determine the interval of convergence.

Working Space

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1.3 Calculus with Power Series

You can integrate and differentiate power series. Let $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$. Recall that $f(x)$ is just a very long polynomial and that the derivative of a polynomial x^n is $n \cdot x^{n-1}$. We can then state that:

$$\frac{d}{dx}f(x) = \frac{d}{dx}\left[c_0 + c_1(x-a)^1 + c_2(x-a)^2 + \cdots + c_n(x-a)^n\right]$$

$$f'(x) = 0 + c_1 + 2c_2(x-a)^1 + \cdots + nc_n(x-a)^{n-1}$$

$$f'(x) = \sum_{n=1}^{\infty} c_n(x-a)^{n-1}$$

which is true when x is in the interval of convergence for the series.

Similarly, we know $\int x^n \, dx = \frac{1}{n+1}x^{n+1}$. We can then say that:

$$\int f(x)\, dx = \int \left[c_0 + c_1(x-a)^1 + c_2(x-a)^2 + \cdots + c_n(x-a)^n\right], dx$$

$$\int f(x)\, dx = C + c_0(x-a) + \frac{c_1}{2}(x-a)^2 + \frac{c_2}{3}(x-a)^3 + \cdots \frac{c_n}{n+1}(x-a)^{n+1}$$

$$\int f(x)\, dx = C + \sum_{n=0}^{\infty} \frac{c_n}{n+1}(x-a)^{n+1}$$

Where C is the integration constant. Again, this is true when x is in the interval of convergence for the series.

Example: Express $\frac{1}{(1-x)^2}$ as a power series by differentiating $\frac{1}{1-x}$.

Solution: Recall that $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ when $|x| < 1$.

$$\frac{1}{1-x} = \sum_{n=1}^{\infty} x^n$$

Differentiating both sides:

$$\frac{d}{dx}\left[\frac{1}{1-x}\right] = \frac{d}{dx}\sum_{n=0}^{\infty} x^n$$

$$(-1) \cdot \frac{1}{(1-x)^2} \cdot \frac{d}{dx}(1-x) = \sum_{n=1}^{\infty} nx^{n-1}$$

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} nx^{n-1}$$

Reindexing to begin at $n = 0$

$$\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} (n+1)x^n$$

Because $\sum_{n=0}^{\infty}$ has a radius of convergence of 1, so does $\sum_{n=0}^{\infty}(n+1)x^n$. We can confirm our series makes sense by plotting the partials sums for $n = 3, 5, and 7$ with the original function (see figure 1.2).

Example: Find a power series representing $\ln(1+x)$.

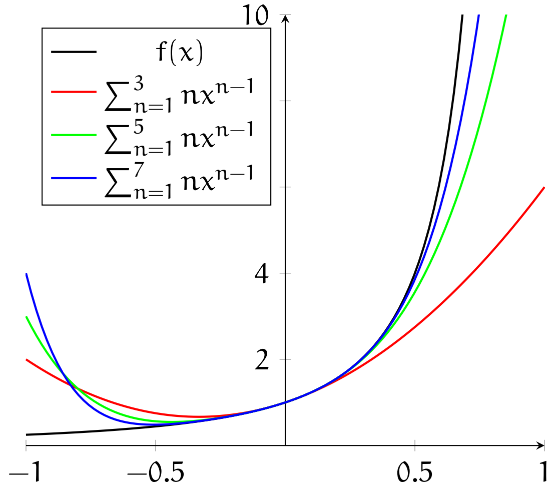

Figure 1.2: The function $f(x) = \frac{1}{(1-x)^2}$ is equal to the power series $\sum_{n=1}^{\infty} nx^{n-1}$

Solution: We know that $\frac{d}{dx}\frac{1}{1-x} = \ln(1-x)$. Replacing x with $-x$, we see that:

$$\frac{1}{1-(-x)} = \frac{1}{1+x} = \sum_{n=0}^{\infty}(-x)^n$$

which converges for $|x| < 1$. We can then integrate both sides:

$$\int \frac{1}{1+x}\,dx = \int \left[\sum_{n=0}^{\infty}(-x)^n\right] dx$$

$$\ln(1+x) = \int \left(1 - x + x^2 - x^3 + \cdots\right) dx$$

$$\ln(1+x) = C + x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots + (-1)^{n-1}\frac{x^n}{n}$$

$$\ln(1+x) = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n} + C$$

when $|x| < 1$. To find C, substitute $x = 0$ and solve:

$$\ln(1+0) = C + \sum_{n=1}^{\infty}(-1)^{n-1}\frac{0^n}{n} = C + 0$$

$$C = \ln 1 = 0$$

So, our final answer is $\ln(1+x) = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}$.

Exercise 3

Find a power series representation for $f(x) = \arctan x$.

Working Space

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Chapter 2

Population Proportion Statistics

Let's say that you are trying to get a candidate elected. The candidate asks you, "What proportion of the voting population is going to vote for me?" So, you go out and ask a random sample of 12 voters. 11 say that they are going to vote for your candidate. What can you tell the candidate?

2.1 Sample Probabilities from Population Proportion

To address these sorts of questions (and there are a lot of them), we start with the opposite question: If we knew what the proportion was in the entire population, what sort of results should we expect in a random sample of just 12?

For example, let's say that 62% of the entire population plans to vote for your candidate. You ask 12 people, "Will you vote for my candidate?" How many will say "Yes"? You don't know — it depends on the sample. For example, there is some chance that you will just happen to choose all 12 from the 48% of the population that does not plan to vote for your candidate.

We can compute the probability of each outcome using the binomial distribution. Let r be the probability that a random person will say, "I plan to vote for your candidate." Let n be the number of people you ask. The probability that exactly k people will say, "I plan to vote for your candidate" is given by:

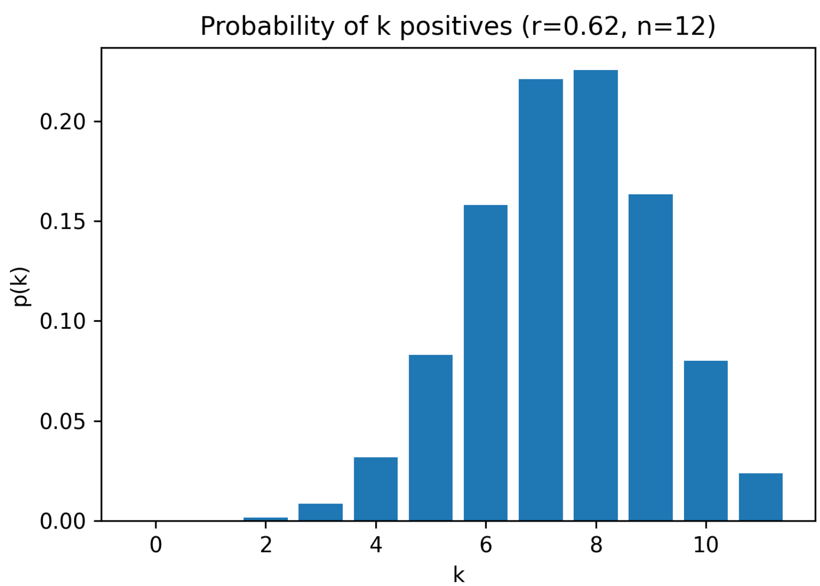
$$p(k) = \binom{n}{k} r^k (1-r)^{n-k}$$

Note that even though most people support your candidate, there is some chance that no one you ask will say that they will vote for your candidate.

Using $r = 0.62$, we can compute the probability of each outcome:

k	p(k)
0	0.000009
1	0.000177
2	0.001593
3	0.008663
4	0.031801
5	0.083017
6	0.158024
7	0.220996
8	0.225358
9	0.163418
10	0.079989
11	0.023729

Looking at this, the most likely outcome is that 8 people will say "Yes." However, although it is the most like, there is still less than a 1 in 4 chance of that outcome. It is very unlikely that less than 2 people will say "Yes." Here is a bar chart of the data



In this section, we knew the proportion of the population (p) and used that to find the probability of each possible number of positives in a random sample (k). Now, we are going to go the other way: You know k, and you are finding the probability of possible values of p.

2.2 Population Proportion from Sample

You ask 12 people if they will vote for your candidate. 9 say "Yes."

Next, you do a thought experiment: "If only 10% of the population were going to vote for my candidate, what is the probability that I would see this outcome?"

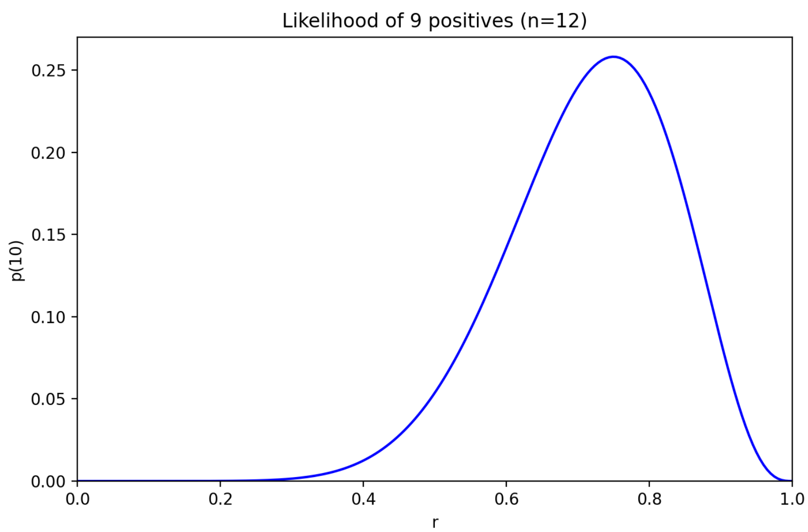
$p(9) = \binom{n}{k} r^k (1-r)^{n-k} = \binom{12}{9}(0.1)^9(0.9)^{12-9} = 0.00000016$

This outcome would be quite unusual. What if 70% of the population were going to vote for your candidate? What is the probability that you would see this outcome?

$p(9) = \binom{n}{k} r^k (1-r)^{n-k} = \binom{12}{9}(0.7)^9(0.3)^{12-9} \approx 0.2397$

In this case, the observed outcome would be a lot less unusual.

So, you decide to plot out the likelihood of this outcome for every possible value of r:



This looks very similar to a probability distribution, but *it is not*. The area under the curve does not integrate to 1.0 — it is significantly less. This is a called a *likelihood*.

However, it still tells us something, right? The maximum likelihood estimator is $9/12 = 0.75$.

2.3 From Likelihood to Probability Density Function

How can we make this likelihood into a probability density function? We use Bayes' Law for continuous probability:

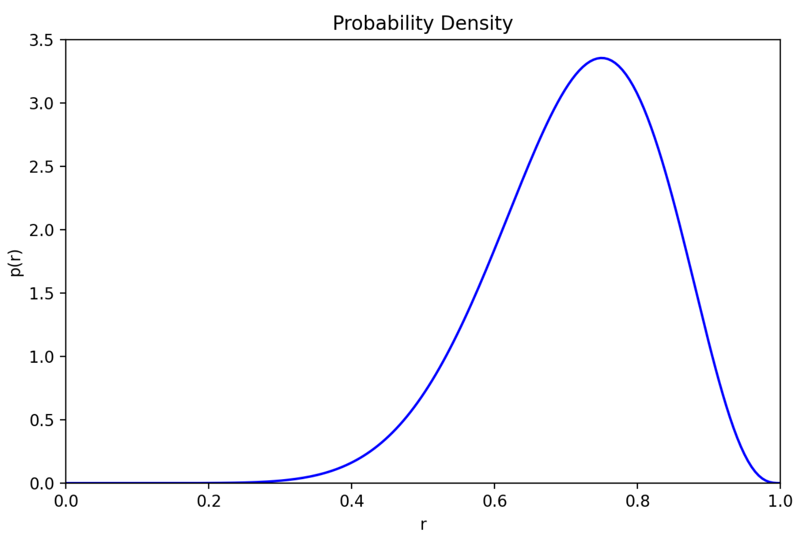
$$p(r|k) = \frac{p(k|r)p(r)}{\int_{r=0}^{1} p(k|r)p(r)dr}$$

In other words, given that we had k positive responses, what is the probability that the proportion of the population that will vote for your candidate is r? The numerator of the fraction is the likelihood scaled up or down by our prior belief about the value of r. What is the denominator of the fraction? For this to be a probability distribution, we need it to integrate to 1. This is taken care of by the denominator.

Let's say we have no prior belief about the value of r. That is, $p(r)$ is the continuous uniform distribution between 0 and 1; thus, $p(r) = 1$ for all possible values of r. Our formula becomes:

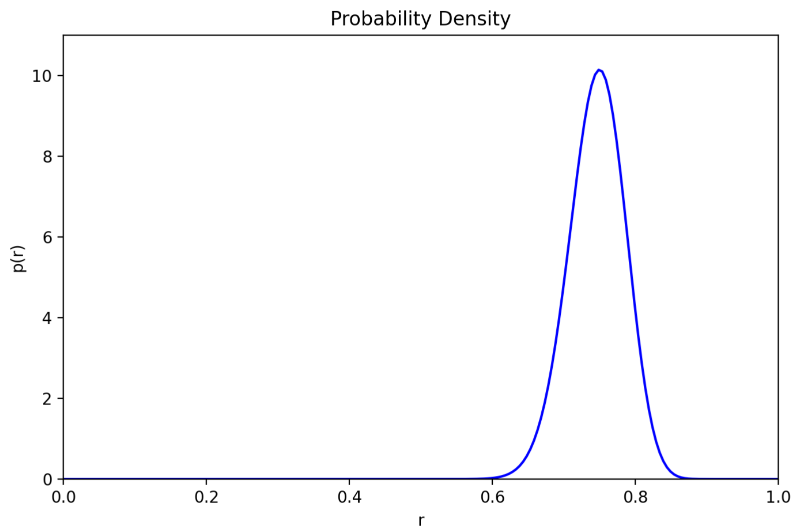
$$p(r|k) = \frac{p(k|r)}{\int_{r=0}^{1} p(k|r)dr}$$

That is the likelihood scaled up so that it integrates to 1. If we plot this, we get:



Here, then, is your report to your candidate: "I asked 12 voters if they were going to vote for you. 9 said yes. Using a uniform prior, here is what I believe about your support in the general population." You also include this graph.

What happens to this graph if you ask 120 voters and 90 say yes?



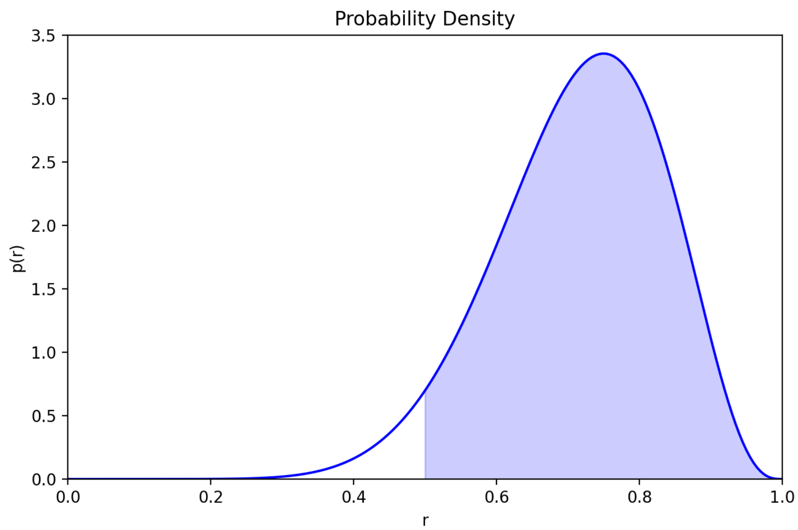
The MLE (0.75) is the same, but because of the much larger sample size, you are more confident when you say ”It is probably close to 75%.”

2.4 Beta Distribution

This probability distribution that you discovered is actually pretty common. It is known as the *beta distribution*.

The beta distribution has two parameters a and b that determine its shape. If you get k positives out of n, then use $a = k + 1$ and $b = n - k + 1$.

When you make your report to your candidate, they will look at your probability distribution with quiet awe and ask ”Based on your sample of 12 people, what is the probability that at least 50% of the population will vote for me?” So, you’d fill in the region and say, ”This area represents that probability.”



Once again, there will be a long silence, followed by a simple question: ”Can you give me a number?” Here is the Python code:

```
import numpy as np
from scipy.stats import beta

# Constants
K =  9
N = 12

# What is the probability r <=0.5?
p_less = beta.cdf(0.5, K +1, N - K +1)

# What is the probability r > 0.5?
p_more = 1.0 - p_less
print(f"I'm {p_more * 100.0:.2f}% sure you will win.")
```

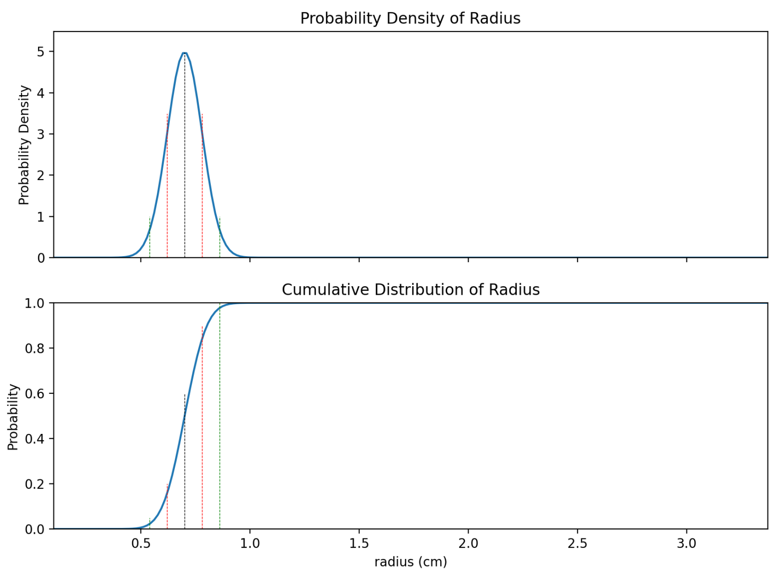
This will give you:

```
I'm 95.39% sure you will win.
```

Chapter 3

Change of Variables

Let's say that we are making ice spheres, and we tell you that the radius of the ice spheres is normally distributed with a mean of 0.7 cm and a standard deviation of 0.08 cm. You can then draw the probability distribution and cumulative distribution for that:



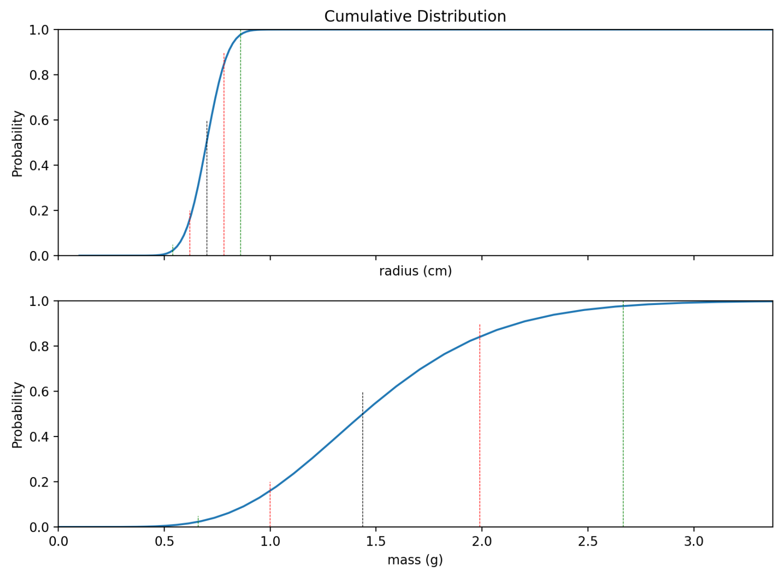
This includes lines indicating the mean and two standard deviations on each side.

Now, let's say we ask you what the cumulative distribution is for the *mass* of the balloons. A cubic centimeter of ice weighs about gram, so if you know the radius of a particular ice sphere, it is easy to compute the mass of it:

$$m = \frac{4}{3}\pi r^3$$

So, for example, if a sphere has a radius of 5cm, its mass in grams is $\frac{4\pi(0.7^3)}{3} \approx 1.44$ g.

Thinking about the graph of the cumulative distribution, if half the balloons have a radius less than 5 cm, then half of the balloons have a mass less than 523.6 g. For each point on the cumulative graph, we can use the radius of that point to compute the corresponding mass — the CDF gets stretched out:



If F is the original cumulative distribution function, and g is the function that maps the new variable (mass, in this case) to the old one (radius), then the new cumulative distribution function H is given by

$$H(m) = F(g(m))$$

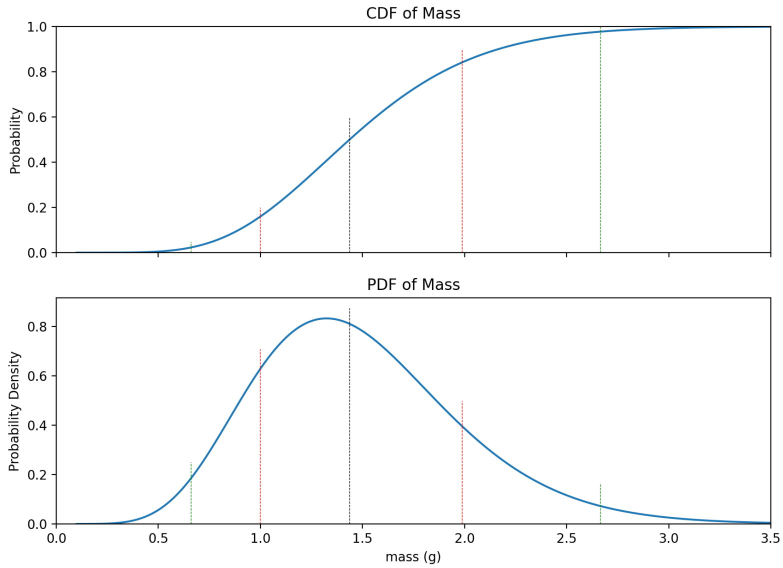
In this case, F is the cumulative function for the normal distribution, with mean 0.7 and standard deviation of 0.08. g maps the mass to the radius:

$$g(m) = \left(\sqrt[3]{\frac{3}{4\pi}}\right) x^{\frac{1}{3}}$$

3.1 Making a Probability Density Function

Now, we know how to calculate a new cumulative distribution function using the new variable. However, we usually want a probability density.

Here is the CDF and the PDF of the mass of the ice spheres:



Reminder: The probability density function is the derivative of the cumulative distribution function. We know the CDF is

$$H(m) = F(g(m))$$

By the chain rule:

$$H'(m) = F'(g(m))g'(m)$$

The function F is the cumulative distribution for the normal distribution with mean 0.7 and standard deviation of 0.08. So, we know its derivative:

$$F'(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

where $\mu = 0.7$ and $\sigma = 0.08$.

We've already said that

$$g(m) = \left(\sqrt[3]{\frac{3}{4\pi}}\right) m^{\frac{1}{3}}$$

Which is easy to differentiate:

$$g'(m) = \left(\frac{1}{3}\right)\left(\sqrt[3]{\frac{3}{4\pi}}\right) m^{-\frac{2}{3}}$$

Here, then, is the code to generate that last plot:

```
import numpy as np
from scipy.stats import norm
import matplotlib.pyplot as plt

# Constants
MEAN_RADIUS = 0.7
STD_RADIUS = 0.08

# Range to plot
MIN_MASS = 0.1
MAX_MASS = 3.5

# Number of points to plot
N = 200

# Needed for radius_for_mass and d_radius_for_mass
C = np.power(3 / (4 * np.pi), 1/3)

# In these three functions, x can
# be a number or a numpy array

def mass_for_radius(x):
    return 4 * np.pi * np.power(x, 3) / 3

def radius_for_mass(x):
    return C * np.power(x, 1/3)

# Derivative of radius_for_mass()
def d_radius_for_mass(x):
    return (C/3) * np.power(x,-2/3)

# Compute mean and 2 standard deviations in each direction
m_mean = mass_for_radius(MEAN_RADIUS)
m_minus_std = mass_for_radius(MEAN_RADIUS - STD_RADIUS)
m_plus_std = mass_for_radius(MEAN_RADIUS + STD_RADIUS)
m_minus_2std = mass_for_radius(MEAN_RADIUS - 2 * STD_RADIUS)
m_plus_2std = mass_for_radius(MEAN_RADIUS + 2 * STD_RADIUS)

# Make N possible values for mass
m_values = np.linspace(MIN_MASS, MAX_MASS, N)

# Compute g(m) for each of these masses
```

```
# That is: What is the radius for each of these masses?
r_values = radius_for_mass(m_values)

# Compute F(g(m)) for each of these masses
# That is: What is the cumulative distribution for each those radii?
cdf_values = norm.cdf(r_values, loc=MEAN_RADIUS, scale=STD_RADIUS)

# Compute g'(m) for each of these masses
dg_values = d_radius_for_mass(m_values)

# What is F'(g(m))g'(m)?
pdf_values = norm.pdf(r_values, loc=MEAN_RADIUS, scale=STD_RADIUS) * dg_values

# Sanity check: It should integrate to a little less then 1.0
dx = (MAX_MASS - MIN_MASS)/N
area_under_curve = pdf_values.sum() * dx
print(f"Integral from {MIN_MASS:.2f} to {MAX_MASS:.2f}: {area_under_curve:.3f}")

# Make a figure with two axes
fig, axs = plt.subplots(nrows=2, sharex=True, figsize=(10, 7), dpi=200)

# Draw the CDF on the second axix
axs[0].set_title("CDF of Mass")
axs[0].set_ylim(bottom=0.0, top=1.0)
axs[0].set_xlim(left=0.0, right=MAX_MASS)
axs[0].set_ylabel("Probability")
axs[0].plot(m_values, cdf_values)

# Add lines for mean,  mean-std, and mean+std
axs[0].vlines(m_minus_2std, 0, 0.05, "g", linestyle="dashed",lw=0.5)
axs[0].vlines(m_minus_std, 0, 0.2, "r", linestyle="dashed",lw=0.5)
axs[0].vlines(m_mean, 0, 0.6, "k", linestyle="dashed",lw=0.5)
axs[0].vlines(m_plus_std, 0, 0.9, "r", linestyle="dashed",lw=0.5)
axs[0].vlines(m_plus_2std, 0, 1.0, "g", linestyle="dashed",lw=0.5)

# How high does the pdf go?
max_density = pdf_values.max()

# Draw the PDF on the second axix
axs[1].set_title("PDF of Mass")
axs[1].set_ylim(bottom=0.0, top=max_density * 1.1)
axs[1].set_xlim(left=0.0, right=MAX_MASS)
axs[1].set_xlabel("mass (g)")
axs[1].set_ylabel("Probability Density")
axs[1].plot(m_values, pdf_values)

# Add lines for mean,  mean-std, and mean+std
axs[1].vlines(m_minus_2std, 0, max_density * .3, "g", linestyle="dashed",lw=0.5)
axs[1].vlines(m_minus_std, 0, max_density * .85 , "r", linestyle="dashed",lw=0.5)
axs[1].vlines(m_mean, 0, max_density * 1.05, "k", linestyle="dashed",lw=0.5)
axs[1].vlines(m_plus_std, 0, max_density * .6, "r", linestyle="dashed",lw=0.5)
axs[1].vlines(m_plus_2std, 0, max_density * .2, "g", linestyle="dashed",lw=0.5)
```

```
fig.savefig("pdf.png")
```

3.2 Decreasing Conversions

The last case (mass and radius) is relatively straightforward, because the function g is always increasing. What if we have a change of variables where g is decreasing? For example, $V = IR$ so $\frac{V}{R} = I$.

Let's say that you work at a lightbulb factory and you sample the lightbulbs to see what their resistance is. You find the resistances of the lightbulbs are normally distributed with a mean of 24 ohms and a standard deviation of 3 ohms. The voltage will be exactly 12 volts. What is the PDF of the currents that will pass through the lightbulbs?

$$I = \frac{12}{R}$$

so

$$g(x) = \frac{12}{x}$$

is the function that will convert the current to resistance. Taking the derivative, we get:

$$g'(i) = -\frac{12}{x^2}$$

Chapter 4

Volumes with Integrals

Suppose we wanted to know the volume of a theoretical irregular shape (we stipulate theoretical because, if you had this object and a large enough container, you could use displacement to determine the volume of the object). [fixme better intro]

4.1 Volume of a Sphere

Below, we will prove the volume of a sphere is given by $\frac{4}{3}\pi r^3$ using the integral method. Suppose we have a sphere of radius r centered at the origin (see figure 4.1).

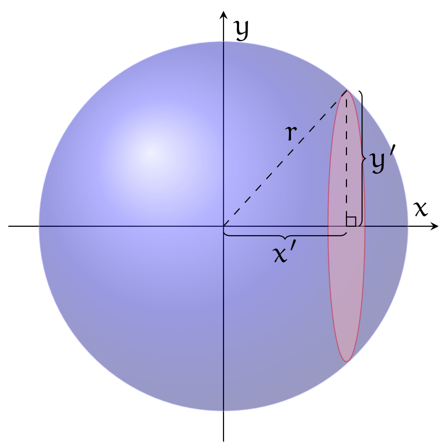


Figure 4.1: A vertical cross-section of a sphere

We begin by taking very thin vertical cross-sections. The radius of the cross-section is the height, y, of the sphere at the horizontal position, x. Since the edges of the cross-section lie on the sphere, we know the edge of the cross-section is distance r from the origin. Applying the Pythagorean theorem, we see that $r^2 = x^2 + y^2$, which implies that $y = \sqrt{r^2 - x^2}$. So, the area of the cross-section is given by $\pi y^2 = \pi(r^2 - x^2)$. If we imagine each cross section as having a width, dx, and taking the sum of all the cross sections from $x = -r$ to $x = r$, we can write an integral equal to the volume of the sphere:

$$V_{\text{sphere}} = \int_{-r}^{r} \pi(r^2 - x^2)\,dx$$

We can then evaluate that integral:

$$V_{\text{sphere}} = \pi \int_{-r}^{r} r^2\,dx - \pi \int_{-r}^{r} x^2\,dx$$

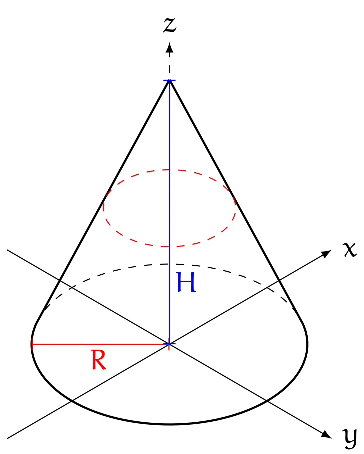
$$V_{\text{sphere}} = \pi \left[r^2 x\right]_{x=-r}^{x=r} - \frac{\pi}{3}\left[x^3\right]_{x=-r}^{x=r}$$

$$V_{\text{sphere}} = \pi \left[r^3 - (-r^3)\right] - \frac{\pi}{3}\left[r^3 - (-r^3)\right]$$

$$V_{\text{sphere}} = 2\pi r^3 - \frac{2\pi}{3} r^3 = \frac{4}{3}\pi r^3$$

Exercise 4

Prove the volume of a regular cone is $\frac{\pi}{3}R^2H$, where R is the radius of the base and H is the height of the cone. (Hint: A cone is a series of decreasing circles stacked on top of each other; see figure below.)



Working Space

Answer on *Page 38*

4.2 Volumes of Solids of Revolution

We can also find the volume of solids made by revolving a graph about the x or y-axis. Suppose the graph $y = \sin x$ from $x = 0$ to $x = \frac{\pi}{2}$ were rotated vertically about the x-axis to form a solid. How could we find the volume of that solid? Well, we can imagine a rectangle of width dx and height y (see figure 4.2)

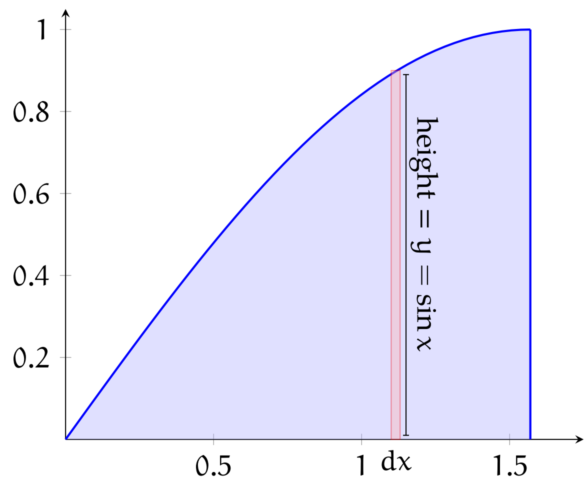


Figure 4.2: A cross section has width dx and height $y = \sin x$

If we rotate the plot vertically about the x-axis, the rectangle becomes a cylinder with radius $y = \sin x$ and height dx (see figure **??**). Therefore, the volume of each cylindrical slice is $V_{slice} = \pi r^2 dx = \pi \cdot \sin^2 x dx$.

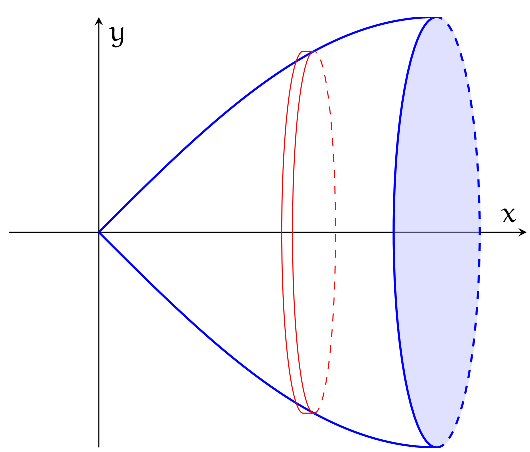


Figure 4.3: When rotated, the cross-section becomes a cylinder with radius $\sin x$ and width dx, which has a total volume of $\pi \sin^2 x dx$

We can find the total volume by integrating from 0 to $\pi/2$:

$$V = \pi \int_0^{\pi/2} \sin^2 x \, dx$$

Recall the half angle formula, $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$. Substituting, we see that:

$$V = \frac{\pi}{2} \int_0^{\pi/2} (1 - \cos 2x) \, dx$$

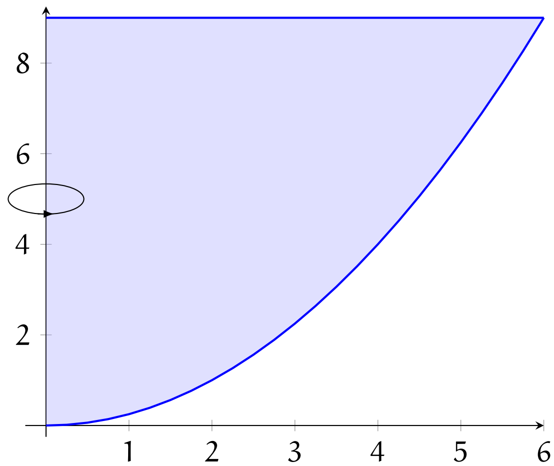
$$V = \frac{\pi}{2} \left(x - \frac{1}{2} \sin 2x \right) \Big|_{x=0}^{x=\pi/2}$$

$$V = \frac{\pi}{2} \left[\left(\pi/2 - \frac{1}{2} \sin \pi \right) - \left(0 - \frac{1}{2} \sin 0 \right) \right]$$

$$V = \frac{\pi}{2} [\pi/2 - 0 - 0 + 0] = \frac{\pi^2}{4}$$

Exercise 5

Find the volume of a solid created by rotating the region bounded by $x = 2\sqrt{y}$, $x = 0$, and $y = 9$ about the y-axis. A graph is shown below.



Working Space

Answer on Page 39

Exercise 6

Let $f(x) = (ax^3 + bx^2 + cx + d)\sqrt{1 - x^2}$. Bird's eggs of various sizes can be modeled by rotating $f(x)$ about the x-axis, with different values of a, b, c, and d defining different sizes and shapes of eggs. For a domestic chicken, $a = -0.02$, $b = 0.03$, $c = 0.12$, and $d = 0.454$. For a mallard duck, $a = -0.06$, $b = 0.04$, $c = 0.1$, and $d = 0.54$. Use a calculator, such as a TI-89 or Wolfram Alpha, to determine which species lays a bigger egg.

Working Space

Answer on Page 40

4.2.1 Using donuts for solids of revolution

Sometimes there is space between the region we are rotating and the line we are rotating it about. Consider the region bounded between $y = 2x$ and $y = x^2$ (see figure 4.4):

When rotated, the slices will take the form of donuts (or washers), the volume of which is $\pi\left(R^2 - r^2\right) dx$, where R is the outer radius and r is the inner radius. Therefore, in this case, the total volume of the rotated region is given by:

$$V = \int_0^2 \pi \left[(2x)^2 - \left(x^2\right)^2\right] dx$$

$$V = \pi \int_0^2 4x^2 - x^4 \, dx = \pi \left[\frac{4}{3}x^3 - \frac{1}{5}x^5\right]_{x=0}^{x=2}$$

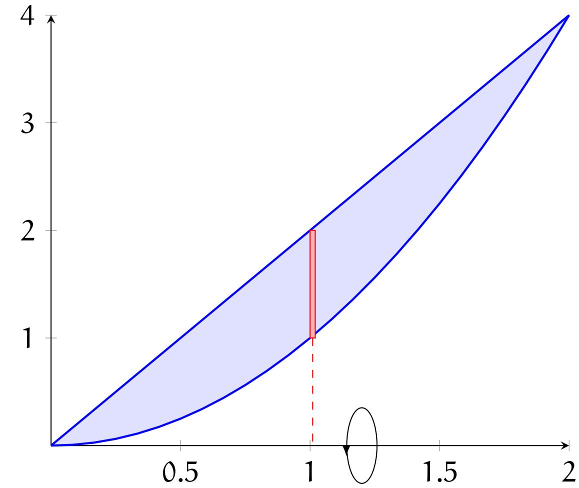


Figure 4.4: When rotated, the slices will become donuts with outer radius $2x$ and inner radius x^2

$$V = \pi \left[\frac{4}{3} 2^3 - \frac{1}{5} 2^5 \right] = \pi \left[\frac{32}{3} - \frac{32}{5} \right]$$

$$V = \frac{64\pi}{15}$$

Exercise 7

What is the volume of the region bounded by $y = x^2$ and $y = 2\sqrt{x}$ when rotated about the y-axis?

Working Space

Answer on Page 40

4.3 Volumes of Other Solids

You can also model a solid as a base defined by a function with cross-sections of specific shapes. Consider the function $y = x^2$ from $x = 0$ to $x = 2$ (see figure 4.5). Suppose the area between the curve, the y-axis, and the line $y = 4$ defines a base and each vertical cross-section is a square. So, the width of the each cross section is dx, the length is $4 - x^2$, and (because they are squares) the height in the z-plane is also $4 - x^2$. The volume of each cross-section is $V_{slice} = \left(4 - x^2\right)^2 dx$ and the total volume of the solid is:

$$V = \int_0^2 \left(4 - x^2\right)^2 dx$$

$$V = \int_0^2 \left(16 - 8x^2 + x^4\right) dx$$

$$V = \left[16x - \frac{8}{3}x^3 + \frac{1}{5}x^5\right]_{x=0}^{x=2}$$

$$V = 16(2) - \frac{8}{3}(2)^3 + \frac{1}{5}(2)^5 = \frac{256}{15} \approx 17.067$$

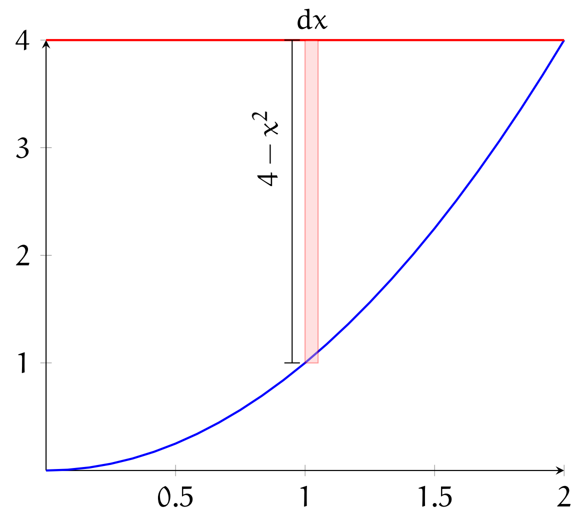


Figure 4.5: $y = x^2$ with a vertical cross-section

You can use a similar method for triangular, semi-circular, or any other shape cross-section. The trick is writing everything in terms of x (when you cross sections are vertical and have width dx) or y (when your cross section are horizontal and have length dy).

Exercise 8

[This question was originally presented as a multiple-choice, calculator-allowed question on the 2012 AP Calculus BC exam.] Let R be the region in the first quadrant bounded above by the graph $y = \ln(3 - x)$, for $0 \leq x \leq 2$. R is the base of a solid for which each cross section perpendicular to the x-axis is square. What is the volume of the solid? Give your answer to 3 decimal places.

Working Space

Answer on Page 42

Exercise 9

Find the volume of a solid whose base is defined by the ellipse $9x^2 + 16y^2 = 25$ and is made up of isosceles-triangular cross-sections perpendicular to the x-axis (with the hypotenuse in the base of the solid).

Working Space

Answer on Page 42

Appendix A

Answers to Exercises

Answer to Exercise 1 (on page 9)

Since this sum has terms to the n^{th} power, we will apply the Root Test, which states a series is convergent if $\lim_{n\to\infty} \sqrt[n]{|a_n|} < 1$.

$$\lim_{n\to\infty} \sqrt[n]{\left|\frac{(x-4)^{2n}}{3^n}\right|} < 1$$

$$\lim_{n\to\infty} \left|\frac{(x-4)^{2n/n}}{3^{n/n}}\right| < 1$$

$$\lim_{n\to\infty} \frac{\left|(x-4)^2\right|}{3} < 1$$

$$(x-4)^2 < 3$$

$$|x-4| < \sqrt{3}$$

Therefore, the radius of convergence is $\sqrt{3}$.

Answer to Exercise 2 (on page 9)

We see that the series is alternating, so we know it involves $(-1)^n$ (we will begin indexing at $n = 0$). The powers of x are given by x^{2n+1} and the denominators are given by $2n+3$. Therefore, the sum in sigma notation is $\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+3}$. Applying the ratio test, the series is convergent when:

$$\lim_{n\to\infty} \left|\frac{(-1)^{n+1}x^{2(n+1)+1}}{2(n+1)+3} \frac{2n+3}{(-1)^n x^{2n+1}}\right| < 1$$

$$\lim_{n\to\infty} \left|\frac{x^{2n+3}}{2n+5} \frac{2n+3}{x^{2n+1}}\right| < 1$$

$$\left|x^2\right| \lim_{n\to\infty} \frac{2n+3}{2n+5} < 1$$

$$\left|x^2\right| < 1$$

$$|x| < 1$$

So, we know that the series is convergent on the open interval $x \in (-1, 1)$. We check the endpoints, $x = -1, 1$ for convergence.

$$\sum_{n=0}^{\infty} \frac{(-1)^n(-1)^{2n+1}}{2n+3} = \sum_{n=0}^{\infty} \frac{(-1)^{3n+1}}{2n+3}$$

When $x = -1$, the series is an alternating series such that $|a_{n+1}| < |a_n|$ and $\lim_{n\to\infty} a_n = 0$. Therefore, the series converges for $x = -1$.

$$\sum_{n=1}^{\infty} (-1)^n \frac{(1)^{2n+1}}{2n+3} = \sum_{n=1}^{\infty} \frac{(-1)^n}{2n+3}$$

which is also an alternating series such that $|a_{n+1}| < |a_n|$ and $\lim_{n\to\infty} a_n = 0$. Therefore, the series converges for $x = 1$ and the interval of convergence is $x \in [-1, 1]$, which can also be written as $-1 \leq x \leq 1$.

Answer to Exercise 3 (on page 12)

Recall that $\frac{d}{dx} \arctan x = \frac{1}{1+x^2}$. Replacing x with $-x^2$, we see that $\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} \left[-x^2\right]^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}$. Then we can also say that $\arctan x = \int \frac{1}{1+x^2}\, dx = \int \left[sum_{n=0}^{\infty} (-1)^n x^{2n}\right] dx$. Evaluating the integral, $\int \left[sum_{n=0}^{\infty} (-1)^n x^{2n}\right] dx = \sum_{n=0}^{\infty} (-1)^n \int x^{2n}\, dx = C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$. Knowing that $\arctan 0 = 0$, we find that $C = 0$ and $\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$.

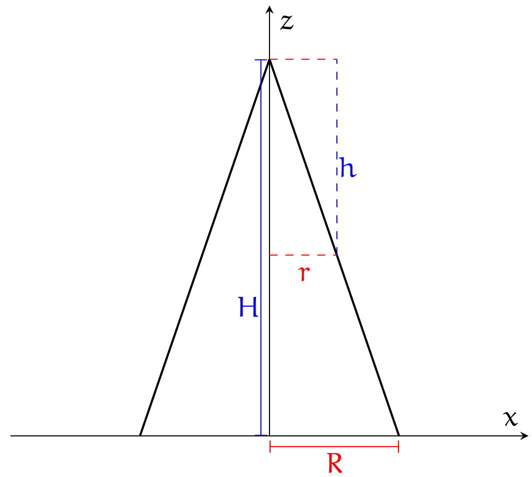
Answer to Exercise 4 (on page 26)

Imagine a side view of the cone (see figure below), an isosceles triangle with height H and base $2R$. If we take horizontal cross-sections, then each cross-section is a circle h from the top with a radius r. Because the triangles are similar (FIXME: better wording/explanation here), we also know that $\frac{H}{h} = \frac{R}{r}$. Therefore, we can define r in terms of h: $r = \frac{hR}{H}$ and the volume of each subsequent cross-section is $\pi r^2 dh = \pi \frac{h^2 R^2}{H^2} dh$. We start with $h = 0$ and end with $h = H$:

$$V_{cone} = \int_0^H \pi \frac{h^2 R^2}{H^2}\, dh = \pi \frac{R^2}{H^2} \int_0^H h^2\, dh$$

$$= \pi \frac{R^2}{H^2} \left[\frac{1}{3} h^3\right]_{h=0}^{h=H} = \pi \frac{R^2}{3H^2} \left[H^3 - 0^3\right]$$

$$= \pi \frac{R^2}{3H^2} H^3 = \frac{\pi}{3} R^2 H$$



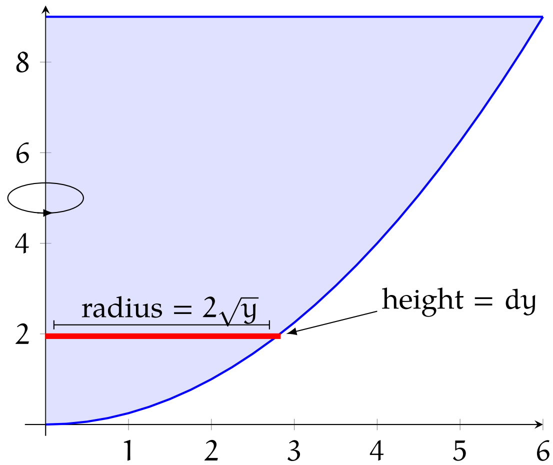
Answer to Exercise 5 (on page 29)

If we are rotating about the y axis, we should make our slices horizontal, so their width is dy (see graph below). Then, the volume of each cylinder is given by $V = \pi r^2 dy$ and the total volume is given by:

$$V = \int_0^9 \pi \left[2\sqrt{y}\right]^2 \, dy$$

$$V = 4\pi \int_0^9 y \, dy = 2\pi y^2 |_{y=0}^{y=9}$$

$$V = 2\pi (9)^2 = 162\pi$$



Answer to Exercise 6 (on page 30)

Since the graph is rotated around the x-axis, we will take vertical slices with width dx, and rotate them to make cylinders with radius $f(x)$ and height dx. The volume of each egg is given by:

$$\int_{-1}^{1} \pi\, [f(x)]^2\ dx$$

To determine our limits of integration, we note that $\sqrt{1-x^2} = 0$ (and therefore, $f(x) = 0$) when $x = \pm 1$.

For the chicken:

$$V_{chickenegg} = \pi \int_{-1}^{1} \left[\left(-0.02x^3 + 0.03x^2 + 0.12x + 0.454\right)\sqrt{1-x^2}\right]^2\ dx$$

For the mallard duck:

$$V_{duckegg} = \pi \int_{-1}^{1} \left[\left(-0.06x^3 + 0.04x^2 + 0.1x + 0.54\right)\sqrt{1-x^2}\right]^2\ dx$$

Using a calculator, we find that $V_{chickenegg} \approx 0.897$ and $V_{duckegg} \approx 1.263$. Therefore, mallard ducks lay larger eggs than chickens do.

Answer to Exercise ?? (on page 32)

First, since we are revolving around the y-axis, we know our slices will have width dy.

We will rewrite the functions as x in terms of y:

$$x = \sqrt{y}$$

$$x = \frac{y^2}{4}$$

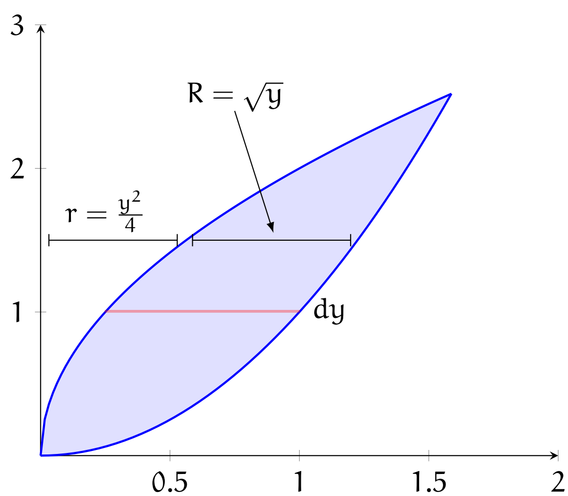
Setting them equal to each other to find the y-value at which they intercept:

$$\sqrt{y} = \frac{y^2}{4}$$

$$4 = \frac{y^2}{\sqrt{y}} = y^{3/2}$$

$$y = \sqrt[3]{4^2} = 2\sqrt[3]{2}$$

Examining a graph (shown below), we see that the outer radius is $x = \sqrt{y}$ and the inner radius is $x = \frac{y^2}{4}$.



So, the total volume of the solid of revolution is given by:

$$V = \pi \int_0^{2\sqrt[3]{2}} (\sqrt{y})^2 - \left(\frac{y^2}{4}\right)^2 \, dy$$

$$V = \pi \int_0^{2\sqrt[3]{2}} \left[y - \frac{y^4}{16}\right] dy$$

$$V = \pi \left[\frac{1}{2}y^2 - \frac{1}{80}y^5\right]_{y=0}^{y=2\sqrt[3]{2}}$$

$$V = \pi \left[\frac{6}{5}2^{2/3}\right] \approx 5.9844$$

Answer to Exercise 8 (on page 34)

If each cross section is a square, then the volume of each cross section is given by $s^2 dx$, where s is the side length of the square. Since the side length is equal to the distance between the graph of y and the x-axis, we can see that $s = y = \ln(3 - x)$. And, therefore, the total volume of all the cross sections is given by $\int_0^2 [\ln(3-x)]^2\, dx$. Using a calculator, this integral evaluates to ≈ 1.029.

Answer to Exercise 9 (on page 35)

Since the cross-sections are perpendicular to the x-axis, they will have width dx and we will integrate across the domain of the ellipse. Setting $y = 0$ to find the domain of the ellipse:

$$9x^2 = 25 \rightarrow x^2 = \frac{25}{9} \rightarrow x = \pm\frac{5}{3}$$

A right isosceles triangle with hypotenuse h has area $\frac{1}{4}h^2$. In this case, each triangle's hypotenuse is given by the distance between the top and bottom of the ellipse. The top of the ellipse if defined by $y = \frac{1}{4}\sqrt{25 - 9x^2}$ and the bottom by $y = -\frac{1}{4}\sqrt{25 - 9x^2}$. Therefore, the length of each hypotenuse is $\frac{1}{2}\sqrt{25 - 9x^2}$.

Then, each cross-section has a total volume of $\frac{1}{4}h^2 dx = \frac{1}{4}\left(\frac{1}{2}\sqrt{25 - 9x^2}\right)^2 dx$ and the volume of the solid is:

$$V_{\text{solid}} = \int_{-5/3}^{5/3} \frac{1}{4}\left(\frac{1}{2}\sqrt{25 - 9x^2}\right)^2 dx$$

$$= \frac{1}{4}\int_{-5/3}^{5/3} \frac{1}{4}\left(25 - 9x^2\right) dx$$

$$= \frac{1}{6}\int_{-5/3}^{5/3} \left(25 - 9x^2\right) dx = \frac{1}{16}\left[25x - 3x^3\right]_{x=-5/3}^{x=5/3}$$

$$= \frac{1}{16}\left[\left(25\left(\frac{5}{3}\right) - 25\left(\frac{-5}{3}\right)\right) - \left(3\left(\frac{5}{3}\right)^3 - 3\left(\frac{-5}{3}\right)^3\right)\right]$$

$$= \frac{1}{16}\left[\frac{250}{3} - \left(\frac{375}{27} + \frac{375}{27}\right)\right] = \frac{1}{16}\left[\frac{250}{3} - \frac{250}{9}\right] = \frac{1}{16}\left[\frac{750}{9} - \frac{250}{9}\right] = \frac{1}{16}\left[\frac{500}{9}\right] = \frac{125}{36}$$

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