Contents

Chapter 1

Euler's Method

How do computers approximate the solution to a differential equation that cannot be explicitly solved? Let's consider the differential equation

$$\frac{dy}{dx} = x + y \text{ with initial condition } y(0) = 1$$

.

This means the solution passes through the point (0, 1). Additionally, the slope of the solution is $\frac{dy}{dx} = 0 + 1 = 1$ at that point. This means we can approximate the solution with the linear function $L(x) = x + 1$ (see figure 1.1). As you can see, near $(0, 1)$ the approximation is good, but as x increases, the divergence between the actual solution and the approximation grows.

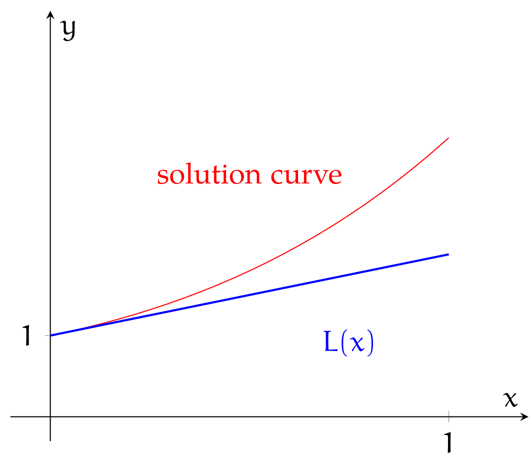


Figure 1.1: A first Euler approximation

How can we make a better approximation? Suppose we stop the first approximation at $x = 1.5$, re-evaluate $\frac{dy}{dx}$, and use that to make a second linear approximation. When $x = 0.5$, $L(x) = 0.5 + 1 = 1.5$. Taking the point $(0.5, 1.5)$, then $\frac{dy}{dx} = 0.5 + 1.5 = 2$. We can then write a second linear approximation, $L_2(x) = 2(x - 0.5) + 1.5 = 2x - 1 + 1.5 = 2x + 0.5$. As you can see (figure 1.2), this new approximation is closer than our first approximation. We call this an approximation with a step size of 0.5.

We can improve this further by taking a step size of 0.25 (see figure 1.3). As the step size decreases and the step number increases, the approximation gets closer and closer to the true solution.

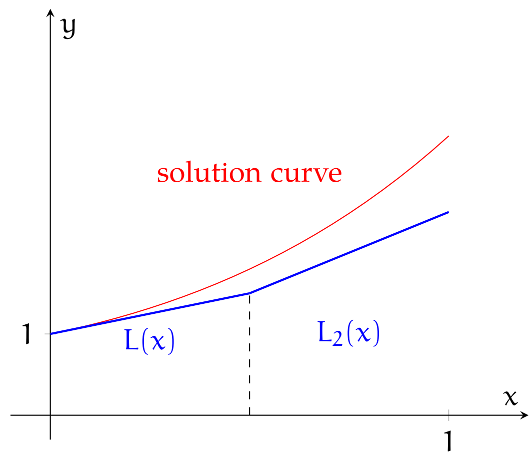


Figure 1.2: An Euler approximation with step size 0.5

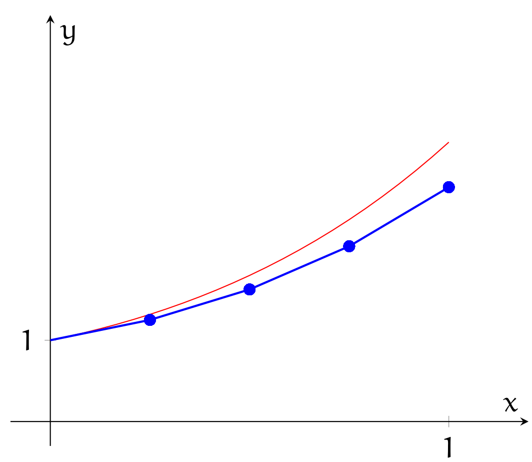


Figure 1.3: An Euler approximation with step size 0.25

In general, Euler's method is a numerical process similar to sketching a solution on a slope field. One begins at the given initial value, proceeds for a short step in the direction indicated by the slope field. You adjust the slope of your approximation based on the value of the slope field at the end of each step.

For a first-order differential equation, let $\frac{dy}{dx} = F(x, y)$ and $y(x_0) = y_0$. If we have step size h, then our successive x-values are $x_1 = x_0 + h$, $x_2 = x_1 + h$, etc. The differential equation tells us that the slope at x_0 is $F(x_0, y_0)$. So, $y_1 = y_0 + hF(x_0, y_0)$ (see figure 1.4).

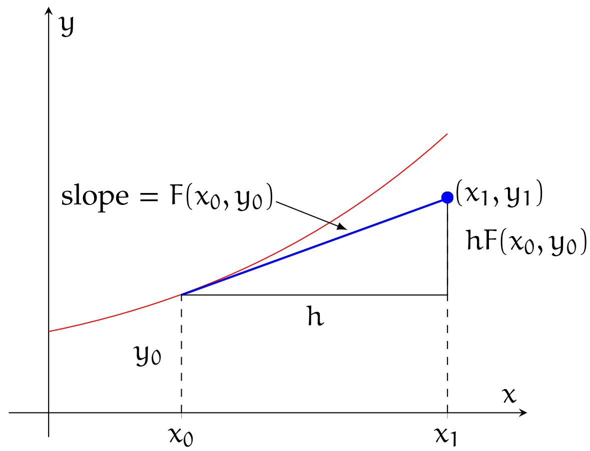


Figure 1.4: Visualization of Euler's method

Continuing, once we have found y_1, we can then define $x_2 = x_1 + h$ and $y_2 = y_1 + hF(x_1, y_1)$. And in general, for an initial-value problem when $\frac{dy}{dx} = F(x, y)$ and $y(x_0) = y_0$, we can make an approximation with step size h where:

$$y_n = y_{n-1} + hF(x_{n-1}, y_{n-1})$$

where $n = 1, 2, 3, \cdots$.

Example: Use Euler's method with a step size of 0.2 to approximate the value of $y(1)$ if $\frac{dy}{dx} = 2x + y$ and $y(0) = 1$.

Solution: We are given $h = 0.2$, $x_0 = 0$, $y_0 = 1$, and $F(x, y) = 2x + y$. This means we will need 5 steps to reach $x_5 = 1$. So, we know that:

$$y_1 = 1 + 0.2[2(0) + 1] = 1 + 0.2[1] = 1.2$$

$$y_2 = 1.2 + 0.2[2(0.2) + 1.2] = 1.2 + 0.2(1.6) = 1.52$$

$$y_3 = 1.52 + 0.2[2(0.4) + 1.52] = 1.984$$

We can continue in this manner. The results are shown in the table:

n	x_n	y_n	$F(x_n, y_n)$
0	0	1	1
1	0.2	1.2	1.6
2	0.4	1.52	2.32
3	0.6	1.984	3.184
4	0.8	2.6208	4.2208
5	1	3.46496	–

Therefore, $y(1) \approx 3.4696$.

Example: This problem was originally presented as a no-calculator, multiple-choice question on the 2012 AP Calculus BC exam. Let $y = f(x)$ be the solution to $\frac{dy}{dx} = x - y$ with initial condition $f(1) = 3$. What is the approximation of $f(2)$ obtained using Euler's method with two steps of equal length starting at $x = 1$?

Solution: The question asks that we use Euler's method two steps. The step size should be $h = \frac{x_2 - x_0}{2} = \frac{2-1}{2} = \frac{1}{2}$. Taking $x_0 = 1$ and $y_0 = 3$, we find that:

$$y_1 = y_0 + h\left[x_0 - y_0\right]$$

$$y_1 = 3 + \frac{1}{2}\left[1 - 3\right]$$

$$y_1 = 3 + \frac{1}{2}\left[-2\right] = 3 - 1 = 2$$

So our intermediate point is $(x_1, y_1) = (\frac{3}{2}, 2)$. Finding y_2:

$$y_2 = y_1 + h\left[x_1 - y_1\right]$$

$$y_2 = 2 + \frac{1}{2}\left[\frac{3}{2} - 2\right]$$

$$y_2 = 2 + \frac{1}{2}\left[\frac{-1}{2}\right] = 2 - \frac{1}{4} = \frac{7}{4}$$

So the approximate value of $f(2)$ is $\frac{7}{4}$.

Exercise 1

Working Space

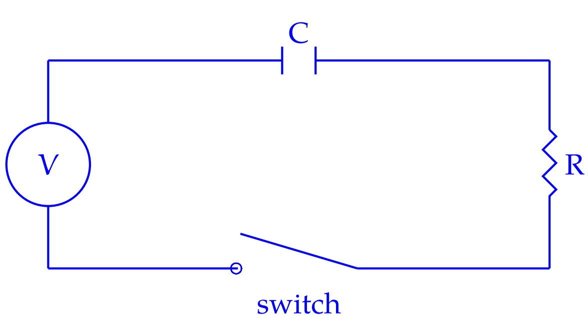
In the previous chapter on slope fields, we discussed the behavior of inductors in electronic circuits. As you may recall, capacitors also exhibit more complex behavior than regular resistors. Consider a circuit with a resistor and capacitor (see figure below). Let the resistor have resistance R ohms and the capacitor have capacitance C farads. By Kirchhoff's law, we know that:

$$RI + \frac{Q}{C} = V$$

where Q is the charge on each side of the capacitor and $\frac{Q}{C}$ is the voltage drop across the capacitor. Recall that current is the change in charge over time. Therefore, $I = \frac{dQ}{dt}$, and we can write the differential equation:

$$R\frac{dQ}{dt} + \frac{1}{C}Q = V$$

When the switch is first closed, there is no charge (that is, $Q(0) = 0$). If the resistor is 5Ω, the battery is 60V, and the capacitor is 0.05F, use Euler's method with a step size of 0.1 to estimate the charge after half a second.



Answer on Page 57

Exercise 2

[This problem was originally presented as a calculator-allowed, free-response question on the 2012 AP Calculus BC exam.] The function f is twice- differentiable for $x > 0$ with $f(1) = 15$ and $f''(1) = 20$. Values of f', the derivative of f, are given for selected values of x in the table below. Use Euler's method, starting at $x = 1$ with two steps of equal size, to approximate $f(1.4)$. Show the computations that lead to your answer.

x	1	1.1	1.2	1.3	1.4
$f'(x)$	8	10	12	13	14.5

Working Space

Answer on Page 60

Chapter 2

Sequences in Calculus

We introduced sequences in a previous chapter. Now, we will examine them in more detail in the context of calculus. You already know about arithmetic and geometric sequences, but not all sequences can be classified as arithmetic or geometric. Take the famous Fibonacci sequence, {1, 1, 2, 3, 5, 8, ...}, which can be explicitly defined as $a_n = a_{n-1} + a_{n-2}$, with $a_1 = a_2 = 1$. There is no common difference or common ratio, so the Fibonacci sequence is not arithmetic or geometric. Another example is $a_n = \sin \frac{n\pi}{6}$, which will cycle through a set of values.

Sequences have many real-world applications, including compound interest and modeling population growth. In later chapters, you will learn that the sum of all the values in a sequence is a series and how to use series to describe functions. In order to be able to do all that, we first need to talk in depth about sequences.

Some sequences are defined explicitly, like $a_n = \sin \frac{n\pi}{6}$, while others are defined recursively, like $a_n = a_{n-1} + a_{n-2}$.

Example: Write the first 5 terms for the explicitly defined sequence $a_n = \frac{n}{n+1}$.

Solution: We can construct a table to keep track of our work:

n	work	a_n
1	$\frac{1}{1+1}$	$\frac{1}{2}$
2	$\frac{2}{2+1}$	$\frac{2}{3}$
3	$\frac{3}{3+1}$	$\frac{3}{4}$
4	$\frac{4}{4+1}$	$\frac{4}{5}$
5	$\frac{5}{5+1}$	$\frac{5}{6}$

So, the first five terms are $\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}$, and $\frac{5}{6}\}$.

Exercise 3

Write the first five terms for each sequence.

1. $a_n = \frac{2^n}{2n+1}$
2. $a_n = \cos\frac{n\pi}{2}$
3. $a_1 = 1$, $a_{n+1} = 5a_n - 3$
4. $a_1 = 6$, $a_{n+1} = \frac{a_n}{n+1}$

Working Space

Answer on Page 57

2.1 Convergence and Divergence

You can visualize a sequence on an xy-plane or a number line. Figures 2.1 and 2.2 show visualizations of the sequence $a_n = \frac{n}{n+1}$. To visualize this on the xy-plane, we take points such that $x = n$ and $y = a_n$, where n is a positive integer. What do you notice about this sequence? As n increases, a_n gets closer and closer to 1.

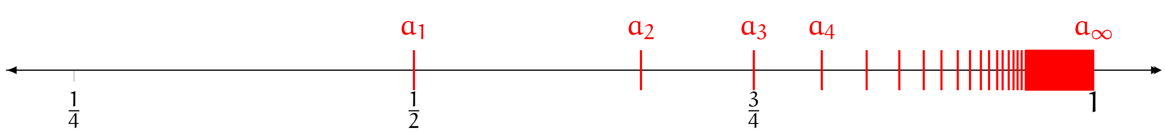


Figure 2.1: $a_n = \frac{n}{n+1}$ on a number line

Because a_n approaches a specific number as $n \to \infty$, we call the series $a_n = \frac{n}{n+1}$ *convergent*. We prove a sequence is convergent by taking the limit as n approaches ∞. If the limit exists and approaches a specific number, the sequence is convergent. If the limit does not exist or approaches $\pm\infty$, the sequence is divergent.

We can see graphically that $\lim_{n\to\infty} \frac{n}{n+1} = 1$, so that sequence is convergent. What about $b_n = \frac{n}{\sqrt{10+n}}$? Is b_n convergent or divergent?

$$\lim_{n\to\infty} \frac{n}{\sqrt{10+n}} = \lim_{n\to\infty} \frac{n/n}{\sqrt{\frac{10}{n^2} + \frac{n}{n^2}}}$$

$$= \lim_{n\to\infty} \frac{1}{\sqrt{\frac{10}{n^2} + \frac{1}{n}}} = \infty$$

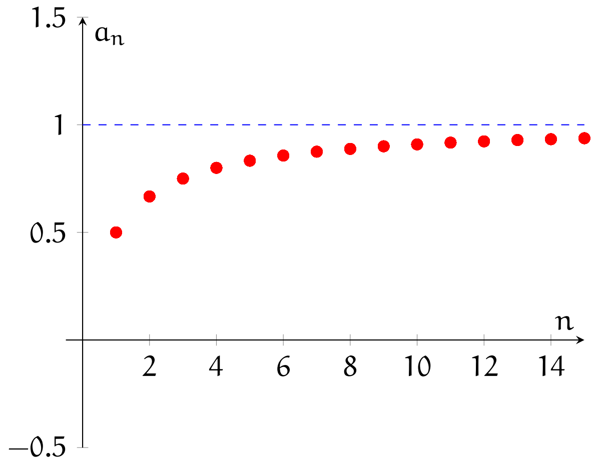


Figure 2.2: $a_n = \frac{n}{n+1}$ on an xy-plane

Therefore, the sequence $b_n = \frac{n}{\sqrt{10+n}}$ is divergent.

Here is another example of a divergent sequence: $c_n = \sin \frac{n\pi}{2}$. The graph is shown in figure 2.3. As you can see, the value of c_n oscillates between 1, 0, and -1 without approaching a specific number. This means that c_n does not approach a particular number as $n \to \infty$ and the sequence is divergent.

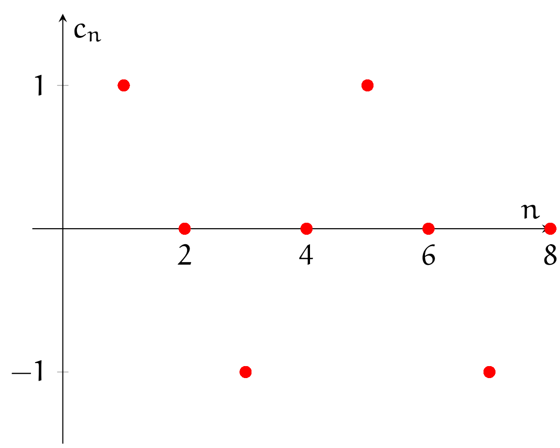


Figure 2.3: $c_n = \sin \frac{n\pi}{2}$ on an xy-plane

Exercise 4

Classify each sequence as convergent or divergent. If the sequence is convergent, find the limit as $n \to \infty$.

1. $a_n = \frac{3+5n^2}{n+n^2}$
2. $a_n = \frac{n^4}{n^3-2n}$
3. $a_n = 2 + (0.86)^n$
4. $a_n = \cos \frac{n\pi}{n+1}$
5. $a_n = \sin n$

Working Space

Answer on Page 58

2.2 Evaluating limits of sequences

Recall that a sequence can be considered a function where the domain is restricted to positive integers. If there is some $f(x)$ such that $a_n = f(n)$ when n is an integer, then $\lim_{n\to\infty} a_n = \lim_{x\to\infty} f(x)$ (see figure 2.4). This means that all the rules that apply to the limits of functions also apply to the limits of sequences, including the Squeeze Theorem and l'Hospital's rule.

Example: What is $\lim_{n\to\infty} \frac{\ln n}{n}$?

Solution: First, we will try to compute the limit directly:

$$\lim_{n\to\infty} \frac{\ln n}{n} =$$

$$\frac{\lim_{n\to\infty} \ln n}{\lim_{n\to\infty} n} = \frac{\infty}{\infty}$$

This is undefined, but fits the criteria for L'Hospital's rule:

$$\lim_{n\to\infty} \frac{\ln n}{n} = \lim_{n\to\infty} \frac{\frac{d}{dn} \ln n}{\frac{d}{dn} n}$$

$$= \lim_{n\to\infty} \frac{\frac{1}{n}}{1} = 0$$

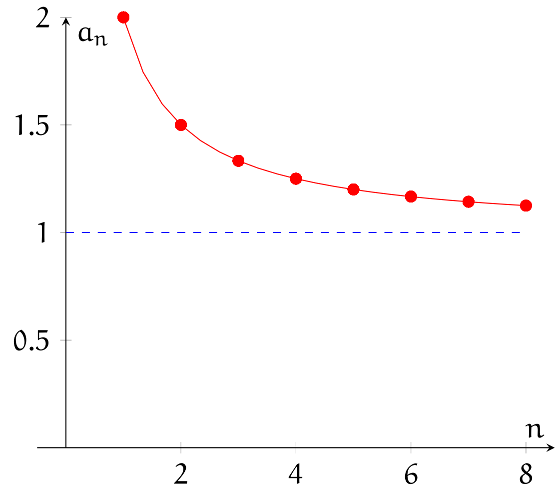


Figure 2.4: The limit of the function is the same as the limit of the sequence

Example: Is the sequence $a_n = \frac{n!}{n^n}$ convergent or divergent?

Solution: First trying to take the limit directly, we see that:

$$\lim_{n\to\infty} \frac{n!}{n^n} = \frac{\infty}{\infty}$$

which is undefined. Because the factorial cannot be described as a continuous function, we can't use L'Hospital's rule. We can examine this sequence graphically (see figure 2.5) and mathematically. We examine it mathematically by writing out a few terms to get an idea of what happens to a_n as n gets large:

$$a_1 = \frac{1!}{1^1} = 1$$

$$a_2 = \frac{2!}{2^2} = \frac{1 \cdot 2}{2 \cdot 2}$$
$$a_3 = \frac{3!}{3^3} = \frac{1 \cdot 2 \cdot 3}{3 \cdot 3 \cdot 3}$$
$$\dots$$
$$a_n = \frac{n!}{n^n} = \frac{1 \cdot 2 \cdot 3 \cdot \dots \cdot n}{n \cdot n \cdot n \cdots n}$$

From examining the graph in figure 2.5, we can guess that $\lim_{n\to\infty} a_n = 0$. Let's prove that mathematically. We can rewrite our expression for a_n as n gets larger:

$$a_n = \frac{n!}{n^n} = \frac{1 \cdot 2 \cdot 3 \cdot \cdots \cdot n}{n \cdot n \cdot n \cdots n} = \frac{1}{n}(\frac{2 \cdot 3 \cdots n}{n \cdot n \cdots n})$$

The expression inside the parentheses is less than 1; therefore, $0 < a_n < \frac{1}{n}$. Since $\lim_{n\to\infty} 0 = 0$ and $\lim_{n\to\infty} \frac{1}{n} = 0$, by Squeeze Theorem, we know that $\lim_{n\to\infty} \frac{n!}{n^n} = 0$. Therefore, the sequence $a_n = \frac{n!}{n^n}$ is convergent.

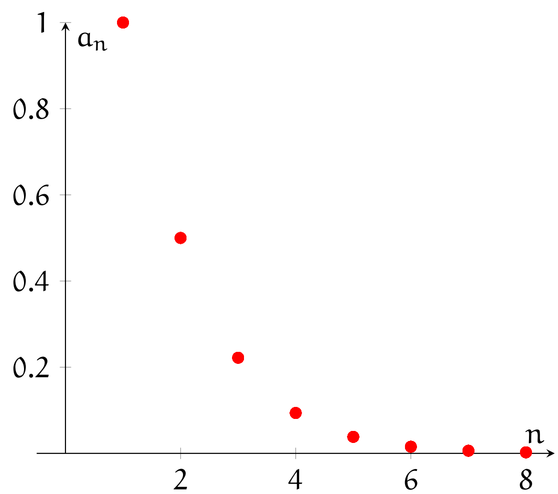


Figure 2.5: $a_n = \frac{n!}{n^n}$

[[FIX ME intro]] If $\lim_{n\to\infty} a_n = L$ and the function f is continuous at L, then $\lim_{n\to\infty} f(a_n) = f(L)$. For example, what is $\lim_{n\to\infty} \sin\frac{\pi}{n}$? Well, we know that $\lim_{n\to\infty} \frac{\pi}{n} = 0$ and that the sine function is continuous at 0. Therefore, $\lim_{n\to\infty} \sin\frac{\pi}{n} = \sin \lim_{n\to\infty} \frac{\pi}{n} = \sin 0 = 0$.

2.3 Monotonic and Bounded sequences

Just like functions, sequences can be increasing or decreasing. A sequence is increasing if $a_n < a_{n+1}$ for $n \geq 1$. Similarly, a sequence is decreasing if $a_n > a_{n+1}$ for $n \geq 1$. If a sequence is strictly increasing or decreasing, it is called *monotonic*.

The sequence $a_n = \frac{1}{n+6}$ is decreasing. We prove this formally by comparing a_n to a_{n+1}:

$$\frac{1}{n+6} > \frac{1}{(n+1)+6} = \frac{1}{n+7}$$

Example: Is the sequence $a_n = \frac{n}{n^2+1}$ increasing or decreasing?

Solution: First, we find an expression for a_{n+1}:

$$a_{n+1} = \frac{n+1}{(n+1)^2+1} = \frac{n+1}{n^2+2n+2}$$

Since the degree of n is greater in the denominator, we have a guess that the sequence is decreasing. To prove this, we check if $a_n > a_{n+1}$ is true:

$$\frac{n}{n^2+1} > \frac{n+1}{n^2+2n+2}$$

We can cross-multiply, because $n > 0$ and the denominators are positive:

$$(n)(n^2+2n+2) > (n+1)(n^2+1)$$

$$n^3+2n^2+2n > n^3+n^2+n+1$$

Subtracting (n^3+n^2+n) from both sides we see that:

$$n^2+n > 1$$

Which is true for all $n \geq 1$. Therefore, $a_n > a_{n+1}$ for all $n \geq 1$ and the sequence is decreasing.

A sequence is *bounded above* if there is some number M such that $a_n \leq M$ for all $n \geq 1$. A sequence is *bounded below* if there is some other number m such that $a_n \geq m$ for all $n \geq 1$. If a sequence is bounded above and below, then it is a *bounded sequence*.

Not all bounded sequences are convergent. Take our earlier example of $a_n = \sin\frac{n\pi}{6}$. This sequence is bounded, since we can say that $-1 \leq a_n \leq 1$ for all n. However, $a_n = \sin\frac{n\pi}{6}$ is divergent because $\lim_{n\to\infty}\sin\frac{n\pi}{6}$ does not exist (see figure 2.6). Additionally, not all monotonic sequences are convergent. Consider $b_n = 2^n$ (shown in figure 2.7). This is monotonically increasing (that is, $b_n > b_{n-1}$ for all n), but $\lim_{n\to\infty}2^n = \infty$ and the sequence is divergent.

A sequence must be convergent if it is **both** monotonic and bounded. Why is this? Recall that to be bounded, a sequence must be bounded above and below, which means there is some m and some M such that $m \leq a_n \leq M$ for all n. If the sequence is increasing, the terms must get close to but not exceed M. Likewise, if the sequence is decreasing, the terms must get close to, but not be less than m.

Example: Is the sequence given by $a_n = 4$ and $a_{n+1} = \frac{1}{2}(a_n+7)$ bounded above, below, both, or neither?

Solution: We start by calculating the first several terms:

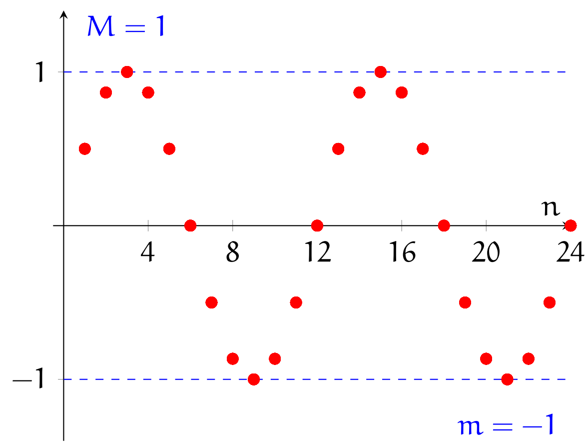


Figure 2.6: The sequence $a_n = \sin \frac{n\pi}{6}$ is bounded and divergent

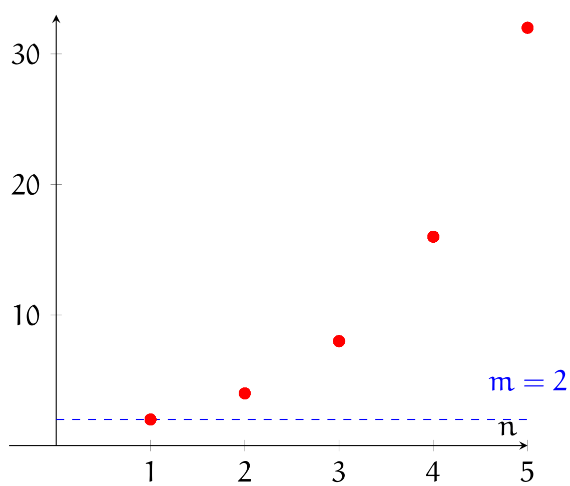


Figure 2.7: The sequence $b_n = 2^n$ is bounded below, monotonically increasing, and divergent

Term	Work	Value
a_1	$a_1 = 4$	4
a_2	$= \frac{1}{2}(4+7)$	5.5
a_3	$= \frac{1}{2}(5.5+7)$	6.25
a_4	$= \frac{1}{2}(6.25+7)$	6.625
a_5	$= \frac{1}{2}(6.625+7)$	6.8125
a_6	$= \frac{1}{2}(6.8125+7)$	6.90625
a_7	$= \frac{1}{2}(6.90625+7)$	6.953125
a_8	$= \frac{1}{2}(6.953125+7)$	6.9765625

The sequence is increasing, so it is bounded below by the initial term, $a_1 = 4$, and we can state that $a_n \geq 4$. Examining the computed terms, we see that $a_n \to 7$ as n grows larger. We can guess that this sequence is bounded above, with $a_n \leq 7$. We can prove this by induction. Suppose that there is some k such that $a_k < 7$ (which is true for a_1, etc.). Then,

$$a_k < 7$$

$$a_k + 7 < 14$$

$$\frac{1}{2}(a_k + 7) < \frac{1}{2}(14)$$

$$a_{k+1} < 7$$

Therefore, $a_n < 7$ for all n and the sequence is bounded above. Because the sequence is monotonic and bounded, we know the sequence is convergent and, therefore, that the limit of a_n as $n \to \infty$ exists.

2.4 Applications of Sequences

2.4.1 Compound Interest

You previously learned about compound interest and modeled the accumulation of compound interest by $P_n = P_0(1+r)^n$, where P_0 is the principal investment, r is the yearly interest rate, and n is the number of elapsed years. This sequence describes the value of an investment accumulating interest, but most people add to their savings on a regular schedule. We can write a sequence to model the value of a savings account that the owner makes regular deposits into.

Example: Suppose you open a savings account with an initial deposit of \$3,000 and you plan to deposit an additional \$1,200 at the end of every year. If your savings account has an annual interest rate of 3.25%, how long will it take you to save \$10,000?

Solution: We can write a recursive definition for the sequence. At the end of each year, the account will gain the interest on the entirety of the previous year's balance plus $1200:

$$P_n = P_{n-1}(1 + 0.0325) + \$1200$$

With an initial investment $P_0 = \$3000$. We can write out the first few terms to find how many years it will take to save $10,000:

Year	Savings
0	$3,000
1	$4,297.50
2	$5,637.17
3	$7,020.38
4	$8,448.54
5	$9.923.12
6	$11,445.62

The accumulation of interest with deposits is better described by a sequence than a function. That iss because the deposits are happening at discrete times, not continuously.

Exercise 5

You invest $1500 at 5%, compounded annually. Write an explicit formula that describes the value of your investment every year. What will your investment be worth after 10 years? Is the sequence convergent or divergent? Explain.

Working Space

Answer on Page 58

2.4.2 Population Growth

Sequences can be used to model a reproducing population that is being occasionally culled from or added to. Similar to compound interest, a population of living things (plants, animals, fungi, etc.) reproducing at a rate r can be modeled with an exponential function:

$$P_n = P_0(1 + r)^n$$

where P_0 is the initial population, r is the yearly reproductive rate, and n is the number of years elapsed.

Example: Suppose the population of deer in a national park is estimated to be 6,500. If the deer reproduce at a rate of 8% per year and wolves hunt and kill 500 deer per year, how many deer will be in the park in 5 years?

Solution: We can write a recursive sequence:

$$P_n = P_{n-1}(1 + 0.08) - 500$$

$$P_0 = 6500$$

And calculate P_5 (we round to the nearest whole number because half of a deer is not a living deer):

Year		Population
1	$6500(1.08) - 500$	6520
2	$6520(1.08) - 500$	6542
3	$6542(1.08) - 500$	6565
4	$6565(1.08) - 500$	6590
5	$6590(1.08) - 500$	6617

There will be 6617 deer in the park after 5 years.

Exercise 6

A farmer keeps his pond stocked with fish. If the fish are eaten by predators at a rate of 5% per month and the farmer can afford to restock the pond with 10 fish every 6 months. If the farmer starts with 100 fish, how many total fish will he have lost to predation after 4 years?

Working Space

Answer on Page 58

Chapter 3

Series

When writing a number with an infinite decimal, such as the Golden Ratio (also known as the Golden Number):

$$\phi = 1.618033988\cdots$$

The decimal system means we can rewrite the Golden Ratio (or any irrational number) as an infinite sum:

$$\phi = 1 + \frac{6}{10} + \frac{1}{10^2} + \frac{8}{10^3} + \frac{0}{10^4} + \frac{3}{10^5} + \cdots$$

You might recall from the chapter on Riemann Sums that we can represent the addition of many (or infinite) with big sigma notation:

$$\sum_{i=1}^{n} a_i$$

where i is the index as discussed in Sequences and n is the number of terms. For infinite sums, $n = \infty$.

3.1 Partial Sums

Let's quickly define a *partial sum*. A partial sum is where we only look at the first n terms of a series. For the general series, $\sum_{i=1}^{n} a_i$, the partial sums are:

$$s_1 = a_1$$

$$s_2 = a_1 + a_2$$

$$s_3 = a_1 + a_2 + a_3$$

$$\cdots$$

$$s_n = a_1 + a_2 + \cdots + a_n = \sum_{i=1}^{n} a_i$$

Example: A series is given by $\sum_{i=1}^{\infty}(\frac{-3}{4})^i$. What is the value of the partial sum s_4?

Solution: s_4 is the sum of the first 4 terms:

$$(\frac{-3}{4})^1 + (\frac{-3}{4})^2 + (\frac{-3}{4})^3 + (\frac{-3}{4})^4$$

$$= \frac{-3}{4} + \frac{9}{16} + \frac{-27}{64} + \frac{81}{256} = \frac{-75}{256}$$

3.2 Reindexing

Sometimes it is necessary to re-index series. This means changing what n the series starts at . In general,

$$\sum_{n=i}^{\infty} a_n = \sum_{n=i+1}^{\infty} a_{n-1} \text{ and } \sum_{n=i}^{\infty} a_n = \sum_{n=i-1}^{\infty} a_{n+1}$$

In other words, to increase the index by 1, you need to replace n with $(n-1)$ and do decrease the index by 1, you need to replace n with $(n+1)$. Let's visualize why this is true (see figure 3.1). Notice that for each series, the terms are the same. This is similar to shifting functions: to move the function to the left on the x-axis, you plot $f(x+1)$, and to move it to the right, $f(x-1)$.

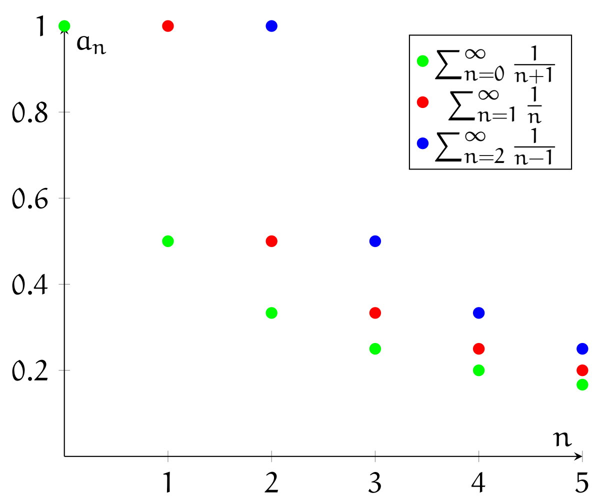


Figure 3.1: $\sum_{n=0}^{\infty} \frac{1}{n+1} = \sum_{n=1}^{\infty} a_n = \sum_{n=2}^{\infty} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$

We can also prove each reindexing rule mathematically. Recall that

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$$

We also know that

$$\sum_{n=2}^{\infty} a_{n-1} = a_{2-1} + a_{3-1} + a_{4-1} + \cdots = a_1 + a_2 + a_3 + \cdots$$

Therefore, $\sum_{n=1}^{\infty} a_n = \sum_{n=2}^{\infty} a_{n-1}$.

Similarly,

$$\sum_{n=0}^{\infty} a_{n+1} = a_{0+1} + a_{1+1} + a_{2+1} + \cdots = a_1 + a_2 + a_3 + \cdots = \sum_{n=1}^{\infty} a_n$$

Example: Reindex the series $\sum_{n=3}^{\infty} \frac{n+1}{n^2-2}$ to begin with $n = 1$.

Solution: We are decreasing the index, so we will use $\sum_{n=i-1}^{\infty} a_{n+1} = \sum_{n=i}^{\infty} a_n$. We will apply this rule twice, to decrease the index from 3 to 1:

$$\sum_{n=2}^{\infty} \frac{(n+1)+1}{(n+1)^2-2} = \sum_{n=2}^{\infty} \frac{n+2}{(n+1)^2-2}$$

$$\sum_{n=1}^{\infty} \frac{(n+1)+2}{[(n+1)+1]^2-2} = \sum_{n=1}^{\infty} \frac{n+3}{(n+2)^2-2}$$

It is easier and faster to be able to reindex a series by more than one step at a time. Using the example above, we can write an even more general rule for reindexing:

$$\sum_{n=i}^{\infty} a_n = \sum_{n=i+j}^{\infty} a_{n-j}$$

where i and j are integers. (Then, to decrease the index, you would choose a j such that $j < 0$.)

3.3 Convergent and Divergent Series

Just like sequences, series can also be convergent or divergent. Consider the series $\sum_{i=1}^{\infty} i$. Given what you already know about the meaning of ”convergent” and ”divergent”, guess whether $\sum_{i=1}^{\infty} i$ is convergent or divergent.

Let’s determine the first few partial sums of the series (shown graphically in figure 3.2):

n	Terms	Partial Sum
1	1	1
2	1+2	3
3	1+2+3	6
4	1+2+3+4	10

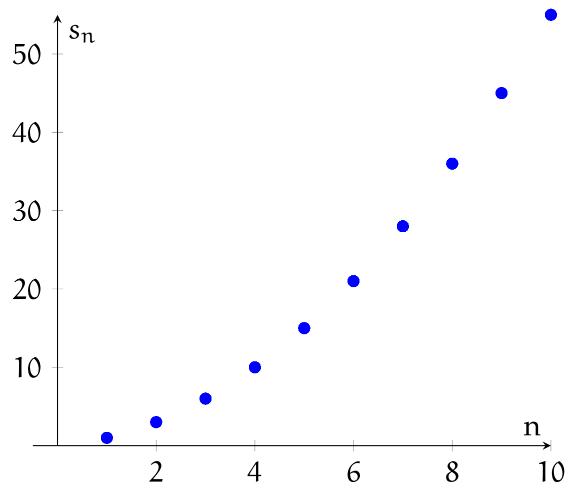


Figure 3.2: For the divergent series $\sum_{i=1}^{n} i$, the value of the partial sum increases to infinity as n increases

As you can see, as n increases, the value of the partial sum increases without approaching a particular value. We can also see that the value of the first n terms summed together is $\frac{n(n+1)}{2}$. This means that as n approaches ∞, the sum also approaches ∞ and the series is divergent.

Obviously, for a series to not become overly large, the values of the terms should decrease as i increases (that is, each subsequent term is smaller than the one before it). Take the series $\sum_{i=1}^{\infty} \frac{1}{2^i}$. As i increases, $\frac{1}{2^i}$ decreases. Let's look at the first few partial sums of this series (shown graphically in figure 3.3):

n	Terms	Partial Sum
1	$\frac{1}{2}$	$\frac{1}{2}$
2	$\frac{1}{2} + \frac{1}{4}$	$\frac{3}{4}$
3	$\frac{1}{2} + \frac{1}{4} + \frac{1}{8}$	$\frac{7}{8}$
4	$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}$	$\frac{15}{16}$

Do you see the pattern? The n^{th} partial sum is equal to $\frac{2^n-1}{2^n} = 1 - \frac{1}{2^n}$. And as n approaches ∞, the partial sum approaches 1. The series $\sum_{i=1}^{\infty} \frac{1}{2^i}$ is convergent.

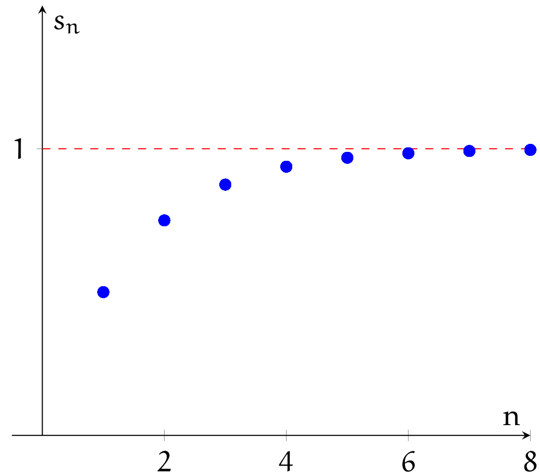


Figure 3.3: For the convergent series $\sum_{i=1}^{n} \frac{1}{2^i}$, the value of the partial sum approaches 1 as n increases

Let's define the sequence $\{s_n\}$, where s_n is the n^{th} partial sum of a series:

$$s_n = \sum_{i=1}^{n} a_i$$

.

If the sequence $\{s_n\}$ is convergent and $\lim_{n\to\infty} s_n$ exists, then the series $\sum_{i=1}^{\infty} a_i$ is also convergent. And if the sequence $\{s_n\}$ is divergent, then the series $\sum_{i=1}^{\infty} a_i$ is also divergent.

Example: Is the harmonic series, $\sum_{n=1}^{\infty} \frac{1}{n}$ convergent or divergent?

Solution: You may think that the series is convergent, since $\lim_{n\to\infty} \frac{1}{n} = 0$. Let's see if we can confirm this. We begin by looking at the partial sums s_2, s_4, s_8, and s_{16}:

$$s_2 = 1 + \frac{1}{2}$$

$$s_4 = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) = 1 + \frac{2}{2}$$

$$s_8 = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) = 1 + \frac{3}{2}$$

$$s_{16} = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \left(\frac{1}{9} + \cdots + \frac{1}{16}\right) >$$
$$1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \left(\frac{1}{16} + \cdots + \frac{1}{16}\right) = 1 + \frac{4}{2}$$

Notice that, in general, $s_{2^n} > 1 + \frac{n}{2}$ for $n > 1$. Taking the limit as $n \to \infty$, we see that

$\lim_{n\to\infty} s_{2^n} > \lim_{n\to\infty} 1 + \frac{n}{2} = \infty$. Therefore, s_{2^n} also approaches ∞ as n gets larger and the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

This example shows a very important point: A series whose terms decrease to zero as n gets large is not necessarily convergent. What we can say, though, is that if the limit as n approaches infinity of the terms of a series does not exist or is not zero, then the series is divergent (i.e., not convergent). This is called the **Test for Divergence**, and we will explore it further in the next chapter.

3.3.1 Properties of Convergent Series

We just saw that if $\lim_{n\to\infty} a_n \neq 0$ then the series $\sum_{n=1}^{\infty} a_n$ diverges. The contrapositive statement gives a property of convergent series:

$$\text{If the series } \sum_{n=1}^{\infty} a_n \text{ is convergent, then } \lim_{n\to\infty} = 0$$

If a series is made of other convergent series, it may be convergent. Recall, if a series is convergent, this means the $\lim_{n\to\infty} \sum_{i=1}^{n} a_i = L$. By the properties of limits, we can also say that the series multiplied by a constant is convergent:

$$\sum_{n=1}^{\infty} ca_n = c \cdot L = c \sum_{n=1}^{\infty} a_n$$

Suppose there is another convergent series such that $\lim_{n\to\infty} \sum_{i=1}^{n} b_i = M$. In this case, the sum of those series is also convergent. That is:

$$\sum_{n=1}^{\infty} (a_n + b_n) = L + M = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$$

Similarly, the difference of the series is convergent:

$$\sum_{n=1}^{\infty} (a_n - b_n) = L - M = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$$

3.4 Geometric Series

A geometric series is the sum of a geometric sequence, and has the form:

$$\sum_{n=1}^{\infty} ar^n \text{ or } \sum_{n=1}^{\infty} ar^{n-1}$$

Where a is some constant and r is the common ratio. For $\sum_{n=1}^{\infty} ar^{n-1}$, a is also the first term.

Example: Write the series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$ in sigma notation.

Solution: We see that the first term is $a = 1$ and the common ratio is $\frac{1}{2}$, so we can write the series:

$$\sum_{n=1}^{\infty} 1(\frac{1}{2})^{n-1} = \sum_{n=1}^{\infty} (\frac{1}{2})^{n-1}$$

When are geometric series convergent? First, let's consider the case where $r = 1$. If this is true, then $s_n = a + a + a + \cdots + a = na$. As n approaches ∞, the sum will approach $\pm\infty$ (depending on whether a is positive or negative), and the series is divergent.

When $r \neq 1$, we can write s_n and rs_n:

$$s_n = a + ar + ar^2 + \cdots + ar^{n-1}$$

$$rs_n = ar + ar^2 + ar^3 + \cdots + ar^n$$

Subtracting rs_n from s_n, we get:

$$s_n - rs_n = (a + ar + ar^2 + \cdots + ar^{n-1}) - (ar + ar^2 + ar^3 + \cdots + ar^{n-1} + ar^n)$$

$$= a - ar^n$$

Solving for s_n, we find:

$$s_n = \frac{a(1 - r^n)}{1 - r}$$

We take the limit as $n \to \infty$ to determine for what values of r the series converges:

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{a(1 - r^n)}{1 - r}$$

$$= \lim_{n\to\infty} \left[\frac{a}{1-r} - \frac{ar^n}{1-r}\right] = \frac{a}{1-r} - \left(\frac{a}{1-r}\right) \lim_{n\to\infty} r^n$$

This introduces the question: When is $\lim_{n\to\infty} r^n$ convergent? From the sequences chapter, we know this limit converges if $|r| < 1$ (that is, $-1 < r < 1$). If this is true, then $\lim_{n\to\infty} r^n = 0$ and

$$\lim_{n\to\infty} s_n = \frac{a}{1-r}$$

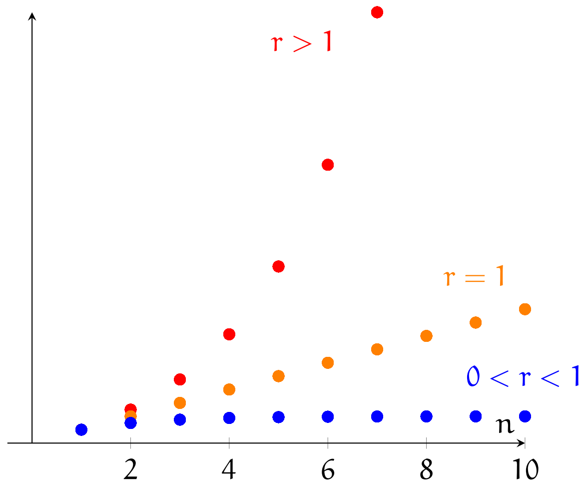


Figure 3.4: Geometric sequences are divergent if $r \geq 1$

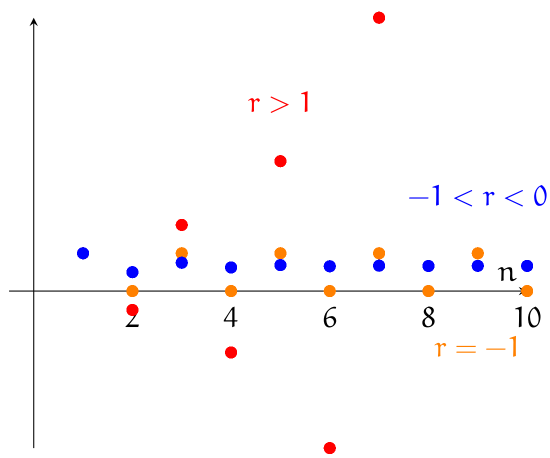


Figure 3.5: Geometric sequences are divergent if $r \leq 1$. Notice that for $r = -1$, the partial sums alternate between the initial term and zero.

(see figures 3.4 and 3.5 for a visual)

Example: Find the sum of the geometric series given by $2 - \frac{2}{3} + \frac{2}{9} - \frac{2}{27} + \cdots$.

Solution: The first term is $a = 2$, and each common ratio is $r = \frac{-1}{3}$. Since $|r| < 1$, we know that the series converges. We can calculate the value of the sum using the geometric series formula:

$$\sum_{i=1}^{\infty} a(r)^{i-1} = \frac{a}{1-r}$$

$$\sum_{i=1}^{\infty} 2(\frac{-1}{3})^{i-1} = \frac{2}{1-\frac{-1}{3}} = \frac{2}{\frac{4}{3}} = \frac{6}{4} = 1.5$$

We can confirm this graphically (see figure 3.6). You can also write out the first several partial sequences. You should find the sums approach 1.5 as n increases.

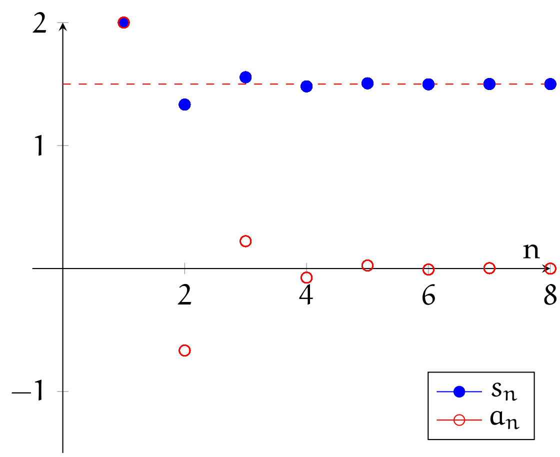


Figure 3.6: the n^{th} term and partial sums of $\sum_{i=1}^{n} 2(\frac{-1}{3})^{i-1}$

Example: What is the value of $\sum_{n=1}^{\infty} 2^{2n}5^{1-n}$

Solution: The key here is to re-write the series in the form $\sum_{n=1}^{\infty} ar^{n-1}$ so we can use the fact that convergent geometric series sum to $\frac{a}{1-r}$.

$$\sum_{n=1}^{\infty} 2^{2n}5^{1-n} = \sum_{n=1}^{\infty} \left(2^2\right)^n \left(\frac{1}{5}\right)^{n-1}$$

$$= \sum_{n=1}^{\infty} 4 \cdot (4)^{n-1} \left(\frac{1}{5}\right)^{n-1} = \sum_{n=1}^{\infty} 4 \cdot \left(\frac{4}{5}\right)^{n-1}$$

Which is in the form $\sum_{n=1}^{\infty} ar^{n-1}$ with $a = 4$ and $r = \frac{4}{5}$. Since $|r| < 1$, the series converges to

$$\frac{a}{1-r} = \frac{4}{1-\frac{4}{5}} = \frac{4}{\frac{1}{5}} = 20$$

Exercise 7

Determine whether the geometric series is convergent or divergent. If it is convergent, find its sum.

1. $3 - 4 + \frac{16}{3} - \frac{64}{9} + \cdots$
2. $2 + 0.5 + 0.125 + 0.03125 + \cdots$
3. $\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{4^n}$
4. $\sum_{n=1}^{\infty} \frac{e^{2n}}{6^{n-1}}$

Working Space

Answer on Page 59

Exercise 8

Find a value of c such that $\sum_{n=0}^{\infty}(1+c)^{-n} = \frac{5}{3}$.

Working Space

Answer on Page 59

Exercise 9

For what values of p does the series $\sum_{n=1}^{\infty} \left(\frac{p}{2}\right)^n$ converge?

Working Space

Answer on Page 60

3.5 p-series

A p-series takes the form $\sum_{n=1}^{\infty} \frac{1}{n^p}$ and converges if $p > 1$ and diverges if $p \leq 1$. We won't prove this here, since it requires the application of a test you will learn about in the next chapter.

Example Write the series $1 + \frac{1}{\sqrt[3]{2}} + \frac{1}{\sqrt[3]{3}} + \frac{1}{\sqrt[3]{4}} + \cdots$. Is it convergent or divergent?

Solution: We see that $a_n = \frac{1}{\sqrt[3]{n}}$, so the infinite series is

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}}$$

. We see that this is a p-series with $p = \frac{1}{3}$. Since $p < 1$, the series is divergent.

Exercise 10

Euler found that the exact sum of the p-series where $p = 2$ is:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

And that the exact sum of the p-series where $p = 4$ is:

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$$

Use this and the properties of convergent series to find the sum of each of the following series:

1. $\sum_{n=1}^{\infty} \frac{n^2+1}{n^4}$
2. $\sum_{n=2}^{\infty} \frac{1}{n^2}$
3. $\sum_{n=3}^{\infty} \frac{1}{(n+1)^2}$
4. $\sum_{n=1}^{\infty} \left(\frac{3}{n}\right)^4$
5. $\sum_{n=1}^{\infty} \left(\frac{4}{n^2} + \frac{3}{n^4}\right)$

Working Space

Answer on Page 60

Exercise 11

For what values of k does the series $\sum_{n=1}^{\infty} \frac{1}{n^{2k}}$ converge?

Working Space

Answer on Page 60

3.6 Alternating Series

An alternating series is one in which the terms alternate between positive and negative . Here is an example:

$$-\frac{1}{2}+\frac{2}{3}-\frac{3}{4}+\frac{4}{5}-\frac{5}{6}+\cdots=\sum_{n=1}^{\infty}(-1)^n\frac{n}{n+1}$$

Alternating series are generally of the form

$$a_n=(-1)^n b_n \text{ or } a_n=(-1)^{n-1}b_n$$

Where b_n is positive (and therefore, $|a_n| = b_n$).

An alternating series is convergent if (i)$b_{n+1} \leq b_n$ and (ii)$\lim_{n\to\infty} b_n = 0$. In other words, we say that if the absolute value of the terms of a series decrease towards zero, then the series converges. This is called the **Alternating Series Test**.

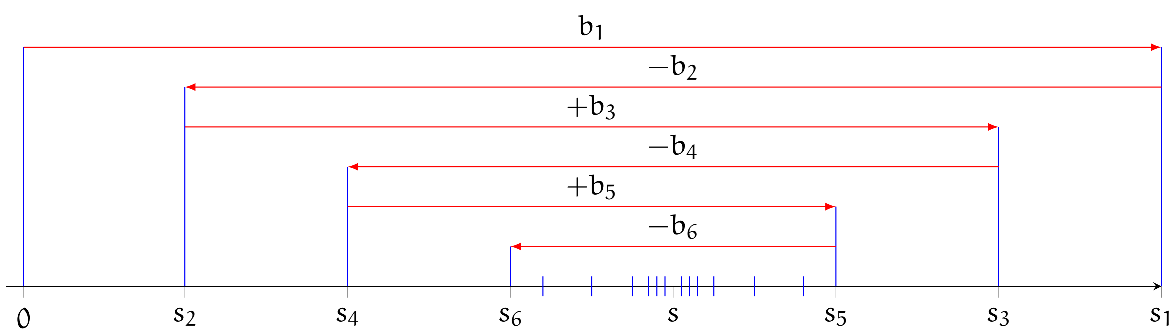


Figure 3.7: As n increases, s_n approaches s

Example: Is the alternating harmonic series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ convergent?

Solution: The Alternating series test states that an alternating series is convergent if

$|a_{n+1}| < |a_n|$:

$$\left|\frac{(-1)^{n-1+1}}{n+1}\right| < \left|\frac{(-1)^{n-1}}{n}\right|$$

$$\frac{1}{n+1} < \frac{1}{n}$$

Since $|a_{n+1}| < |a_n|$ and the series is alternating, $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ is convergent.

Exercise 12

Test the following alternating series for convergence:

1. $\sum_{n=1}^{\infty} \frac{(-1)^n 3n}{4n-1}$
2. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^3+1}$
3. $\sum_{n=1}^{\infty} (-1)^{n-1} e^{2/n}$

Working Space

Answer on Page 60

Chapter 4

Convergence Tests for Series

4.1 Test for Divergence

Recall from the previous chapter that if the terms of a series do not approach zero as n approaches infinity, then the series is divergent. This is the Test for Divergence, and there are two possible outcomes. For a series $\sum_{n=1}^{\infty} a_n$:

$$\text{If } \lim_{n\to\infty} a_n \neq 0\text{, then the series diverges}$$

$$\text{If } \lim_{n\to\infty} a_n = 0\text{, then the test is inconclusive}$$

It is important to remember that the Test for Divergence cannot tell us conclusively that a series converges. Rather, it only identifies series that are divergent.

Example: Apply the Test for Divergence to the series $\sum_{n=1}^{\infty} \sqrt{n}$ and $\sum_{n=1}^{\infty} \frac{1}{n}$

Solution: $\lim_{n\to\infty} \sqrt{n} = \infty \neq 0$. Therefore, the series $\sum_{n=1}^{\infty} \sqrt{n}$ is divergent.

$\lim_{n\to\infty} \frac{1}{n} = 0$. Therefore, the series $\sum_{n=1}^{\infty} \frac{1}{n}$ may be divergent or convergent. This is the harmonic series, which we proved to be divergent in the previous chapter. This is a good example that demonstrates that just because $\lim_{n\to\infty} a_n = 0$ does not mean the series is convergent.

4.2 The Integral Test

We were able to determine the exact value of some infinite series because it was possible to write the n^{th} partial sum, s_n, in terms of n. For example, we determined that the n^{th} partial sum of $\sum_{i=1}^{n} \frac{1}{2^i}$ is $s_n = 1 - \frac{1}{2^n}$. However, it is not always possible to do this. How can we estimate the value of an infinite series in cases where we can't explicitly write s_n in terms of n?

Consider the series $\sum_{i=1}^{\infty} \frac{1}{i^2}$. The first few terms are:

$$\sum_{i=1}^{\infty} \frac{1}{2^i} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots$$

The series is decreasing, but is it convergent? Let's plot this series on an xy-plane (see figure 4.1).

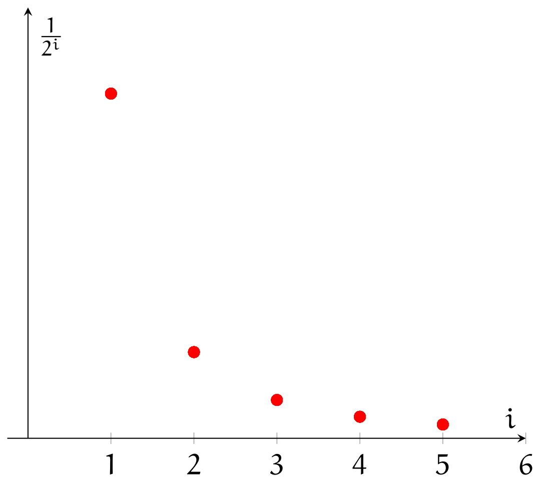


Figure 4.1: The first 5 terms of $\sum_{i=1}^{\infty} \frac{1}{2^i}$

We can overlay the function $y = \frac{1}{2^x}$ (figure 4.2). We can draw rectangles of width 1 and height $\frac{1}{x^2}$ (see figure 4.3). The area of the first n rectangles is equal to the n^{th} partial sum.

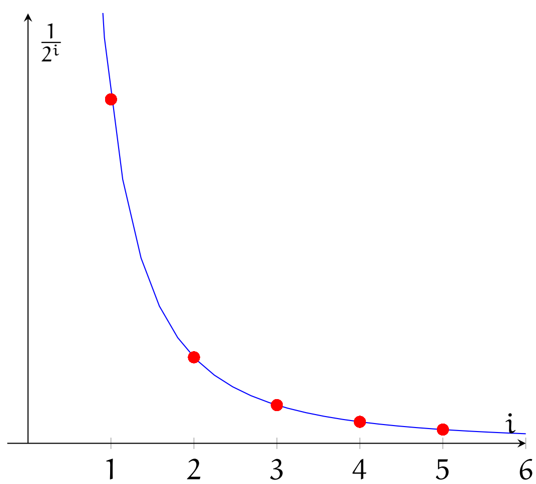


Figure 4.2: The first 5 terms of $\sum_{i=1}^{\infty} \frac{1}{2^i}$ lie on the curve $y = \frac{1}{x^2}$

This should remind you of a Riemann sum. Since the total area of the rectangles is less than the area under the curve, we can state:

$$\sum_{i=1}^{\infty} \frac{1}{2^i} < \int_0^{\infty} \frac{1}{x^2}\, dx$$

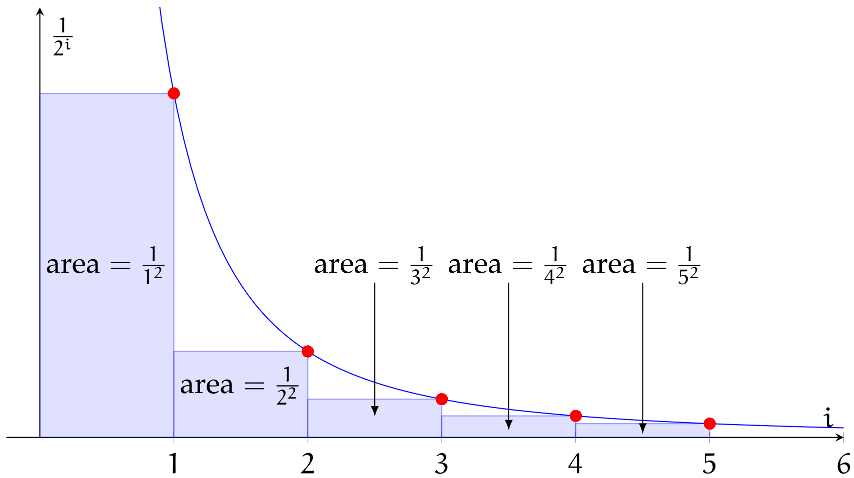


Figure 4.3: The partial sum $\sum_{i=1}^{n=5} \frac{1}{2^i}$ is equal to the area of the rectangles

We can exclude the first rectangle and also state that:

$$\sum_{i=1}^{\infty} \frac{1}{2^i} < 1 + \int_1^{\infty} \frac{1}{x^2}\,dx$$

We can evaluate this integral:

$$\int_1^{\infty} \frac{1}{x^2}\,dx = \lim_{t\to\infty} \left[\int_1^t \frac{1}{x^2}\,dx\right]$$

$$= \lim_{t\to\infty} \frac{-1}{x}\Big|_{x=1}^{t} = \lim_{t\to\infty} \left(\frac{-1}{t}\right) - \frac{-1}{1} = 0 - (-1) = 1$$

Therefore:

$$\sum_{i=1}^{\infty} \frac{1}{2^i} < 1 + 1 = 2$$

This means the series $\sum_{i=1}^{\infty} \frac{1}{2^i}$ is bounded above. Since the series is also monotonic (each term is positive, so the value of the sum increases as n increases), we can state that the sum is convergent!

Let's look at a divergent example: $\sum_{i=1}^{\infty} \frac{1}{\sqrt{x}}$. Again, we will make a visual, but this time we will draw rectangles that lie above the curve $y = \frac{1}{\sqrt{x}}$ (see figure 4.4). In this case, $\sum_{i=1}^{\infty} \frac{1}{\sqrt{x}} > \int_1^{\infty} \frac{1}{\sqrt{x}}\,dx$. Let's evaluate the integral:

$$\int_1^{\infty} \frac{1}{\sqrt{x}}\,dx = \lim_{t\to\infty} \left[\int_1^t \frac{1}{\sqrt{x}}\,dx\right]$$

$$= \lim_{t\to\infty} \left[2\sqrt{x}\right]_{x=1}^{t} = \lim_{t\to\infty} \left(2\sqrt{t}\right) - 2\sqrt{1} = \infty - 2 \to \text{divergent}$$

Since the integral diverges to infinity and the series is greater than the integral, the series must also diverge to infinity. This is another case where a monotonic decreasing series is not convergent!

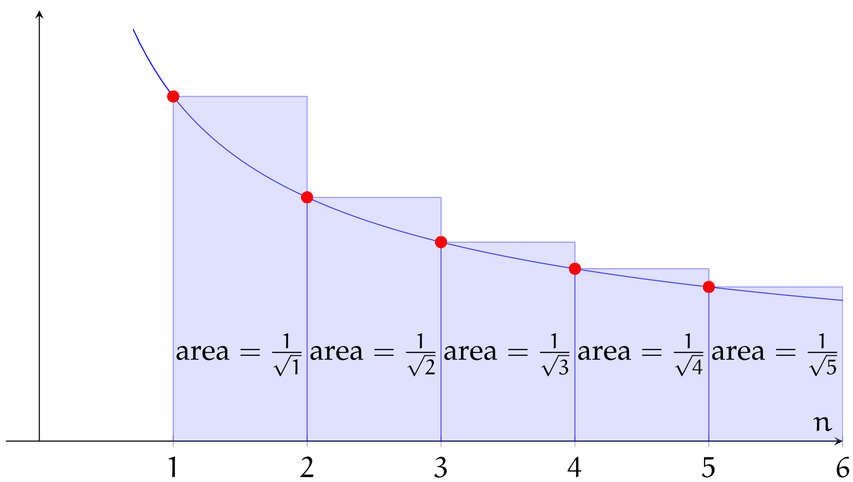


Figure 4.4: $\sum_{i=1}^{\infty} \frac{1}{\sqrt{x}} > \int_1^{\infty} \frac{1}{\sqrt{x}}\, dx$

This leads us to the **Integral Test**. If f is a continuous, positive, decreasing function on the interval $x \in [1, \infty)$ and $a_n = f(n)$, then the series $\sum_{n=1}^{\infty} a_n$ converges if and only if $\int_1^{\infty} f(x)\, dx$ is convergent. Subsequently, if $\int_1^{\infty} f(x)\, dx$ is divergent, then the series is also divergent.

Example: Is the series $\sum_{i=1}^{\infty} \frac{1}{n^2+1}$ convergent or divergent?

Solution: To apply the integral test, we define $f(x) = \frac{1}{x^2+1}$, which is a positive, decreasing function on the interval $x \in [1, \infty)$.

$$\int_1^{\infty} \frac{1}{x^2+1}\, dx = \lim_{t\to\infty} \int_1^{t} \frac{1}{x^2+1}\, dx$$

$$= \lim_{t\to\infty} \left[\arctan x\right]_{x=1}^{t} = \lim_{t\to\infty} (\arctan t) - \arctan 1 = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

Because the integral $\int_1^{\infty} \frac{1}{x^2+1}\, dx$ converges, so does the series $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$.

Exercise 13

Use the integral test to determine if the following series are convergent or divergent.

1. $\sum_{n=1}^{\infty} 2n^{-3}$
2. $\sum_{n=1}^{\infty} \frac{5}{3n-1}$
3. $\sum_{n=1}^{\infty} \frac{n}{3n^2+1}$

Working Space

Answer on Page 61

Exercise 14

Apply the Integral Test to show that p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ are convergent only when $p > 1$ (hint: consider the cases $p \leq 0$, $0 < p < 1$, $p = 1$ and $p > 1$).

Working Space

Answer on Page 61

4.2.1 Using Integrals to Estimate the Value of a Series

Recall that $\sum_{i=1}^{\infty} a_i = a_1 + a_2 + a_3 + \cdots = s$ and that the n^{th} partial sum, often represented as s_n, is $s_n = a_1 + a_2 + \cdots + a_{n-1} + a_n$. We can then define the n^{th} remainder $R_n = s - s_n$. Expanding s and s_n, we see that:

$$R_n = [a_1 + a_2 + \cdots + a_{n-1} + a_n + a_{n+1} + \cdots] - [a_1 + a_2 + \cdots + a_{n-1} + a_n]$$

$$R_n = [a_1 - a_1] + [a_2 - a_2] + \cdots + [a_{n-1} - a_{n-1}] + [a_n - a_n] + a_{n-1} + a_{n-2} + \cdots$$

$$R_n = a_{n+1} + a_{n+2} + a_{n+3} + \cdots$$

Just like the integral test, suppose there is some continuous, positive, decreasing function, such that $a_n = f(n)$. We can then represent R_n as the right Riemann sum with width $\Delta x = 1$ from $x = n$ to ∞. Since the rectangles are below the curve (see figure 4.5), we can state that $R_n \leq \int_n^{\infty} f(x)\, dx$.

Similarly, we can represent R_n as the left Riemann sum with width $\Delta x = 1$ from $x = n + 1$ to ∞. This time, the rectangles are above the curve (see figure 4.6), and we can state that $R_n \geq \int_{n+1}^{\infty} f(x)\, dx$. Putting this all together, we have an estimate for the remainder, R_n, from the integral test:

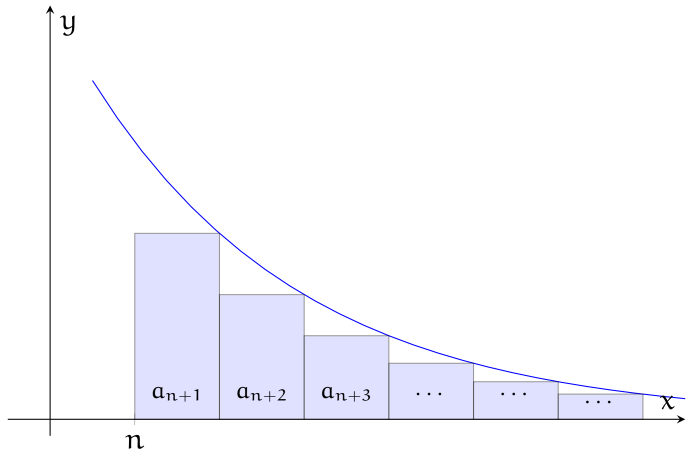


Figure 4.5: $R_n \leq \int_n^\infty f(x)\,dx$

Suppose there is a function such that $f(k) = a_k$, where f is a continuous, positive, decreasing function for $x \geq n$ and $\sum a_n$ is convergent. Then, $\int_{n+1}^\infty f(x)\,dx \leq R_n \leq \int_n^\infty f(x)\,dx$, where R_n is $s - s_n$.

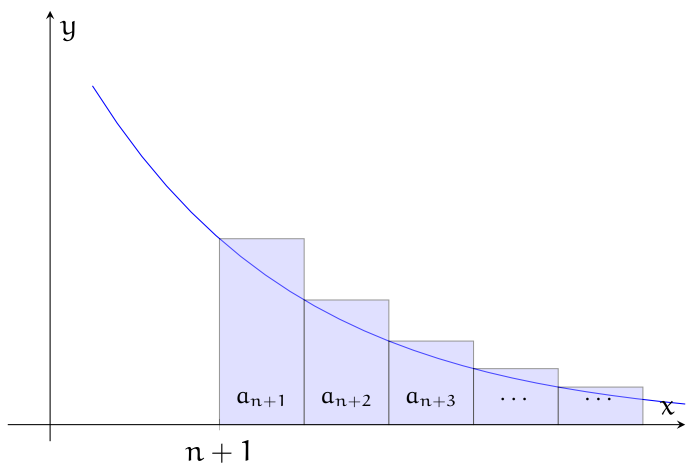


Figure 4.6: $R_n \geq \int_{n+1}^\infty f(x)\,dx$

Example: Approximate the sum of the series $\sum_{n=1}^\infty \frac{3}{n^3}$ by finding the 10^{th} partial sum. Estimate the error of this approximation.

Solution: Using a calculator, you can find the 10^{th} partial sum:

$$\sum_{n=1}^{10} \frac{3}{n^3} = \frac{3}{1^3} + \frac{3}{2^3} + \frac{3}{3^3} + \cdots + \frac{3}{10^3} \approx 3.593 = s_{10}$$

Recall that the remainder, R_{10}, is the difference between the actual sum, s, and the partial sum, s_{10}. Using the integral test to estimate the remainder, we can state that:

$$R_{10} \leq \int_{10}^{\infty} \frac{3}{x^3}\,dx = \frac{3}{2(10)^2} = \frac{3}{200} = 0.015$$

Therefore, the size of the error is at most 0.015.

Example: How many terms are required for the error to be less than 0.0001 for the sum presented above?

Solution: We are looking for an n such that $R_n \leq 0.0001$. Recalling that $R_n \leq \int_n^{\infty} \frac{3}{x^3}\,dx$, we need to find an n such that $\int_n^{\infty} \frac{3}{x^3}\,dx \leq 0.0001$.

$$\int_n^{\infty} \frac{3}{x^3}\,dx \leq 0.0001$$
$$\frac{-1}{6x^2}\Big|_{x=n}^{\infty} \leq 0.0001$$
$$\lim_{x\to\infty} \frac{-1}{6x^2} - \frac{-1}{6n^2} \leq 0.0001$$
$$0 + \frac{1}{6n^2} = \frac{1}{6n^2} \leq 0.0001$$
$$1 \leq 0.0006n^2$$
$$1667 \leq n^2$$
$$40.8 \leq n \to n = 41$$

Therefore, $s - s_{41} \leq 0.0001$ and the partial sum $\Sigma_{n=1}^{41} \frac{3}{n^3}$ is less than 0.0001 from the value of the infinite sum $\sum_{n=1}^{\infty} \frac{3}{n^3}$.

Exercise 15

Working Space

1. Find the partial sum s_{10} of the series $\sum_{n=1}^{\infty} \frac{1}{n^4}$.
2. Estimate the error from using s_{10} as an approximation of the series.
3. Use $s_n + \int_{n+1}^{\infty} \frac{1}{x^4}\,dx \leq s \leq s_n + \int_{n}^{\infty} \frac{1}{x^4}\,dx$ to give an improved estimate of the sum.
4. The actual value of $\sum_{n=1}^{\infty} \frac{1}{n^4}$ is $\frac{\pi^4}{90}$. Compare your estimate with the actual value.
5. Find a value of n such that s_n is within 0.00001 of the sum.

Answer on Page 62

4.3 Comparison Tests

In comparison tests, we compare a series to a known convergent or divergent series. Take the series $\sum_{n=1}^{\infty} \frac{1}{3^n+3}$. This is similar to $\sum_{n=1}^{\infty} \frac{1}{3^n}$, which is a geometric series that converges to $\frac{1}{2}$. Notice that:

$$\frac{1}{3^n+3} < \frac{1}{3^n}$$

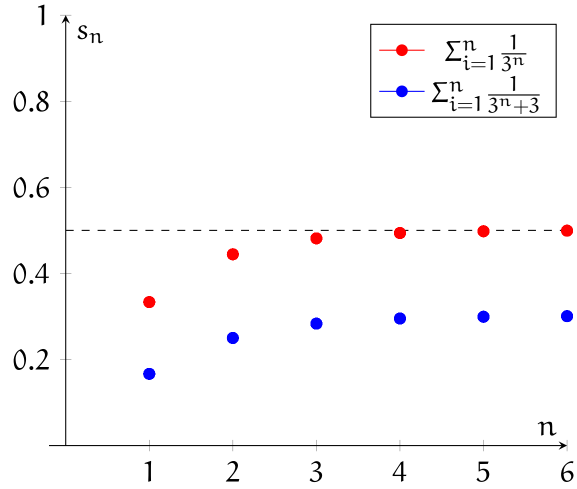


Figure 4.7: $\sum_{i=1}^{n} \frac{1}{3^n+3} < \sum_{i=1}^{n} \frac{1}{3^n}$ for all n

Which implies that

$$\sum_{n=1}^{\infty} \frac{1}{3^n + 3} < \sum_{n=1}^{\infty} \frac{1}{3^n}$$

Since $\sum_{n=1}^{\infty} \frac{1}{3^n}$ is convergent, it follows that $\sum_{n=1}^{\infty} \frac{1}{3^n+3}$ is also convergent (see figure 4.7). As you can see, since $\sum_{n=1}^{\infty} \frac{1}{3^n}$ approaches $\frac{1}{2}$, $\sum_{n=1}^{\infty} \frac{1}{3^n+3}$ must be $\leq \frac{1}{2}$ and therefore convergent.

4.3.1 The Direct Comparison Test

F For the **Direct Comparison Test**, we compare the terms a_n to b_n directly. Take $\sum a_n$ and $\sum b_n$ to be series with positive terms. Then,

1. If $a_n \leq b_n$ and $\sum b_n$ is convergent, then $\sum a_n$ is also convergent.
2. If $a_n \geq b_n$ and $\sum b_n$ is divergent, then $\sum a_n$ is also divergent.

We already discussed above why the first part is true. The second part follows a similar argument: If a_n is greater than b_n, then you can imagine that as $\sum b_n$ grows and diverges, it is pushing upwards on $\sum a_n$, meaning that $\sum a_n$ must also diverge. Consider the series $\sum_{n=1}^{\infty} \frac{2 \ln n}{n}$. For $n \geq 2$, $2 \ln n > 1$, and therefore if $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, then $\sum_{n=1}^{\infty} \frac{2 \ln n}{n}$ must also diverge. We recognize the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent. Therefore, $\sum_{n=1}^{\infty} \frac{2 \ln n}{n}$ is also divergent (see figure 4.8).

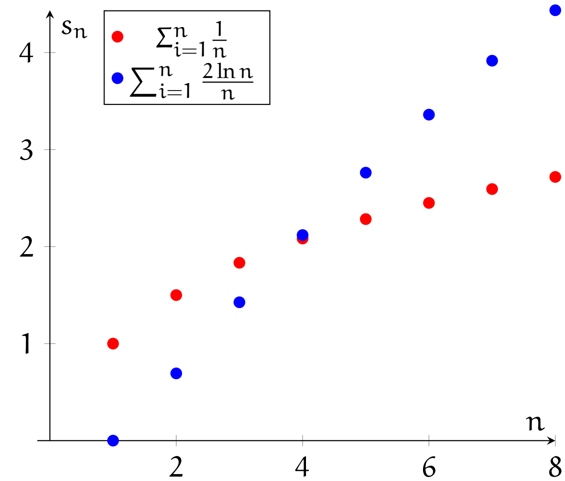


Figure 4.8: $\sum_{i=1}^{n} \frac{2 \ln n}{n} > \sum_{i=1}^{n} \frac{1}{n}$ for $n \geq 4$

4.3.2 The Limit Comparison Test

Consider the series $\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$. We may want to compare this to the convergent series $\sum_{n=1}^{\infty} \frac{1}{2^n}$. The direct comparison test isn't helpful here, since $\frac{1}{2^n-1} > \frac{1}{2^n}$, so $\sum_{n=1}^{\infty} \frac{1}{2^n}$ doesn't put a cap on $\sum_{n=1}^{\infty} \frac{1}{2^n-1}$ like our earlier example (see figure 4.7). In a case such as this, we can use the **Limit Comparison Test**, which states that:
If $\sum a_n$ and $\sum b_n$ are series with positive terms and $\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$, then either both series converge or both series diverge.

Let's apply this to the series $\sum_{n=1}^{\infty} \frac{1}{2^n-1}$. We know that $\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges, since it is a geometric series with $r < 1$.

$$\lim_{n\to\infty} \frac{\frac{1}{2^n-1}}{\frac{1}{2^n}} = \lim_{n\to\infty} \frac{1}{2^n - 1} \cdot \frac{2^n}{1}$$

$$= \lim_{n\to\infty} \frac{2^n}{2^n - 1} = \lim_{n\to\infty} \frac{1}{1 - 1/2^n} = \frac{1}{1-0} = 1 > 0$$

Therefore, by the Limit Comparison Test, $\sum_{n=1}^{\infty} \frac{1}{2^n-1}$ converges.

In general, comparison tests are most useful for series resembling geometric or p-series. When choosing a p-series to compare the unknown series to, choose p such that the order of your p series is the same as the order of the unknown series.

Example: What p-series should one compare the series $\sum_{n=1}^{\infty} \frac{\sqrt{n^3+1}}{3n^3+4n^2+2}$ to?

Solution: We can determine the order of $\frac{\sqrt{n^3+1}}{3n^3+4n^2+2}$ by looking at the highest-order terms in the numerator and denominator:

$$\frac{\sqrt{n^3}}{n^3} = \frac{n^{3/2}}{n^3} = \frac{1}{n^{3/2}}$$

So, we should compare $\sum_{n=1}^{\infty} \frac{\sqrt{n^3+1}}{3n^3+4n^2+2}$ to the convergent p-series $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$.

Example: Is $\sum_{n=1}^{\infty} \frac{\sqrt{n^3+1}}{3n^3+4n^2+2}$ convergent or divergent?

Solution: We have already determine that we should compare this series to $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$. To apply the limit test, we need to evaluate

$$\lim_{n\to\infty} \frac{\frac{\sqrt{n^3+1}}{3n^3+4n^2+2}}{\frac{1}{n^{3/2}}}$$

$$= \lim_{n\to\infty} \frac{n^{3/2}\sqrt{n^3+1}}{3n^3+4n^2+2}$$

$$= \lim_{n\to\infty} \frac{\sqrt{n^6+n^3}}{3n^3+4n^2+2} = \frac{1}{3} > 0$$

Therefore, by the Limit Comparison Test, $\sum_{n=1}^{\infty} \frac{\sqrt{n^3+1}}{3n^3+4n^2+2}$ is convergent because the p-series $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ is convergent.

Exercise 16

Use the Comparison Test or the Limit Comparison Test to determine if the following series are convergent or divergent.

1. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+1}}$
2. $\sum_{n=1}^{\infty} \frac{9^n}{3+10^n}$
3. $\sum_{n=1}^{\infty} \frac{n \sin^2 n}{1+n^3}$

Working Space

Answer on Page 62

4.4 Ratio and Root Tests for Convergence

4.4.1 Absolute Convergence

Suppose there is a series $\sum_{n=1}^{\infty} a_n$, then there is a corresponding series $\sum_{n=1}^{\infty} |a_n| = |a_1| + |a_2| + |a_3| + \cdots$. If $\sum_{n=1}^{\infty} |a_n|$ is convergent, then the series $\sum_{n=1}^{\infty} a_n$ is called **absolutely convergent**.

Example: Consider the alternating series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = 1 - \frac{1}{2^2} + \frac{1}{3^2} + \cdots$$

Is this series absolutely convergent?

Solution: We examine the corresponding series where we take the absolute value of each term:

$$\sum_{n=1}^{\infty}\left|\frac{(-1)^{n-1}}{n^2}\right| = \sum_{n=1}^{\infty}\frac{1}{n^2}$$

We can identify $\sum_{n=1}^{\infty}\frac{1}{n^2}$ as a convergent p-series. Since $\sum_{n=1}^{\infty}\frac{1}{n^2}$ is convergent, we can state that $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2}$ is absolutely convergent.

Example Is the convergent series $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}$ absolutely convergent?

Solution We consider the sum of the absolute values of the terms:

$$\sum_{n=1}^{\infty}\left|\frac{(-1)^{n-1}}{n}\right| = \sum_{n=1}^{\infty}\frac{1}{n}$$

You should recognize this as the harmonic series, which is divergent. When a series is convergent but the corresponding series of absolute values is not, we call it **conditionally convergent**.

We won't prove the theorem here, but it is useful to know that if a series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, then it is convergent. You can prove this yourself using the Comparison Test.

Exercise 17

Is the series given by

$$\frac{\cos 1}{1^2} + \frac{\cos 2}{2^2} + \frac{\cos 3}{3^3} + \cdots$$

convergent or divergent?

Working Space

Answer on Page 63

Exercise 18

Determine whether each of the following series is absolutely or conditionally convergent.

1. $\sum_{n=1}^{\infty} \frac{(-1)^n}{3n+2}$
2. $\sum_{n=1}^{\infty} \frac{\sin n}{4^n}$
3. $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{2n}{n^2+4}$

Working Space

Answer on Page 63

4.4.2 The Ratio Test

The ratio test compares the $(n+1)^{\text{th}}$ term of a series to the n^{th} term and takes the limit as $n \to \infty$ of the absolute value of this ratio:

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L$$

There are three possible outcomes of the ratio test:

1. If $L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).
2. If $L = 1$, then the ratio test is inconclusive and we cannot draw any conclusions about whether $\sum_{n=1}^{\infty} a_n$ is convergent or divergent.
3. If $L > 1$ or $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

Example: Apply the ratio test to determine if $\sum_{n=1}^{\infty}(-1)^n \frac{n^3}{3^n}$ is convergent or divergent.

Solution:

$$\lim_{n\to\infty}\left|\frac{\frac{(-1)^{n+1}(n+1)^3}{3^{n+1}}}{\frac{(-1)^n n^3}{3^n}}\right| = \lim_{n\to\infty}\frac{(n+1)^3}{3^{n+1}}\cdot\frac{3^n}{n^3}$$

$$= \lim_{n\to\infty}\frac{(n+1)^3}{n^3}\cdot\frac{3^n}{3\cdot 3^n}$$

$$= \lim_{n\to\infty}\left(\frac{n+1}{n}\right)\cdot\frac{1}{3} = \frac{1}{3}\lim_{n\to\infty}\left(\frac{n+1}{n}\right) = \frac{1}{3}\cdot 1 = \frac{1}{3}$$

Since $L < 1$, the series $\sum_{n=1}^{\infty}(-1)^n \frac{n^3}{3^n}$ is absolutely convergent.

The ratio test is most useful for series that contain factorials, constants raised to the n^{th} power, or other products.

Exercise 19

[This question was originally presented as a multiple-choice, no-calculator problem on the 2012 AP Calculus BC exam.] Which of the following series are convergent?

1. $\sum_{n=1}^{\infty}\frac{8^n}{n!}$
2. $\sum_{n=1}^{\infty}\frac{n!}{n^{100}}$
3. $\sum_{n=1}^{\infty}\frac{n+1}{(n)(n+2)(n+3)}$

Working Space

Answer on Page 64

Exercise 20

[This question was originally presented as a multiple-choice, calculator-allowed problem on the 2012 AP Calculus BC exam.]If the series $\Sigma_{n=1}^{\infty} a_n$ converges and $a_n > 0$ for all n, which of the following statements must be true? Explain why.

1. $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 0$
2. $|a_n| < 1$ for all n
3. $\sum_{n=1}^{\infty} a_n = 0$
4. $\sum_{n=1}^{\infty} n a_n$ diverges
5. $\sum_{n=1}^{\infty} \frac{a_n}{n}$ converges

Working Space

Answer on Page 64

4.4.3 Root Test

The root test examines the behavior of the n^{th} root of a_n as $n \to \infty$. Similar to the ratio test, there are three possible outcomes:

1. If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, and therefore convergent.
2. If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$ or $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.
3. If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L = 1$, then the Root Test is inconclusive.

The root test is best when there is a term or terms raised to the n^{th} power. Consider the series $\sum_{n=1}^{\infty} \left(\frac{2n+3}{3n+2}\right)^n$:

Example: Is the series $\sum_{n=1}^{\infty} \left(\frac{2n+3}{3n+2}\right)^n$ convergent or divergent?

Solution: Since a_n consists of terms raised to the n^{th} power, we will apply the root test for convergence:

$$\lim_{n\to\infty} \sqrt[n]{\left|\left(\frac{2n+3}{3n+2}\right)^n\right|} = \lim_{n\to\infty} \frac{2n+3}{3n+2} = \frac{2}{3} < 1$$

Therefore, by the root test, the series $\sum_{n=1}^{\infty} \left(\frac{2n+3}{3n+2}\right)^n$ is convergent.

Exercise 21

Use the Root Test to determine whether the following series are convergent or divergent.

1. $\sum_{n=1}^{\infty} \left(\frac{3n^2+1}{n^2-4}\right)^n$
2. $\sum_{n=1}^{\infty} \frac{(-1)^n}{(\ln n)^n}$
3. $\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^{n^2}$

Working Space

Answer on Page 65

4.5 Strategies for Testing Series

When testing series for convergence, we want to choose a test based on the form of the series. While you may by tempted to try each test one-by-one until you find an answer, this quickly becomes cumbersome and time-consuming. Additionally, if you plan to take an AP Calculus exam, you need to be able to quickly choose an appropriate test as to conserve the time you have available for the exam. Here are some tips:

1. Check if the series is a p-series ($\sum_{n=1}^{\infty} \frac{1}{n^p}$). If so, then if $p > 1$, the series converges.

Otherwise, the series diverges.

2. If the series is not a p-series, check to see if you can write it as a geometric series ($\sum_{n=1}^{\infty} ar^{n-1}$ or $\sum_{n=1}^{\infty} ar^{n}$). Recall that geometric series are convergent if $|r| < 1$ and divergent otherwise.

3. If the series can't be written as a p-series or geometric series, but has a similar form, consider the comparison tests (the Direct Comparison Test and the Limit Comparison Test). When choosing a p-series to compare your series to, follow the guidelines outlined in the Comparison Tests section above.

4. If you can see at a glance that $\lim_{n\to\infty} a_n \neq 0$, then apply the Test for Divergence to show the series is divergent. REMEMBER: $\lim_{n\to\infty} a_n \neq 0$ implies the series $\sum_{n=1}^{\infty} a_n$ is divergent, but $\lim_{n\to\infty} a_n = 0$ **does not** necessarily imply the series $\sum_{n=1}^{\infty} a_n$ is convergent.

5. If the series is alternating (has $(-1)^n$ or $(-1)^{n-1}$ in the term), the Alternating Series test may provide an answer.

6. The Ratio Test is excellent for series with factorials, other products, or constants to the n^{th} power. Remember that the Ratio Test will be inconclusive for p-series, rational functions of n, and algebraic functions of n.

7. If a_n is of the form $(b_n)^n$, use the Root Test.

8. If $a_n = f(n)$ where $f(n)$ is continuous, positive, and decreasing and you can evaluate $\int_1^{\infty} f(x)\,dx$, use the Integral Test.

You don't need to treat this as a checklist, where you check for every condition. Rather, you should use this as a guide to quickly determine the convergence test most likely to be useful.

Exercise 22

Choose an appropriate test to determine if the series is convergent of divergent. Apply the test and classify the series as convergent or divergent.

1. $\sum_{n=1}^{\infty} \frac{e^n}{n^2}$
2. $\sum_{n=1}^{\infty} \frac{3^n n^2}{n!}$
3. $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$
4. $\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2}$
5. $\sum_{n=1}^{\infty} \left(\sqrt[n]{2} - 1\right)^n$

Working Space

Answer on Page 65

Appendix A

Answers to Exercises

Answer to Exercise 1 (on page 7)

Substituting the given values, we find that $(5)\frac{dQ}{dT} + \frac{1}{0.05}Q = 60$. Solving for $\frac{dQ}{dt}$:

$$(5)\frac{dQ}{dt} + (20)Q = 60$$

$$\frac{dQ}{dt} + 4Q = 12$$

$$\frac{dQ}{dt} = 12 - 4Q$$

We also know that $Q(0) = 0$. Using Euler's method with step size $h = 0.1$, $Q(0.1) \approx Q(0) + h\,[12 - 4Q(0)] = 0 + 0.1\,[12 - 4(0)] = 1.2$. And $Q(0.2) \approx Q(0.1) + h\,[12 - 4Q(0.1)] = 1.2 + 0.1\,[12 - 4(1.2)] = 1.92$. And $Q(0.3) \approx Q(0.2) + h\,[12 - 4Q(0.2)] = 1.92 + 0.1\,[12 - 4(1.92)] = 2.352$. And $Q(0.4) \approx Q(0.3) + h\,[12 - 4Q(0.3)] = 2.352 + 0.1\,[12 - 4(2.352)] = 2.6112$. And finally, $Q(0.5) \approx Q(0.4) + h\,[12 - 4Q(0.4)] = 2.6112 + 0.1\,[12 - 4(2.6112)] = 2.76672$. Because we are finding a charge, the unit it Coulombs (C), so our final answer is $Q(0.5) \approx 2.77C$.

Answer to Exercise 10 (on page 34)

We are given $x_0 = 1$ and $x_2 = 1.4$. Therefore we will use step size $h = \frac{1.4-1}{2} = \frac{0.4}{2} = 0.2$. Taking $x_0 = 1$ and $y_0 = f(1) = 15$, we find y_1: $y_1 = y_0 + h \cdot f'(x_0) = 15 + 0.2 \cdot f'(1) = 15 + 0.2(8) = 15 + 1.6 = 16.6$. And then $y_2 = y_1 + h \cdot f'(x_1) = 16.6 + 0.2 \cdot f'(1.2) = 16.6 + 0.2(12) = 16.6 + 2.4 = 19$. Therefore, $f(1.4) \approx 19$.

Answer to Exercise 3 (on page 12)

1. $\frac{2}{3}, \frac{4}{5}, \frac{8}{7}, \frac{16}{9}, \frac{32}{11}$
2. 0, -1, 0, 1, 0
3. 1, 2, 7, 32, 157
4. 6, 3, 1, $\frac{1}{4}$, $\frac{1}{20}$

Answer to Exercise 4 (on page 14)

1. convergent, 5
2. divergent
3. convergent, 2
4. convergent, -1
5. divergent

Answer to Exercise 5 (on page 20)

Out principal is $P = 1500$ and the interest rate is $r = 0.06$. After n years, your investment will be worth $a_n = 1500(1.06)^n$. For $n = 10$, your investment will be valued at $a_{10} = \$1500(1.06)^{10} = \$2686.27$ (that's over \$1000 in interest!). To determine if the sequence is convergent or divergent, we examine the limit as $n \to \infty$:

$$\lim_{n\to\infty} 1500(1.06)^n = 1500 \cdot \lim_{n\to\infty}(1.06)^n = 1500 \cdot \infty = \infty$$

The sequence is divergent.

Answer to Exercise 6 (on page 21)

The number of fish in the pond is:

$$P_n = P_{n-1}(0.95)^6 + 50$$

$$P_0 = 100$$

where n is the number of 6-month periods that have passed. The four-year period is given by $1 \leq n \leq 8$. The amount lost to predation every 6 months is given by $P_{n-1}(1 - 0.95^6)$.

n	Fish Population	Lost to Predators
0	100	
1	84	26
2	71	22
3	62	19
4	56	17
5	51	15
6	48	14
7	45	13
8	43	12

Adding up all the fish lost to predators, we find that over 4 years, the farmer loses 138 fish.

Answer to Exercise 7 (on page 32)

1. We need to identify a and r. If we use the form $\sum_{n=1}^{\infty} ar^{n-1}$, then $a = 3$. To find the common ratio, we can evaluate $\frac{a_{n+1}}{a_n} = \frac{-4}{3}$. We can then write the series as $\sum_{n=1}^{\infty} 3\left(\frac{-4}{3}\right)^{n-1}$. In this case, $r = \frac{-4}{3}$ and $|r| \geq 1$, and therefore the series is divergent.
2. Following the process outlined above, we see that $a = 2$ and $r = \frac{1}{4}$. Therefore, the series is $\sum_{n=1}^{\infty} 2\left(\frac{1}{4}\right)^{n-1}$. Since $|r| < 1$, the series converges to $\frac{a}{1-r} = \frac{2}{1-1/4} = \frac{2 \cdot 4}{3} = \frac{8}{3}$
3. We need to rewrite the series into a standard from in order to identify a and r:
$$\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{4^n} = \sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{4(4)^{n-1}} = \sum_{n=1}^{\infty} \frac{1}{4}\left(\frac{-3}{4}\right)^{n-1}$$
So $r = \frac{-3}{4}$ and $|r| < 1$. Therefore, the series converges to $\frac{1/4}{1-(-3/4)} = \frac{1}{4} \cdot \frac{4}{7} = \frac{1}{7}$
4. We need to rewrite the series into a standard from in order to identify a and r:
$$\sum_{n=1}^{\infty} \frac{e^{2n}}{6^{n-1}} = \sum_{n=1}^{\infty} \frac{(e^2)^n}{6^{n-1}} = \sum_{n=1}^{\infty} \frac{(e^2)(e^2)^{n-1}}{6^{n-1}} = \sum_{n=1}^{\infty} e^2\left(\frac{e^2}{6}\right)^{n-1}$$
Therefore, $r = \frac{e^2}{6} \approx 1.232$. Since $|r| > 1$, the series diverges.

Answer to Exercise 8 (on page 32)

We want to rewrite this as a geometric series of the form $\sum_{n=i}^{\infty} ar^{n-1}$, so we can use the fact that the sum of a convergent geometric series is $\frac{a}{1-r}$. $\sum_{n=0}^{\infty}(1+c)^{-n} = \sum_{n=0}^{\infty}\left(\frac{1}{1+c}\right)^n =$

$\sum_{n=1}^{\infty}\left(\frac{1}{1+c}\right)^{n-1}$. This is a geometric series with $a = 1$ and $r = \frac{1}{1+c}$. So, the value of the series is $\frac{1}{1-\frac{1}{1+c}} = \frac{1}{\frac{c}{c+1}} = \frac{c+1}{c}$. Setting this equal to $\frac{5}{3}$ and solving for c, we find that $c = \frac{3}{2}$.

Answer to Exercise 9 (on page 32)

$-2 < p < 2$ Let's rewrite this geometric series into standard form: $\sum_{n=1}^{\infty}\left(\frac{p}{2}\right)^n = \sum_{n=1}^{\infty}\frac{p}{2}\left(\frac{p}{2}\right)^n$ which means $a = \frac{p}{2}$ and $r = \frac{p}{2}$. We know that geometric series converge if $|r| < 1$, so we set up an inequality and solve for p:

$$\left|\frac{p}{2}\right| < 1$$
$$-1 < \frac{p}{2} < 1$$
$$-2 < p < 2$$

Answer to Exercise 10 (on page 34)

1. Separating the terms, we see that $\sum_{n=1}^{\infty}\frac{n^2+1}{n^4} = \sum_{n=1}^{\infty}\left(\frac{n^2}{n^4}+\frac{1}{n^4}\right) = \sum_{n=1}^{\infty}\frac{1}{n^2} + \sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^2}{6}+\frac{\pi^4}{90}$
2. Notice that this series starts at $n = 2$. By the properties of series, we know that $\sum_{n=1}^{\infty}a_n = a_1 + \sum_{n=2}^{\infty}a_n$. Therefore, $\sum_{n=2}^{\infty}\frac{1}{n^2} = \sum_{n=1}^{\infty}\left(\frac{1}{n^2}\right) - \frac{1}{1^2} = \frac{\pi^2}{6} - 1$
3. We can begin by reindexing this series: $\sum_{n=3}^{\infty}\frac{1}{(n+1)^2} = \sum_{n=4}^{\infty}\frac{1}{n^2}$. Similar to the previous problem, we also know that $\sum_{n=4}^{\infty}\frac{1}{n^2} = \sum_{n=1}^{\infty}\left(\frac{1}{n^2}\right) - \left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}\right) = \frac{\pi^2}{6} - \frac{49}{36}$
4. We can rewrite this series as $\sum_{n=1}^{\infty}\left(\frac{3}{n}\right)^4 = \sum_{n=1}^{\infty}(3^4)\frac{1}{n^4} = 81\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{81\pi^4}{90} = \frac{9\pi^4}{10}$
5. We can re-write the series as $\sum_{n=1}^{\infty}\left(\frac{4}{n^2}+\frac{3}{n^4}\right) = \sum_{n=1}^{\infty}\frac{4}{n^2} + \sum_{n=1}^{\infty}\frac{3}{n^4} = 4\sum_{n=1}^{\infty}\frac{1}{n^2} + 3\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{4\pi^2}{6}+\frac{3\pi^4}{90} = \frac{2\pi^2}{3}+\frac{\pi^4}{30}$

Answer to Exercise 11 (on page 35)

This is a p-series where $p = 2k$. We know that p-series converge for $p > 1$: $2k > 1 \rightarrow k > \frac{1}{2}$.

Answer to Exercise 12 (on page 36)

1. The series is convergent if $\left|\frac{(-1)^{n+1}3(n+1)}{4(n+1)-1}\right| < \left|\frac{(-1)^n 3n}{4n-1}\right|$ if $\frac{3n+3}{4n+4-1} < \frac{3n}{4n-1}$ and if $\frac{3n+3}{4n+3} < \frac{3n}{4n-1}$ if $(3n+3)(4n-1) < (3n)(4n+3)$ if $12n^2+12n-3n-3 < 12n^2+9n$ if $-3 < 0$ which is true. Therefore, $\sum_{n=1}^{\infty} \frac{(-1)^n 3n}{4n-1}$ is convergent.

2. The series is convergent if $\left|(-1)^{n+1+1}\frac{(n+1)^2}{(n+1)^3+1}\right| < \left|(-1)^{n+1}\frac{n^2}{n^3+1}\right|$, which is true if $\frac{(n+1)^2}{(n+1)^3+1} < \frac{n^2}{n^3+1}$ if $(n+1)^2(n^3+1) < (n^2)((n+1)^3+1)$ if $(n^2+2n+1)(n^3) < (n^2)(n^3+3n^2+3n+1+1)$ if $n^5+2n^4+n^3 < n^5+3n^4+3n^3+2n^2$, which is true for all $n \geq 1$. Therefore, $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n^2}{n^3+1}$ is convergent.

3. The series is convergent if $\left|(-1)^{n-1+1}e^{2/(n+1)}\right| < \left|(-1)^{n-1}e^{2/n}\right|$, which is true if $e^{2/(n+1)} < e^{2/n}$, which is true if $\frac{2}{n+1} < \frac{2}{n}$ which is true for all $n \geq 1$. Therefore, $\sum_{n=1}^{\infty}(-1)^{n-1}e^{2/n}$ is convergent.

Answer to Exercise 13 (on page 41)

1. The function $2x^{-3}$ is positive and decreasing for $x \in [1,\infty)$. $\int_1^{\infty} 2x^{-3}\,dx = \lim_{t\to\infty}\int_1^t 2x^{-3}\,dx = \lim_{t\to\infty}\left[-x^{-2}\right]_{x=1}^{t} = \lim_{t\to\infty}(-t^{-2}) - -(1)^{-2} = 0+1 = 1$. Since the integral $\int_1^{\infty} 2x^{-3}\,dx$ converges, the series $\sum_{n=1}^{\infty} 2n^{-3}$ is also convergent.

2. The function $\frac{5}{3x+1}$ is positive and decreasing for $x \in [1,\infty)$. $\int_1^{\infty}\frac{5}{3x-1}\,dx = \lim_{t\to\infty}\int_1^t \frac{5}{3x-1}\,dx$ Using u-substitution to evaluate the integral, we set $u = 3x-1$ and find that $du = 3dx \to dx = \frac{du}{3}$. Substituting, $\int_1^t \frac{5}{3x-1}\,dx = \int_{x=1}^{x=t}\frac{5}{3}\frac{1}{u}\,du$. Evaluating the integral, $\int_{x=1}^{x=t}\frac{5}{3}\frac{1}{u}\,du = \frac{5}{3}\ln u|_{x=1}^{x=t} = \frac{5}{3}\ln 3x+1|_1^t$. Substituting this back into the limit, $\int_1^{\infty}\frac{5}{3x-1}\,dx = \lim_{t\to\infty}\frac{5}{3}\ln 3x+1|_1^t = \lim_{t\to\infty}[\frac{5}{3}\ln 3t+1] - \frac{5}{3}\ln 4 = \infty - \frac{5}{3}\ln 4 = \infty$. Therefore, the integral $\int_1^{\infty}\frac{5}{3x-1}\,dx$ is divergent and so is the series $\sum_{n=1}^{\infty}\frac{5}{3n-1}$.

3. The function $\frac{x}{3x^2+1}$ is positive and decreasing for $x \in [1,\infty)$. $\int_1^{\infty}\frac{x}{3x^2+1}\,dx = \lim_{t\to\infty}\int_1^t \frac{x}{3x^2+1}\,dx$. Applying the substitution $u = 3x^2+1$ and $\frac{du}{6} = x\,dx$, we see that $\lim_{t\to\infty}\int_1^t \frac{x}{3x^2+1}\,dx = \lim_{t\to\infty}\int_{x=1}^{x=t}\frac{1}{6u}\,du = \lim_{t\to\infty}\frac{1}{6}\ln u|_{x=1}^{x=t} = \lim_{t\to\infty}\frac{1}{6}\ln 3x^2+1|_1^t = \lim_{t\to\infty}\left[\frac{1}{6}\ln 3t^2+1\right] - \frac{1}{6}\ln 4 = \infty$. Therefore, the integral $\int_1^{\infty}\frac{x}{3x^2+1}\,dx$ is divergent, and so is the series $\sum_{n=1}^{\infty}\frac{n}{3n^2+1}$.

Answer to Exercise 14 (on page 42)

1. If $p \leq 0$, then $\lim_{n\to\infty}\frac{1}{n^p} \neq 0$, and the series fails the Test for Divergence. Therefore, a p-series is divergent if $p \leq 0$.

2. If $p > 0$, then $f(x) = \frac{1}{x^p}$ is continuous, positive, and decreasing on the interval $x \in [1,\infty)$, and we can apply the integral test. So, we want to know, when is $\int_1^{\infty}\frac{1}{x^p}\,dx$

convergent? When $p = 1$, $\int_1^\infty \frac{1}{x^p}\,dx = \ln x|_{x=1}^{x=\infty} = \lim_{t\to\infty} \ln t - \ln 1 = \infty$ and the integral and p-series are both divergent.

3. What about when $0 < p < 1$? In this case, the integral $\int_1^\infty \frac{1}{x^p}\,dx = \lim_{t\to\infty} \int_1^t x^{-p}\,dx = \lim_{t\to\infty} \frac{1}{1-p} x^{1-p}|_{x=1}^{x=t} = \lim_{t\to\infty} \frac{1}{1-p}\frac{1}{x^{p-1}} = \left(\frac{1}{1-p}\right)\left[\lim_{t\to\infty}\left(\frac{1}{t^{p-1}}\right) - 1\right]$. When $0 < p < 1$, then $1 - p > 0$ is positive and $\lim_{t\to\infty} \frac{1}{t^{p-1}} = \lim_{t\to\infty} t^{1-p} = \infty$ and the integral diverges. Therefore, p-series are divergent for $0 < p < 1$.

4. When $p > 1$, then $\int_1^\infty \frac{1}{x^p}\,dx = \left(\frac{1}{1-p}\right)\left[\lim_{t\to\infty}\left(\frac{1}{t^{p-1}}\right) - 1\right]$. When $p > 1$, $p-1 > 0$ and $\lim_{t\to\infty} \frac{1}{t^{p-1}} = 0$. Therefore, $\int_1^\infty \frac{1}{x^p}\,dx$ converges to $\frac{1}{p-1}$ when $p > 1$, and therefore the p-series is convergent when $p > 1$.

Answer to Exercise 15 (on page 45)

1. $s_{10} = \frac{1}{1^4} + \frac{1}{2^4} + \cdots + \frac{1}{10^4} \approx 1.082037$.
2. $R_{10} \leq \int_{10}^\infty \frac{1}{x^4}\,dx = \frac{-1}{3x^3}|_{x=10}^\infty = \lim_{x\to\infty} \frac{-1}{3x^3} - \frac{-1}{3\cdot 10^3} = \frac{1}{3000} = 0.000333$. Therefore, the error is less than 0.000333.
3. Given $s_{10} \approx 1.082037$, we can say that $1.082037 + \int_{n+1}^\infty \frac{1}{x^4}\,dx \leq s \leq 1.082037 + \int_n^\infty \frac{1}{x^4}\,dx$. Using a calculator to evaluate each integral, we see that: $1.082037 + 0.000250 \leq s \leq 1.082037 + 0.000333$ and therefore the sum is between 1.082287 and 1.082370.
4. Writing the actual value as a decimal, $\frac{\pi^4}{90} \approx 1.082323$, which is in the estimate window from the previous part.
5. We are looking for an n such that $\int_n^\infty \frac{1}{x^4}\,dx \leq 0.00001$. $\lim_{x\to\infty} \frac{-1}{3x^3} - \frac{-1}{3n^3} = \frac{1}{3n^3} \leq 0.00001$. $100,000 \leq 3n^3$. $33,333.33 \leq n^3$. $32.183 \leq n$. Since n must be an integer, $n = 33$ gives $R_n \leq 0.00001$.

Answer to Exercise 16 (on page 49)

1. This is similar to $\sum_{n=1}^\infty \frac{1}{n}$, which is divergent. Unfortunately, $\frac{1}{n} > \frac{1}{\sqrt{n^2+1}}$, so we can't use the direct comparison test. We will try the limit comparison test:

$$\lim_{n\to\infty}\left(\frac{\frac{1}{\sqrt{n^2+1}}}{\frac{1}{n}}\right) = \lim_{n\to\infty}\left(\frac{1}{\sqrt{n^2+1}}\cdot\frac{n}{1}\right) = \lim_{n\to\infty}\frac{n}{\sqrt{n^2+1}} = \lim_{n\to\infty}\frac{1}{\sqrt{1+1/n^2}} = \frac{1}{1+0} = 1 > 0$$

Therefore, since $\sum_{n=1}^\infty \frac{1}{n}$ diverges, so does $\sum_{n=1}^\infty \frac{1}{\sqrt{n^2+1}}$.

2. This series is similar to the convergent geometric series $\sum_{n=1}^{\infty}\left(\frac{9}{10}\right)^n$. Given that:

$$\left(\frac{9}{10}\right) = \frac{9^n}{10^n} < \frac{9^n}{3+10^n}$$

Since $\frac{9^n}{3+10^n} < \left(\frac{9}{10}\right)^n$ and $\sum_{n=1}^{\infty}\left(\frac{9}{10}\right)^n$ is convergent, by the direct comparison test, $\sum_{n=1}^{\infty}\frac{9^n}{3+10^n}$ is also convergent.

3. We can compare this to the convergent p-series $\sum_{n=1}^{\infty}\frac{1}{n^2}$. Noting that $\sin^2 n \leq 1$:

$$\frac{n\sin^2 n}{1+n^3} < \frac{n\sin^2 n}{n^3} \leq \frac{n}{n^3} = \frac{1}{n^2}$$

Because $\frac{n\sin^2 n}{1+n^3} \leq \frac{1}{n^2}$ for all $n \geq 1$ and $\sum_{n=1}^{\infty}\frac{1}{n^2}$ is convergent, we can state by the direct comparison test that $\sum_{n=1}^{\infty}\frac{n\sin^2 n}{1+n^3}$ is also convergent.

Answer to Exercise 17 (on page 50)

We can write the series as $\sum_{n=1}^{\infty}\frac{\cos n}{n^2}$. Since n is real, we know that $n^2 > 0$ and we can say that $\sum_{n=1}^{\infty}\left|\frac{\cos n}{n^2}\right| = \sum_{n=1}^{\infty}\frac{|\cos n|}{n^2}$. Additionally, $|\cos n| \leq 1$ for all n, and therefore $\frac{|\cos n|}{n^2} \leq \frac{1}{n^2}$. We know the series $\sum_{n=1}^{\infty}\frac{1}{n^2}$ is convergent. And since we have shown that $\sum_{n=1}^{\infty}\frac{|\cos n|}{n^2} \leq \sum_{n=1}^{\infty}\frac{1}{n^2}$, by the comparison test $\sum_{n=1}^{\infty}\frac{|\cos n|}{n^2}$ is convergent. Therefore, $\sum_{n=1}^{\infty}\frac{\cos n}{n^2}$ is absolutely convergent and therefore convergent.

Answer to Exercise 18 (on page 51)

1. Conditionally Convergent. $\sum_{n=1}^{\infty}\left|\frac{(-1)^n}{3n+2}\right| = \sum_{n=1}^{\infty}\frac{1}{3n+2}$ Applying the integral test to this sum: $\int_1^{\infty}\frac{1}{3x+2}\,dx = \lim_{t\to\infty}\int_1^t\frac{1}{3x+2}\,dx = \left[\frac{1}{3}\ln 3x+2\right]_{x=1}^{t} = \lim_{t\to\infty}[\ln 3x+2] - \ln 3(1) - 2 = \infty - 0 = \infty$. Since $\int_1^{\infty}\frac{1}{3x+2}\,dx$ is divergent, $\sum_{n=1}^{\infty}\frac{1}{3n+2}$ is divergent, and $\sum_{n=1}^{\infty}\frac{(-1)^n}{3n+2}$ is conditionally convergent.

2. Absolutely Convergent. $\sum_{n=1}^{\infty}\left|\frac{\sin n}{4^n}\right| \leq \sum_{n=1}^{\infty}\frac{1}{4^n}$. Applying the integral test to $\sum_{n=1}^{\infty}\frac{1}{4^n}$: $\int_1^{\infty}\frac{1}{4^x}\,dx = \lim_{t\to\infty}\frac{-1}{4^x\ln 4}\Big|_{x=1}^{t} = \lim_{t\to\infty}\left[\frac{-1}{4^t\ln 4}\right] - \frac{-1}{4^1\ln 4} = 0 + \frac{1}{4\ln 4} = \frac{1}{4\ln 4}$. Since $\int_1^{\infty}\frac{1}{4^x}\,dx$ is convergent, the series $\sum_{n=1}^{\infty}\frac{1}{4^n}$ is also convergent. And since $\sum_{n=1}^{\infty}\left|\frac{\sin n}{4^n}\right| \leq \sum_{n=1}^{\infty}\frac{1}{4^n}$, $\sum_{n=1}^{\infty}\left|\frac{\sin n}{4^n}\right|$ is also convergent, which shows that $\sum_{n=1}^{\infty}\frac{\sin n}{4^n}$ is absolutely convergent.

3. Conditionally Convergent. We are asking if the series $\sum_{n=1}^{\infty}\left|(-1)^{n-1}\frac{2n}{n^2+4}\right|$ is convergent. $\sum_{n=1}^{\infty}\left|(-1)^{n-1}\frac{2n}{n^2+4}\right| = \sum_{n=1}^{\infty}\frac{2n}{n^2+4}$ We will apply the Limit Comparison test

and compare this series to the known, divergent series $\sum_{n=1}^{\infty}\frac{1}{n}$. $\lim_{n\to\infty}\frac{\frac{2n}{n^2+4}}{\frac{1}{n}} = \lim_{n\to\infty}\frac{2n^2}{n^2+4} = 2 > 0$. Therefore, by the Limit Comparison test, $\sum_{n=1}^{\infty}\left|(-1)^{n-1}\frac{2n}{n^2+4}\right|$ is divergent AND $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{2n}{n^2+4}$ is conditionally convergent.

Answer to Exercise 19 (on page 52)

Series 1 and 3 converge

1. We apply the ratio test: $\lim_{n\to\infty}\left|\frac{\frac{8^{n+1}}{(n+1)!}}{\frac{8^n}{n!}}\right| = \lim_{n\to\infty}\frac{8\cdot 8^n}{(n+1)(n!)}\cdot\frac{n!}{8^n} = \lim_{n\to\infty}\frac{8}{n+1} = 0$. Therefore, the series converges.

2. We apply the ratio test: $\lim_{n\to\infty}\left|\frac{\frac{(n+1)!}{(n+1)^{100}}}{\frac{n!}{n^{100}}}\right| = \lim_{n\to\infty}\frac{(n+1)n!}{(n+1)^{100}}\cdot\frac{n^{100}}{n!} = \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^{100}\cdot(n+1) = \lim_{n\to\infty}\frac{n^{100}}{(n+1)^{99}} = \infty$. Therefore, the series diverges.

3. We apply the comparison test: $\frac{n+1}{(n)(n+2)(n+3)} = \frac{n}{(n)(n+2)(n+3)} + \frac{1}{(n)(n+2)(n+3)} = \frac{1}{(n+2)(n+3)} + \frac{1}{(n)(n+2)(n+3)} = \frac{1}{n^2+5n+6} + \frac{1}{n^3+5n^2+6n} \leq \frac{1}{n^2}+\frac{1}{n^3}$. The series $\sum_{n=1}^{\infty}\frac{1}{n^2}$ and $\sum_{n=1}^{\infty}\frac{1}{n^3}$ are both convergent, because they are p-series with $p > 1$. Having established that $\sum_{n=1}^{\infty}\frac{n+1}{(n)(n+2)(n+3)} \leq \sum_{n=1}^{\infty}\frac{1}{n^2}+\frac{1}{n^3}$ and that $\sum_{n=1}^{\infty}\frac{1}{n^2}+\frac{1}{n^3}$ converges, by the comparison test we can state that $\sum_{n=1}^{\infty}\frac{n+1}{(n)(n+2)(n+3)}$ converges.

Answer to Exercise 20 (on page 53)

1. This is not necessarily true. For a convergent series, the result of the ratio test is $L < 1$, so the limit could be $\neq 0$.

2. This is not necessarily true. Consider the geometric series $\sum_{n=1}^{\infty}2(\frac{1}{2})^{n-1}$. This series is convergent because the common ratio is less than one, but the first term is $2(\frac{1}{2})^0 = 2 > 1$.

3. This is not necessarily true. Again, consider the geometric series $\sum_{n=1}^{\infty}2(\frac{1}{2})^{n-1}$, which converges to $4 \neq 0$.

4. This is not necessarily true. Consider the p-series $\sum_{n=1}^{\infty}\frac{1}{n^4}$. Then the series $\sum_{n=1}^{\infty}n\frac{1}{n^4} = \sum_{n=1}^{\infty}\frac{1}{n^3}$ is convergent.

5. This must be true. By the comparison test, $\sum_{n=1}^{\infty}\frac{a_n}{n} \leq \sum_{n=1}^{\infty}a_n$. Since $\sum_{n=1}^{\infty}a_n$ converges, so much $\sum_{n=1}^{\infty}na_n$.

Answer to Exercise 21 (on page 54)

1. $\lim_{n=1}^{\infty} \sqrt[n]{\left|\left(\frac{3n^2+1}{n^2-4}\right)^n\right|} = \lim_{n=1}^{\infty} \frac{3n^2+1}{n^2-4} = 3 > 1$. Therefore, the series $\sum_{n=1}^{\infty} \left(\frac{3n^2+1}{n^2-4}\right)^n$ is divergent.

2. $\lim_{n\to\infty} \sqrt[n]{\left|\frac{(-1)^n}{(\ln n)^n}\right|} = \lim_{n\to\infty} \sqrt[n]{\frac{1}{(\ln n)^n}} = \lim_{n\to\infty} \frac{1}{\ln n} = \frac{1}{\infty} = 0 < 1$. Therefore, the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{(\ln n)^n}$ is convergent.

3. $\lim_{n\to\infty} \sqrt[n]{\left|\left(1+\frac{1}{n}\right)^{n^2}\right|} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e > 1$. Therefore, $\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^{n^2}$ is divergent.

Answer to Exercise 22 (on page 56)

1. Divergent. Since there is a constant to the n^{th} power and an algebraic function of n, we will try the Ratio Test. $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{e^{n+1}}{(n+1)^2} \cdot \frac{n^2}{e^n} = \lim_{n\to\infty} \frac{e^n \cdot e}{e^n} \cdot \left(\frac{n}{n+1}\right)^2 = \lim_{n\to\infty} e \cdot \left(\frac{n}{n+1}\right)^2 = e \cdot 1^2 = e > 1$. Therefore, $\sum_{n=1}^{\infty} \frac{e^n}{n^2}$ is divergent.

2. Convergent. Since there is a factorial, we will try the Ratio Test. $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{3^{n+1}(n+1)^2}{(n+1)!} \cdot \frac{n!}{3^n n^2} = \lim_{n\to\infty} \frac{3 \cdot 3^n}{3^n} \cdot \frac{n!}{(n+1)n!} \cdot \left(\frac{n+1}{n}\right)^2 = \lim_{n\to\infty} \frac{3(n+1)^2}{(n+1)n^2} = \lim_{n\to\infty} \frac{3(n+1)}{n^2} = 0 < 1$. Therefore, the series $\sum_{n=1}^{\infty} \frac{3^n n^2}{n!}$ is convergent.

3. Divergent. Since $\int_2^{\infty} \frac{1}{x\sqrt{\ln x}}\,dx$ can be integrated, we will apply the integral test. $\int_2^{\infty} \frac{1}{x\sqrt{\ln x}}\,dx = \lim_{t\to\infty} \int_2^t \frac{1}{x\sqrt{\ln x}}\,dx$. Setting $u = \ln x$, then $du = \frac{dx}{x}$ and $\frac{1}{x\sqrt{\ln x}}dx = \frac{1}{\sqrt{u}}du$. Then we can say that $\int_2^{\infty} \frac{1}{x\sqrt{\ln x}}\,dx = \lim_{t\to\infty} \int_{x=2}^{x=t} \frac{1}{\sqrt{u}}\,du = \lim_{t\to\infty} \left(\frac{-1}{2}\right) \sqrt{u}\Big|_{x=2}^{x=t} = \lim_{t\to\infty} \left(\frac{-1}{2}\right) \sqrt{\ln x}_2^t = \left(\frac{-1}{2}\right) \lim_{t\to\infty} \sqrt{\ln t} - \left(\frac{-1}{2}\right) \sqrt{\ln 2} = \infty$. Since the integral diverges, so does the series.

4. Convergent. Since this series has terms to the n^{th} power, we will try the Root Test. $\lim_{n\to\infty} \sqrt[n]{\left|\left(\frac{n}{n+1}\right)^{n^2}\right|} = \lim_{n\to\infty} \sqrt[n]{\left(\frac{n}{n+1}\right)^{n+2}} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^{\frac{n^2}{n}} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n = \lim_{n\to\infty} \left(\frac{1}{1+\frac{1}{n}}\right)^n = \frac{1}{\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n} = \frac{1}{e} < 1$ Therefore, by the root test, the series is convergent.

5. Convergent. This series also has terms raised to the n^{th} power, we will try the Root Test again. $\lim_{n\to\infty} \sqrt[n]{\left|\left(\sqrt[n]{2}-1\right)^n\right|} = \lim_{n\to\infty} \sqrt[n]{\left(\sqrt[n]{2}-1\right)^n} = \lim_{n\to\infty} \left(\sqrt[n]{2}-1\right)^{n/n} = \lim_{n\to\infty} \left(\sqrt[n]{2}-1\right) = \lim_{n\to\infty} 2^{1/n} - 1 = 1 - 1 = 0 < 1$. Therefore, the series con-

verges.

Index