



CONTENTS

1	Double Integrals in Other Coordinate Systems	3
1.1	When Polar Coordinates Help	3
1.2	Slicing a Region into Polar Rectangles	3
1.3	A Worked Example	5
2	Mass, Charge, and Center of Mass	9
2.1	Total Mass and Charge	9
2.2	Center of Mass	13
3	Moments, Surface Area, and Average Value	17
3.1	Moment of Inertia	17
3.1.1	Radius of Gyration	19
3.2	Surface Area	21
3.3	Average Value	26
3.4	Expected Values for Two Variables	27
4	Multivariate Distributions	29
4.1	Multivariate Distributions	29
4.2	The Multivariate Normal Distribution	29
4.2.1	The Covariance Matrix	31
4.2.2	Multivariate Normal Probability Density	33
4.3	Joint Density Functions	35
5	Numerical Double Integration	41
5.1	Riemann Sums on 2-Dimensional Domains	42

5.2	Numerical Integration in Python	44
6	Triple Integrals	47
6.1	From Double to Triple Integrals	47
6.2	Iterated Triple Integrals	47
6.3	Cylindrical Coordinates	48
6.4	Spherical Coordinates	48
6.5	Choosing a Coordinate System	49
A	Answers to Exercises	51
	Index	65

Double Integrals in Other Coordinate Systems

A dartboard, a pizza, the ripples from a stone in a pond: some shapes are made of circles, and forcing them into x and y makes the math miserable. Polar coordinates describe them naturally. This chapter shows how to set up double integrals in polar coordinates, including the one surprising extra factor you must not forget.

1.1 When Polar Coordinates Help

Consider a region shaped like half of a ring (see figure ??). Describing the region in Cartesian coordinates is complicated; you would have to split it into three regions. However, in polar coordinates, we can describe the whole region in one statement:

$$D = \{(r, \theta) \mid 1 \leq r \leq 4, 0 \leq \theta \leq \pi\}$$

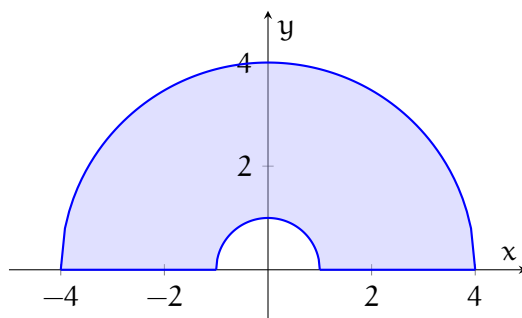


Figure 1.1: A semi-circular ring

1.2 Slicing a Region into Polar Rectangles

There are many instances where a region is simpler to describe in polar coordinates, so how do we take double integrals in polar coordinates? Suppose we want to integrate some function, $f(x, y)$, over a polar rectangle described by $D = \{(r, \theta) \mid a \leq r \leq b, \alpha \leq \theta \leq \beta\}$ (see figure 1.2). Similar to Cartesian coordinates, we can divide this region into many smaller polar rectangles, with each subrectangle defined by $D_{ij} = \{(r, \theta) \mid r_{i-1} \leq r \leq$

r_i , $\theta_{j-1} \leq \theta \leq \theta_j$. The center of each subrectangle has polar coordinates (r_i^*, θ_j^*) , where:

$$r_i^* = \frac{1}{2} (r_{i-1} + r_i)$$

$$\theta_j^* = \frac{1}{2} (\theta_{j-1} + \theta_j)$$

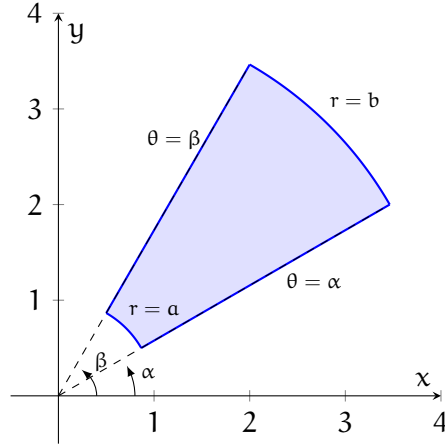


Figure 1.2: A polar rectangle described by $D = \{(r, \theta) \mid a \leq r \leq b, \alpha \leq \theta \leq \beta\}$

Each subrectangle is a larger radius sector minus a smaller radius sector, each with the same central angle, $\Delta\theta = \theta_j - \theta_{j-1}$. The total area of each subrectangle is given by:

$$\Delta A_i = \frac{1}{2} (r_i)^2 \Delta\theta - \frac{1}{2} (r_{i-1})^2 \Delta\theta = \frac{1}{2} (r_i^2 - r_{i-1}^2) \Delta\theta$$

Substituting $(r_i^2 - r_{i-1}^2) = (r_i + r_{i-1})(r_i - r_{i-1})$, we see that:

$$\Delta A_i = \frac{1}{2} (r_i + r_{i-1})(r_i - r_{i-1}) \Delta\theta$$

Recall that we have defined $r_i^* = \frac{1}{2} (r_{i-1} + r_i)$. Additionally, $\Delta r = r_i - r_{i-1}$. Substituting this, we find a simplified expression for the area of each subrectangle:

$$\Delta A_i = r_i^* \Delta r \Delta\theta$$

Therefore, the Riemann sum of $f(x, y)$ over the region is:

$$\sum_{i=1}^m \sum_{j=1}^n f(r_i^* \cos \theta_j^*, r_i^* \sin \theta_j^*) \Delta A_i$$

(Recall that to convert from Cartesian to polar coordinates, we use $x = r \cos \theta$ and $y = r \sin \theta$). Substituting for ΔA_i :

$$= \sum_{i=1}^m \sum_{j=1}^n f(r_i^* \cos \theta_j^*, r_i^* \sin \theta_j^*) r_i^* \Delta r \Delta \theta$$

Taking the limit as m and n go to ∞ , the Riemann sum becomes the double integral:

$$\int_{\alpha}^{\beta} \int_a^b f(r \cos \theta, r \sin \theta) r \, dr \, d\theta$$

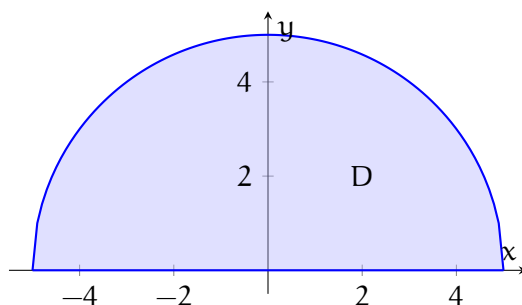
Notice the extra factor of r . A polar rectangle far from the origin is bigger than one near the origin with the same Δr and $\Delta \theta$, and the r accounts for that. Forgetting it is the most common mistake with polar integrals.

Therefore, if f is continuous on the polar rectangle $a \leq r \leq b$, $\alpha \leq \theta \leq \beta$, then:

$$\iint_D f(x, y) \, dA = \int_{\alpha}^{\beta} \int_a^b f(r \cos \theta, r \sin \theta) r \, dr \, d\theta$$

1.3 A Worked Example

Example: Evaluate $\iint_D x^2 y \, dA$, where D is the semi-circle shown below.



Solution: Since the region is a semi-circle with radius 5, we can describe D as $D = \{(r, \theta) \mid 0 \leq r \leq 5, 0 \leq \theta \leq \pi\}$. Therefore,

$$\begin{aligned} \iint_D x^2 y \, dA &= \int_0^{\pi} \int_0^5 (r \cos \theta)^2 (r \sin \theta) r \, dr \, d\theta \\ &= \int_0^{\pi} \int_0^5 r^4 \cos^2 \theta \sin \theta \, dr \, d\theta \end{aligned}$$

$$\begin{aligned}
&= \int_0^\pi \cos^2 \theta \sin \theta \left[\frac{1}{5} r^5 \right]_{r=0}^{r=5} d\theta \\
&= \int_0^\pi \cos^2 \theta \sin \theta \frac{5^5}{5} d\theta = 625 \int_0^\pi \cos^2 \theta \sin \theta d\theta
\end{aligned}$$

Using u -substitution, let $u = \cos \theta$. Then $-du = \sin \theta d\theta$, therefore:

$$\begin{aligned}
625 \int_0^\pi \cos^2 \theta \sin \theta d\theta &= 625 \int_{\theta=0}^{\theta=\pi} -u^2 du \\
&= -625 \frac{1}{3} u^3 \Big|_{\theta=0}^{\theta=\pi} = -625 \frac{1}{3} (\cos^3 \theta) \Big|_{\theta=0}^{\theta=\pi} \\
&= -\frac{625}{3} [(-1)^3 - (1)^3] = -\frac{625}{3} (-2) = \frac{1250}{3}
\end{aligned}$$

Exercise 1 Changing to Polar Coordinates

Evaluate the following iterated integrals by converting to polar coordinates:

Working Space

- $\int_0^2 \int_0^{\sqrt{4-x^2}} e^{-x^2-y^2} dy dx$
- $\int_0^{1/2} \int_{\sqrt{3}y}^{\sqrt{1-y^2}} xy^2 dx dy$
- $\int_0^2 \int_0^{\sqrt{2x-x^2}} \sqrt{x^2+y^2} dy dx$

Answer on Page 51

Exercise 2 Using Polar Coordinates in Multiple Integration

Working Space

Find the volume of the solid that lies under the surface $z = 4 - x^2 - y^2$ and above the xy -plane.

Answer on Page 53

Exercise 3 **The volume of a pool**

A circular swimming pool has a 40-ft diameter. The depth of the pool is constant along the north-south axis and increases from 3 feet at the west end to 10 feet at the east end. What is the total volume of water in the pool?

Working Space

Answer on Page 54

Mass, Charge, and Center of Mass

Try balancing a slice of pizza on one fingertip. If the toppings are spread evenly, the balance point is easy to find. Pile the pepperoni on one side and the balance point shifts. Double integrals let you compute exactly where it goes, as well as the total mass, the total electric charge, or the total of anything else spread over a region.

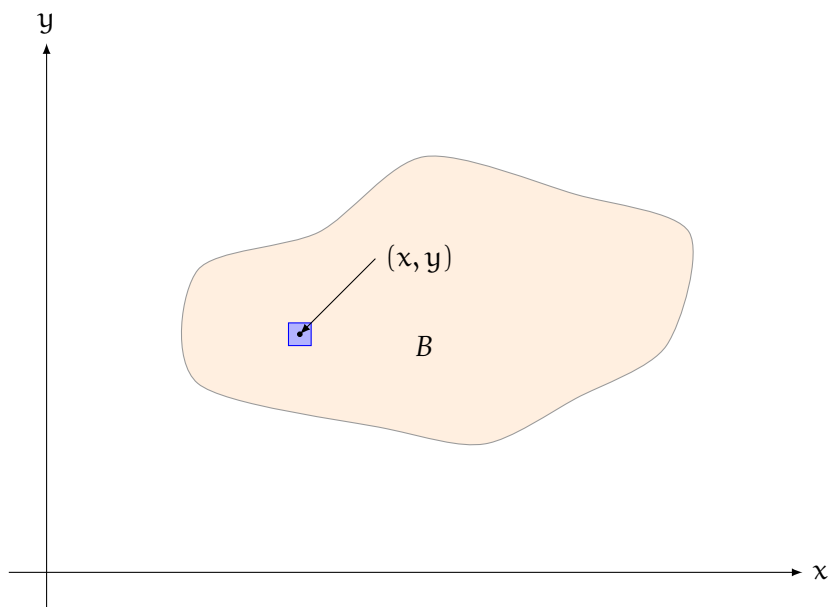
2.1 Total Mass and Charge

Suppose there is a thin layer (called a *lamina*) with a variable density that occupies an area B (see figure 2.1). Further, let the density of the lamina be described by a function, $\rho(x, y)$, which is continuous over B . For some small rectangle centered at (x, y) , the density is given by:

$$\rho(x, y) = \frac{\Delta m}{\Delta A}$$

where Δm is the mass of the small rectangle and ΔA is the area. Then the mass of the rectangle is given by:

$$\Delta m = \rho(x, y)\Delta A$$

Figure 2.1: A generic lamina that occupies the region B

We can find the mass of the entire lamina by dividing it into many of these small rectangles and adding the masses of all the rectangles (see figure 2.2). Pick a sample point (x_{ij}^*, y_{ij}^*) in each rectangle R_{ij} . The mass of the part of the lamina in R_{ij} is approximately $\rho(x_{ij}^*, y_{ij}^*)\Delta A$. Adding all these masses yields:

$$m_{\text{total}} \approx \sum_{i=1}^m \sum_{j=1}^n \rho(x_{ij}^*, y_{ij}^*)\Delta A$$

Taking the limit as $m, n \rightarrow \infty$ increases the number of rectangles to yield the true total mass:

$$m_{\text{total}} = \lim_{m, n \rightarrow \infty} \sum_{i=1}^m \sum_{j=1}^n \rho(x_{ij}^*, y_{ij}^*)\Delta A = \iint_B \rho(x, y) \, dA$$

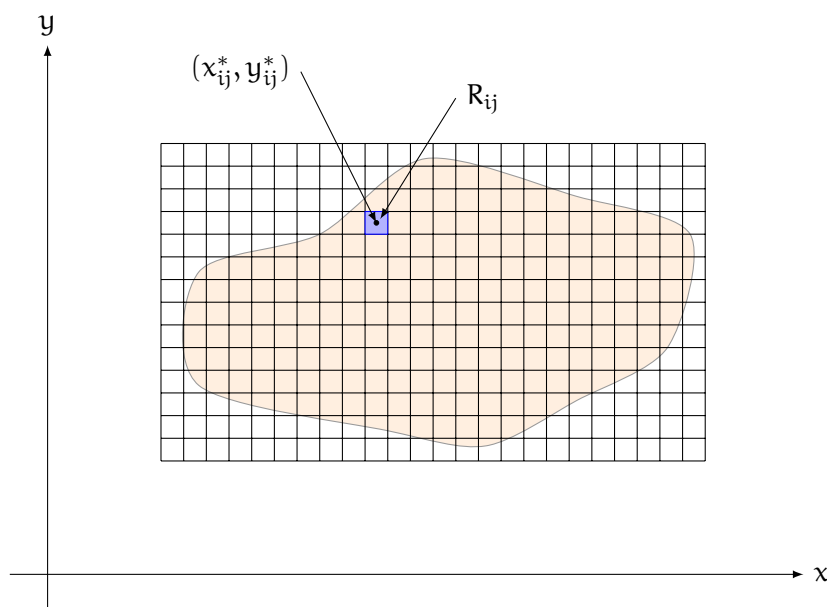


Figure 2.2: A generic lamina divided into many rectangles

Example: Find the total mass of a lamina that occupies the region $D = \{(x, y) \mid 1 \leq x \leq 3, 1 \leq y \leq 4\}$ with a density function $\rho(x, y) = 3y^2$.

Solution: We know that the total mass is given by:

$$\iint_D 3y^2 \, dA$$

Applying Fubini's Theorem, we see that:

$$\begin{aligned} \iint_D 3y^2 \, dA &= \int_1^3 \int_1^4 3y^2 \, dy \, dx \\ &= \int_1^3 \left[y^3 \right]_{y=1}^{y=4} \, dx = \int_1^3 [4^3 - 1^3] \, dx \\ &= \int_1^3 63 \, dx = 63x \Big|_{x=1}^{x=3} = 126 \end{aligned}$$

Exercise 4 Finding Total Mass

Find the mass of the lamina that occupies the region, D , and has the given density function, ρ .

Working Space

1. $D = \{(x, y) \mid 0 \leq x \leq 4, 0 \leq y \leq 3\}; \rho(x, y) = 1 + x^2 + y^2$
2. D is the triangular region with vertices $(0, 0)$, $(2, 1)$, $(0, 3)$; $\rho(x, y) = x + y$

Answer on Page 54

This method applies not only to mass density, but any other type of density. Some examples could include animals per acre of forest, cells per square centimeter of petri dish, or people per city block. A density physicists are often interested in is charge density (that is, the amount of charge, Q , per unit area). Charge is measured in coulombs (C). Often, charge density is given by a function, $\sigma(x, y)$, in units of coulombs per area (such as cm^2 or m^2). If there is some region, D , with charge distributed across it such that the charge density can be described by a continuous function, $\sigma(x, y)$, then the total charge, Q , is given by:

$$Q = \iint_D \sigma(x, y) \, dA$$

Example: Charge is distributed over the region B shown in figure 2.3 such that the charge density is given by $\sigma(x, y) = xy$, measured in C/m^2 . Find the total charge.

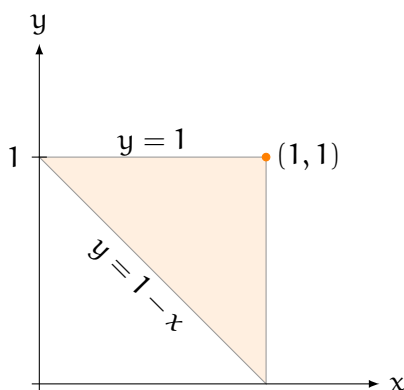


Figure 2.3: A triangular region over which charge is distributed such that $\sigma(x, y) = xy$

Solution: We know that total charge is given by:

$$Q = \iint_B xy \, dA$$

Examining figure 2.3, we see that:

$$\begin{aligned}
 \iint_B xy \, dA &= \int_0^1 \int_{1-x}^1 xy \, dy \, dx \\
 &= \int_0^1 \frac{x}{2} \left[y^2 \right]_{y=1-x}^{y=1} dx = \int_0^1 \frac{x}{2} \left[1^2 - (1-x)^2 \right] dx \\
 &= \frac{1}{2} \int_0^1 x (1 - 1 + 2x - x^2) dx = \frac{1}{2} \int_0^1 x (2x - x^2) dx \\
 &= \frac{1}{2} \int_0^1 2x^2 - x^3 dx = \frac{1}{2} \left[\frac{2}{3}x^3 - \frac{1}{4}x^4 \right]_{x=0}^{x=1} = \frac{1}{2} \left(\frac{2}{3} - \frac{1}{4} \right) = \frac{5}{24}C
 \end{aligned}$$

2.2 Center of Mass

For a lamina of variable density in the xy -plane, the coordinates of the center of mass, $(\bar{x}, \bar{y})$, are given by:

$$\begin{aligned}
 \bar{x} &= \frac{1}{m} \iint_D x \rho(x, y) \, dA \\
 \bar{y} &= \frac{1}{m} \iint_D y \rho(x, y) \, dA
 \end{aligned}$$

where m is the total mass and ρ is the density of the lamina as a function of x and y .

Example: Find the center of mass of a triangular lamina with vertices at $(0, 0)$, $(2, 0)$, and $(0, 1)$ and a density function $\rho(x, y) = 2 + x + 3y$.

Solution: We begin by visualizing the region so we can determine if it is type I or type II:

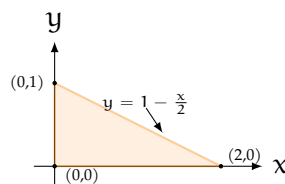


Figure 2.4: Region of integration for the triangular lamina.

Recall that the total mass is given by $m = \iint_D \rho(x, y) \, dA$. As shown above, we can define $D = \{(x, y) \mid 0 \leq x \leq 2, 0 \leq y \leq 1 - x/2\}$:

$$m = \int_0^2 \int_0^{1-x/2} (2 + x + 3y) \, dy \, dx = \int_0^2 \left[2y + xy + \frac{3}{2}y^2 \right]_{y=0}^{y=1-x/2} dx$$

$$\begin{aligned}
&= \int_0^2 \left[2\left(1 - \frac{x}{2}\right) + x\left(1 - \frac{x}{2}\right) + \frac{3}{2}\left(1 - \frac{x}{2}\right)^2 \right] dx = \int_0^2 \left[\frac{7}{2} - \frac{3x}{2} - \frac{x^2}{8} \right] dx \\
&= \left[\frac{7x}{2} - \frac{3x^2}{4} - \frac{x^3}{24} \right]_{x=0}^{x=2} = \frac{7(2)}{2} - \frac{3(4)}{4} - \frac{8}{24} = 7 - 3 - \frac{1}{3} = \frac{11}{3}
\end{aligned}$$

Finding $\bar{x}$:

$$\begin{aligned}
\bar{x} &= \frac{1}{m} \iint_D x(2 + x + 3y) \, dA \\
\bar{x} &= \frac{3}{11} \int_0^2 \int_0^{1-x/2} [2x + x^2 + 3xy] \, dy \, dx \\
\bar{x} &= \frac{3}{11} \int_0^2 \left[2xy + x^2y + \frac{3}{2}xy^2 \right]_{y=0}^{y=1-x/2} dx \\
\bar{x} &= \frac{3}{11} \int_0^2 \left[\frac{7x}{2} - \frac{3x^2}{2} - \frac{x^3}{8} \right] dx \\
\bar{x} &= \frac{3}{11} \left[\frac{7x^2}{4} - \frac{x^3}{2} - \frac{x^4}{32} \right]_{x=0}^{x=2} \\
\bar{x} &= \frac{3}{11} \left[\frac{7(4)}{4} - \frac{8}{2} - \frac{16}{32} \right] = \frac{3}{11} \left(7 - 4 - \frac{1}{2} \right) = \frac{3}{11} \left(\frac{5}{2} \right) = \frac{15}{22}
\end{aligned}$$

We can similarly find $\bar{y}$:

$$\begin{aligned}
\bar{y} &= \frac{1}{m} \iint_D y(2 + x + 3y) \, dA \\
\bar{y} &= \frac{3}{11} \int_0^2 \int_0^{1-x/2} [2y + xy + 3y^2] \, dy \, dx \\
\bar{y} &= \frac{3}{11} \int_0^2 \left[y^2 + \frac{x}{2}y^2 + y^3 \right]_{y=0}^{y=1-x/2} dx \\
\bar{y} &= \frac{3}{11} \int_0^2 \left[\left(1 - \frac{x}{2}\right)^2 + \frac{x}{2}\left(1 - \frac{x}{2}\right)^2 + \left(1 - \frac{x}{2}\right)^3 \right] dx \\
\bar{y} &= \frac{3}{11} \int_0^2 \left[2 - 2x + \frac{x^2}{2} \right] dx = \frac{3}{11} \left[2x - x^2 + \frac{x^3}{6} \right]_{x=0}^{x=2} \\
\bar{y} &= \frac{3}{11} \left[2(2) - 2(2) + \frac{8}{6} \right] = \frac{3}{11} \left(\frac{4}{3} \right) = \frac{4}{11}
\end{aligned}$$

Therefore, the center of mass $(\bar{x}, \bar{y})$ is $(\frac{15}{22}, \frac{4}{11})$. Compare this with the center of a *uniform* triangle with these vertices, which is the average of the vertices, $(\frac{2}{3}, \frac{1}{3})$. The density grows with x and even faster with y , so the balance point shifts up and to the right.

Exercise 5 **Center of Mass**

Find the center of mass of:

1. A lamina that occupies the area enclosed by the curves $y = 0$ and $y = 2 \sin x$ from $0 \leq x \leq \pi$ if its density is given by $\rho(x, y) = x$.
2. The region D if $D = \{(x, y) \mid 0 \leq x \leq 4, 0 \leq y \leq 3\}$; $\rho(x, y) = 1 + x^2 + y^2$
3. The triangular region D with vertices $(0, 0)$, $(2, 1)$, $(0, 3)$; $\rho(x, y) = x + y$

Working Space

Answer on Page 55

Moments, Surface Area, and Average Value

A figure skater spins faster by pulling in their arms. The mass has not changed, but where it sits has, and that changes how hard it is to spin. That resistance to spinning is the moment of inertia, and double integrals compute it for any flat shape. In this chapter we also use double integrals to find the area of curved surfaces and the average value of a function over a region.

3.1 Moment of Inertia

We can also use double integrals to find the **moment of inertia** of a lamina about a particular axis (we will extend this to three-dimensional objects in the Triple Integrals chapter). Recall that the moment of inertia for a particle with mass m a distance r from the axis of rotation is mr^2 . Dividing a lamina into small pieces, we see that the moment of inertia of each piece about the x -axis is:

$$(y_{ij}^*)^2 \rho(x_{ij}^*, y_{ij}^*) \Delta A$$

Where x_{ij}^* and y_{ij}^* are the x - and y -coordinates of the small piece. The moment of inertia of the entire lamina about the x -axis is then the sum of all the individual moments:

$$I_x = \lim_{m,n \rightarrow \infty} \sum_{i=1}^m \sum_{j=1}^n (y_{ij}^*)^2 \rho(x_{ij}^*, y_{ij}^*) \Delta A = \iint_D y^2 \rho(x, y) \, dA$$

Similarly, the moment of inertia of a lamina about the y -axis is:

$$I_y = \lim_{m,n \rightarrow \infty} \sum_{i=1}^m \sum_{j=1}^n (x_{ij}^*)^2 \rho(x_{ij}^*, y_{ij}^*) \Delta A = \iint_D x^2 \rho(x, y) \, dA$$

Example: Find the moment of inertia of a square centered at the origin with side length r and constant density ρ about the x -axis.

Solution: We can describe the square as the region bounded by $D = \{(x, y) \mid -r/2 \leq x \leq r/2, -r/2 \leq y \leq r/2\}$ with density function $\rho(x, y) = \rho$. Therefore, the moment of inertia

about the x -axis is given by:

$$\begin{aligned}
 I_x &= \int_{-r/2}^{r/2} \int_{-r/2}^{r/2} \rho y^2 \, dy \, dx \\
 &= \rho \int_{-r/2}^{r/2} \left[\frac{1}{3} y^3 \right]_{y=-r/2}^{y=r/2} dx \\
 &= \frac{\rho}{3} \int_{-r/2}^{r/2} \left[\frac{r^3}{8} - \left(-\frac{r^3}{8} \right) \right] dx \\
 &= \frac{\rho}{3} \int_{-r/2}^{r/2} \left[\frac{r^3}{4} \right] dx = \frac{r^3 \rho}{12} \int_{-r/2}^{r/2} 1 \, dx \\
 &= \frac{r^3 \rho}{12} [x]_{x=-r/2}^{x=r/2} = \frac{r^3 \rho}{12} \cdot r = \frac{r^4 \rho}{12}
 \end{aligned}$$

We can also find the moment of inertia about the origin, I_o . This is the moment of inertia for an object rotating in the xy -plane about the origin. The moment of inertia about the origin is the sum of the moments of inertia about the x - and y -axes:

$$I_o = I_x + I_y = \iint_D (x^2 + y^2) \rho(x, y) \, dA$$

Example: Find the moment of inertia about the origin of a disk with density $\rho(x, y) = b$, centered at the origin, with a radius of a . Show that this is equal to the expected moment of inertia, $\frac{1}{2}MR^2$, where M is the total mass of the disk and R is the radius of the disk.

Solution: Since we are examining a circle about the origin, the region can be described in polar coordinates as $D = \{(r, \theta) \mid 0 \leq r \leq a, 0 \leq \theta \leq 2\pi\}$. Converting from Cartesian coordinates to polar coordinates:

$$\begin{aligned}
 I_o &= \iint_D (x^2 + y^2) b \, dA = \int_0^a \int_0^{2\pi} r (r^2) b \, d\theta \, dr \\
 &= \int_0^a r^3 \, dr \cdot \int_0^{2\pi} b \, d\theta = \frac{a^4}{4} \cdot (2\pi b) = \frac{\pi a^4 b}{2}
 \end{aligned}$$

The total mass of this disk is the density, b , multiplied by the area, πa^2 . Therefore,

$$R = a$$

$$M = \pi a^2 b$$

Substituting into the result of our double integral, we see that:

$$\frac{\pi a^4 b}{2} = (\pi a^2 b) \cdot \left(\frac{a^2}{2}\right) = M \cdot \frac{R^2}{2} = \frac{1}{2}MR^2$$

3.1.1 Radius of Gyration

When modeling rotating objects, it can be helpful to have a simplified model. A spinning, continuous object can be modeled as a point mass by using the lamina's *radius of gyration*. The radius of gyration of a lamina about the origin is a radius, R , such that:

$$mR^2 = I_o$$

where m is the mass of the lamina and I_o is the moment of inertia of the lamina about the origin. Essentially, we are finding a radius such that if the lamina were shrunk down to a point mass and rotated about the axis at that radius, the moment of inertia would be the same.

We can also find radii of gyration about the x - and y -axes:

$$m\bar{\bar{y}}^2 = I_x$$

$$m\bar{\bar{x}}^2 = I_y$$

About the origin, $R = \sqrt{\bar{\bar{x}}^2 + \bar{\bar{y}}^2}$.

Example: Find the radius of gyration about the y -axis for a disk with density $\rho(x, y) = y$ if the disk has radius 2 and is centered at $(0, 2)$.

Solution: We are ultimately looking for a radius such that $m\bar{\bar{x}}^2 = I_y$, so we need to know the mass, m , and the moment of inertia about the y -axis, I_y . First, let us find the total mass, m , of the disk. We can describe the disk in polar coordinates as $D = \{(r, \theta) \mid 0 \leq r \leq 4 \sin \theta, 0 \leq \theta \leq \pi\}$, and therefore the mass is given by:

$$\begin{aligned} m &= \iint_D y \, dA = \int_0^\pi \int_0^{4 \sin \theta} r (r \sin \theta) \, dr \, d\theta \\ &= \int_0^\pi \sin \theta \int_0^{4 \sin \theta} r^2 \, dr \, d\theta = \frac{1}{3} \int_0^\pi \sin \theta [4 \sin \theta]^3 \, d\theta \\ &= \frac{64}{3} \int_0^\pi \sin^4 \theta \, d\theta = \frac{64}{3} \int_0^\pi \left(\frac{1 - \cos 2\theta}{2} \right)^2 \, d\theta = \frac{64}{3} \left(\frac{1}{2} \right)^2 \int_0^\pi (1 - 2 \cos 2\theta + \cos^2 2\theta) \, d\theta \end{aligned}$$

$$\begin{aligned}
&= \frac{16}{3} \left[(\theta - \sin 2\theta) \Big|_{\theta=0}^{\theta=\pi} + \int_0^{\pi} \frac{1 + \cos 4\theta}{2} d\theta \right] = \frac{16}{3} \left[\pi + \frac{1}{2} \left(\theta + \frac{1}{4} \sin 4\theta \right) \Big|_{\theta=0}^{\theta=\pi} \right] \\
&= \frac{16}{3} \left[\pi + \frac{\pi}{2} \right] = \frac{16}{3} \left(\frac{3\pi}{2} \right) = 8\pi
\end{aligned}$$

Now that we have found the mass, let us find the moment of inertia, I_y :

$$\begin{aligned}
I_y &= \iint_D x^2 y \, dA = \int_0^{\pi} \int_0^{4 \sin \theta} r (r \cos \theta)^2 (r \sin \theta) \, dr \, d\theta \\
&= \int_0^{\pi} \left[\cos^2 \theta \sin \theta \int_0^{4 \sin \theta} r^4 \, dr \right] d\theta = \int_0^{\pi} \cos^2 \theta \sin \theta \left[\frac{1}{5} r^5 \right]_{r=0}^{r=4 \sin \theta} d\theta \\
&= \frac{1024}{5} \int_0^{\pi} \cos^2 \theta \sin^6 \theta \, d\theta = \frac{1024}{5} \int_0^{\pi} \left(\frac{1 + \cos 2\theta}{2} \right) \left(\frac{1 - \cos 2\theta}{2} \right)^3 d\theta \\
&= \frac{1024}{5} \left(\frac{1}{2} \right)^4 \int_0^{\pi} (1 + \cos 2\theta) (1 - \cos 2\theta) (1 - \cos 2\theta)^2 d\theta \\
&= \frac{64}{5} \int_0^{\pi} (1 - \cos^2 2\theta) (1 - 2 \cos 2\theta + \cos^2 2\theta) d\theta \\
&= \frac{64}{5} \int_0^{\pi} 1 - 2 \cos 2\theta + \cos^2 2\theta - \cos^2 2\theta + 2 \cos^3 2\theta - \cos^4 2\theta d\theta \\
&= \frac{64}{5} \int_0^{\pi} 1 - 2 \cos 2\theta + 2 \cos 2\theta (1 - \sin^2 2\theta) - \left(\frac{1 + \cos 4\theta}{2} \right)^2 d\theta \\
&= \frac{64}{5} \int_0^{\pi} 1 + 2 \cos 2\theta \sin^2 2\theta - \frac{1}{4} (1 + 2 \cos 4\theta + \cos^2 4\theta) d\theta \\
&= \frac{64}{5} \int_0^{\pi} 1 + 2 \cos 2\theta \sin^2 2\theta - \frac{1}{4} - \frac{\cos 4\theta}{2} - \frac{1}{4} \left(\frac{1 + \cos 8\theta}{2} \right) d\theta \\
&= \frac{64}{5} \left[\frac{5\theta}{8} + \frac{1}{3} \sin^3 2\theta - \frac{\sin 4\theta}{8} - \frac{\sin 8\theta}{64} \right]_{\theta=0}^{\theta=\pi} = \frac{64}{5} \cdot \frac{5\pi}{8} = 8\pi
\end{aligned}$$

We have found that $m = 8\pi$ and $I_y = 8\pi$. Substituting to find the radius of gyration:

$$m \bar{\bar{x}}^2 = I_y$$

$$(8\pi) \bar{\bar{x}}^2 = 8\pi$$

$$\bar{\bar{x}} = 1$$

Therefore, the radius of gyration about the y-axis is $\bar{\bar{x}} = 1$

Exercise 6 **Moments of Inertia and Radii of Gyration**

Find the requested moment of inertia and radius of gyration of the lamina with the given density function.

Working Space

1. about the x -axis, $D = \{(x, y) \mid 1 \leq x \leq 4, 0 \leq y \leq 3\}$, $\rho(x, y) = xy$.
2. about the y -axis, D is enclosed by the curves $y = 0$ and $y = 2 \cos x$ for $-\pi/2 \leq x \leq \pi/2$, $\rho(x, y) = y$.
3. about the origin, $D = \{(r, \theta) \mid 1 \leq r \leq 2, 0 \leq \theta \leq \pi\}$, $\rho(r, \theta) = r$.

Answer on Page 57

3.2 Surface Area

We have already seen how to find the areas of surfaces of revolution using single-variable calculus. Now, we will use multivariable calculus to find the surface area of a generic, two-variable function, $z = f(x, y)$. Suppose a surface, S , is defined by the continuous, partially differentiable function, $f(x, y)$, over a rectangular region, R (see figure 3.1).

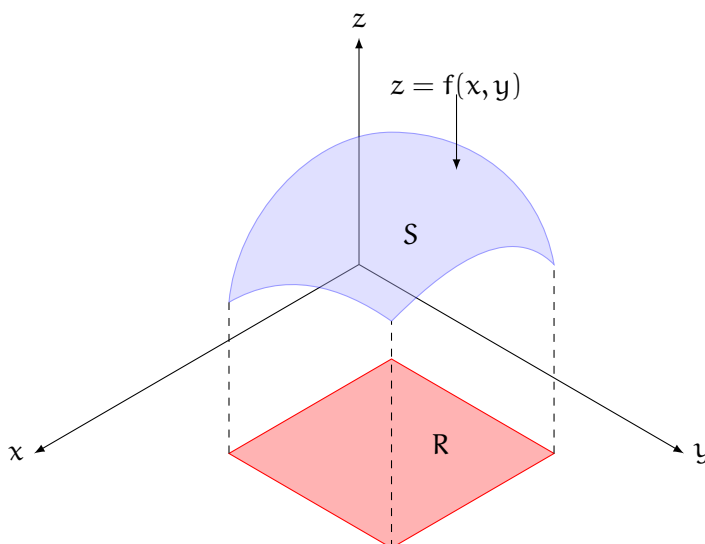


Figure 3.1: The graph of f over the region R creates a surface, S

We begin by dividing the region, R , into sub-rectangles, R_{ij} , each with area $\Delta A = \Delta x \Delta y$. Then, projecting upwards from the point closest to the origin, $(x_i, y_j, 0)$, we find a point on the surface, $P_{ij} = (x_i, y_j, f(x_i, y_j))$. Next, there is a small plane, ΔT_{ij} , tangent to the surface at P_{ij} , and the area of the tangent plane is approximately the same as the area of the surface over the sub-rectangle R_{ij} (see figure 3.2).

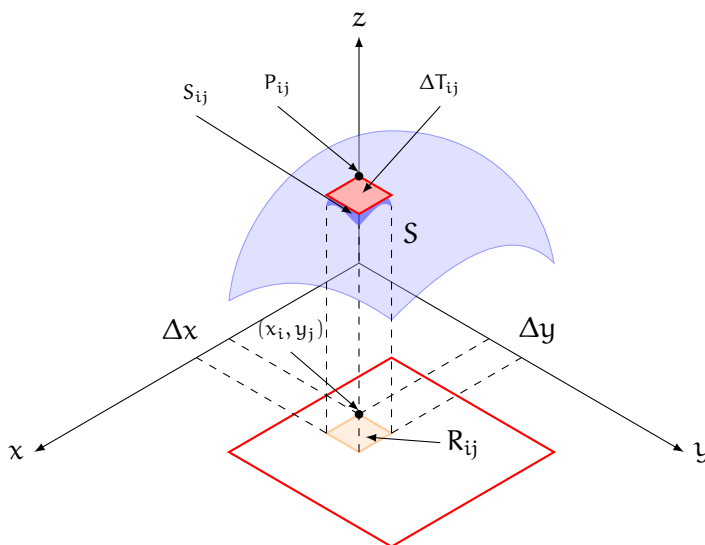


Figure 3.2: The tangent surface, ΔT_{ij} , is approximately the same surface area as the surface, S_{ij} , over the sub-rectangle, R_{ij}

It follows that the total surface area of the surface, S , is the sum of all these little tangent

surfaces as the number of tangent surfaces approaches infinity:

$$A(S) = \lim_{m,n \rightarrow \infty} \sum_{i=1}^m \sum_{j=1}^n \Delta T_{ij}$$

How can we find an expression for ΔT_{ij} ? We will define two vectors, $\mathbf{a}$ and $\mathbf{b}$, that are equal to the sides of ΔT_{ij} (see figure 3.3). Geometrically, the area of ΔT_{ij} is the absolute value of the cross product of the two vectors. Mathematically,

$$\Delta T_{ij} = |\mathbf{a} \times \mathbf{b}|$$

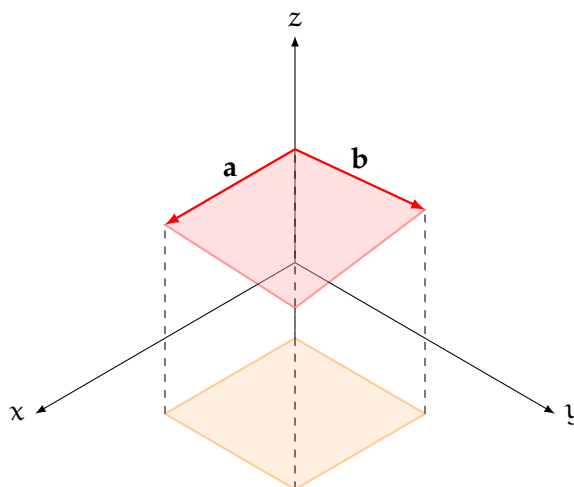


Figure 3.3: The vectors $\mathbf{a}$ and $\mathbf{b}$ define the sides of the tangent surface ΔT_{ij}

Recall the three unit vectors: $\mathbf{i}$ in the x -direction, $\mathbf{j}$ in the y -direction, and $\mathbf{k}$ in the z -direction. We can then describe $\mathbf{a}$ and $\mathbf{b}$ in terms of $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$:

$$\mathbf{a} = \Delta x \mathbf{i} + f_x(x_i, y_j) \Delta x \mathbf{k}$$

$$\mathbf{b} = \Delta y \mathbf{j} + f_y(x_i, y_j) \Delta y \mathbf{k}$$

(Recall that f_x is the partial derivative of $f(x, y)$ with respect to x , and f_y is the partial derivative with respect to y .) This is true because f_x gives the slope of a tangent line parallel to the x -axis, and f_y parallel to the y -axis. We then find an expression for $|\mathbf{a} \times \mathbf{b}|$ (we have omitted some details here):

$$\mathbf{a} \times \mathbf{b} = -f_x(x_i, y_j) \Delta x \Delta y \mathbf{i} - f_y(x_i, y_j) \Delta x \Delta y \mathbf{j} + \Delta x \Delta y \mathbf{k}$$

Substituting $\Delta A = \Delta x \Delta y$:

$$\mathbf{a} \times \mathbf{b} = [-f_x(x_i, y_j)\mathbf{i} - f_y(x_i, y_j)\mathbf{j} + \mathbf{k}] \Delta A$$

To find the area of ΔT_{ij} , we need to find the length of $\mathbf{a} \times \mathbf{b}$. Recall that we can use the Pythagorean theorem to find the length of a vector. For a 3-dimensional vector $\mathbf{v} = r\mathbf{i} + s\mathbf{j} + t\mathbf{k}$, its length is given by:

$$|\mathbf{v}| = \sqrt{r^2 + s^2 + t^2}$$

Applying this, we find the length of $\mathbf{a} \times \mathbf{b}$ (which is the same as the area of ΔT_{ij}) is:

$$\begin{aligned} \Delta T_{ij} &= |[-f_x(x_i, y_j)\mathbf{i} - f_y(x_i, y_j)\mathbf{j} + \mathbf{k}] \Delta A| \\ &= \sqrt{(-f_x(x_i, y_j)\Delta A)^2 + (-f_y(x_i, y_j)\Delta A)^2 + (\Delta A)^2} \\ &= \sqrt{[f_x(x, y)]^2 + [f_y(x, y)]^2 + 1} \Delta A \end{aligned}$$

So, the area of the entire surface over region R is:

$$A(S) = \lim_{m, n \rightarrow \infty} \sum_{i=1}^m \sum_{j=1}^n \sqrt{[f_x(x, y)]^2 + [f_y(x, y)]^2 + 1} \Delta A$$

This is the definition of a double integral; therefore, the surface area of a two-variable function, $f(x, y)$ over a region, R , where f_x and f_y are continuous, is:

$$A(S) = \iint_R \sqrt{[f_x(x, y)]^2 + [f_y(x, y)]^2 + 1} \, dA$$

Using the notation of partial derivatives, this is also expressed as:

$$A(S) = \iint_R \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} \, dA$$

Example: Find the surface area of the part of the surface $z = 2 - y^2$ that lies over the triangle whose vertices are at $(0, 0)$, $(0, 4)$, and $(3, 4)$.

Solution: We can define $R = \{(x, y) \mid 0 \leq x \leq \frac{3}{4}y, 0 \leq y \leq 4\}$. Additionally,

$$\frac{\partial z}{\partial x} = 0$$

$$\frac{\partial z}{\partial y} = -2y$$

Therefore, the area of the surface that lies above R is:

$$\begin{aligned} A(S) &= \iint_R \sqrt{1 + 0^2 + (-2y)^2} \, dA = \int_0^4 \int_0^{\frac{3}{4}y} \sqrt{1 + 4y^2} \, dx \, dy \\ &= \int_0^4 \sqrt{1 + 4y^2} [x]_{x=0}^{x=\frac{3}{4}y} \, dy = \frac{3}{4} \int_0^4 y \sqrt{1 + 4y^2} \, dy \end{aligned}$$

Let $u = 1 + 4y^2$, then $du = (8y)dy$ and $(y)dy = \frac{du}{8}$. Substituting:

$$\begin{aligned} A(S) &= \frac{3}{4} \int_{y=0}^{y=4} \frac{1}{8} \sqrt{u} \, du = \frac{3}{32} \left[\frac{2}{3} u^{3/2} \right]_{y=0}^{y=4} \\ &= \frac{1}{16} \left[(1 + 4y^2)^{3/2} \right]_{y=0}^{y=4} = \frac{1}{16} \left[(65)^{3/2} - 1 \right] \approx 32.69 \end{aligned}$$

Exercise 7 Surface Area of Two-Variable Functions

Find the area of the surface.

Working Space

1. The part of the plane $9x + 6y - 3z + 6 = 0$ that lies above the rectangle $[2, 6] \times [1, 4]$.
2. The part of the paraboloid $z = 2x^2 + 2y^2$ that lies under the plane $z = 32$.
3. The part of the surface $z = 3xy$ that lies in the cylinder $x^2 + y^2 = 4$.

Answer on Page 59

3.3 Average Value

Recall that the average value of a one-variable function over the interval $x \in [a, b]$ is given by:

$$f_{\text{ave}} = \frac{1}{b-a} \int_a^b f(x) \, dx$$

For a two-variable function, the average value over a region, R , is given by:

$$f_{\text{ave}} = \frac{1}{A(R)} \iint_R f(x, y) \, dA$$

Where $A(R)$ is the area of the two-dimensional region.

Example: Find the average value of $f(x, y) = xy^2$ over the rectangle with vertices at $(-2, 0)$, $(-2, 4)$, $(2, 4)$, and $(2, 0)$.

Solution: The rectangular region has an area of $(2 - (-2)) \cdot (4 - 0) = 4 \cdot 4 = 16$. Therefore, the average value is given by:

$$\begin{aligned} f_{\text{ave}} &= \frac{1}{16} \iint_R xy^2 \, dA = \frac{1}{16} \int_{-2}^2 \int_0^4 xy^2 \, dy \, dx \\ &= \frac{1}{16} \int_{-2}^2 \frac{x}{3} y^3 \Big|_{y=0}^{y=4} \, dx = \frac{1}{16} \int_{-2}^2 \frac{x}{3} (4^3) \, dx = \frac{4}{3} \int_{-2}^2 x \, dx \\ &= \frac{4}{3} \left(\frac{1}{2} \right) x^2 \Big|_{x=-2}^{x=2} = 0 \end{aligned}$$

Exercise 8 Average Value

Find the average value of the function over the region D :

1. $f(x, y) = x \sin y$, $D = [0, 2] \times [-\pi/2, \pi/2]$
2. $f(x, y) = x + y$, D is the circle with radius 1 centered at $(1, 0)$
3. $f(x, y) = xy$, D is the triangle with vertices at $(0, 0)$, $(2, 0)$, $(2, 2)$

Working Space

Answer on Page 60

3.4 Expected Values for Two Variables

Recall that for one variable, X , the expected value of X (its mean) is:

$$\mu = \int_{-\infty}^{\infty} xf(x) dx$$

Where $f(x)$ is the probability density function for X . For two random variables X and Y with joint density $f(x, y)$, the expected value of each is:

$$\mu_x = \iint_{\mathbb{R}^2} xf(x, y) dA$$

$$\mu_y = \iint_{\mathbb{R}^2} yf(x, y) dA$$

Example: X and Y are random variables with joint density function:

$$f(x, y) = \begin{cases} 0.1e^{-(0.2x+0.5y)}, & \text{if } x \geq 0, y \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

What are the expected values of X and Y ?

Solution: The expected value of X is:

$$\begin{aligned} \mu_x &= \int_0^{\infty} \int_0^{\infty} x(0.1)e^{-(0.2x+0.5y)} dy dx \\ &= \frac{1}{10} \int_0^{\infty} x(-2)e^{-(0.2x+0.5y)} \Big|_{y=0}^{y=\infty} dx = -\frac{1}{5} \int_0^{\infty} (x)e^{-0.2x} \left[e^{-0.5y} \right]_{y=0}^{y=\infty} dx \\ &= -\frac{1}{5} \int_0^{\infty} xe^{-0.2x} [e^{-\infty} - e^0] dx = -\frac{1}{5} \int_0^{\infty} xe^{-0.2x} [0 - 1] dx \\ &= \frac{1}{5} \int_0^{\infty} xe^{-0.2x} dx \end{aligned}$$

Applying integration by parts,

$$\begin{aligned} \mu_x &= \frac{1}{5} \left[-5xe^{-0.2x} \Big|_{x=0}^{x=\infty} - \int_0^{\infty} (-5)e^{-0.2x} dx \right] \\ &= \left[-xe^{-0.2x} \right]_{x=0}^{x=\infty} + \int_0^{\infty} e^{-0.2x} dx = -5e^{-0.2x} \Big|_{x=0}^{x=\infty} \\ &= -5(e^{-\infty} - e^0) = -5(0 - 1) = 5 \end{aligned}$$

And the expected value of Y is:

$$\begin{aligned}\mu_y &= \int_0^\infty \int_0^\infty y(0.1)e^{-(0.2x+0.5y)} dx dy \\ &= \frac{1}{10} \int_0^\infty (y)e^{-0.5y} \int_0^\infty (e^{-0.2x}) dx dy = \frac{1}{10} \int_0^\infty (y)e^{-0.5y} \left[-5e^{-0.2x}\right]_{x=0}^{x=\infty} dy \\ &= \frac{1}{10} \int_0^\infty (y)e^{-0.5y} (0+5) dy = \frac{1}{2} \int_0^\infty ye^{-0.5y} dy\end{aligned}$$

Applying integration by parts:

$$\begin{aligned}\mu_y &= \frac{1}{2} \left[(-2)ye^{-0.5y} \Big|_{y=0}^{y=\infty} - \int_0^\infty (-2)e^{-0.5y} dy \right] \\ &= \left[-ye^{-0.5y} \right]_{y=0}^{y=\infty} + (-2)e^{-0.5y} \Big|_{y=0}^{y=\infty} = (-2) \left(e^{-\infty} - e^0 \right) = -2(0-1) = 2\end{aligned}$$

Therefore, the expected value of X is 5 and of Y is 2.

Exercise 9 Expected Values

The joint density function for a pair of random variables, X and Y is

$$f(x, y) = \begin{cases} \frac{1}{4}y(1+2x), & \text{if } 0 \leq x \leq 1, 0 \leq y \leq 2 \\ 0, & \text{otherwise} \end{cases}$$

What are the expected values of X and Y?

Working Space

Answer on Page 61

Multivariate Distributions

Taller people tend to weigh more. Students who sleep more tend to score higher. Snails with bigger shells tend to be heavier. To describe two variables at once, including how they move together, you need a distribution of two variables. In this chapter, you will meet the multivariate normal distribution, with real snail data, and joint density functions, which use the double integrals you have just learned.

4.1 Multivariate Distributions

The world of probability and statistics does not limit itself to the study of single variables. Often, we are interested in the interconnections, relationships, and associations among several variables. In such a scenario, the univariate distributions that we have studied so far become inadequate. To comprehend the joint behavior of these variables and to uncover the underlying patterns of dependency, we must turn to the realm of multivariate distributions.

In this section, we look at the most important example: the multivariate normal distribution, along with covariance, which measures how two variables change together.

4.2 The Multivariate Normal Distribution

Let us say that you are gathering statistics on a species of snail. You have found 89 snails. Every snail has been weighed and had its shell measured. You have created a table like this:

Weight	Diameter
4.4769 grams	2.2692 cm
5.4755 grams	2.1973 cm
4.1183 grams	2.52928 cm
...	...
3.0522 grams	1.7822 cm

You have been told that for any particular species of snail, the weight and shell diameter are typically normally distributed. So, you compute the mean of each:

$$\bar{x} = \frac{\sum_{i=1}^n x_i}{n}$$

Your snails have a mean mass of 4.25 grams and a mean shell diameter of 2.35 centimeters. Now you know where the center of each normal distribution will be.

To know how wide each normal distribution is, you need to compute the variance:

$$\sigma^2 = \frac{\sum_{i=1}^n (x_i - \bar{x})^2}{n}$$

(Reminder: The standard deviation is the square root of σ^2)

Plotting this data is shown in Figure 4.1.

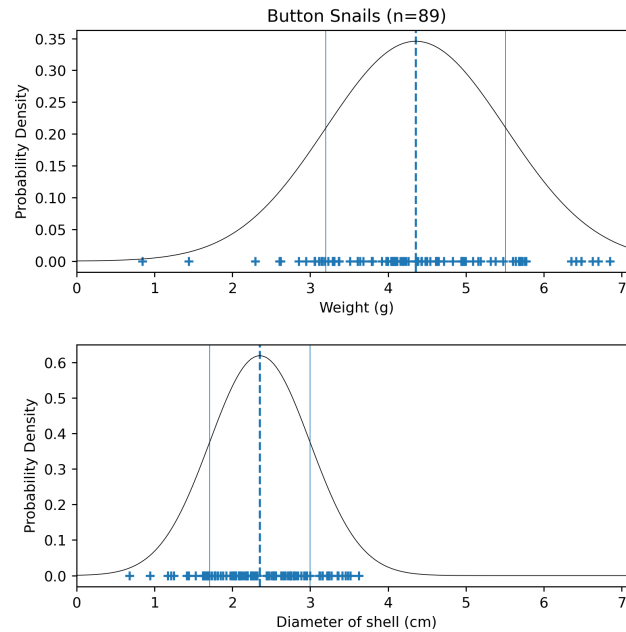


Figure 4.1: Probability density functions of weight and diameter.

Nice! However, then you think: "Hmm. Maybe the mass and the diameter of the shell are related. It seems like a heavy snail might tend to have a larger shell." So, you make a scatter plot:

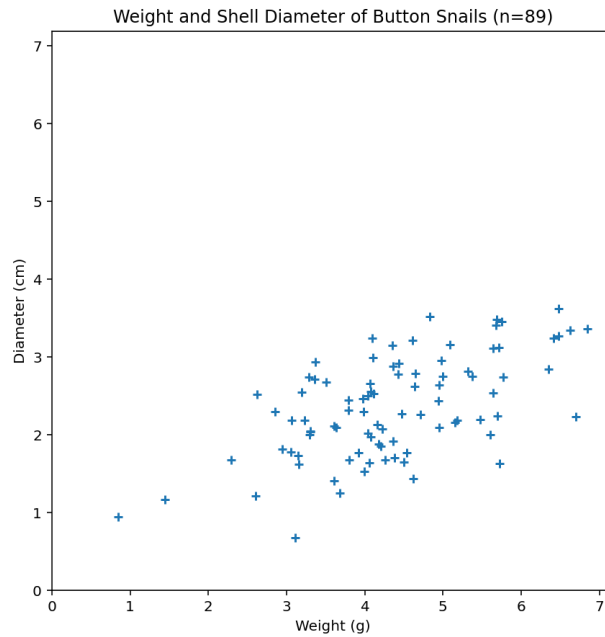


Figure 4.2: A scatter plot with weight of the snail horizontally and diameter of the shell on the y-axis.

Sure enough, there is some correlation between the mass of the snail and the diameter of its shell.

There is a form of the normal distribution that can deal with multiple variables and their correlations. This is known as the *multivariate normal* (or multivariate Gaussian) distribution.

In a multivariate normal distribution, the mean (often denoted μ) is a vector containing the mean of every variable. For your snails, $\mu = [4.25, 2.35]$.

Just as in the single-variable normal distribution, measurements near the mean are what you are most likely to observe. In a single-variable normal distribution, the standard deviation tells us how fast that likelihood falls off as we move away from the mean. In multivariate normal distributions, we need something a little more expressive: a matrix.

4.2.1 The Covariance Matrix

We can think of the data as a list of vectors. For example, if we have n snails and d properties that we have measured, we would get a list like this:

$$\begin{bmatrix} x_{1,1} & x_{1,2} & \dots & x_{1,d} \\ x_{2,1} & x_{2,2} & \dots & x_{2,d} \\ \dots & \dots & & \dots \\ x_{n,1} & x_{n,2} & \dots & x_{n,d} \end{bmatrix}$$

Each row represents one snail. Each column represents one property.

Note that to make μ (the mean vector), you just take the mean of each column of this data matrix. (For convenience, we will use μ_i to refer to the mean of column i .)

We usually use Σ (the uppercase Greek sigma) to represent a covariance matrix. Σ is a $d \times d$ matrix. The entries on the diagonal of Σ are just the variance of each property. So, we can calculate the entry at row j , column j like this:

$$\Sigma_{j,j} = \frac{\sum_{i=1}^n (x_{i,j} - \mu_j)^2}{n}$$

(Yes, we are using Σ to represent both summation and the covariance matrix here. It can be confusing, but you will be able to tell from context how it is being used.)

The other entries, $\Sigma_{j,k}$ where $j \neq k$, are the *covariance* between property j and property k . If the covariance is a positive number, property j and property k are correlated. For example, in your snails, weight and diameter have a positive covariance: If the snail is heavier than average, it tends to have a diameter that is larger than average.

If the covariance is a negative number, when property j is greater than average, property k tends to be less than average. For example, the number of times a restaurant mops the floor in a week has a negative covariance with how much bacteria lives on the floor.

How do we compute the covariance?

$$\Sigma_{j,k} = \frac{\sum_{i=1}^n (x_{i,j} - \mu_j)(x_{i,k} - \mu_k)}{n}$$

Note that $\Sigma_{j,k} = \Sigma_{k,j}$, so the matrix is symmetric.

Computing Mean and Covariance in Python

If you have a numpy array where each row represents one sample and each column represents one property, it is really easy to compute μ and Σ :

```
1 # Read in the data matrix, each row represents one snail
2 snails = ...
3
4 # Compute mu
5 mean_vector = snails.mean(axis=0)
6 print(f"Mean = {mean_vector}")
7
8 # Compute Sigma
9 covariance_matrix = np.cov(snails, rowvar=False)
10 print(f"Covariance = {covariance_matrix}")
```

4.2.2 Multivariate Normal Probability Density

Now that we have good estimates of μ and Σ , how can we compute the probability density?

When you were working with just one variable, x , you computed the probability density using the mean μ and the variance σ^2 like this:

$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

The probability density function of a d -dimensional multivariate normal distribution with a mean of μ and a covariance matrix of Σ is given by:

$$p(\mathbf{x}) = \frac{1}{\sqrt{(2\pi)^d |\Sigma|}} \exp\left(-\frac{1}{2}(\mathbf{x} - \mu)^T \Sigma^{-1} (\mathbf{x} - \mu)\right)$$

If we draw lines to show where μ is and contours to show lines of equal probability density, we get a plot shown in Figure 4.3.

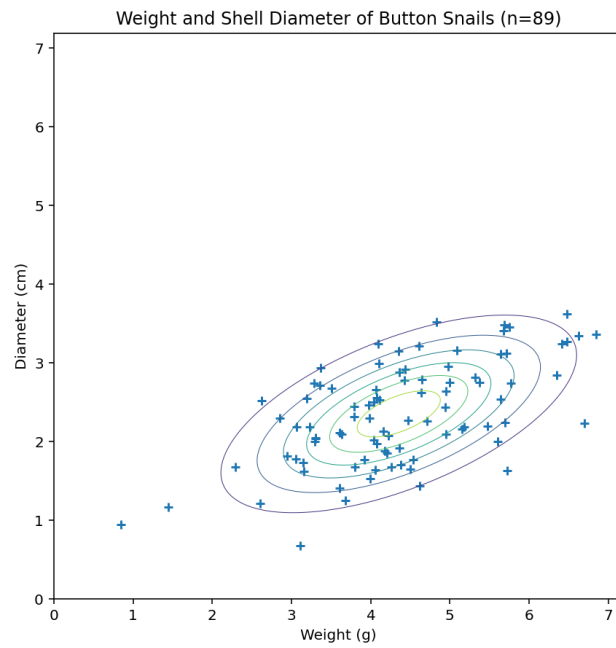


Figure 4.3: Contour plot of the snail data.

Or, we can do a 3D plot, as shown in Figure 4.4.

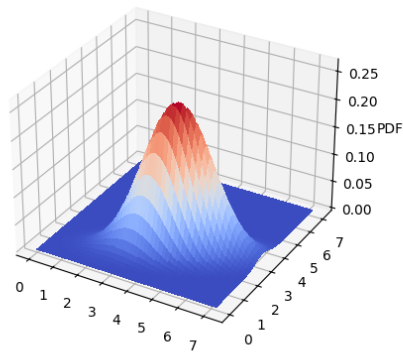


Figure 4.4: 3D plot of the probability density function.

A probability density always integrates to 1.0, so the volume under the surface must be 1.0.

Multivariate Normal Probability Density in Python

The `scipy` library has a class that represents the multivariate normal. Here is how you could compute the probability density for a particular snail:

```

1 import numpy as np
2 from scipy.stats import multivariate_normal
3
4 ...Compute mean_vector and covariance_matrix...
5
6 # Get the probability density at [3 grams, 2 cm]
7 x = np.array([3.0, 2.0])
8 pd = multivariate_normal.pdf(x, mean=mean_vector, cov=covariance_matrix)
9 print(f"The probability density at {x} is {pd}")

```

Alternatively, maybe you would like to generate weights and diameters for a fictional population of 700 snails:

```

1 new_snails = multivariate_normal.rvs(mean=mean_vector, cov=covariance_matrix,
   ↪ size=700)
2 print(f"My fictional snails:\n{new_snails}")

```

4.3 Joint Density Functions

Recall that you have learned about *probability density functions* in a previous chapter. For some variable, X , with probability density function $f(x)$, the likelihood that X lies between a and b is given by:

$$P(a \leq X \leq b) = \int_a^b f(x) dx$$

We can extend this to two random variables, X and Y . A function that describes the distribution of both variables is called a *joint density function*. In particular, the probability that X and Y lie in a region D is:

$$P((X, Y) \in D) = \iint_D f(x, y) dA$$

Where $f(x, y)$ is the two-variable joint density function. Just like with one variable, all probabilities are positive and the sum of the probabilities of all possibilities must be 1. Mathematically,

$$f(x, y) \geq 0$$

$$\iint_{\mathbb{R}} f(x, y) \, dA = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \, dx \, dy = 1$$

Example: Suppose the joint density function for X and Y is given by:

$$f(x, y) = \begin{cases} C(3x + y), & \text{if } 0 \leq x \leq 15, 0 \leq y \leq 15 \\ 0, & \text{otherwise} \end{cases}$$

where C is a constant. Find the value of C and the likelihood that $X \geq Y$.

Solution: To find C , we take advantage of the fact that the sum of all probabilities is 1. Therefore,

$$\begin{aligned} 1 &= \int_0^{15} \int_0^{15} C(3x + y) \, dx \, dy = \int_0^{15} C \left[\frac{3}{2}x^2 + xy \right]_{x=0}^{x=15} dy \\ &= \int_0^{15} C \left[\frac{675}{2} + 15y \right] dy = C \left[\frac{675}{2}y + \frac{15}{2}y^2 \right]_{y=0}^{y=15} \\ &= C \left[\frac{10125}{2} + \frac{3375}{2} \right] = 6750C = 1 \end{aligned}$$

Which implies that $C = \frac{1}{6750}$.

To find the probability that $X \geq Y$, we note that this is represented by the region $R = \{(x, y) \mid 0 \leq x \leq 15, 0 \leq y \leq x\}$. Then the likelihood that $X \geq Y$ is given by:

$$\begin{aligned} \frac{1}{6750} \int_0^{15} \int_0^x (3x + y) \, dy \, dx &= \frac{1}{6750} \int_0^{15} \left[3xy + \frac{1}{2}y^2 \right]_{y=0}^{y=x} dx \\ &= \frac{1}{6750} \int_0^{15} \left(3x^2 + \frac{1}{2}x^2 \right) dx = \frac{1}{6750} \left(\frac{7}{2} \right) \int_0^{15} x^2 dx \\ &= \frac{7}{13500} \left(\frac{1}{3} \right) \left[x^3 \right]_{x=0}^{x=15} = \frac{7}{13500} \left(\frac{1}{3} \right) 15^3 = \frac{7 \cdot 3375}{3 \cdot 13500} = \frac{7}{12} \end{aligned}$$

And this answer makes sense, as it is less than 1.

Exercise 10 Joint Density Functions

The joint density function for a pair of random variables, X and Y is

$$f(x, y) = \begin{cases} ky(1 + 2x), & \text{if } 0 \leq x \leq 1, 0 \leq y \leq 2 \\ 0, & \text{otherwise} \end{cases}$$

Working Space

1. Find the value of the constant, k .
2. Find $P(X \leq \frac{1}{2}, Y \leq 1)$.
3. Find $P(X + Y \geq 1)$.

Answer on Page 62

Often, two variables are independent of each other. If both variables are random, the joint density function for the two variables is the product of the two individual probability distributions. For example, if the probability distribution of an individual's height, H , is given by $f_1(h)$ and the probability distribution of an individual's blood pressure, B , is given by $f_2(b)$, then the joint probability function for height and blood pressure is

$$f(h, b) = f_1(h)f_2(b)$$

Example: Sally is spending the day at Six Flags Over Georgia and is looking forward to riding several roller coasters. The wait time for the Great American Scream machine, X , is modeled by density function

$$f_1(x) = \begin{cases} 0, & \text{if } x < 0 \\ \frac{1}{45}e^{-x/45}, & \text{if } x \geq 0 \end{cases}$$

And the wait time for the Georgia Scorcher, Y , is modeled by density function

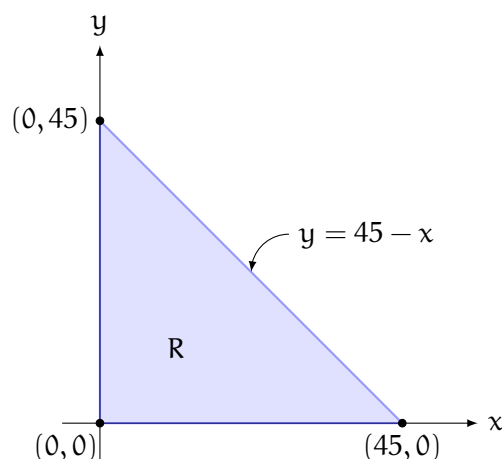
$$f_2(y) = \begin{cases} 0, & \text{if } y < 0 \\ \frac{1}{30}e^{-y/30}, & \text{if } y \geq 0 \end{cases}$$

What is the probability that she spends less than 45 minutes total waiting for the two rides?

Solution: Since the variables are independent, the joint density function for both wait times is:

$$f(x, y) = f_1(x)f_2(y) = \begin{cases} \frac{1}{1350}e^{-x/45}e^{-y/30}, & \text{if } x \geq 0, y \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

We want to know the probability that $X + Y < 45$, which is represented by the region, R , shown below.



We can see that $R = \{(x, y) \mid 0 \leq x \leq 45, 0 \leq y \leq 45 - x\}$. Therefore, the probability that Sally's total wait time is less than 45 minutes is given by:

$$\begin{aligned}
 P(X + Y < 45) &= \frac{1}{1350} \int_0^{45} \int_0^{45-x} e^{-x/45} e^{-y/30} dy dx \\
 &= \frac{-30}{1350} \int_0^{45} \left(e^{-x/45} \right) \left[e^{-y/30} \right]_{y=0}^{y=45-x} dx \\
 &= -\frac{1}{45} \int_0^{45} \left(e^{-x/45} \right) \left[e^{(x-45)/30} - 1 \right] dx \\
 &= -\frac{1}{45} \int_0^{45} e^{(x-135)/90} - e^{-x/45} dx \\
 &= -\frac{1}{45} \left[90e^{(x-135)/90} + 45e^{-x/45} \right]_{x=0}^{x=45} \\
 &= -\frac{1}{45} \left[90 \left(e^{(45-135)/90} - e^{(0-135)/90} \right) + 45 \left(e^{-45/45} - e^{-0/45} \right) \right] \\
 &= -\frac{1}{45} \left[90 \left(e^{-1} - e^{-3/2} \right) + 45 \left(e^{-1} - 1 \right) \right] \\
 &= -\frac{1}{45} \left[135e^{-1} - 90e^{-3/2} - 45 \right] = 1 + 2e^{-3/2} - 3e^{-1} \approx 0.343
 \end{aligned}$$

So, there is an approximately 34.3% chance Sally will wait less than 45 minutes to ride both roller coasters.

Exercise 11 Snow Days

In order for it to snow, two conditions must be met: the temperature must be low enough and the humidity must be

Working Space

high enough. The probability function for the temperature, T , in Celsius on a winter day is given by:

$$f_1(t) = \begin{cases} \frac{1}{80} [\cos(\frac{\pi t}{40}) + 1], & -40 \leq t \leq 40 \\ 0, & \text{otherwise} \end{cases}$$

And the probability function for the humidity, H , in percent on the same winter day is:

$$f_2(h) = \begin{cases} \frac{6(100h-h^2)}{1,000,000}, & 0 \leq h \leq 100 \\ 0, & \text{otherwise} \end{cases}$$

If the temperature must be below freezing and the humidity above 75% for snow to form, what is the probability it snows?

Answer on Page 63

Numerical Double Integration

In the Multivariate Distributions chapter, we gave an example of the multivariate normal distribution: the mass and shell diameter of a population of snails.

Starting with a table of measurements, we estimated that the mean vector μ was $[4.36g \ 2.35cm]$.

We then computed the covariance matrix:

$$\Sigma = \begin{bmatrix} 1.33 & 0.443 \\ 0.443 & 0.416 \end{bmatrix}$$

Plotting the data points, the mean, and the equal-density contours looked like this:

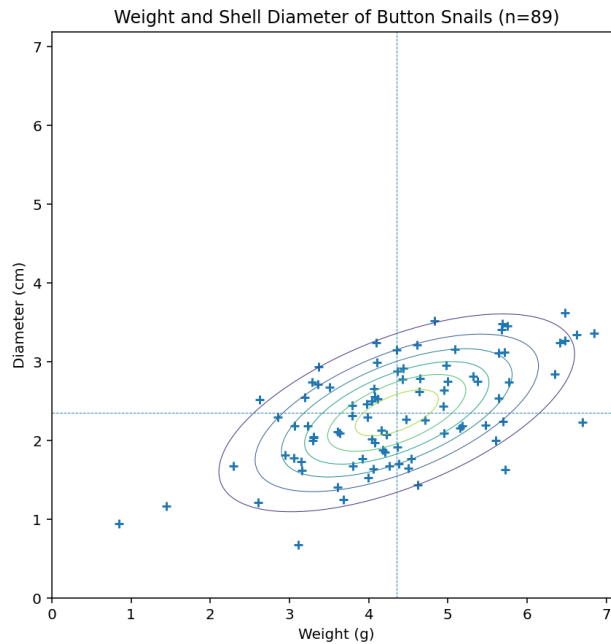


Figure 5.1: Equal density contours plotted with the snail data.

We have a great formula for computing the probability density for any mass/diameter combination:

$$p(\mathbf{x}) = \frac{1}{\sqrt{(2\pi)^d |\Sigma|}} \exp \left(-\frac{1}{2} (\mathbf{x} - \boldsymbol{\mu})^T \Sigma^{-1} (\mathbf{x} - \boldsymbol{\mu}) \right)$$

Can you answer the following question? “If I pick a random snail off the floor of the ocean, what is the probability that its mass is between 3 and 4 grams and its diameter is between 1.5 and 2.0 centimeters?”

If we think of the probability density as a surface, this question is really “What is the volume under the surface in the rectangular patch $3 \leq x_1 \leq 4$ and $1.5 \leq x_2 \leq 2.0$?” Which you should recognize as a double integration problem:

$$P = \int_{1.5}^{2.0} \int_3^4 \frac{1}{\sqrt{(2\pi)^d |\Sigma|}} \exp \left(-\frac{1}{2} (\mathbf{x} - \boldsymbol{\mu})^T \Sigma^{-1} (\mathbf{x} - \boldsymbol{\mu}) \right) dx_1 dx_2$$

Sadly, the multivariate normal density has no elementary antiderivative, so we cannot write this integral in closed form.

Instead, we use Riemann sums to find an approximate solution. This is known as *numerical integration*.

(After spending so much time learning the techniques for integration, it may be disappointing to hear this: For a lot of real-world problems, there is no way to find an antiderivative, so we end up doing numerical integration much more often than most people realize.)

5.1 Riemann Sums on 2-Dimensional Domains

When doing Riemann sums on a function that takes a single real number, you summed the area of the rectangles under the function to approximate the integral:

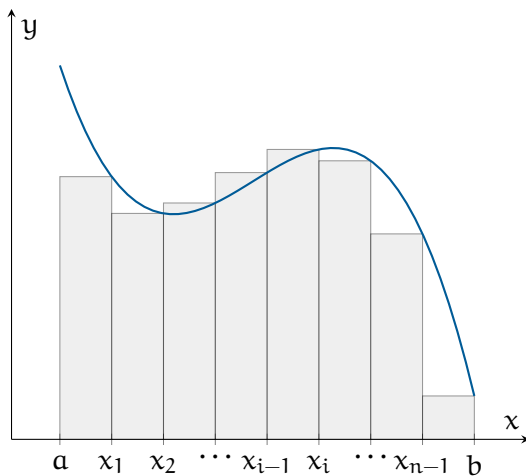


Figure 5.2: A representative function divided into n rectangles of equal width, with rectangle height determined by the right endpoint of the subinterval

Here you are finding the volume under a function that takes two variables (the probability density is based on the mass and diameter of the shell).

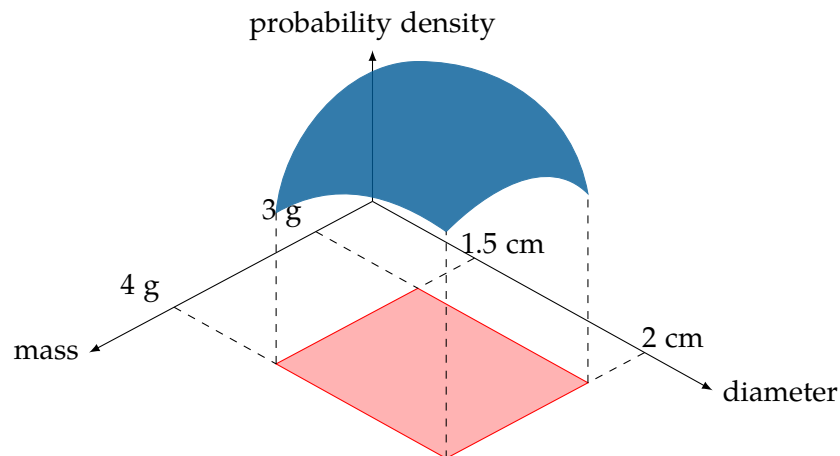


Figure 5.3: The probability is the volume under a surface for a region

To do the Riemann sum, we can break the range of the mass into n_1 equally sized intervals and the range of the diameter into n_2 equally sized intervals. For the diagram, we will just break each range into three, but you will get a more accurate estimate as the intervals get smaller.

Now you will be calculating the volume of rectangular solids and summing up those volumes. Here are a few of the rectangular volumes:

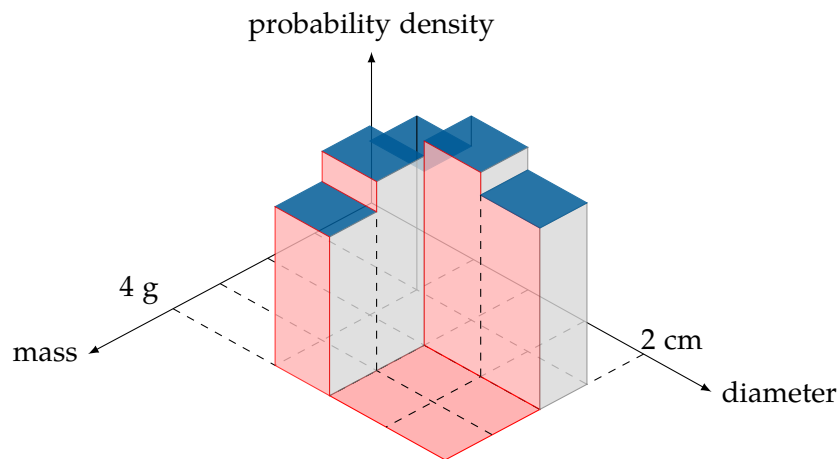


Figure 5.4: Riemann Sum

5.2 Numerical Integration in Python

Here is the code. It uses a midpoint Riemann sum, so each small square is evaluated at its center:

```

1  import numpy as np
2  from scipy.stats import multivariate_normal
3
4  # pip3 install scipy
5  weight_lower_limit = 3.0
6  weight_upper_limit = 4.0
7  weight_slices = 100
8
9  diameter_lower_limit = 1.5
10 diameter_upper_limit = 2.0
11 diameter_slices = 100
12
13 # What is the average weight and diameter?
14 mean_vector = np.array([4.35559489, 2.3526593 ])
15 print(f"Mean [weight, diameter] = {mean_vector}")
16
17 # Do heavier snails tend to have bigger shells?
18 covariance_matrix = np.array([[1.33099714, 0.44309754],
19                               [0.44309754, 0.41603925]])
20 print(f"Covariance = \n{covariance_matrix}")
21
22 # Create a multivariate normal distribution
23 rv = multivariate_normal(mean_vector, covariance_matrix)
24
25 delta_weight = (weight_upper_limit - weight_lower_limit) / weight_slices
26 delta_diameter = (diameter_upper_limit - diameter_lower_limit) / diameter_slices
27

```

```

28 total = 0.0
29 # Step through each different weight
30 for i in range(weight_slices):
31     # What is the weight in the middle of this slice?
32     current_weight = weight_lower_limit + (i + 0.5) * delta_weight
33
34     for j in range(diameter_slices):
35         # What is the diameter in the middle of this slice?
36         current_diameter = diameter_lower_limit + (j + 0.5) * delta_diameter
37
38         # What is the probability density there?
39         prob_density = rv.pdf((current_weight, current_diameter))
40
41         # What is the volume under that for this tiny square
42         total += prob_density * delta_weight * delta_diameter
43
44 print(
45     f"\nThe probability that the weight is between {weight_lower_limit} and
46     ↪ {weight_upper_limit}"
47 )
48 print(
49     f"and that the diameter is between {diameter_lower_limit} and
50     ↪ {diameter_upper_limit}"
51 )
52 print(f"is about {total * 100.0:.4f}%")

```

This should get you the following output:

```

1 > python3 num_integration.py
2 Mean [weight, diameter] = [4.35559489 2.3526593 ]
3 Covariance =
4 [[1.33099714 0.44309754]
5  [0.44309754 0.41603925]]
6
7 The probability that the weight is between 3.0 and 4.0
8 and that the diameter is between 1.5 and 2.0
9 is about 7.8316%

```

How good is this estimate? Using only $10 \times 10 = 100$ squares instead of 100×100 gives 7.8364%, already within 0.005 percentage points. SciPy can also compute this particular probability directly from the cumulative distribution function, and it gets 7.8316%, matching our 10,000-square estimate to four decimal places. The midpoint rule converges quickly for smooth functions like this one.

Exercise 12

Estimate $\iint_R (x^2 + y^2) \, dA$ over $R =$

Working Space

$[0, 1] \times [0, 1]$ by hand, using a midpoint Riemann sum with a 2×2 grid. Compare your answer with the exact value.

Answer on Page 63

Triple Integrals

How much does a scoop of ice cream weigh if it is denser at the bottom where it has started to melt? How much air is inside a dome-shaped tent? Questions about solids need one more dimension of adding up. That is what triple integrals do.

6.1 From Double to Triple Integrals

A double integral adds up a function over a flat region. A *triple integral* adds up a function over a three-dimensional solid E :

$$\iiint_E f(x, y, z) \, dV$$

If $f = 1$, the triple integral gives the volume of E . If f is a density, it gives the mass of E .

6.2 Iterated Triple Integrals

We evaluate a triple integral one variable at a time, from the inside out, just as we did for double integrals. For a rectangular box $0 \leq x \leq 1$, $0 \leq y \leq 2$, $0 \leq z \leq 3$:

$$\int_0^1 \int_0^2 \int_0^3 xyz \, dz \, dy \, dx = \int_0^1 x \, dx \cdot \int_0^2 y \, dy \cdot \int_0^3 z \, dz = \frac{1}{2} \cdot 2 \cdot \frac{9}{2} = \frac{9}{2}$$

Because the limits are constants, the integral splits into a product of three one-variable integrals.

When the limits depend on other variables, the order matters. Consider the solid E in the first octant bounded by the plane $x + y + z = 1$. Its volume is

$$\int_0^1 \int_0^{1-x} \int_0^{1-x-y} dz \, dy \, dx$$

The innermost integral gives $1 - x - y$. The middle integral gives $\frac{(1-x)^2}{2}$. The outer integral gives $\int_0^1 \frac{(1-x)^2}{2} \, dx = \frac{1}{6}$. The volume is $\frac{1}{6}$.

6.3 Cylindrical Coordinates

Many solids are round around an axis. For those, use cylindrical coordinates, where $x = r \cos \theta$, $y = r \sin \theta$, and z stays the same. The volume element becomes $dV = r \, dz \, dr \, d\theta$. The extra factor of r comes from the area of a thin ring, $2\pi r \, dr$.

For a cylinder of radius 2 and height 3:

$$\int_0^{2\pi} \int_0^2 \int_0^3 r \, dz \, dr \, d\theta = 2\pi \cdot \int_0^2 3r \, dr = 2\pi \cdot 6 = 12\pi$$

This matches the formula $\pi r^2 h = \pi \cdot 4 \cdot 3 = 12\pi$. Think of a can of soup: its volume is the area of the round top times the height. The triple integral gives the same answer.

Example: Find the volume of an upside-down cone whose tip is at the origin and whose top is the disk of radius 1 at height 1.

Solution: The cone's side is $z = r$. For each point (r, θ) in the unit disk, z runs from the side up to the top, $r \leq z \leq 1$:

$$\int_0^{2\pi} \int_0^1 \int_r^1 r \, dz \, dr \, d\theta = 2\pi \int_0^1 r(1-r) \, dr = 2\pi \left(\frac{1}{2} - \frac{1}{3} \right) = \frac{\pi}{3}$$

This matches $\frac{1}{3}\pi r^2 h$ with $r = h = 1$.

6.4 Spherical Coordinates

For solids that are round in every direction, use spherical coordinates. Let ρ be the distance from the origin, φ the angle down from the positive z -axis, and θ the angle around the z -axis as before:

$$x = \rho \sin \varphi \cos \theta, \quad y = \rho \sin \varphi \sin \theta, \quad z = \rho \cos \varphi$$

The volume element is $dV = \rho^2 \sin \varphi \, d\rho \, d\varphi \, d\theta$.

For the unit ball:

$$\int_0^{2\pi} \int_0^\pi \int_0^1 \rho^2 \sin \varphi \, d\rho \, d\varphi \, d\theta = 2\pi \cdot 2 \cdot \frac{1}{3} = \frac{4\pi}{3}$$

This is the familiar volume formula for a sphere, $\frac{4}{3}\pi r^3$, with $r = 1$.

Example: A ball of radius 1 is denser near its center, with density $\delta = 2 - \rho$ (in g/cm^3 , with ρ in cm). Find its mass.

Solution:

$$m = \int_0^{2\pi} \int_0^\pi \int_0^1 (2 - \rho) \rho^2 \sin \varphi \, d\rho \, d\varphi \, d\theta = 2\pi \cdot 2 \cdot \left(\frac{2}{3} - \frac{1}{4} \right) = \frac{5\pi}{3} \approx 5.24 \text{ g}$$

A ball with uniform density 1 would have mass $\frac{4\pi}{3} \approx 4.19$ g, so the extra density near the center adds about a gram.

6.5 Choosing a Coordinate System

Use rectangular coordinates when the limits are flat planes. Use cylindrical coordinates when the solid is round around one axis, such as a cylinder, a cone, or a paraboloid $z = x^2 + y^2$. Use spherical coordinates when the solid is round around a point, such as a ball or a sphere.

Exercise 13 Evaluate a box integral

Evaluate $\int_0^1 \int_0^1 \int_0^1 (x + y + z) \, dz \, dy \, dx$.

Working Space

Answer on Page 63

Exercise 14 Volume by cylindrical coordinates

Use cylindrical coordinates to find the volume of the solid inside the cylinder $r = 1$, between $z = 0$ and $z = 4$.

Working Space

Answer on Page 63

Exercise 15 Volume of a larger ball

Use spherical coordinates to find the vol-

Working Space

ume of a ball of radius 2.

Answer on Page 64

Exercise 16 **Order of integration**

Write the volume of the solid in the first octant under $x + y + z = 1$ as an iterated integral in the order $dx \, dy \, dz$.

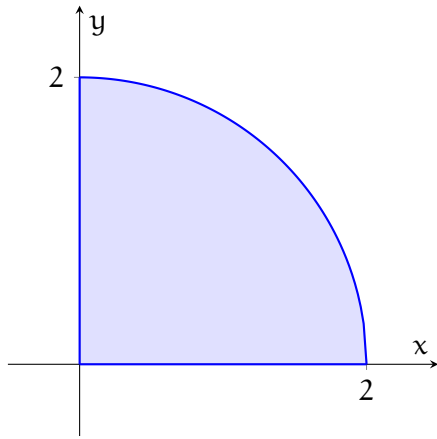
Working Space

Answer on Page 64

Answers to Exercises

Answer to Exercise 1 (on page 6)

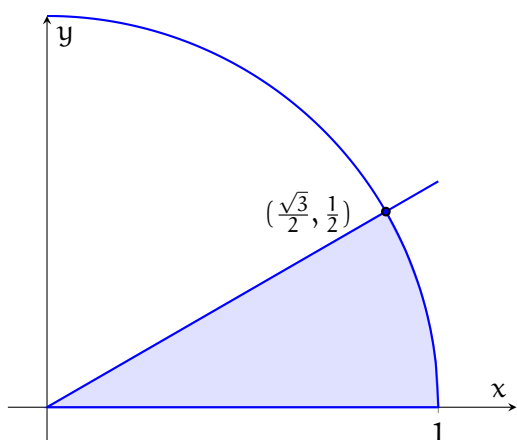
1. Let us visualize the region in the xy -plane:



The region is a quarter-circle that can be described with $D = \{(r, \theta) \mid 0 \leq r \leq 2, 0 \leq \theta \leq \pi/2\}$. We can then rewrite the integral in polar coordinates:

$$\begin{aligned}
 \int_0^2 \int_0^{\sqrt{4-x^2}} e^{-x^2-y^2} dy dx &= \int_0^{\pi/2} \int_0^2 r e^{-r^2} dr d\theta \\
 &= \int_0^{\pi/2} \left[-\frac{1}{2} e^{-r^2} \right]_{r=0}^{r=2} d\theta = \int_0^{\pi/2} \left(-\frac{1}{2} \right) [e^{-4} - 1] d\theta \\
 &= \frac{1}{2} \int_0^{\pi/2} 1 - e^{-4} d\theta = \frac{1}{2} \left(1 - \frac{1}{e^4} \right) \int_0^{\pi/2} 1 d\theta \\
 &= \frac{1}{2} \left(1 - \frac{1}{e^4} \right) \theta \Big|_{\theta=0}^{\theta=\pi/2} = \frac{\pi}{4} \left(1 - \frac{1}{e^4} \right)
 \end{aligned}$$

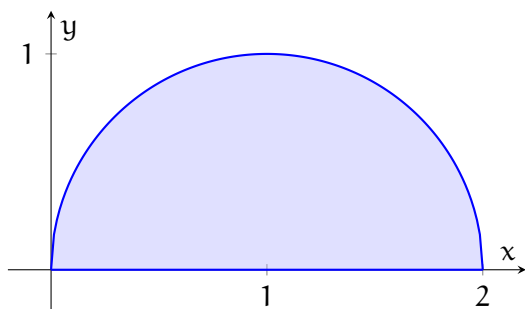
2. The region is bounded by the x -axis, the line $y = x/\sqrt{3}$, and the circle $x^2 + y^2 = 1$:



We see that the region defined in polar coordinates is $D = \{(r, \theta) \mid 0 \leq r \leq 1, 0 \leq \theta \leq \pi/6\}$. And therefore:

$$\begin{aligned} \int_0^{1/2} \int_{\sqrt{3}y}^{\sqrt{1-y^2}} xy^2 \, dx \, dy &= \int_0^{\pi/6} \int_0^1 r (r \cos \theta) (r \sin \theta)^2 \, dr \, d\theta \\ &= \int_0^{\pi/6} [\cos \theta \sin^2 \theta] \, d\theta \cdot \int_0^1 r^4 \, dr \\ &= \left(\frac{1}{3} \sin^3 \theta \Big|_{\theta=0}^{\theta=\pi/6} \right) \cdot \left(\frac{1}{5} r^5 \Big|_{r=0}^{r=1} \right) \\ &= \frac{1}{15} \cdot \left(\frac{1}{2} \right)^3 = \frac{1}{120} \end{aligned}$$

3. Visualizing the region:



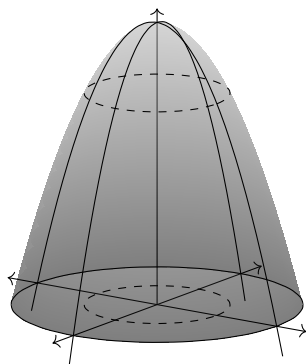
We see that the region is the top half of a circle of radius 1 centered at $(1, 0)$. In polar coordinates, this region is $D = \{(r, \theta) \mid 0 \leq r \leq 2 \cos \theta, 0 \leq \theta \leq \pi/2\}$. And therefore:

$$\begin{aligned} \int_0^2 \int_0^{\sqrt{2x-x^2}} \sqrt{x^2+y^2} \, dy \, dx &= \int_0^{\pi/2} \int_0^{2 \cos \theta} r \sqrt{r^2} \, dr \, d\theta \\ &= \int_0^{\pi/2} \int_0^{2 \cos \theta} r^2 \, dr \, d\theta = \int_0^{\pi/2} \frac{1}{3} [r^3]_{r=0}^{r=2 \cos \theta} \, d\theta \end{aligned}$$

$$\begin{aligned}
&= \frac{8}{3} \int_0^{\pi/2} \cos^3 \theta \, d\theta = \frac{8}{3} \int_0^{\pi/2} \cos \theta (1 - \sin^2 \theta) \, d\theta \\
&= \frac{8}{3} \left[\int_0^{\pi/2} \cos \theta \, d\theta - \int_0^{\pi/2} \cos \theta \sin^2 \theta \, d\theta \right] \\
&= \frac{8}{3} \left[(\sin \theta)_{\theta=0}^{\theta=\pi/2} - \left(\frac{1}{3} \sin^3 \theta \right)_{\theta=0}^{\theta=\pi/2} \right] \\
&= \frac{8}{3} \left[(1 - 0) - \frac{1}{3} (1^3 - 0^3) \right] = \frac{8}{3} \cdot \frac{2}{3} = \frac{16}{9}
\end{aligned}$$

Answer to Exercise 2 (on page 6)

We are finding the volume of the solid that lies under the surface $z = 4 - x^2 - y^2$ and above the xy -plane.



We can use polar coordinates to simplify the double integral. In polar coordinates, $x = r \cos(\theta)$ and $y = r \sin(\theta)$, so $x^2 + y^2 = r^2$. The volume under the surface and above the xy -plane is given by

$$V = \int \int (4 - r^2) r \, dr \, d\theta, \quad (1.1)$$

where r ranges from 0 to 2 (since $4 - r^2 \geq 0$ if $0 \leq r \leq 2$) and θ ranges from 0 to 2π .

Hence,

$$\begin{aligned} V &= \int_0^{2\pi} \int_0^2 (4r - r^3) \, dr \, d\theta \\ &= \int_0^{2\pi} \left[2r^2 - \frac{1}{4}r^4 \right]_0^2 \, d\theta \\ &= \int_0^{2\pi} (8 - 4) \, d\theta \\ &= \int_0^{2\pi} 4 \, d\theta \\ &= [4\theta]_0^{2\pi} \\ &= 8\pi. \end{aligned}$$

So, the volume of the solid is 8π cubic units.

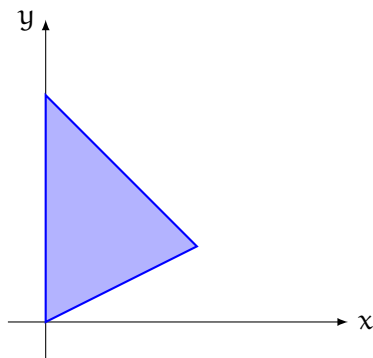
Answer to Exercise 3 (on page 7)

Let us describe the footprint of the pool as a 20-foot radius circle centered at the origin (that is, a region $D = \{(r, \theta) \mid 0 \leq r \leq 20, 0 \leq \theta \leq 2\pi\}$). Further, let us take north-south as parallel to the y -axis and east-west as parallel to the x -axis. So, the depth of water is then given by $z = f(x, y) = \frac{7}{40}x + \frac{13}{2}$ over the footprint of the pool. The total volume of water is given by:

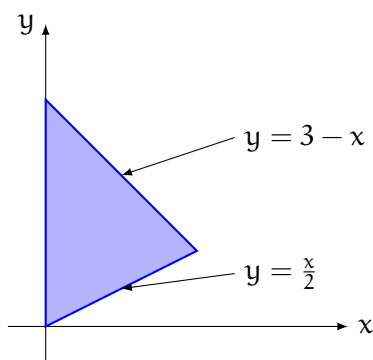
$$\begin{aligned} &\int_0^{2\pi} \int_0^{20} r \left(\frac{7}{40}r \cos \theta + \frac{13}{2} \right) \, dr \, d\theta \\ &= \int_0^{2\pi} \int_0^{20} \left[\frac{7}{40}r^2 \cos \theta + \frac{13}{2}r \right] \, dr \, d\theta \\ &= \int_0^{2\pi} \left[\frac{7 \cos \theta}{40} \int_0^{20} r^2 \, dr + \frac{13}{2} \int_0^{20} r \, dr \right] \, d\theta \\ &= \int_0^{2\pi} \left[\frac{7 \cos \theta}{40} \left(\frac{1}{3}r^3 \right)_{r=0}^{r=20} + \frac{13}{2} \left(\frac{1}{2}r^2 \right)_{r=0}^{r=20} \right] \, d\theta \\ &= \int_0^{2\pi} \left[\frac{1400}{3} \cos \theta + 1300 \right] \, d\theta = \left[\frac{1400}{3} \sin \theta + 1300\theta \right]_{\theta=0}^{\theta=2\pi} \\ &= 2600\pi \text{ cubic feet} \end{aligned}$$

Answer to Exercise 4 (on page 11)

1. $\iint_D (1 + x^2 + y^2) \, dA = \int_0^4 \int_0^3 (1 + x^2 + y^2) \, dy \, dx = \int_0^4 [y + x^2 y + \frac{1}{3} y^3]_{y=0}^{y=3} \, dx = \int_0^4 [3 + 3x^2 + \frac{1}{3}(3)^3] \, dx$
 $= \int_0^4 (12 + 3x^2) \, dx = [12x + x^3]_{x=0}^{x=4} = 12(4) + 4^3 = 112$
2. First, let us visualize this region, since it is not a rectangle:



We slice the triangle vertically, so we need equations for the two sides that do not lie on the y -axis.



We see that we can describe region D as $D = \{(x, y) \mid 0 \leq x \leq 2, \frac{x}{2} \leq y \leq 3 - x\}$. Therefore $\iint_D (x + y) \, dA = \int_0^2 \int_{x/2}^{3-x} (x + y) \, dy \, dx = \int_0^2 [xy + \frac{1}{2}y^2]_{y=x/2}^{y=3-x} \, dx = \int_0^2 [(x(3-x)) - (x(x/2)) + \frac{1}{2}((3-x)^2 - (\frac{x}{2})^2)] \, dx$
 $= \int_0^2 \left[\left(3x - x^2 - \frac{x^2}{2}\right) + \frac{1}{2} \left(9 - 6x + x^2 - \frac{x^2}{4}\right) \right] \, dx = \int_0^2 \left[-x^2 - \frac{x^2}{2} + \frac{x^2}{2} - \frac{x^2}{8} + 3x - 3x + \frac{9}{2} \right] \, dx$
 $= \int_0^2 \left(-\frac{9x^2}{8} + \frac{9}{2} \right) \, dx = \left[-\frac{9x^3}{24} + \frac{9x}{2} \right]_{x=0}^{x=2} = \frac{9(2)}{2} - \frac{3(8)}{8} = 9 - 3 = 6$

Answer to Exercise 5 (on page 15)

1. First, we find the total mass: $m = \int_0^\pi \int_0^{2\sin x} x \, dy \, dx = \int_0^\pi [xy]_{y=0}^{y=2\sin x} \, dx = \int_0^\pi 2x \sin x \, dx$.

We apply integration by parts to evaluate the integral: $\int_0^\pi 2x \sin x \, dx = (-2x \cos x) \Big|_{x=0}^{x=\pi} + \int_0^\pi 2 \cos x \, dx = [-2\pi(-1)] - (0) + \sin x \Big|_{x=0}^{x=\pi} = 2\pi + \sin \pi - \sin 0 = 2\pi$

Now that we know $m = 2\pi$, we can find $\bar{x}$ and $\bar{y}$: $\bar{x} = \frac{1}{2\pi} \int_0^\pi \int_0^{2\sin x} x \cdot x \, dy \, dx = \frac{1}{2\pi} \int_0^\pi x^2 y \Big|_{y=0}^{y=2\sin x} \, dx = \frac{1}{2\pi} \int_0^\pi x^2 (2 \sin x) \, dx = \frac{1}{\pi} \int_0^\pi x^2 \sin x \, dx$.

Applying integration by parts: $\frac{1}{\pi} \int_0^\pi x^2 \sin x \, dx = \frac{1}{\pi} [x^2(-\cos x)|_{x=0}^{x=\pi} - \int_0^\pi 2x(-\cos x) \, dx]$
 $= \frac{1}{\pi} [(-\pi^2 \cos \pi) + 2 \int_0^\pi x \cos x \, dx] = \frac{1}{\pi} [\pi^2 + 2 \int_0^\pi x \cos x \, dx] = \pi + \frac{2}{\pi} \int_0^\pi x \cos x \, dx.$

Applying integration by parts again: $\pi + \frac{2}{\pi} \int_0^\pi x \cos x \, dx = \pi + \frac{2x \sin x}{\pi}|_{x=0}^{x=\pi} - \frac{2}{\pi} \int_0^\pi \sin x \, dx$
 $= \pi - \frac{2}{\pi} \int_0^\pi \sin x \, dx = \pi + \frac{2}{\pi} [\cos x]|_{x=0}^{x=\pi} = \pi + \frac{2}{\pi} [\cos \pi - \cos 0] = \pi + \frac{2}{\pi} (-1 - 1) = \pi - \frac{4}{\pi} = \bar{x}$

And finding $\bar{y}$: $\bar{y} = \frac{1}{2\pi} \int_0^\pi \int_0^{2 \sin x} y \cdot x \, dy \, dx = \frac{1}{2\pi} \int_0^\pi [\frac{1}{2}xy^2]_{y=0}^{y=2 \sin x} \, dx = \frac{1}{4\pi} \int_0^\pi x [2 \sin x]^2 \, dx$
 $= \frac{1}{4\pi} \int_0^\pi 4x \sin^2 x \, dx = \frac{1}{\pi} \int_0^\pi x \sin^2 x \, dx = \frac{1}{\pi} \int_0^\pi x \frac{1 - \cos(2x)}{2} \, dx = \frac{1}{2\pi} \int_0^\pi x \, dx - \frac{1}{2\pi} \int_0^\pi x \cos(2x) \, dx$
 $= \frac{1}{2\pi} [\frac{x^2}{2}]_{x=0}^{x=\pi} - \frac{1}{2\pi} [\frac{1}{2}x \sin(2x)|_{x=0}^{x=\pi} - \frac{1}{2} \int_0^\pi \sin(2x) \, dx] = \frac{1}{2\pi} (\frac{\pi^2}{2}) - \frac{1}{2\pi} [\pi \sin(2\pi) - 0 + \frac{1}{2} \cos(2x)|_{x=0}^{x=\pi}]$
 $= \frac{\pi}{4} - \frac{1}{2\pi} [\frac{1}{2}(\cos 2\pi - \cos 0)] = \frac{\pi}{4}$

Therefore, the center of mass is found at $(\bar{x}, \bar{y}) = (\pi - \frac{4}{\pi}, \frac{\pi}{4})$

2. We know from a previous question that the total mass of this lamina is 112 (see *Finding Total Mass*).

Finding $\bar{x}$: $\bar{x} = \frac{1}{112} \int_0^4 \int_0^3 x(1 + x^2 + y^2) \, dy \, dx = \frac{1}{112} \int_0^4 \int_0^3 (x + x^3 + xy^2) \, dy \, dx$
 $= \frac{1}{112} \int_0^4 [xy + \frac{x^3}{3}y + \frac{x}{3}y^3]_{y=0}^{y=3} \, dx = \frac{1}{112} \int_0^4 [3x + 3x^3 + 9x] \, dx = \frac{3}{112} \int_0^4 [4x + x^3] \, dx =$
 $\frac{3}{112} [2x^2 + \frac{x^4}{4}]_{x=0}^{x=4} = \frac{3}{112} [2(4)^2 - 2(0)^2 + \frac{4^4}{4} - \frac{0^4}{4}] = \frac{3}{112} [32 + 64] = \frac{3 \cdot 96}{112} = \frac{3 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3}{2 \cdot 2 \cdot 7 \cdot 2 \cdot 2} = \frac{18}{7}$

Finding $\bar{y}$: $\bar{y} = \frac{1}{112} \int_0^4 \int_0^3 y(1 + x^2 + y^2) \, dy \, dx = \frac{1}{112} \int_0^4 \int_0^3 [y + x^2y + y^3] \, dy \, dx$
 $= \frac{1}{112} \int_0^4 [\frac{y^2}{2} + \frac{x^2y^2}{2} + \frac{y^4}{4}]_{y=0}^{y=3} \, dx = \frac{1}{112} \int_0^4 [\frac{3^2}{2} + \frac{3^2x^2}{2} + \frac{3^4}{4}] \, dx = \frac{1}{112} \int_0^4 [\frac{99}{4} + \frac{9}{2}x^2] \, dx =$
 $\frac{1}{112} [\frac{99}{4}x + \frac{3}{2}x^3]_{x=0}^{x=4} = \frac{3}{224} [\frac{33}{2}(4) + 4^3] = \frac{3}{224} (66 + 64) = \frac{3 \cdot 130}{224} = \frac{3 \cdot 65}{112} = \frac{195}{112}$

Therefore, the center of mass of the rectangular region D is $(\frac{18}{7}, \frac{195}{112})$

3. We know from a previous question (see *Finding Total Mass*) that the total mass of D is 6, and it can be described as $D = \{(x, y) \mid 0 \leq x \leq 2, \frac{x}{2} \leq y \leq 3 - x\}$

Finding $\bar{x}$:

$\bar{x} = \frac{1}{6} \int_0^2 \int_{x/2}^{3-x} x(x + y) \, dy \, dx = \frac{1}{6} \int_0^2 \int_{x/2}^{3-x} (x^2 + xy) \, dy \, dx = \frac{1}{6} \int_0^2 [x^2y + \frac{x}{2}y^2]_{y=x/2}^{y=3-x} \, dx =$
 $\frac{1}{6} \int_0^2 [x^2(3 - x - \frac{x}{2}) + \frac{x}{2}((3 - x)^2 - (\frac{x}{2})^2)] \, dx = \frac{1}{6} \int_0^2 [3x^2 - x^3 - \frac{x^3}{2} + \frac{x}{2}(9 - 6x + x^2 - \frac{x^2}{4})] \, dx$
 $= \frac{1}{6} \int_0^2 [3x^2 - \frac{3}{2}x^3 + \frac{x}{2}(9 - 6x + \frac{3}{4}x^2)] \, dx = \frac{1}{6} \int_0^2 [3x^2 - \frac{3}{2}x^3 + \frac{9}{2}x - 3x^2 + \frac{3}{8}x^3] \, dx = \frac{1}{6} \int_0^2 [\frac{9}{2}x - \frac{9}{8}x^3] \, dx$
 $= \frac{1}{6} [\frac{9}{4}x^2 - \frac{9}{32}x^4]_{x=0}^{x=2} = \frac{1}{6} [\frac{9 \cdot 4}{4} - \frac{9 \cdot 16}{32}] = \frac{1}{6} [9 - \frac{9}{2}] = \frac{1}{6} \cdot \frac{9}{2} = \frac{9}{12} = \frac{3}{4}$

And finding $\bar{y}$:

$\bar{y} = \frac{1}{6} \int_0^2 \int_{x/2}^{3-x} y(x + y) \, dy \, dx = \frac{1}{6} \int_0^2 \int_{x/2}^{3-x} (xy + y^2) \, dy \, dx = \frac{1}{6} \int_0^2 [\frac{x}{2}y^2 + \frac{1}{3}y^3]_{y=x/2}^{y=3-x} \, dx$
 $= \frac{1}{6} \int_0^2 [\frac{x}{2}((3 - x)^2 - (\frac{x}{2})^2) + \frac{1}{3}((3 - x)^3 - (\frac{x}{2})^3)] \, dx$
 $= \frac{1}{6} \int_0^2 [\frac{x}{2}(9 - 6x + x^2 - \frac{x^2}{4}) + \frac{1}{3}(27 - 27x + 9x^2 - x^3 - \frac{x^3}{8})] \, dx$
 $= \frac{1}{6} \int_0^2 [\frac{x}{2}(9 - 6x + \frac{3x^2}{4}) + \frac{1}{3}(27 - 27x + 9x^2 - \frac{9x^3}{8})] \, dx$
 $= \frac{1}{6} \int_0^2 [\frac{9}{2}x - 3x^2 + \frac{3}{8}x^3 + 9 - 9x + 3x^2 - \frac{3}{8}x^3] \, dx = \frac{1}{6} \int_0^2 [9 - \frac{9}{2}x] \, dx = \frac{1}{6} [9x - \frac{9}{4}x^2]_{x=0}^{x=2}$
 $= \frac{3}{6} [3(2) - \frac{3}{4}(2)^2] = \frac{1}{2} (6 - 3) = \frac{1}{2} \cdot 3 = \frac{3}{2}$

Therefore, the center of mass is $(\bar{x}, \bar{y}) = (\frac{3}{4}, \frac{3}{2})$

Answer to Exercise 6 (on page 21)

1.

$$\begin{aligned} I_x &= \iint_D y^2 \rho(x, y) \, dA = \int_1^4 \int_0^3 y^2 (xy) \, dy \, dx \\ &= \int_1^4 \int_0^3 xy^3 \, dy \, dx = \int_1^4 x \left[\frac{1}{4} y^4 \right]_{y=0}^{y=3} dx = \frac{81}{4} \int_1^4 x \, dx \\ &= \frac{81}{4} \left[\frac{1}{2} x^2 \right]_{x=1}^{x=4} = \frac{81}{4} \cdot \frac{15}{2} = \frac{1215}{8} \end{aligned}$$

To find the radius of gyration, first we need to find the total mass:

$$\begin{aligned} m &= \iint_D \rho(x, y) \, dA = \int_1^4 \int_0^3 xy \, dy \, dx \\ &= \int_1^4 \frac{x}{2} [y^2]_{y=0}^{y=3} dx = \frac{9}{2} \int_1^4 x \, dx = \frac{9}{2} \cdot \frac{15}{2} = \frac{135}{4} \end{aligned}$$

Finding the radius of gyration about the x -axis:

$$\begin{aligned} I_x &= m \bar{y}^2 \\ \frac{1215}{8} &= \left(\frac{135}{4} \right) \bar{y}^2 \\ \bar{y}^2 &= \frac{9}{2} \\ \bar{y} &= \frac{3}{\sqrt{2}} \end{aligned}$$

2.

$$\begin{aligned} I_y &= \iint_D x^2 \rho(x, y) \, dA = \int_{-\pi/2}^{\pi/2} \int_0^{2 \cos x} x^2 y \, dy \, dx \\ &= \int_{-\pi/2}^{\pi/2} x^2 \left[\frac{1}{2} y^2 \right]_{y=0}^{y=2 \cos x} dx = \frac{1}{2} \int_{-\pi/2}^{\pi/2} x^2 (2 \cos x)^2 dx \\ &= 2 \int_{-\pi/2}^{\pi/2} x^2 \cos^2 x \, dx = 2 \int_{-\pi/2}^{\pi/2} x^2 \left(\frac{1 + \cos 2x}{2} \right) dx = \int_{-\pi/2}^{\pi/2} x^2 dx + \int_{-\pi/2}^{\pi/2} x^2 \cos 2x \, dx \\ &= \frac{1}{3} [x^3]_{x=-\pi/2}^{x=\pi/2} + \frac{1}{2} x^2 \sin 2x \Big|_{x=-\pi/2}^{x=\pi/2} - \int_{-\pi/2}^{\pi/2} \frac{1}{2} \sin 2x (2x) \, dx \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{3} \left[\left(\frac{\pi}{2} \right)^3 - \left(\frac{-\pi}{2} \right)^3 \right] - \int_{-\pi/2}^{\pi/2} x \sin 2x \, dx \\
&= \frac{1}{3} \left(\frac{2\pi^3}{8} \right) - \left(\left[-\frac{1}{2} x \cos 2x \right]_{x=-\pi/2}^{\pi/2} - \int_{-\pi/2}^{\pi/2} \left(-\frac{1}{2} \cos 2x \right) dx \right) \\
&= \frac{\pi^3}{12} + \left(\frac{1}{2} \left(\frac{\pi}{2} \right) \cos(\pi) - \frac{1}{2} \left(\frac{-\pi}{2} \right) \cos(-\pi) \right) - \left[\frac{1}{4} \sin 2x \right]_{x=-\pi/2}^{\pi/2} \\
&= \frac{\pi^3}{12} + \frac{\pi}{4}(-1) + \frac{\pi}{4}(-1) = \frac{\pi^3}{12} - \frac{\pi}{2}
\end{aligned}$$

In order to find the radius of gyration, we need to first know the total mass:

$$\begin{aligned}
m &= \iint_D \rho(x, y) \, dA = \int_{-\pi/2}^{\pi/2} \int_0^{2 \cos x} y \, dy \, dx \\
&= \int_{-\pi/2}^{\pi/2} \frac{1}{2} y^2 \Big|_{y=0}^{y=2 \cos x} \, dx = \frac{1}{2} \int_{-\pi/2}^{\pi/2} 4 \cos^2 x \, dx = \int_{-\pi/2}^{\pi/2} 1 + \cos 2x \, dx \\
&= \left[x + \frac{1}{2} \sin 2x \right]_{x=-\pi/2}^{\pi/2} = \pi
\end{aligned}$$

We can then find the radius of gyration about the y-axis:

$$\begin{aligned}
m \bar{\bar{x}}^2 &= I_y \\
\pi \bar{\bar{x}}^2 &= \frac{\pi^3}{12} - \frac{\pi}{2} \\
\bar{\bar{x}} &= \sqrt{\frac{\pi^2}{12} - \frac{1}{2}}
\end{aligned}$$

3.

$$\begin{aligned}
I_o &= \iint_D (x^2 + y^2) \rho(x, y) \, dA = \int_1^2 \int_0^\pi r(r^2) r \, d\theta \, dr \\
&= \int_1^2 r^4 \theta \Big|_{\theta=0}^{\theta=\pi} \, dr = \pi \int_1^2 r^4 \, dr = \frac{\pi}{5} r^5 \Big|_{r=1}^{r=2} = \frac{\pi}{5} (2^5 - 1) = \frac{\pi}{5} (31) = \frac{31\pi}{5}
\end{aligned}$$

We find the total mass:

$$\begin{aligned}
m &= \iint_D \rho(x, y) \, dA = \int_1^2 \int_0^\pi r^2 \, d\theta \, dr = \int_1^2 r^2 \theta \Big|_{\theta=0}^{\theta=\pi} \, dr \\
&= \pi \int_1^2 r^2 \, dr = \frac{\pi}{3} r^3 \Big|_{r=1}^{r=2} = \frac{\pi}{3} (2^3 - 1) = \frac{\pi}{3} (7) = \frac{7\pi}{3}
\end{aligned}$$

To find the radius of gyration about the origin:

$$\begin{aligned}
 mR^2 &= I_o \\
 \left(\frac{7\pi}{3}\right) R^2 &= \frac{31\pi}{5} \\
 R^2 &= \frac{31}{5} \cdot \frac{3}{7} = \frac{93}{35} \\
 R &= \sqrt{\frac{93}{35}}
 \end{aligned}$$

Answer to Exercise 7 (on page 25)

1. Rearranging the formula for the plane, we find that $z = 3x + 2y + 2$. Therefore, $\partial z / \partial x = 3$ and $\partial z / \partial y = 2$. Then the surface area is given by:

$$\begin{aligned}
 A(S) &= \int_2^6 \int_1^4 \sqrt{1 + 3^2 + 2^2} \, dy \, dx = \int_2^6 \sqrt{14} y \Big|_{y=1}^{y=4} \, dx \\
 &= \int_2^6 3\sqrt{14} \, dx = 3\sqrt{14} x \Big|_{x=2}^{x=6} = 12\sqrt{14}
 \end{aligned}$$

2. The paraboloid intersects the plane when $2x^2 + 2y^2 = 32$, which is the circle of radius 4 centered at the origin. So, we are looking for the area of the surface $z = 2x^2 + 2y^2$ that lies above the region $R = \{(r, \theta) \mid 0 \leq r \leq 4, 0 \leq \theta \leq 2\pi\}$. The surface area is:

$$\begin{aligned}
 A(S) &= \iint_R \sqrt{1 + (4x)^2 + (4y)^2} \, dA = \int_0^4 \int_0^{2\pi} r \sqrt{1 + (4r \cos \theta)^2 + (4r \sin \theta)^2} \, d\theta \, dr \\
 &= \int_0^4 \int_0^{2\pi} r \sqrt{1 + 16r^2} \, d\theta \, dr = \int_0^4 r \sqrt{1 + 16r^2} [\theta]_{\theta=0}^{\theta=2\pi} \, dr \\
 &= \int_0^4 2\pi r \sqrt{1 + 16r^2} \, dr
 \end{aligned}$$

Let $u = 1 + 16r^2$, then $du = 32r(dr)$ and $r(dr) = du/32$. Substituting:

$$\begin{aligned}
 A(S) &= \frac{2\pi}{32} \int_{r=0}^{r=4} \sqrt{u} \, du = \frac{\pi}{16} \left(\frac{2}{3}\right) \left[u^{3/2}\right]_{r=0}^{r=4} \\
 &= \frac{\pi}{24} \left[(1 + 16r^2)^{3/2}\right]_{r=0}^{r=4} = \frac{\pi}{24} \left[(257)^{3/2} - 1\right] \approx 539.2
 \end{aligned}$$

3. The region, R , we are interested in is the circle of radius 2 centered at the origin of the xy -plane, described by $R = \{(r, \theta) \mid 0 \leq r \leq 2, 0 \leq \theta \leq 2\pi\}$. Noting that $\partial z / \partial x = 3y$

and $\partial z/\partial y = 3x$, we see that the surface area is given by:

$$\begin{aligned} A(S) &= \iint_R \sqrt{1 + (3y)^2 + (3x)^2} \, dA = \int_0^2 \int_0^{2\pi} r \sqrt{1 + 9r^2 \sin^2 \theta + 9r^2 \cos^2 \theta} \, d\theta \, dr \\ &= \int_0^2 \int_0^{2\pi} r \sqrt{1 + 9r^2} \, d\theta \, dr = 2\pi \int_0^2 r \sqrt{1 + 9r^2} \, dr \end{aligned}$$

Let $u = 1 + 9r^2$, then $du = 18r(dr)$, which means that $r(dr) = du/18$. Substituting:

$$\begin{aligned} A(S) &= \frac{2\pi}{18} \int_{r=0}^{r=2} \sqrt{u} \, du = \frac{\pi}{9} \left(\frac{2}{3} \right) \left[u^{3/2} \right]_{r=0}^{r=2} \\ &= \frac{2\pi}{27} \left[(1 + 9(4))^{3/2} - 1 \right] = \frac{2\pi}{27} \left[(37)^{3/2} - 1 \right] \approx 52.14 \end{aligned}$$

Answer to Exercise 8 (on page 26)

1. The area of D is $A = 2\pi$. Since $\sin y$ is odd and $[-\pi/2, \pi/2]$ is symmetric about 0, the integral of $\sin y$ over that interval is 0. So the average value is:

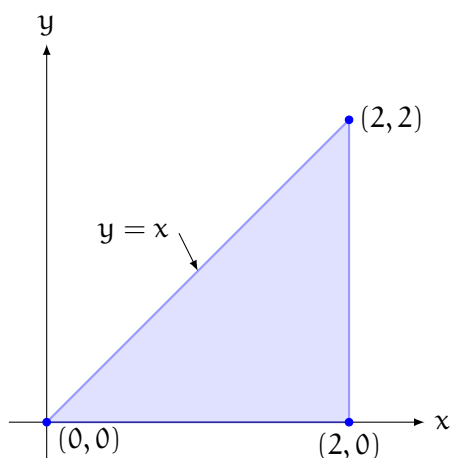
$$\begin{aligned} \frac{1}{2\pi} \iint_D x \sin y \, dA &= \frac{1}{2\pi} \int_0^2 \int_{-\pi/2}^{\pi/2} x \sin y \, dy \, dx = \frac{1}{2\pi} \int_0^2 x [-\cos y]_{y=-\pi/2}^{y=\pi/2} \, dx \\ &= \frac{1}{2\pi} \int_0^2 x \cdot 0 \, dx = 0 \end{aligned}$$

2. Since D is a circle of radius $r = 1$, the area is $A = \pi r^2 = \pi$. D can be described with $D = \{(r, \theta) \mid 0 \leq r \leq 2 \cos \theta, -\pi/2 \leq \theta \leq \pi/2\}$. Therefore, the average value of $f(x, y) = x + y$ over D is:

$$\begin{aligned} f_{\text{ave}} &= \frac{1}{\pi} \iint_D (x + y) \, dA = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \int_0^{2 \cos \theta} r \cdot (r \cos \theta + r \sin \theta) \, dr \, d\theta \\ &= \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} (\cos \theta + \sin \theta) \left[\int_0^{2 \cos \theta} r^2 \, dr \right] \, d\theta \\ &= \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} (\cos \theta + \sin \theta) \cdot \left[\frac{1}{3} r^3 \right]_{r=0}^{r=2 \cos \theta} \, d\theta \\ &= \frac{8}{3\pi} \int_{-\pi/2}^{\pi/2} (\cos \theta + \sin \theta) \cos^3 \theta \, d\theta = \frac{8}{3\pi} \int_{-\pi/2}^{\pi/2} (\cos^4 \theta + \sin \theta \cos^3 \theta) \, d\theta \\ &= \frac{8}{3\pi} \left[\int_{-\pi/2}^{\pi/2} \left(\frac{1 + \cos 2\theta}{2} \right)^2 \, d\theta - \left[\frac{1}{4} \cos^4 \theta \right]_{\theta=-\pi/2}^{\theta=\pi/2} \right] \end{aligned}$$

$$\begin{aligned}
&= \frac{2}{3\pi} \int_{-\pi/2}^{\pi/2} (1 + 2\cos 2\theta + \cos^2 2\theta) d\theta = \frac{2}{3\pi} \left[(\theta + \sin 2\theta) \Big|_{\theta=-\pi/2}^{\theta=\pi/2} + \int_{-\pi/2}^{\pi/2} \frac{1 + \cos 4\theta}{2} d\theta \right] \\
&= \frac{2}{3\pi} \left[\pi + \frac{1}{2} \left(\theta + \frac{1}{4} \sin 4\theta \right) \Big|_{\theta=-\pi/2}^{\theta=\pi/2} \right] = \frac{2}{3\pi} \left[\pi + \frac{1}{2} (\pi) \right] = \frac{2}{3\pi} \left(\frac{3\pi}{2} \right) = 1
\end{aligned}$$

3. Let us visualize D :



So, D can be described $D = \{(x, y) \mid 0 \leq x \leq 2, 0 \leq y \leq x\}$. Additionally, D has area $A = \frac{1}{2} (2^2) = 2$. Therefore, the average value of $f(x, y) = xy$ over D is:

$$\begin{aligned}
f_{\text{ave}} &= \frac{1}{2} \iint_D (xy) dA = \frac{1}{2} \int_0^2 \int_0^x (xy) dy dx \\
&= \frac{1}{2} \int_0^2 x \left[\frac{1}{2} y^2 \right]_{y=0}^{y=x} dx = \frac{1}{4} \int_0^2 x^3 dx = \frac{1}{4} \left[\frac{1}{4} x^4 \right]_{x=0}^{x=2} \\
&= \frac{1}{16} (2^4) = 1
\end{aligned}$$

Answer to Exercise 9 (on page 28)

Finding the expected value of X :

$$\begin{aligned}
\mu_x &= \int_0^1 \int_0^2 \frac{1}{4} xy (1 + 2x) dy dx = \frac{1}{4} \int_0^1 x (1 + 2x) \int_0^2 y dy dx \\
&= \frac{1}{4} \int_0^1 x (1 + 2x) \left[\frac{1}{2} y^2 \right]_{y=0}^{y=2} dx = \frac{1}{4} \int_0^1 x (1 + 2x) (2) dx \\
&= \frac{1}{2} \int_0^1 x + 2x^2 dx = \frac{1}{2} \left[\frac{1}{2} x^2 + \frac{2}{3} x^3 \right]_{x=0}^{x=1}
\end{aligned}$$

$$= \frac{1}{2} \left(\frac{1}{2} + \frac{2}{3} \right) = \frac{1}{2} \left(\frac{7}{6} \right) = \frac{7}{12}$$

And finding the expected value of Y :

$$\begin{aligned} \mu_y &= \int_0^2 \int_0^1 \frac{1}{4} y^2 (1 + 2x) \, dx \, dy = \frac{1}{4} \int_0^2 y^2 \int_0^1 (1 + 2x) \, dx \, dy \\ &= \frac{1}{4} \int_0^2 y^2 \left[x + x^2 \right]_{x=0}^{x=1} dy = \frac{1}{4} \int_0^2 y^2 (2) \, dy \\ &= \frac{1}{2} \int_0^2 y^2 \, dy = \frac{1}{2} \left[\frac{1}{3} y^3 \right]_{y=0}^{y=2} = \frac{1}{2} \left(\frac{8}{3} \right) = \frac{4}{3} \end{aligned}$$

Answer to Exercise 10 (on page 36)

1.

$$\begin{aligned} 1 &= \int_0^1 \int_0^2 ky (1 + 2x) \, dy \, dx = k \left[\int_0^1 1 + 2x \, dx \right] \cdot \left[\int_0^2 y \, dy \right] \\ &= k \left[\left(x + x^2 \right)_{x=0}^{x=1} \right] \cdot \left[\frac{1}{2} (y^2)_{y=0}^{y=2} \right] = k (2) \left(\frac{1}{2} \right) (4) = 4k \end{aligned}$$

If $4k = 1$, then $k = 1/4$.

2.

$$\begin{aligned} P(X \leq \tfrac{1}{2}, Y \leq 1) &= \int_0^{1/2} \int_0^1 \frac{1}{4} y (1 + 2x) \, dy \, dx \\ &= \frac{1}{4} \left[\int_0^{1/2} (1 + 2x) \, dx \right] \cdot \left[\int_0^1 y \, dy \right] = \frac{1}{4} \left[\left(x + x^2 \right)_{x=0}^{x=1/2} \right] \cdot \left[\frac{1}{2} (y^2)_{y=0}^{y=1} \right] \\ &= \frac{1}{4} \left(\frac{1}{2} + \frac{1}{4} \right) \left(\frac{1}{2} \right) (1) = \frac{3}{32} \end{aligned}$$

3. If $X + Y \geq 1$, then $1 - X \leq Y \leq 2$ and

$$\begin{aligned} P(X + Y \geq 1) &= \int_0^1 \int_{1-x}^2 \frac{1}{4} y (1 + 2x) \, dy \, dx \\ &= \frac{1}{4} \int_0^1 (1 + 2x) \frac{1}{2} [y^2]_{y=1-x}^{y=2} \, dx = \frac{1}{8} \int_0^1 (1 + 2x) (2^2 - (1-x)^2) \, dx \\ &= \frac{1}{8} \int_0^1 (1 + 2x) (3 + 2x - x^2) \, dx = \frac{1}{8} \int_0^1 3 + 8x + 3x^2 - 2x^3 \, dx \end{aligned}$$

$$= \frac{1}{8} \left[3x + 4x^2 + x^3 - \frac{1}{2}x^4 \right]_{x=0}^{x=1} = \frac{1}{8} \left[3 + 4 + 1 - \frac{1}{2} \right] = \frac{1}{8} \cdot \frac{15}{2} = \frac{15}{16}$$

Answer to Exercise 11 (on page 38)

The humidity density must integrate to 1. Since $\int_0^{100} (100h - h^2) dh = 166,667$, the constant is $6/10^6$, so $f_2(h) = \frac{6(100h - h^2)}{10^6}$ on $0 \leq h \leq 100$. Because temperature and humidity are independent, the joint density is $f(t, h) = f_1(t)f_2(h)$.

The probability that snow forms is the product of the two separate probabilities:

$$P(T \leq 0, 75 \leq H \leq 100) = \left[\int_{-40}^0 \frac{1}{80} \left(\cos \frac{\pi t}{40} + 1 \right) dt \right] \left[\int_{75}^{100} \frac{6(100h - h^2)}{10^6} dh \right]$$

The first integral is $\frac{1}{80} \left[\frac{40}{\pi} \sin \frac{\pi t}{40} + t \right]_{-40}^0 = \frac{1}{80}(40) = \frac{1}{2}$. The second is $\frac{6}{10^6} \left[50h^2 - \frac{h^3}{3} \right]_{75}^{100} = \frac{6}{10^6} \cdot \frac{78125}{3} = \frac{5}{32}$.

So $P = \frac{1}{2} \cdot \frac{5}{32} = \frac{5}{64} \approx 0.078$. There is about a 7.8% chance it snows.

Answer to Exercise 12 (on page 45)

The midpoints are $(0.25, 0.25)$, $(0.25, 0.75)$, $(0.75, 0.25)$, and $(0.75, 0.75)$, where f equals 0.125, 0.625, 0.625, and 1.125. Each square has area 0.25, so the estimate is $0.25(0.125 + 0.625 + 0.625 + 1.125) = 0.625$. The exact value is $\int_0^1 \int_0^1 (x^2 + y^2) dy dx = \frac{1}{3} + \frac{1}{3} = \frac{2}{3} \approx 0.667$. Even four squares get within about 6%.

Answer to Exercise 13 (on page 49)

Each of the three terms contributes $\frac{1}{2}$, so the total is $\frac{3}{2}$.

Answer to Exercise 14 (on page 49)

$$\int_0^{2\pi} \int_0^1 \int_0^4 r dz dr d\theta = 2\pi \cdot \frac{1}{2} \cdot 4 = 4\pi.$$

Answer to Exercise 15 (on page 49)

$$\int_0^{2\pi} \int_0^\pi \int_0^2 \rho^2 \sin \varphi \, d\rho \, d\varphi \, d\theta = 2\pi \cdot 2 \cdot \frac{8}{3} = \frac{32\pi}{3}.$$

Answer to Exercise 16 (on page 50)

$$\int_0^1 \int_0^{1-z} \int_0^{1-y-z} dx \, dy \, dz = \frac{1}{6}.$$



INDEX

- moment of inertia, [17](#)
- average value
 - two variables, [26](#)
- center of mass
 - double integral, [13](#)
- correlation, [31](#)
- covariance, [29](#), [32](#)
 - formula, [32](#)
- covariance matrix, [32](#)
- cylindrical coordinates
 - triple integral, [48](#)
- density
 - other types, [12](#)
- double integral, [42](#)
- Gaussian distribution, [31](#)
- iterated integral
 - triple, [47](#)
- joint density function, [35](#)
- mass of lamina, [9](#)
- mean vector, [31](#)
- moment of inertia
 - lamina, [17](#)
- multivariate distribution, [29](#)
- multivariate normal, [31](#)
 - probability density, [35](#)
- multivariate normal distribution, [31](#), [41](#)
- numerical integration, [42](#)
 - python, [44](#)
- polar double integrals, [3](#)
- polar rectangle, [3](#)
- radius of gyration, [19](#)
- Riemann sum, [42](#)
 - two dimensions, [42](#)
- spherical coordinates
 - triple integral, [48](#)
- surface area
 - double integral, [21](#)
- tangent plane, [22](#)
- total charge, [12](#)
- triple integral, [47](#)
- volume
 - double integral in polar, [6](#)