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Chapter 1

u-Substitution

U-Substitution, also known as the method of substitution, is a technique used to simplify the process of finding antiderivatives and integrals of complicated functions. The method is similar to the chain rule for differentiation in reverse.

Suppose we have an integral of the form:

$$\int f(g(x)) \cdot g'(x)\,dx \tag{1.1}$$

The u-substitution method suggests letting a new variable u be equal to the inside function $g(x)$, i.e.,

$$u = g(x) \tag{1.2}$$

Next, the differential of u, du, is given by:

$$du = g'(x)\,dx \tag{1.3}$$

Substituting u and du back into the integral gives us a simpler integral:

$$\int f(u)\,du \tag{1.4}$$

This new integral can often be simpler to evaluate. Once the antiderivative of $f(u)$ is found, we can substitute $u = g(x)$ back into the antiderivative to get the antiderivative of the original function in terms of x.

The method of u-substitution is a powerful tool for evaluating integrals, especially when combined with other techniques like integration by parts, partial fractions, and trigonometric substitutions.

Example: Find $\int 2x^2 \cos\left(x^3 - 3\right) dx$.

Solution: Integrating $\cos\left(x^3 - 3\right)$ isn't so straightforward, so let's try the substitution

$u = x^3 - 3$. Then:

$$du = 3x^2 \; dx$$

We don't have $3x^2$ in the integral, but we do have $2x^2$:

$$\frac{2}{3} du = 2x^2 \; dx$$

Substituting:

$$\int 2x^2 \cos\left(x^3 - 3\right) dx = \int \frac{2}{3} \cos(u) \, du$$
$$= \frac{2}{3} \sin(u) + C$$

Now that we have the antiderivative of $f(u)$, we can back-substitute in for u:

$$\frac{2}{3} \sin(u) + C = \frac{2}{3} \sin\left(x^3 - 3\right) + C$$

We can check our answer by taking its derivative: we should get the original integrand back:

$$\frac{d}{dx}\left[\frac{2}{3} \sin\left(x^3 - 3\right) + C\right] = \frac{2}{3} \cos\left(x^3 - 3\right) \cdot \left[\frac{d}{dx}\left(x^3 - 3\right)\right]$$
$$= \frac{2}{3} \cos\left(x^3 - 3\right) \cdot \left(3x^2\right) = 2x^2 \cos\left(x^3 - 3\right)$$

Sometimes, the right substitution takes a little thinking. Consider the following example:

Example: Find $\int \sqrt{x^2 - 1} \; x^5 \, dx$.

Solution: We can guess that $u = x^2 - 1$ could be an appropriate substitution, as that is what is under the square root. What to do with x^5? First, let's look at how the u-substitution for $x^2 - 1$ works out:

$$u = x^2 - 1$$
$$du = 2x \; dx$$
$$\frac{du}{2} = x \; dx$$

Then we will need to use one of the x's in x^5 for the square root u-substitution. What can we do with the remaining x^4? Well, we see that if $u = x^2 - 1$, then $u + 1 = x^2$ and therefore $(u+1)^2 = x^4$. Substituting this all in:

$$\int \sqrt{x^2 - 1} \; x^5 \, dx = \int x^4 \sqrt{x^2 - 1} \; x \, dx$$

$$= \frac{1}{2} \int (u+1)^2 \sqrt{u}\, du$$

We can expand this to find the antiderivative:

$$= \frac{1}{2} \int \left(u^2 + 2u + 1\right) u^{1/2}\, du = \frac{1}{2} \int u^{5/2} + 2u^{3/2} + u^{1/2}\, du$$

$$= \frac{1}{2} \left[\frac{2}{7}u^{7/2} + \frac{4}{5}u^{5/2} + \frac{2}{3}u^{3/2}\right] + C$$

$$= \frac{1}{7}u^{7/2} + \frac{2}{5}u^{5/2} + \frac{1}{3}u^{3/2} + C$$

$$= \frac{1}{7}\left(x^2 - 1\right)^{7/2} + \frac{2}{5}\left(x^2 - 1\right)^{5/2} + \frac{1}{3}\left(x^2 - 1\right)^{3/2} + C$$

Exercise 1 Indefinite Integrals and u-substitution

Use u-substitution to evaluate the following indefinite integrals. Confirm your answer by taking the derivative of the result.

1. $\int \sin x \sqrt{1 + \cos x}\, dx$
2. $\int \frac{\cos(\pi/x)}{x^2}\, dx$
3. $\int 2x^2 \left(9 - x^3\right)^{2/3}\, dx$
4. $\int 3x^2 \sqrt{1 + x}\, dx$
5. $\int \frac{3x^2}{x^3 - 1}\, dx$

Working Space

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1.1 The Substitution Rule for Definite Integrals

How to we use u-substitution for definite integrals? We will apply the fundamental theorem of calculus to answer this question. We define f and F such that F is the antiderivative of f. Then:

$$\int_a^b f(g(x)) \cdot g'(x)\,dx = F(g(x))|_a^b = F(g(b)) - F(g(a))$$

This represents the method of finding the indefinite antiderivative and evaluating from the original limits of integration.

We can also see that:

$$F(g(b)) - F(g(a)) = F(u)|_{g(a)}^{g(b)} = \int_{g(a)}^{g(b)} f(u)\,du$$

Therefore, if g' is continuous on $[a, b]$ and f is continuous on the range of $u = g(x)$, then:

$$\int_a^b f(g(x)) \cdot g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

This represents changing the limits of integration into the new variable, u, then evaluating the integral. While the second method is preferable, both methods yield the same answer.

Example: Evaluate $\int_0^5 \sqrt{3x+1}\,dx$ using both methods outlined above.

Solution: We start with the first method. We will use the substitution $u = 3x + 1$, and therefore $du/3 = dx$:

$$\int_0^5 \sqrt{3x+3}\,dx = \frac{1}{3}\int_{x=0}^{x=5} \sqrt{u}\,du$$

(We write the limits as $x = \cdots$ to remind us the limits are for x, not u.)

$$\frac{1}{3}\int_{x=0}^{x=5} \sqrt{u}\,du = \frac{1}{3}\left[\frac{2}{3}u^{3/2}\right]_{x=0}^{x=5}$$

Now we substitute back in for u and evaluate:

$$\frac{2}{9}(3x+1)^{3/2}|_0^5 = \frac{2}{9}(16)^{3/2} - \frac{2}{9}(1)^{3/2}$$

$$= \frac{2}{9}(64-1) = \frac{2 \cdot 63}{9} = 14$$

Let's compare this to the second, preferred method. We already know the u-substitution we'll make, so next we need to find $g(0)$ and $g(5)$ (recall that we choose u such that $u = g(x)$):

$$g(x) = 3x + 1$$

$$g(0) = 1$$
$$g(5) = 16$$

Now we can make our substitution *and* change the limits of integration:

$$\int_0^5 \sqrt{3x+1}\,dx = \frac{1}{3}\int_1^{16} \sqrt{u}\,du$$
$$= \frac{1}{3}\left[\frac{2}{3}u^{3/2}\right]_1^{16} = \frac{2}{9}\left[16^{3/2} - 1^{3/2}\right]$$
$$= \frac{2}{9}(64-1) = 14$$

With the second method, we get the same answer in fewer steps.

Exercise 2 Definite Integrals and u-substitution

Use u-substitution to evaluate the following definite integrals.

1. $\int_0^{\pi/2} \cos x \sin(\sin x)\, dx$
2. $\int_0^{13} \frac{1}{\sqrt[3]{(1+2x)^2}}\, dx$
3. $\int_1^2 \frac{e^{1/x}}{x^2}\, dx$
4. $\int_0^{\pi/6} \frac{\sin x}{\cos^2 x}\, dx$
5. $\int_0^4 \frac{x}{\sqrt{1+2x}}\, dx$

Working Space

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CHAPTER 2

Calculus with Polar Coordinates

We have been working in Cartesian coordinates, which are rectangular, with x representing the horizontal position and y representing the vertical position. Another way to represent a position in 2D space is with **polar coordinates**. In this coordinate system, the first number and dependent variable is r, which represents how far the point is from the origin. The second number is θ, which represents the degrees of rotation from the the x axis (see figure ??).

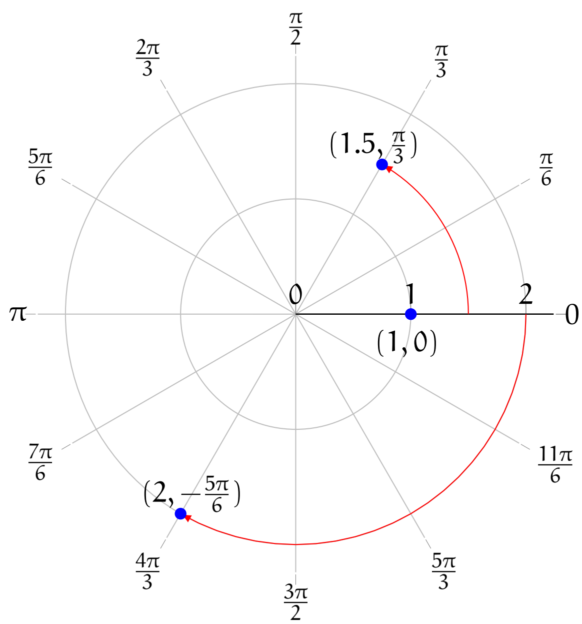

Figure 2.1: Polar coordinates give a degree of rotation, θ, and a distance from the origin, r, in the form of (r, θ)

2.1 Derivatives of Polar Functions

Consider the cardioid $r = 2 + \sin\theta$ (see figure ??). What is the slope of the line tangent to the curve at $\theta = \frac{\pi}{2}$?

From a visual inspection, we can guess that the slope of the tangent line is zero. Let's prove this mathematically:

First, recall that to convert polar coordinates to Cartesian coordinates, we can use the

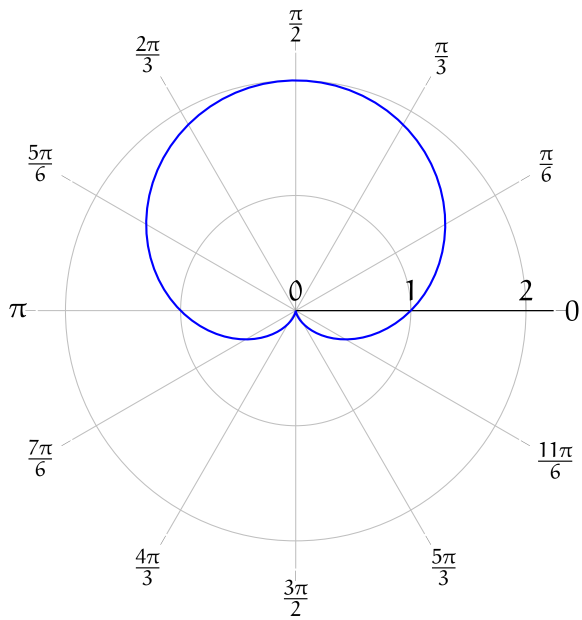

Figure 2.2: $r = 2 + \sin\theta$

trigonometric identities:

$$x = r\cos\theta$$
$$y = r\sin\theta$$

So, we can write the parametric equation:

$$x = [2 + \sin\theta]\cos\theta$$
$$y = [2 + \sin\theta]\sin\theta$$

Recall from parametric equations that we can use implicit differentiation to find $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$$

Finding $\frac{dy}{d\theta}$ and $\frac{dx}{d\theta}$:

$$\frac{dy}{d\theta} = \frac{d}{d\theta}\left(2\sin\theta + \sin^2\theta\right) = 2\cos\theta + 2\sin\theta\cos\theta$$
$$\frac{dx}{d\theta} = \frac{d}{d\theta}\left(2\cos\theta + \sin\theta\cos\theta\right) = \cos^2\theta - \sin^2\theta - 2\sin\theta$$

Substituting $\theta = \frac{\pi}{2}$, we find that:

$$\frac{dy}{d\theta} = 2(0) + 2(1)(0) = 0$$

$$\frac{dx}{d\theta} = (0)^2 - (1)^2 - 2(1) = -3$$

Therefore,

$$\frac{dy}{dx} = \frac{0}{-3} = 0$$

Which is the result we expected from examining the graph of $r = 2 + \sin\theta$.

So, in general for polar equations,

Tangent to a Polar Function

For a polar function, $r = f(\theta)$, the slope of a tangent line is given by:

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$$

Where $y = r \cdot \sin\theta$ and $x = r \cdot \cos\theta$

Exercise 3

[This problem was originally presented as a no-calculator, multiple-choice question on the 2012 AP Calculus BC exam.] What is the slope of the line tangent to the polar curve $r = 1 + 2\sin\theta$ at $\theta = 0$?

Working Space

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Exercise 4

Find the slope of the tangent line to the given polar curve at the value of θ specified. Use this to write an equation for the tangent line in Cartesian coordinates.

1. $r = \frac{2}{3}\cos\theta$, $\theta = \frac{\pi}{6}$
2. $r = \frac{1}{2\theta}$, $\theta = \frac{\pi}{2}$
3. $r = 2 + 3\cos\theta$, $\theta = \frac{2\pi}{3}$

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2.2 Integrals of Polar Functions

Similar to Cartesian functions, an integral of a polar function tells us the area within the function. We say "within" as opposed to "under" because a polar function describes how far from the origin the graph is based on the angle. Consider the graph of $r = 2\sin\theta$ (figure 2.3). Geometrically, we expect the area inside the curve to be $\pi r^2 = \pi$. However, this is not the result we get from directly integrating the function (we only integrate from $\theta = 0$ to $\theta = \pi$ because the circle is complete when θ reaches π):

$$\int_0^\pi 2\sin\theta \, d\theta = -2\cos\theta|_{\theta=0}^{\theta=\pi}$$

$$= -2\left[\cos\pi - \cos 0\right] = -2\left[-1 - 1\right] = 4 \neq \pi$$

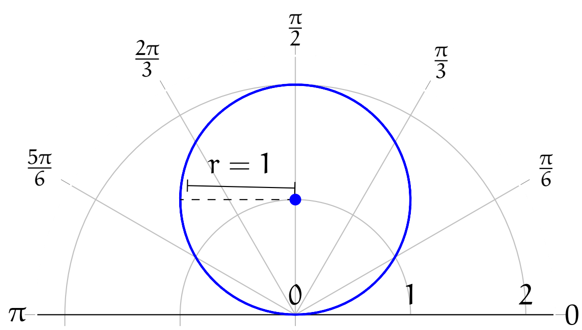


Figure 2.3: The graph of $r = 2\sin\theta$ is a circle of radius 1 centered at $(1, \frac{\pi}{2})$

Clearlym something else is happening here. We can just take the integral of a Cartesian function because the area of a rectangle is the base times the height. When integrating Cartesian functions, the base is given by the dx and the height by the function, $f(x)$. In polar coordinates, the integral sweeps across a θ interval, making a wedge, not a rectangle.

Let us consider a generic polar function, shown in figure 2.4

Suppose we are interested in a specific region, bounded by $a \leq \theta \leq b$ (see figure 2.5).

We can divide the region into many small sectors. Then, each small sector has a central angle $\Delta\theta$ and a radius $r(\theta_i^*)$, where $\theta_{i-1} < \theta_i^* < \theta_i$ (see figure 2.6).

What is the area of the i^{th} sector? Recall from the chapter on circles that the area of a sector with angle θ and length r is $A = \frac{1}{2}r^2\theta$. Substituting, we see the area of the i^{th} sector

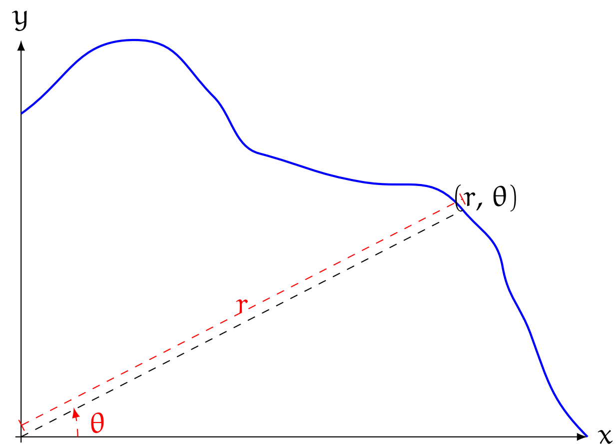


Figure 2.4: A generic polar function

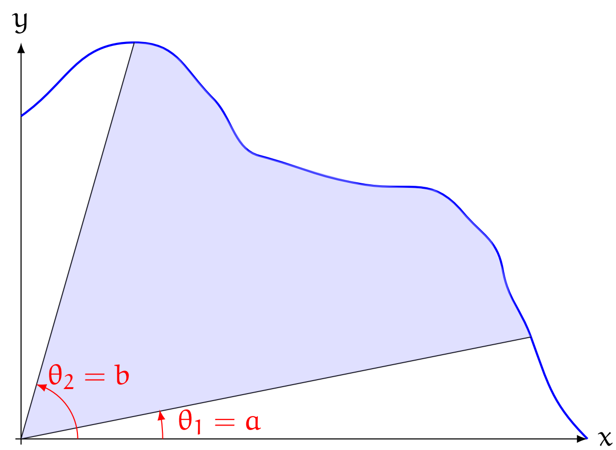


Figure 2.5: A generic polar with a region from $\theta = a$ to $\theta = b$ highlighted

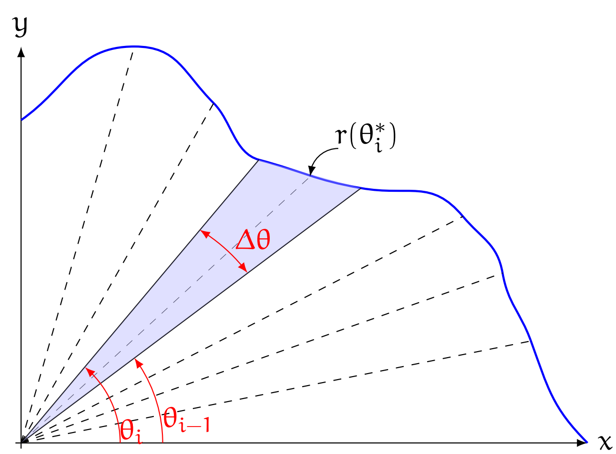


Figure 2.6: A single sector from θ_{i-1} to θ_i

is:

$$A_i = \frac{1}{2}\left[r(\theta_i^*)\right]^2 d\theta$$

Therefore, the total area of the whole sector from $\theta = a$ to $\theta = b$ is the limit as the number of sectors approaches infinity of sum of the areas of all the small sectors:

$$A = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{2}\left[r(\theta_i^*)\right]^2 \Delta\theta$$

Does this look familiar? It is the definition of an integral!

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{2}\left[r(\theta_i^*)\right]^2 \Delta\theta = \int_a^b \frac{1}{2}\left[r(\theta)\right]^2\, d\theta$$

Area of a Polar Function

The area of a polar function is given by the integral

$$\int_a^b \frac{1}{2} r^2\, d\theta$$

Where r is a function of θ.

We can check this with the example from the beginning of the section. Recall that the polar function $r = 2\sin\theta$ graphs a circle with a radius of 1. Therefore, we expect the area enclosed by the graph of $r = 2\sin\theta$ from $\theta = 0$ to $\theta = \pi$ to be π:

$$A = \frac{1}{2}\int_0^\pi \left[2\sin\theta\right]^2\, d\theta$$

$$A = 2\int_0^\pi \sin^2\theta\, d\theta = 2\int_0^\pi \left[\frac{1-\cos 2\theta}{2}\right]\, d\theta$$

$$A = \int_0^\pi \left[1-\cos 2\theta\right]\, d\theta = \left[\theta - \frac{1}{2}\sin 2\theta\right]_{\theta=0}^{\theta=\pi}$$

$$A = [\pi - 0] - [0 - 0] = \pi$$

Which is the expected result, confirming our formula for the area within a polar function.

Example: The graph of $r = 3\sin 2\theta$ is shown below. What is the total area enclosed by the graph?

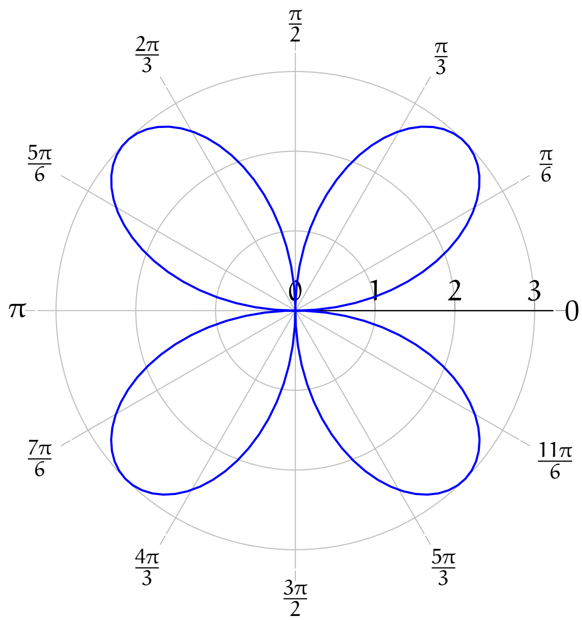


Figure 2.7: $r = 3\sin 2\theta$

Solution: Since each lobe is symmetric to the others, we can find the area of one lobe and multiply it by four. To find the area of one lobe, we need to determine an interval for θ that defines one lobe. You can imagine each lobe being draw out from the center and then back in. So, we will find where $r = 0$:

$$0 = 3\sin 2\theta$$

$$\sin 2\theta = 0$$

$$2\theta = n\pi$$

$$\theta = \frac{n\pi}{2}$$

Taking the first two solutions, $\theta = 0$ and $\theta = \frac{\pi}{2}$, as our limits of integration, we see that the area of one lobe is:

$$A_{\text{lobe}} = \frac{1}{2}\int_0^{\pi/2} [3\sin 2\theta]^2 \, d\theta$$

$$A_{\text{lobe}} = \frac{9}{2}\int_0^{\pi/2} \sin^2 2\theta \, d\theta$$

Applying the half-angle formula $\sin^2\theta = \frac{1-\cos 2\theta}{2}$, we see that:

$$A_{\text{lobe}} = \frac{9}{2}\int_0^{\pi/2} \frac{1-\cos 4\theta}{2} \, d\theta = \frac{9}{4}\int_0^{\pi/2} 1 - \cos 4\theta \, d\theta$$

$$= \frac{9}{4}\left[\theta - \frac{1}{4}\sin 4\theta\right]_{\theta=0}^{\theta=\pi/2} = \frac{9}{4}\left(\frac{\pi}{2} - 0\right) - \frac{9}{4}\left(\frac{1}{4}\right)(\sin 2\pi - \sin 0)$$
$$= \frac{9\pi}{8} - \frac{9}{16}(0) = \frac{9\pi}{8}$$

Since the area of one lobe is $\frac{9\pi}{8}$, the area of all four lobes is $\frac{9\pi}{2}$.

2.2.1 Area between polar curves

Consider the circle $r = 6\sin\theta$ and the cardioid $r = 2 + 2\sin\theta$. How can we find the area that lies inside the circle, but outside the cardioid (see figure 2.8)? First, let's find where these curves intersect. This will determine the limits of any integrals we take.

$$6\sin\theta = 2 + 2\sin\theta$$
$$3\sin\theta = 1 = \sin\theta$$
$$2\sin\theta = 1$$
$$\sin\theta = \frac{1}{2}$$
$$\theta = \frac{\pi}{6}, \frac{5\pi}{6}$$

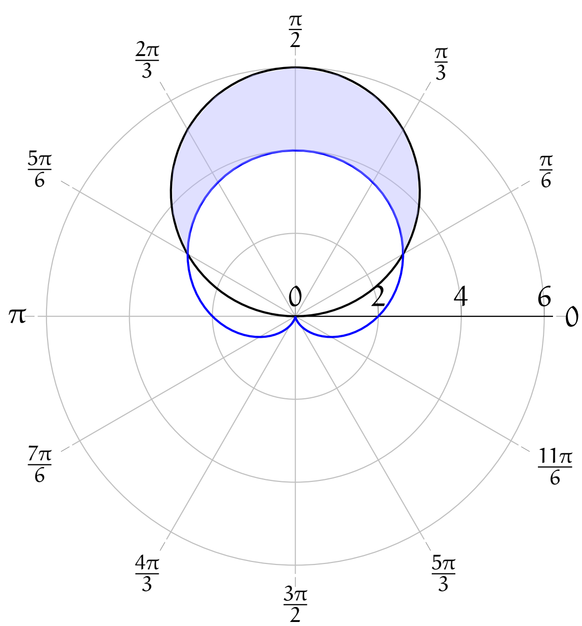


Figure 2.8: The area inside $r = 6\sin\theta$ and outside of $r = 2 + 2\sin\theta$ is highlighted

Recall that for Cartesian functions, to find the area between two curves, we subtract the area under the lower curve from the total area under the higher curve. In polar coordinates, we want to subtract the area in the inner curve from the total area in the outer curve. In this case, the outer curve is $r = 6\sin\theta$ and the inner curve is $r = 2 + 2\sin\theta$. We have already found our limits of integration ($\frac{\pi}{6} \leq \theta \leq \frac{5\pi}{6}$), so we set up and evaluate our integral:

$$\begin{aligned}
A_{\text{between}} &= \frac{1}{2}\int_{\pi/6}^{5\pi/6} [4\sin\theta]^2 \, d\theta - \frac{1}{2}\int_{\pi/6}^{5\pi/6} [2 + 2\sin\theta]^2 \, d\theta \\
&= \frac{1}{2}\int_{\pi/6}^{5\pi/6} \left[16 sin^2\theta - 4 - 8\sin\theta - 4\sin^2\theta\right] \, d\theta \\
&= \int_{\pi/6}^{5\pi/6} \left[6\sin^2\theta - 4\sin\theta - 2\right] \, d\theta \\
&= \int_{\pi/6}^{5\pi/6} [3\,(1 - \cos 2\theta) - 4\sin\theta - 2] \, d\theta \\
&= \int_{\pi/6}^{5\pi/6} [1 - 3\cos 2\theta - 4\sin\theta] \, d\theta \\
&= \left[\theta - \frac{3}{2}\sin 2\theta + 4\cos\theta\right]_{\theta=\pi/6}^{\theta=5\pi/6}
\end{aligned}$$

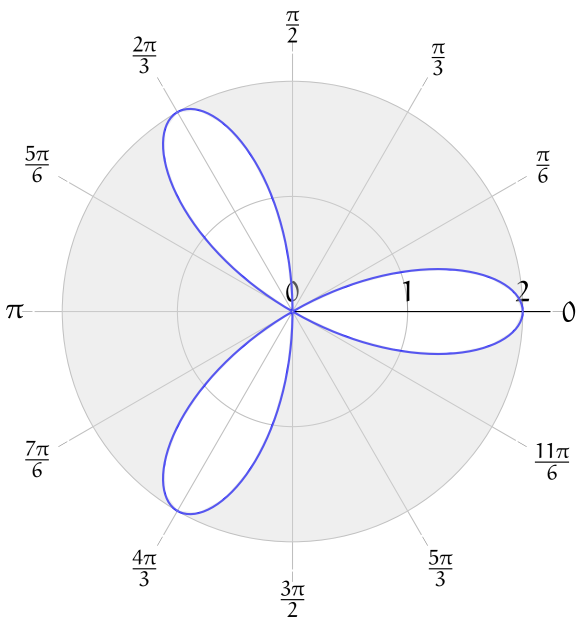
$$= \left[\frac{5\pi}{6} - \frac{\pi}{6}\right] - \left[\frac{3}{2}\sin\left(2 \cdot \frac{5\pi}{6}\right) - \frac{3}{2}\sin\left(2 \cdot \frac{\pi}{6}\right)\right] + \left[4\cos\frac{5\pi}{6} - 4\cos\frac{\pi}{6}\right]$$

$$\begin{aligned}
&= \frac{4\pi}{6} - \left[\frac{3}{2} \cdot -\frac{\sqrt{3}}{2} - \frac{3}{2} \cdot \frac{\sqrt{3}}{2}\right] + \left[4 \cdot -\frac{\sqrt{3}}{2} - 4 \cdot \frac{\sqrt{3}}{2}\right] \\
&= \frac{2\pi}{3} + \frac{3\sqrt{3}}{2} - 4\sqrt{3} = \frac{2\pi}{3} + \frac{3\sqrt{3} - 8\sqrt{3}}{2} = \frac{2\pi}{3} - \frac{5\sqrt{3}}{2}
\end{aligned}$$

(*Note*: Because these polar functions are symmetric about the y-axis, we could have also taken the integral from $\theta = \frac{\pi}{6}$ to $\theta = \frac{\pi}{2}$ and doubled the result. We leave it as an exercise for the student to show this works.)

Exercise 5

[This question was originally presented as a multiple-choice, calculator- allowed problem on the 2012 AP Calculus BC exam.] The figure below shows the graphs of polar curves $r = 2\cos 3\theta$ and $r = 2$. What is the sum of the areas of the shaded regions to three decimal places?



Working Space

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Exercise 6

Find the area of the region bounded by the given curve and angles.

1. $r = e^{\theta/2}$, $\pi/4 \leq \theta \leq \pi/2$
2. $r = 2\sin\theta + \cos 2\theta$, $0 \leq \theta \leq \pi$
3. $r = 4 + 3\sin\theta$, $-\pi/2 \leq \theta \leq \pi/2$

Working Space

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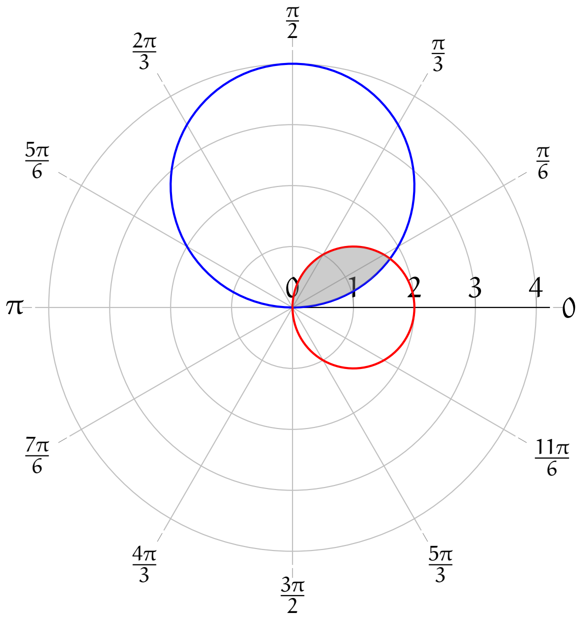
Exercise 7

Find the area of the region that lies between the curves $r = 4\sin\theta$ and $r = 2\cos\theta$. A graph is shown below.



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CHAPTER 3

Differential Equations

Differential equations are equations involving an unknown function and its derivatives. They play a crucial role in mathematics, physics, engineering, economics, and other disciplines due to their ability to describe change over time or in response to changing conditions.

3.1 Ordinary Differential Equations

An ordinary differential equation (ODE) involves a function of a single independent variable and its derivatives. The order of an ODE is determined by the order of the highest derivative present in the equation. An example of a first-order ODE is:

$$\frac{dy}{dx} + y = x \tag{3.1}$$

Here, y is the function of the independent variable x, and $\frac{dy}{dx}$ represents its first derivative.

A real-world example of the application of differential equations is an oscillating spring (or any harmonic motion). When a spring is stretched, the restoring force (the force pulling or pushing it back to its neutral position) is proportional to the distance by which the spring has been stretched (see figure ??). Mathematically, we say that

$$\text{restoring force} = -kx$$

where k is the positive spring constant (the stiffer a spring, the greater k).

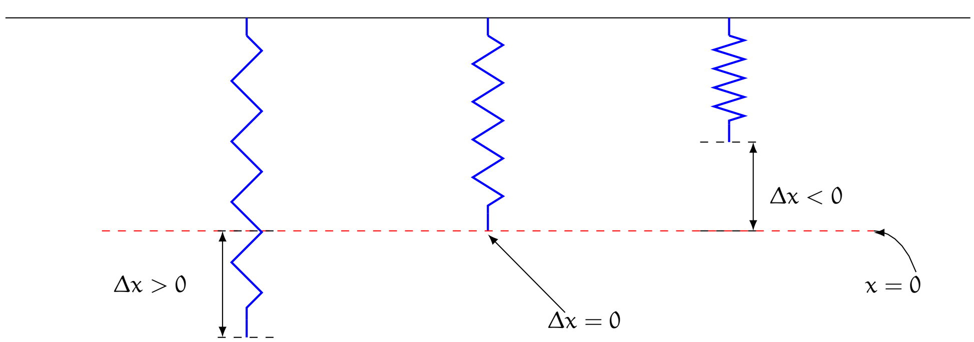


Figure 3.1: A spring can have a positive or negative displacement

Recall that Newton's Second Law tells us that force is equal to mass times acceleration, and that acceleration is the second derivative of position. We can then write the differential equation:

$$m\frac{d^2x}{dt^2} = -kx$$

This is called a **second-order differential equation**, because it involves second-order derivatives. The order of a differential equation is the same as the highest order of derivative in the equation. We can further rewrite the equation to isolate the second derivative:

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x$$

In everyday language, this is saying that the second derivative is proportional to the original function, just negative. There are two trigonometric functions that have this property, take a second to see if you remember and write down your guess.

The sine and cosine functions both have the property $\frac{d^2x}{dt^2} \propto -x(t)$ (recall that $\propto$ means "proportional to").

Example: Assuming $x(t)$ is a sine function, solve the second-order differential equation $\frac{d^2x}{dt^2} = \frac{-k}{m}x$.

Solution: Let $x(t) = \sin Ct$. Then $\frac{dx}{dt} = C\cos Ct$ and $\frac{d^2x}{dt^2} = -C^2\sin t$. This implies that $C^2 = \frac{k}{m}$ and $C = \pm\sqrt{\frac{k}{m}}$. So, a solution to the differential equation $\frac{d^2x}{dt^2} = \frac{-k}{m}x$ is $x(t) = \sin\sqrt{\frac{k}{m}}t$.

3.1.1 Population Growth

Another real-world application of differential equations is modeling population growth. Under ideal conditions (unlimited food, no predators, disease-free, etc.), the population of a species grows at a rate proportional to the current population size. We can identify two variables:

$$t = \text{ time (the independent variable)}$$

$$P = \text{ the number pf individuals in the population (the dependent variable)}$$

So, what is the rate of growth? Recall that a rate is change over time. In that case, the rate of growth is given by $\frac{dP}{dt}$. If the rate of growth is proportional to the population, then we can write a first-order differential equation:

$$\frac{dP}{dt} = kP$$

where k is a proportionality constant. This is called **natural growth** or **logarithmic growth**. To find a solution, we must answer the question: What function's derivative is a constant multiple of itself? Recall that we have seen that the derivative of the exponential function e^{kt} is ke^{kt}. Setting $P(t) = Ce^{kt}$ (where C is some constant), we see that the derivative is $\frac{dP}{dt} = kCe^{kt} = kP(t)$ (see figure 3.2). You can determine C from initial conditions.

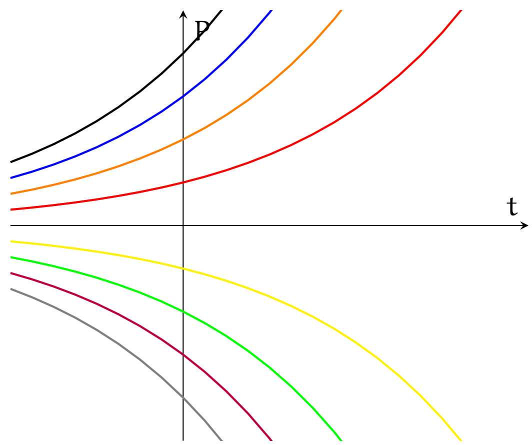


Figure 3.2: Several solutions to $\frac{dP}{dt} = kP$

Example: Suppose a population of bacteria has an initial population of 100 bacteria. If the bacteria's growth rate is given by $\frac{dP}{dt} = 2P$ (where t is in hours), how many bacteria are present after 4 hours?

Solution: We have seen that the solution to $\frac{dP}{dt} = 2P$ is $P(t) = Ce^{2t}$. We can then use the given initial condition to find C:

$$P(0) = 100 = Ce^{2 \cdot 0} = C \cdot 1 = C$$

Which means that the complete solution is:

$$P(t) = 100e^{2t}$$

To answer the question, we need to find $P(4)$:

$$P(4) = 100e^{2 \cdot 4} = 100e^{8} \approx 298096$$

As stated above, this model works well for populations under specific, ideal conditions. However, there are very few environments in which these conditions are met. Real animals suffer from disease, are hunted by predators, and have limited food supplies. Most environments have a maximum number of animals they can support, which ecologists call a **carrying capacity**. Let us call the carrying capacity of an environment M. So, the population growth can be modeled by the logistic differential equation:

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$$

This is called a **logistic differential growth model**. Notice that if P is small, then $\frac{dP}{dt} \approx kP$. This makes sense: If the population is very small compared to the carrying capacity, the conditions are nearly ideal, and so growth should be nearly ideal too. On the other hand, if the population ever goes *above* the carrying capacity, the $\frac{dP}{dt} < 0$ and the population will decrease back below the carrying capacity (see figure 3.3). Notice that if the initial population is $P_0 = M$, then $\frac{dP}{dt} = kP(1-1) = 0$ and the population is stable at $P(t) = M$. We call this an **equilibrium solution**. Can you logically find the other equilibrium solution?

If there are no animals to begin with, then there are none to reproduce, and $P(t) = 0$. This is the other equilibrium solution. Notice that when the population is in equilibrium, then the rate of change is zero. Mathematically, to find equilibrium solutions, we can set $\frac{dP}{dt} = 0$ and solve for P.

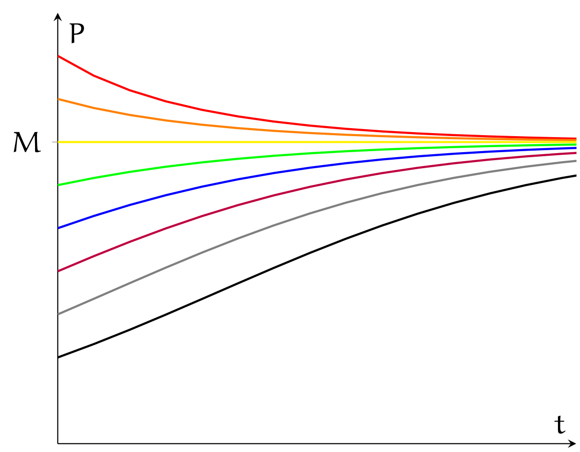


Figure 3.3: Several solutions to $\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$

Exercise 8

A population is modeled by the differential equation $\frac{dP}{dt} = 1.2P\left(1 - \frac{P}{4200}\right)$.

1. What is the carrying capacity of the environment?
2. For what values of P is the population increasing?
3. For what values of P is the population decreasing?
4. What are the equilibrium solutions?

Working Space

Answer on Page 48

Exercise 9

[This problem was originally presented as a calculator-allowed, free response question on the 2012 AP Calculus BC exam.] Let k be a positive constant. Which of the following is a logistic differential equation?
(a) $\frac{dy}{dt} = kt$
(b) $\frac{dy}{dt} = ky$
(c) $\frac{dy}{dt} = kt(1-t)$
(d) $\frac{dy}{dt} = ky(1-t)$
(e) $\frac{dy}{dt} = ky(1-y)$

Working Space

Answer on Page 49

3.1.2 Separable Differential Equations

Sometimes, differential equations can be explicitly solved. A first-order differential equation is separable if $\frac{dy}{dx}$ can be written as a function of x times a function of y. Symbolically, a differential equation is separable if it takes the form

$$\frac{dy}{dx} = g(x)f(y)$$

The equations may be solvable by separating the x from the y and integrating each side. For our generic form, we can separate the variables thusly if $f(y) \neq 0$:

$$\frac{dy}{dx}\frac{1}{f(y)} = g(x)$$

$$\frac{1}{f(y)}dy = g(x)dx$$

Integrating both sides:

$$\int \frac{1}{f(y)}\,dy = \int g(x)\,dx$$

.

Let's look at the example $\frac{dy}{dx} = \frac{x^2}{y}$. We can separate the variables by multiplying both sides by $y\,dx$:

$$y\,dy = x^2 dx$$

Integrating both sides:

$$\int y\,dy = \int x^2\,dx$$

$$\frac{1}{2}y^2 + C_1 = \frac{1}{3}x^3 + C_2$$

We can combine the constants by defining $C = C_2 - C_1$. Making this substitution and solving for y, we find:

$$y^2 = \frac{2}{3}x^3 + 2C$$

$$y = \sqrt{\frac{2}{3}x^3 + 2C}$$

Noting that $2C$ is also a constant (which we will call K for convenience), we find the general solution is

$$y = \sqrt{\frac{2}{3}x^3 + K}$$

A graph showing the solution for several values of K is in figure 3.4.

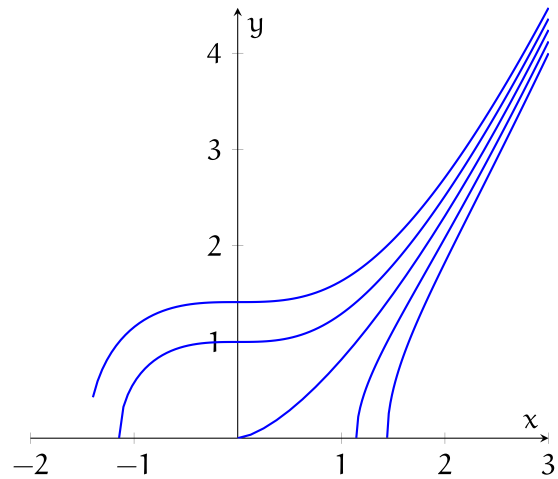


Figure 3.4: Several possible solutions to $\frac{dy}{dx} = \frac{x^2}{y}$

It is not always possible to solve for y explicitly in terms of x. The practice problem below is an example of this.

Exercise 10

Solve the differential equation $\frac{dy}{dx} = \frac{3x^2}{2y+\sin y}$.

Working Space

Answer on Page 49

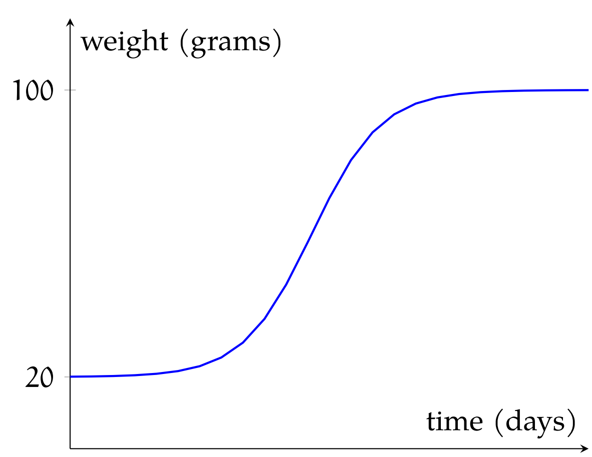
Exercise 11

[This problem was originally presented as a calculator-allowed, free response question on the 2012 AP Calculus BC exam.] The rate at which a baby bird gains mass is proportional to the difference between its adult mass and its current mass. At time $t = 0$, when the bird is first weighed, its mass is 20 grams. If $B(t)$ is the mass of the bird, in grams, at time t days after it is first weighed, then

$$\frac{dB}{dt} = \frac{1}{5}(100 - B)$$

Let $y = B(t)$ be the solution to the differential equation with initial condition $B(0) = 20$.

1. Is the bird gaining mass faster when it masses 40 grams or when it masses 70 grams? Explain your reasoning.
2. Find $\frac{d^2B}{dt^2}$ in terms of B. Use it to explain why the graph of B cannot resemble the graph shown below.
3. Use separation of variables to find $y = B(t)$, the particular solution to the differential equation with initial condition $B(0) = 20$.



Working Space

Answer on Page 49

Exercise 12

[This problem was originally presented as a no-calculator, multiple-choice question on the 2012 AP Calculus BC exam.] If $P(t)$ is the size of a population at time t, which of the following differential equations describes *linear* growth in the size of the population?
(a) $\frac{dP}{dt} = 200$
(b) $\frac{dP}{dt} = 200t$
(c) $\frac{dP}{dt} = 100t^2$
(d) $\frac{dP}{dt} = 200P$
(e) $\frac{dP}{dt} = 100P^2$

Working Space

Answer on Page 49

3.2 Partial Differential Equations

Partial differential equations (PDEs), on the other hand, involve a function of multiple independent variables and their partial derivatives. An example of a PDE is the heat equation, a second-order PDE:

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2} \tag{3.2}$$

In this equation, $u = u(x, t)$ is a function of the two independent variables x and t, $\frac{\partial u}{\partial t}$ is the first partial derivative of u with respect to t, and $\frac{\partial^2 u}{\partial x^2}$ is the second partial derivative of u with respect to x.

CHAPTER 4

Slope Fields

While separable differential equations are solvable, most differential equations are not separable. In fact, it is impossible to obtain an explicit formula as a solution to most differential equations. How do computers solve these, then? They start with a given quantity (usually initial conditions) and perform many small calculations to estimate the behavior of the solution. We can do this graphically with slope fields (also called direction fields), which allow us to visualize the family of solutions to the differential equation.

4.1 Drawing Slope Fields

When a differential equation is in the form

$$y' = f(x, y)$$

we can use the coordinates (x, y) to determine the slope of a solution to the differential equation at that coordinate. Take $y' = x + y$ as an example. According to this differential equation, a solution that passes through the point $(1, 1)$ would have a slope of 2. We can represent this with a small tick of slope 2 at the $(1, 1)$ (see figure 4.1).

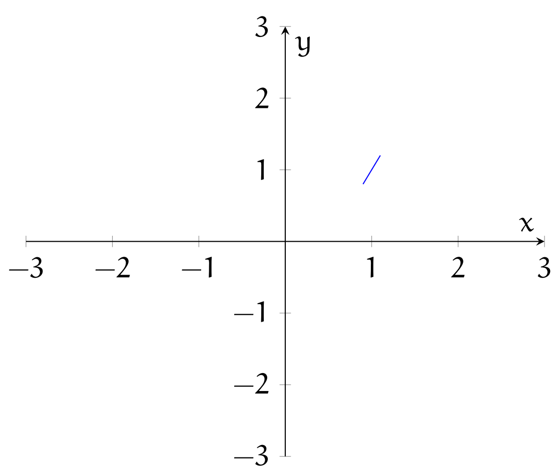


Figure 4.1: A solution to $y' = x + y$ that passes through $(1, 1)$ will have a slope of 2 at that point

Continuing, we want to choose coordinates that are easy to determine the slope. Notice that $y' = 0$ when $-x = y$, so let's go ahead and fill those ticks in (see figure 4.2):

We can repeat this process for all the coordinates shown, resulting in a slope field (see

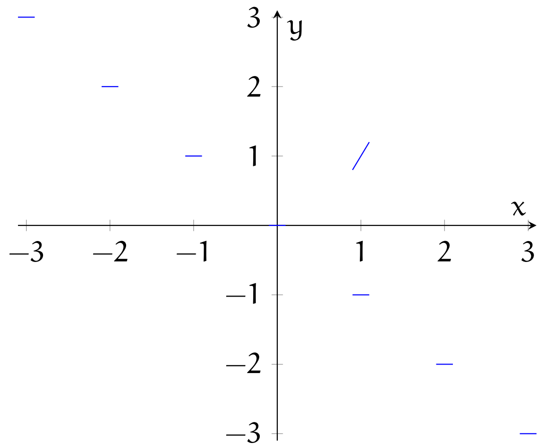


Figure 4.2: Solutions to $y' = x + y$ that lie on the line $y = -x$ will have a slope of 0.

figure 4.3).

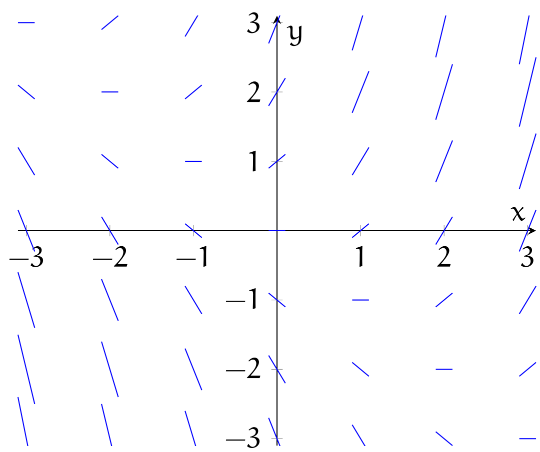


Figure 4.3: Slope field of $y' = x + y$

4.2 Sketching solutions on slope fields

If you are given an initial condition or a known point in the solution to the differential equation, you can begin sketching a curve on the slope field. Start at the given point and draw parallel to the nearby slopes. For example, suppose we know that particular solution to $y' = x + y$ passes through the point $(1, 0)$. Begin by extending the dash at $(1, 0)$ (see figure 4.4), changing the slope of your sketched solution to be approximately parallel to the nearby slopes (see figure 4.5).

While this method doesn't yield an exact, formulaic solution to the differential equation, it does allow us to visualize solutions and generally describe the behavior of any solutions. Sketching solutions in this way is logically similar to Euler's method for finding numerical

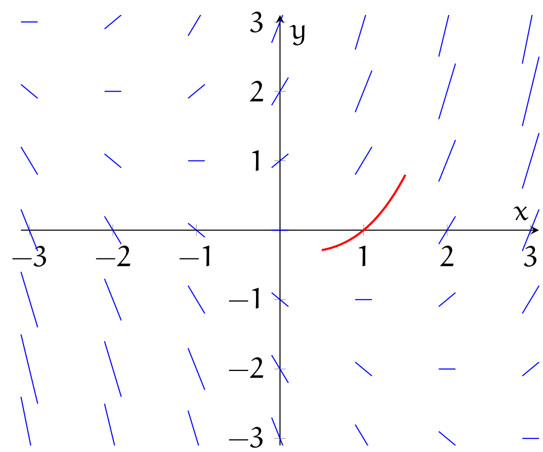


Figure 4.4: To begin sketching a solution to the differential equation, start at the point given as part of the solution

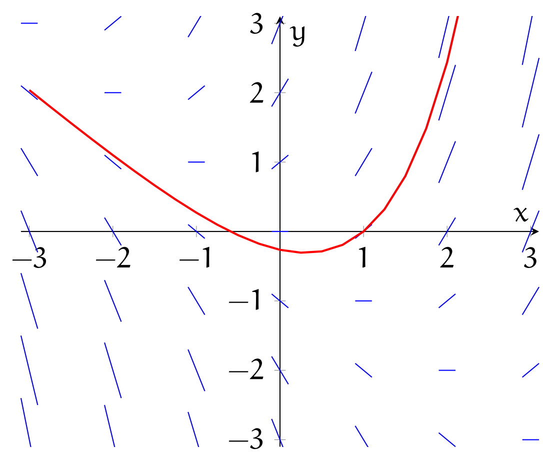


Figure 4.5: To sketch a solution to the differential equation, draw a function parallel to the nearby slopes that passes through the given point in the particular solution

approximations of solutions to differential equations, which we will discuss more in the next chapter.

4.3 Example: Application of Differential Equations to Electronics

Think back to the chapter on DC circuits. You learned that Ohm's Law relates voltage (electromotive force), current, and resistance for simple DC circuits:

$$V = IR$$

Simple resistors have a constant resistance, so once the voltage source (battery) is connected, the current is constant. There are other electronic components, such as inductors and capacitors, that behave differently. When current changes in an inductor, a voltage drop is induced across the inductor. This is described by the differential equation:

$$V = -L\frac{dI}{dt}$$

Where L is inductance, measured in henries (H), of the inductor. Consider, then, a circuit consisting of a constant-voltage battery, a fixed resistor, and an inductor (shown in figure 4.6). Since Kirchoff's Law states that the sum of the voltage drops across each component must equal the voltage supplied by the battery, we can write a differential equation to describe the circuit:

$$V = L\frac{dI}{dt} + RI$$

Where the current, I, is a function of time, t.

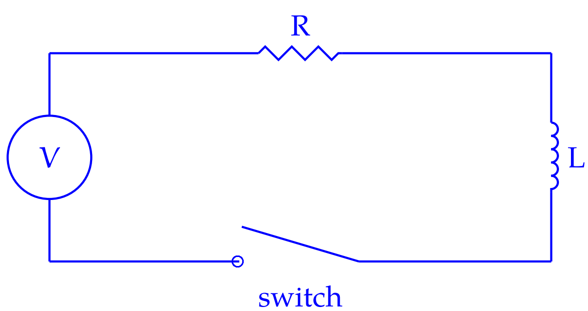


Figure 4.6: A simple circuit with a battery, resistor, inductor, and switch

Example: If the resistor is 12Ω, the inductance is 4H, and the battery supplies a constant voltage of 60V:

1. Draw a slope field for the differential equation describing the current in the circuit.
2. Describe the expected behavior of the current over a long period of time.

3. Identify any equilibrium solutions.
4. If the initial current at $t = 0$ is $I(0) = 0$, sketch the particular solution to the differential equation on the slope field.

Solution: Substituting the given values into the differential equation and rearranging to isolate $\frac{dI}{dt}$, we get $\frac{dI}{dt} = 15 - 3I$. Notice that the current is not dependent on time. When the slope is only dependent on the value of the function (as in this case), we call this an **autonomous differential equation**. This means that the slope will be the same of all values of t for a given I. The slope field is shown in figure 4.7.

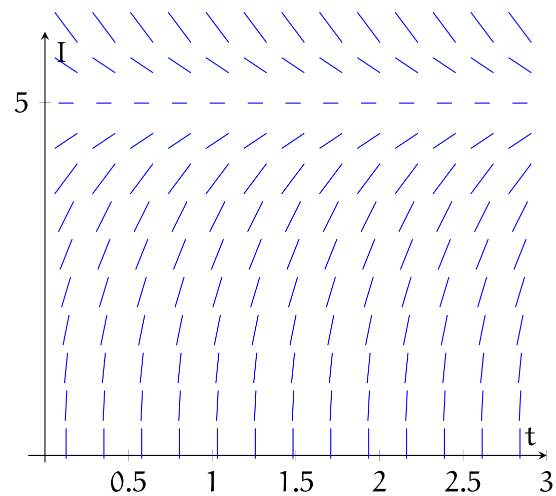


Figure 4.7: Slope field for the differential equation $\frac{dI}{dt} = 15 - 3I$

Examining the slope field, we see that the solutions tend towards $I(t) = 5$, which suggests that over an extended period of time, the current will approach 5 amperes. Similarly, if the initial current were 5 amperes, then the current would be constant at 5 amperes. Therefore, $I(t) = 5$ is an equilibrium solution. A sketch of the solution with $I(0) = 0$ is shown in figure 4.8.

4.4 Practice

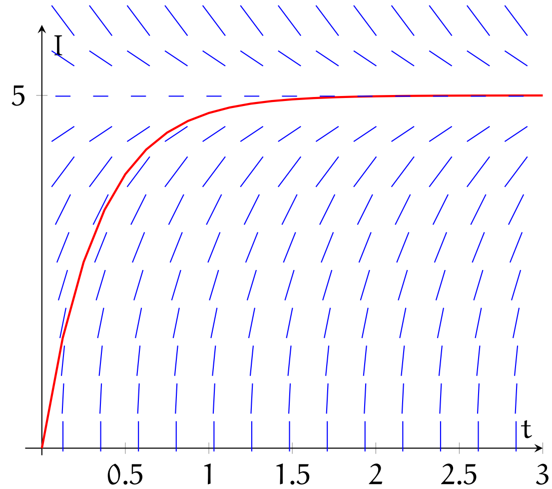


Figure 4.8: Slope field for the differential equation $\frac{dI}{dt} = 15 - 3I$

Exercise 13

Sketch the slope field for the differential equation $y' = x + y^2$. Use your slope field to sketch a solution that passes through the point $(0, 0)$.

Working Space

Answer on Page 50

APPENDIX A

Answers to Exercises

Answer to Exercise 1 (on page 6)

1. Let $u = 1 + \cos x$. Then $du = -\sin x\ dx$ and $-du = \sin x\ dx$. Substituting:

$$\int \sin x \sqrt{1+\cos x}\, dx = \int -\sqrt{u}\, du = -\frac{2}{3}u^{3/2} + C$$

$$= -\frac{2}{3}(1+\cos x)^{3/2} + C$$

Taking the derivative:

$$\frac{d}{dx}\left[-\frac{2}{3}(1+\cos x)^{3/2} + C\right] = -\frac{2}{3}\left[\frac{d}{dx}(1+\cos x)^{3/2} + \frac{d}{dx}C\right]$$

$$= -\frac{2}{3}\left[\frac{3}{2}(1+\cos x)^{1/2} \cdot \frac{d}{dx}(1+\cos x)\right] = -1\sqrt{1+\cos x}\cdot(-\sin x) = \sin x\sqrt{1+\cos x}$$

2. Let $u = \pi/x$. Then $du = (-\pi/x^2)dx$ and $-du/\pi = (1/x^2)dx$. Substituting:

$$\int \frac{\cos(\pi/x)}{x^2}\, dx = -\frac{1}{\pi}\int \cos u\, du = -\frac{1}{\pi}\sin u + C = -\frac{1}{\pi}\sin(\pi/x) + C$$

Taking the derivative:

$$\frac{d}{dx}\left[-\frac{1}{\pi}\sin(\pi/x) + C\right] = -\frac{1}{\pi}\left[\frac{d}{dx}\sin(\pi/x) + \frac{d}{dx}C\right]$$

$$= -\frac{1}{\pi}\left[\cos(\pi/x)\cdot\frac{d}{dx}\left(\frac{\pi}{x}\right)\right] = -\frac{1}{\pi}\left[\cos(\pi/x)\cdot\left(\frac{-\pi}{x^2}\right)\right]$$

$$= \frac{\cos(\pi/x)}{x^2}$$

3. Let $u = 9 - x^3$. Then $du = -3x^2\ dx$ and $-\frac{2}{3}du = 2x^2\ dx$. Substituting:

$$\int 2x^2\left(9-x^3\right)^{2/3}\ dx = -\frac{2}{3}\int (u)^{2/3}\ du$$

$$= -\frac{2}{3}\left[\frac{3}{5}u^{5/3} + C\right] = -\frac{2}{5}\left(9-x^3\right)^{5/3} + C$$

Taking the derivative:

$$\frac{d}{dx}\left[-\frac{2}{5}\left(9-x^3\right)^{5/3}+C\right] = -\frac{2}{5}\left[\frac{d}{dx}\left(9-x^3\right)^{5/3}\right]+\frac{d}{dx}C$$

$$= -\frac{2}{5}\left[\frac{5}{3}\left(9-x^3\right)^{2/3}\cdot\frac{d}{dx}\left(9-x^3\right)\right] = -\frac{2}{3}\left(9-x^3\right)^{2/3}\left(-3x^2\right) = 2x^2\left(9-x^3\right)^{2/3}$$

4. Let $u = 1+x$. Then $du = dx$. Additionally, $u-1 = x$ and $x^2 = (u-1)^2$. Substituting:

$$\int 3x^2\sqrt{1+x}\,dx = \int 3\left(u-1\right)^2\sqrt{u}\,du = 3\int\left(u^2-2u+1\right)\sqrt{u}\,du$$

$$= 3\int u^{5/2}-2u^{3/2}+u^{1/2}\,du = 3\left[\frac{2}{7}u^{7/2}-\frac{2\cdot 2}{5}u^{5/2}+\frac{2}{3}u^{3/2}+C\right]$$

$$= \frac{6}{7}u^{7/2}-\frac{12}{5}u^{5/2}+2u^{3/2}+C$$

$$= \frac{6}{7}\left(1+x\right)^{7/2}-\frac{12}{5}\left(1+x\right)^{5/2}+2\left(1+x\right)^{3/2}+C$$

Taking the derivative:

$$\frac{d}{dx}\left[\frac{6}{7}\left(1+x\right)^{7/2}-\frac{12}{5}\left(1+x\right)^{5/2}+2\left(1+x\right)^{3/2}+C\right]$$

$$= \frac{6}{7}\left(\frac{7}{2}\left(1+x\right)^{5/2}\right)-\frac{12}{5}\left(\frac{5}{2}\left(1+x\right)^{3/2}\right)+2\left(\frac{3}{2}\left(1+x\right)^{1/2}\right)$$

$$= 3\left(1+x\right)^{5/2}-6\left(1+x\right)^{3/2}+3\left(1+x\right)^{1/2}$$

$$= \left[3\left(1+x\right)^2-6\left(1+x\right)+3\right]\left(1+x\right)^{1/2}$$

$$= \left[3\left(1+2x+x^2\right)-6-6x+3\right]\sqrt{1+x}$$

$$= \left[3+6x+3x^2-6-6x+3\right]\sqrt{1+x} = 3x^2\sqrt{1+x}$$

5. Let $u = x^3-1$. Then $du = 3x^2\,dx$. Substituting:

$$\int\frac{3x^2}{x^3-1}\,dx = \int\frac{1}{u}\,du = \ln u + C$$

$$= \ln\left(x^3-1\right)+C$$

Taking the derivative:

$$\frac{d}{dx}\left[\ln\left(x^3-1\right)+C\right] = \frac{d}{dx}\ln\left(x^3-1\right)+\frac{d}{dx}C$$

$$= \frac{1}{x^3 - 1} \cdot \left[\frac{d}{dx} \left(x^3 - 1 \right) \right] = \frac{3x^2}{x^3 - 1}$$

Answer to Exercise 2 (on page 9)

1. Let $g(x) = u = \sin x$. Then $du = \cos x \, dx$. Additionally, $g(0) = \sin 0 = 0$ and $g(\pi/2) = \sin(\pi/2) = 1$. Substituting and changing the limits of integration:

$$\int_0^{\pi/2} \cos x \sin(\sin x) \, dx = \int_0^1 \sin u \, du = -\cos u|_0^1 = \cos 0 - \cos 1 = 1 - \cos 1$$

2. Let $g(x) = u = 1 + 2x$. Then $du = 2 \, dx$ and $\frac{du}{2} = dx$. Additionally, $g(0) = 1$ and $g(13) = 1 + 2(13) = 27$. Substituting and changing the limits of integration:

$$\int_0^{13} \frac{1}{\sqrt[3]{(1+2x)^2}} \, dx = \frac{1}{2} \int_1^{27} \frac{1}{\sqrt[3]{u^2}} \, du = \frac{1}{2} \int_1^{27} u^{-2/3} \, du$$

$$= \frac{1}{2} \left[3u^{1/3} \right]_1^{27} = \frac{3}{2} \left[\sqrt[3]{27} - \sqrt[3]{1} \right] = \frac{3}{2} (3 - 1) = 3$$

3. Let $g(x) = u = 1/x$. Then $du = (-1/x^2) dx$ and $-du = dx/x^2$. Additionally, $g(1) = 1$ and $g(2) = 1/2$. Substituting and changing the limits of integration:

$$\int_1^2 \frac{e^{1/x}}{x^2} \, dx = -\int_1^{1/2} e^u \, du = \int_{1/2}^1 e^u \, du$$

$$= e^u|_{1/2}^1 = e - \sqrt{e}$$

4. Let $g(x) = u = \cos x$. Then $du = -\sin x \, dx$ and $-du = \sin x \, dx$. Additionally, $g(0) = \cos 0 = 1$ and $g(\pi/6) = \cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}$. Substituting and changing the limits of integration:

$$\int_0^{\pi/6} \frac{\sin x}{\cos^2 x} \, dx = -\int_1^{\sqrt{3}/2} \frac{1}{u^2} \, du = \int_{\sqrt{3}/2}^1 \frac{1}{u^2} \, du$$

$$= -\frac{1}{u}\Big|_{\sqrt{3}/2}^1 = \frac{2}{\sqrt{3}} - 1 = \frac{2\sqrt{3} - 3}{3}$$

5. Let $g(x) = u = 1 + 2x$. Then $du = 2 \, dx$ and $\frac{du}{2} = dx$. And if $u = 1 + 2x$, then $x = \frac{u-1}{2}$. Additionally, $g(0) = 1$ and $g(4) = 9$. Substituting and changing the limits of integration:

$$\int_0^4 \frac{x}{\sqrt{1+2x}} \, dx = \frac{1}{2} \int_1^9 \frac{\frac{u-1}{2}}{\sqrt{u}} \, du = \frac{1}{4} \int_1^9 \frac{u-1}{\sqrt{u}} \, du$$

$$= \frac{1}{4}\int_1^9 \sqrt{u} - \frac{1}{\sqrt{u}}\, du = \frac{1}{4}\left[\frac{2}{3}u^{3/2} - 2u^{1/2}\right]_1^9$$

$$= \frac{1}{4}\left[\frac{2}{3}\left(9^{3/2} - 1^{3/2}\right) - 2\left(\sqrt{9} - \sqrt{1}\right)\right] = \frac{1}{4}\left[\frac{2}{3}(26) - 2(2)\right]$$

$$= \frac{1}{4}\left[\frac{52}{3} - 4\right] = \frac{1}{4}\left[\frac{52-12}{3}\right] = \frac{1}{4}\left[\frac{40}{3}\right] = \frac{10}{3}$$

Answer to Exercise 3 (on page 14)

Recall that for a polar function, $\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$. We also know that $x = r\cos\theta$, which equals $[1 + 2\sin\theta] \cdot \cos\theta = \cos\theta + 2\sin\theta\cos\theta$ in this case. We also know that $y = r \cdot \sin\theta$, which equals $[1 + 2\sin\theta] \cdot \sin\theta = \sin\theta + 2\sin^2\theta$ in this case. Taking the derivative with respect to θ:

$$\frac{dy}{d\theta} = \frac{d}{d\theta}\left[\sin\theta + 2\sin^2\theta\right]$$
$$\frac{dy}{d\theta} = \cos\theta + 4\sin\theta\cos\theta$$

And

$$\frac{dx}{d\theta} = \frac{d}{d\theta}\left[\cos\theta + 2\sin\theta\cos\theta\right]$$
$$\frac{dx}{d\theta} = -\sin\theta - 2\sin^2\theta + 2\cos^2\theta$$

Evaluating each at $\theta = 0$:

$$\frac{dy}{d\theta} = \cos 0 + 4\sin 0\cos 0 = 1 + 0 = 1$$

$$\frac{dx}{d\theta} = -\sin 0 - 2\sin^2 0 + 2\cos^2 0 = 0 - 0 + 2 = 2$$

Therefore, $\frac{dr}{d\theta} = \frac{dy/d\theta}{dx/d\theta} = \frac{1}{2}$

Answer to Exercise 4 (on page 15)

1. Answer: slope $= -\frac{\sqrt{3}}{3}$ and an equation for the tangent line is $y - \frac{\sqrt{3}}{6} = -\frac{\sqrt{3}}{3}\left(x - \frac{1}{2}\right)$.

Explanation: $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\frac{d}{d\theta} r \cdot \sin\theta}{\frac{d}{d\theta} r \cdot \cos\theta} = \frac{\frac{d}{d\theta}\left(\frac{2}{3}\cos\theta\sin\theta\right)}{\frac{d}{d\theta}\left(\frac{2}{3}\cos^2\theta\right)} = \frac{\frac{2}{3}(\cos^2\theta - \sin^2\theta)}{\frac{2}{3}(-2\cos\theta\sin\theta)} = \frac{\sin^2\theta - \cos^2\theta}{2\cos\theta\sin\theta}$

Substituting $\theta = \frac{\pi}{6}$: $\frac{dy}{dx} = \frac{\sin^2\pi/6 - \cos^2\pi/6}{2\cos\pi/6\sin\pi/6} = \frac{(1/2)^2 - \left(\sqrt{3}/2\right)^2}{2\left(\sqrt{3}/2\right)(1/2)} = \frac{1/4 - 3/4}{\sqrt{3}/2} = \frac{-1/2}{\sqrt{3}/2} = \frac{-1}{2} \cdot \frac{2}{\sqrt{3}} = -\frac{\sqrt{3}}{3}$

To write an equation for a line, we need a Cartesian point. First, we find r at $\theta = \frac{\pi}{6}$: $r = \frac{2}{3}\cos\left(\frac{\pi}{6}\right) = \frac{2}{3} \cdot \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{3}$. So the point the tangent passes through is the polar coordinate $\left(\frac{\sqrt{3}}{3}, \frac{\pi}{6}\right)$. We convert this to Cartesian coordinates: $x = r\cos\theta = \frac{\sqrt{3}}{3} \cdot \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{3} \cdot \frac{\sqrt{3}}{2} = \frac{3}{6} = \frac{1}{2}$ And $y = r\sin\theta = \frac{\sqrt{3}}{3} \cdot \sin\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{3} \cdot \frac{1}{2} = \frac{\sqrt{3}}{6}$

So, an equation for a line with slope $-\frac{\sqrt{3}}{3}$ that passes through Cartesian coordinate $\left(\frac{1}{2}, \frac{\sqrt{3}}{6}\right)$ is: $y - \frac{\sqrt{3}}{6} = -\frac{\sqrt{3}}{3}\left(x - \frac{1}{2}\right)$

2. Answer: slope $= \frac{2}{\pi}$ and an equation for the tangent line is $y - \frac{1}{\pi} = \frac{2}{\pi}x$

   Explanation: $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\frac{d}{d\theta}\left(\frac{\sin\theta}{2\theta}\right)}{\frac{d}{d\theta}\left(\frac{\cos\theta}{2\theta}\right)} = \frac{\frac{\theta\cos\theta - \sin\theta}{2\theta^2}}{-\frac{\theta\sin\theta + \cos\theta}{2\theta^2}} = \frac{\sin\theta - \theta\cos\theta}{\theta\sin\theta + \cos\theta}$.

   Substituting $\theta = \frac{\pi}{2}$: $\frac{dy}{dx} = \frac{\sin\frac{\pi}{2} - \left(\frac{\pi}{2}\right)\cos\frac{\pi}{2}}{\left(\frac{\pi}{2}\right)\sin\frac{\pi}{2} + \cos\frac{\pi}{2}} = \frac{1 - \left(\frac{\pi}{2}\right)\cdot 0}{\left(\frac{\pi}{2}\right)\cdot 1 + 0} = \frac{1}{\frac{\pi}{2}} = \frac{2}{\pi}$

   To write an equation for a line, we need a Cartesian point. First, we find r at $\theta = \frac{\pi}{2}$: $r = \frac{1}{2\theta} = \frac{1}{2\frac{\pi}{2}} = \frac{1}{\pi}$. So the tangent line passes through the point with polar coordinates $\left(\frac{1}{\pi}, \frac{\pi}{2}\right)$. We convert this to Cartesian coordinates: $x = r \cdot \cos\theta = \frac{1}{\pi} \cdot \cos\frac{\pi}{2} = \frac{1}{\pi} \cdot 0 = 0$ and $y = r \cdot \sin\theta = \frac{1}{\pi} \cdot \sin\frac{\pi}{2} = \frac{1}{\pi} \cdot 1 = \frac{1}{\pi}$.

   So, an equation for a line with slope $\frac{2}{\pi}$ that passes through Cartesian coordinate $\left(0, \frac{1}{\pi}\right)$ is $y - \frac{1}{\pi} = \frac{2}{\pi}x$

3. Answer: slope $= -\frac{5}{\sqrt{3}}$ and an equation for the tangent line is $y - \frac{\sqrt{3}}{4} = \left(-\frac{5}{\sqrt{3}}\right)\left(x + \frac{1}{4}\right)$.

   Explanation: $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\frac{d}{d\theta}[(2+3\cos\theta)\cdot\sin\theta]}{\frac{d}{d\theta}[(2+3\cos\theta)\cdot\cos\theta]} = \frac{\cos\theta(2+3\cos\theta) - 3\sin^2\theta}{(2+3\cos\theta)\cdot(-\sin\theta) + \cos\theta(-3\sin\theta)} = \frac{\cos\theta(2+3\cos\theta) - 3\sin^2\theta}{-2\sin\theta(1+3\cos\theta)}$.

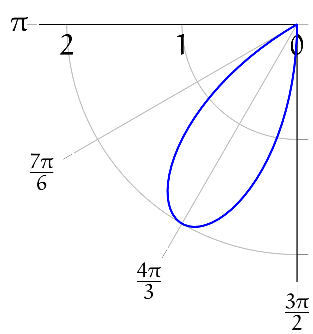
   Substituting $\theta = \frac{2\pi}{3}$: $\frac{dy}{dx} = \frac{\cos\frac{2\pi}{3}\left(2+3\cos\frac{2\pi}{3}\right) - 3\sin^2\frac{2\pi}{3}}{-2\sin\frac{2\pi}{3}\left(1+3\cos\frac{2\pi}{3}\right)} = \frac{\left(-\frac{1}{2}\right)\left(2+3\left(-\frac{1}{2}\right)\right) - 3\left(\frac{\sqrt{3}}{2}\right)^2}{-2\left(\frac{\sqrt{3}}{2}\right)\left(1+3\left(-\frac{1}{2}\right)\right)} = \frac{\left(-\frac{1}{2}\right)\left(2-\frac{3}{2}\right) - 3\left(\frac{3}{4}\right)}{-\sqrt{3}\left(1-\frac{3}{2}\right)} = \frac{\left(-\frac{1}{2}\right)\left(\frac{1}{2}\right) - \frac{9}{4}}{-\sqrt{3}\left(-\frac{1}{2}\right)} = \frac{-\frac{1}{4} - \frac{9}{4}}{\frac{\sqrt{3}}{2}} = -\frac{\frac{10}{4}}{\frac{\sqrt{3}}{2}} = -\frac{10\cdot 2}{4\cdot\sqrt{3}} = -\frac{5}{\sqrt{3}}$

   To write an equation for a tangent line, we need a Cartesian point. First, we find r at $\theta = \frac{2\pi}{3}$: $r = 2 + 3\cos\frac{2\pi}{3} = 2 + 3\left(\frac{-1}{2}\right) = 2 - \frac{3}{2} = \frac{1}{2}$. So the tangent line passes through polar coordinate $\left(\frac{1}{2}, \frac{2\pi}{3}\right)$. We convert this to Cartesian coordinates: $x = r\cos\theta = \frac{1}{2}\cos\frac{2\pi}{3} = \frac{1}{2}\left(-\frac{1}{2}\right) = -\frac{1}{4}$ and $y = r\sin\theta = \frac{1}{2}\sin\frac{2\pi}{3} = \frac{1}{2}\left(\frac{\sqrt{3}}{2}\right) = \frac{\sqrt{3}}{4}$.

   So, an equation with slope $-\frac{\sqrt{5}}{3}$ that passes through the Cartesian coordinate $\left(-\frac{1}{4}, \frac{\sqrt{3}}{4}\right)$ is: $y - \frac{\sqrt{3}}{4} = \left(-\frac{5}{\sqrt{3}}\right)\left(x + \frac{1}{4}\right)$

Answer to Exercise 5 (on page 22)

We know the area of the circle is $\pi r^2 = \pi(2)^2 = 4\pi$. To find the area of the shaded regions, we need to subtract the area of the trefoil from the area of the circle. The trefoil has three equal areas. We can find the area of the leaf that is formed on the interval $\frac{\pi}{6} \leq \theta \leq \frac{\pi}{2}$ (see figure below).



The area of one leaf of the trefoil is given by $\frac{1}{2}\int_{\pi/6}^{\pi/2} [2\cos 3\theta]^2 \, d\theta$. Using a calculator, the area of one leaf is ≈ 1.0472. The area of the circle is given by $\pi r^2 = \pi(2)^2 \approx 12.5664$. The area of the shaded region is the area of the circle minus three times the area of a single leaf: $12.5664 - 3 \cdot 1.0472 = 9.4248 \approx 9.425$.

Answer to Exercise 6 (on page 23)

1. Answer: $A = e^{\pi/8}\left(e^{\pi/8} - 1\right)$

   Explanation: $A = \frac{1}{2}\int_{\pi/4}^{\pi/2} \left[e^{\theta/2}\right] \, , d\theta = \frac{1}{2} \cdot 2\left[e^{\theta/2}\right]_{\theta=\pi/4}^{\theta=\pi/2} = e^{\pi/4} - e^{\pi/8} = e^{\pi/8}\left(e^{\pi/8} - 1\right) \approx$ 0.712

2. Answer: The area is $\frac{1}{2}\left[e^{\pi/2} - e^{\pi/4}\right] \approx 1.309$

   Explanation: $A = \frac{1}{2}\int_{\pi/4}^{\pi/2} \left[e^{\theta/2}\right]^2 \, d\theta = \frac{1}{2}\int_{\pi/4}^{\pi/2} e^{\theta} \, d\theta = \frac{1}{2}e^{\theta}\Big|_{\theta=\pi/4}^{\theta=\pi/2} = \frac{1}{2}\left[e^{\pi/2} - e^{\pi/4}\right] \approx$ 1.309

3. Answer: $A = \frac{41}{4}\pi \approx 32.201$

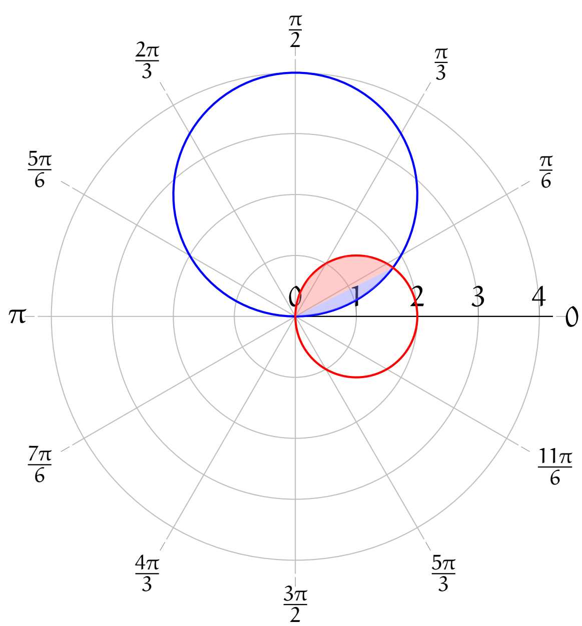
   Explanation: $A = \frac{1}{2}\int_{-\pi/2}^{\pi/2} [4 + 3\sin\theta]^2 \, d\theta = \frac{1}{2}\int_{-\pi/2}^{\pi/2} \left[16 + 24\sin\theta + 9\sin^2\theta\right] \, d\theta =$

$\int_{-\pi/2}^{\pi/2} 8\,d\theta + 12\int_{-\pi/2}^{\pi/2} \sin\theta\,d\theta + \frac{9}{2}\int_{-\pi/2}^{\pi/2} \sin^2\theta\,d\theta = [8\theta]_{\theta=-\pi/2}^{\theta=\pi/2} + 12\,[-\cos\theta]_{\theta=-\pi/2}^{\theta=\pi/2} + \frac{9}{2}\int_{-\pi/2}^{\pi/2} \frac{1-\cos 2\theta}{2}\,d\theta = 8\left[\left(\frac{\pi}{2}\right) - \left(\frac{-\pi}{2}\right)\right] + 12\left[\left(-\cos\frac{\pi}{2}\right) - \left(-\cos\frac{-\pi}{2}\right)\right] + \frac{9}{4}\int_{-\pi/2}^{\pi/2} 1\,d\theta - \frac{9}{4}\int_{-\pi/2}^{\pi/2} \cos 2\theta\,d\theta = 8\pi + 12\,(0-0) + \frac{9}{4}\,[\theta]_{\theta=-\pi/2}^{\theta=\pi/2} - \frac{9}{4}\left[\frac{1}{2}\sin 2\theta\right]_{\theta=-\pi/2}^{\theta=\pi/2} = 8\pi + \frac{9}{4}\left[\left(\frac{\pi}{2}\right) - \left(-\frac{\pi}{2}\right)\right] - \frac{9}{8}\left[\sin\left(2\cdot\frac{\pi}{2}\right) - \sin\left(2\cdot -\frac{\pi}{2}\right)\right] = 8\pi + \frac{9}{4}\pi - \frac{9}{8}\,[\sin(\pi) - \sin(-\pi)] = \frac{41}{4}\pi - \frac{9}{8}\,[0-(-0)] = \frac{41}{4}\pi \approx 32.201$

Answer to Exercise 7 (on page 24)

Answer: The area between the circles is approximately 0.96174.

Explanation: Examining the graph, we see that the region we are interested in is the area within $r = 4\sin\theta$ from $\theta = 0$ to $\theta = \theta_i$ plus the area within $r = 2\cos\theta$ from $\theta = \theta_i$ to $\theta = \frac{\pi}{2}$, where θ_i is the angle where the two curves intersect. Examine the graph below to see why this is true.



Setting the equations equal to each other to find θ_i:

$$4\sin\theta_i = 2\cos\theta_i$$

$$\frac{\sin\theta_i}{\cos\theta_i} = \tan\theta_i = \frac{2}{4}$$

$$\theta_i = \arctan 1/2 \approx 0.464$$

So, the total area between the circles is:

$$\frac{1}{2}\int_0^{\theta_i} [4\sin\theta]^2 \, d\theta + \frac{1}{2}\int_{\theta_i}^{\pi/2} [2\cos\theta]^2 \, d\theta$$

$$= 8\int_0^{\theta_i} \sin^2\theta \, d\theta + 2\int_{\theta_i}^{\pi/2} \cos^2\theta \, d\theta$$

$$= 8\int_0^{\theta_i} \frac{1}{2}(1-\cos 2\theta) \, d\theta + 2\int_{\theta_i}^{\pi/2} \frac{1}{2}(1+\cos 2\theta) \, d\theta$$

$$= 4\int_0^{\theta_i} (1-\cos 2\theta) \, d\theta + \int_{\theta_i}^{\pi/2} (1+\cos 2\theta) \, d\theta$$

$$= 4\left[\theta - \frac{1}{2}\sin 2\theta\right]_{\theta=0}^{\theta=\theta_i} + \left[\theta + \frac{1}{2}\sin 2\theta\right]_{\theta=\theta_i}^{\theta=\pi/2}$$

$$= 4\left[(\theta_i - 0) - \frac{1}{2}(\sin 2\theta_i - \sin 0)\right] + \left[\left(\frac{\pi}{2} - \theta_i\right) + \frac{1}{2}(\sin\pi - \sin 2\theta_i)\right]$$

$$= 4\left[\theta_i - \frac{1}{2}\sin 2\theta_i\right] + \frac{\pi}{2} - \theta_i - \frac{1}{2}\sin 2\theta_i$$

$$= 4\theta_i - 2\sin 2\theta_i + \frac{\pi}{2} - \theta_i - \frac{1}{2}\sin 2\theta_i = 3\theta_i - \frac{5}{2}\sin 2\theta_i + \frac{\pi}{2}$$

Substituting $\theta_i = \arctan 1/2 \approx 0.464$:

$$= 3(0.464) - \frac{5}{2}\sin 0.927 + \frac{\pi}{2} \approx 0.96174$$

Answer to Exercise 8 (on page 29)

1. 4200
2. Logically, we can say that the population will increase if it is below the carrying capacity (that is, $P < 4200$), but we can also prove it mathematically: $\frac{dP}{dt} < 0 \rightarrow 1.2P\left(1 - \frac{P}{4200}\right) < 0 \rightarrow P\left(1 - \frac{P}{4200}\right) < 0$. Since we are talking about population, we can assume that $P > 0$ and continue: $1 - \frac{P}{4200} < 0 \rightarrow 1 < \frac{P}{4200} \rightarrow 4200 < P$, which is the result we expected.
3. Similarly, we know the population should be decreasing when P is greater than the carrying capacity of 4200.
4. The equilibrium solutions can be found by setting $\frac{dP}{dt} = 0$ and solving. The solutions are $P(t) = 0$ and $P(t) = 4200$.

Answer to Exercise 9 (on page 29)

Recall that logistic differential equations are of the form $\frac{dy}{dt} = ky(1 - \frac{y}{m})$ where y is a function and t is the independent variable. (e) is the only logistic differential equation, with $m = 1$.

Answer to Exercise 10 (on page 31)

$$\begin{aligned}
\frac{dy}{dx}dx &= \frac{3x^2}{2y + \sin y}dx \\
(2y + \sin y)(dy) &= \frac{3x^2}{2y + \sin y}(2y + \sin y)(dx) \\
(2y + \sin y)dy &= (3x^2)dx \\
\int 2y\, dy + \int \sin y\, dy &= \int 3x^2\, dx \\
y^2 - \cos y &= x^3 + C
\end{aligned}$$

Answer to Exercise 11 (on page 32)

1. Since $\frac{dB}{dt}$ depends only on B, we can use the given masses to find the rate of growth for each mass. $\frac{dB}{dt}(40) = \frac{1}{5}(100 - 40) = \frac{1}{5}(60) = 12$ and $\frac{dB}{dt}(70) = \frac{1}{5}(100 - 70) = \frac{1}{5}(30) = 6$. Since $\frac{dB}{dt}$ is greater when $B = 40$, the baby bird is gaining mass faster when it has a mass of 40 grams.

2. $\frac{d^2B}{dt^2} = \frac{d}{dt}\left(\frac{dB}{dt}\right) = \frac{d}{dt}\left[\frac{1}{5}(100 - B)\right] = \frac{1}{5}\left(-\frac{dB}{dt}\right) = \frac{-1}{5}\left[\frac{1}{5}(100 - B)\right] = -\frac{1}{25}(100 - B)$. For $20 < B < 100$, $\frac{d^2B}{dt^2} < 0$ and the graph of B should be concave down. The graph shown has a concave up portion, so it cannot represent $B(t)$.

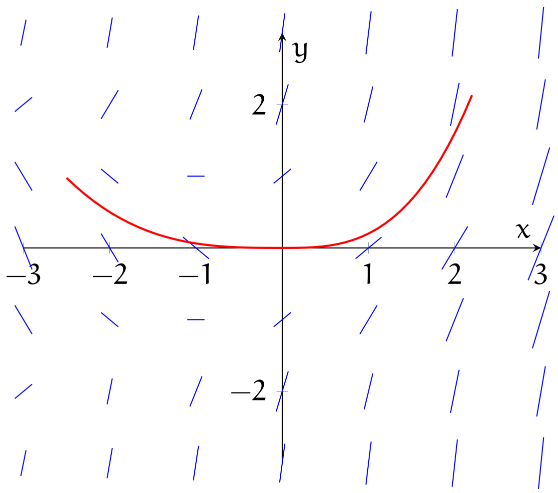
3. $\frac{dB}{dt} = \frac{1}{5}(100 - B) \to \frac{dB}{100-B} = \frac{1}{5}dt \to \int(100 - B)\, dB = \int \frac{1}{5}\, dt \to -\ln 100 - B = \frac{t}{5} + C \to e^{\frac{-t}{5}+C} = 100 - B \to ke^{\frac{-t}{5}} = 100 - B \to B(t) = 100 - ke^{\frac{-t}{5}}$. Setting $B(0) = 20$ to find k: $20 = 100 - ke^0 \to 20 = 100 - k \to k = 80$. So, the particular solution is $B(t) = 100 - 80e^{\frac{-t}{5}}$

Answer to Exercise 12 (on page 34)

(A). (a), (b), and (c) are all separable equations. But only the solution to A is linear

$(P(t) = 200t + C)$. (d) is logarithmic, or natural growth and (e) is also not linear.

Answer to Exercise 13 (on page 40)



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