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Fourier Series and Signals

A piano and a violin can play the same note, at the same loudness, and you can still tell them apart with your eyes closed. In the Sound chapter, you learned why: each instrument adds its own mix of overtones to the fundamental frequency. In 1807, Joseph Fourier made a much bolder claim. *Every* repeating signal, no matter how jagged, can be built by adding up sine and cosine waves. A square wave, a heartbeat, the hum of a refrigerator: all of them are sums of pure tones.

Fourier's idea now runs inside almost every device you own. Your phone uses it to compress music, your earbuds use it to cancel noise, and your speakers use it to send each sound to the right driver. This chapter shows how to find the recipe of sines inside a signal, and what you can do once you have it.

1.1 Building a Wave from Sines

Start with a square wave: a signal that jumps between -1 and 1 , like a switch flipping on and off. It is about as far from a smooth sine wave as a signal can be. Yet look at what happens when we add these sine waves:

$$\frac{4}{\pi} \left(\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \frac{\sin 7x}{7} + \dots \right)$$

```

1 import numpy as np
2 import matplotlib.pyplot as plt
3
4 x = np.linspace(-np.pi, np.pi, 1000)
5
6 def partial_sum(x, terms):
7     """The first `terms` nonzero terms of the square wave's Fourier series."""
8     total = np.zeros_like(x)
9     for k in range(terms):
10         n = 2 * k + 1 # only odd n appear
11         total += 4 / (n * np.pi) * np.sin(n * x)
12     return total
13
14 plt.plot(x, np.sign(x), "k", linewidth=2, label="square wave")
15 for terms in [1, 3, 15]:
16     plt.plot(x, partial_sum(x, terms), label=f"{terms} term" + ("s" if terms > 1
17     ↪ else ""))
18 plt.xlabel("x")
19 plt.legend()

```

```
plt.savefig("square_wave.png")
```

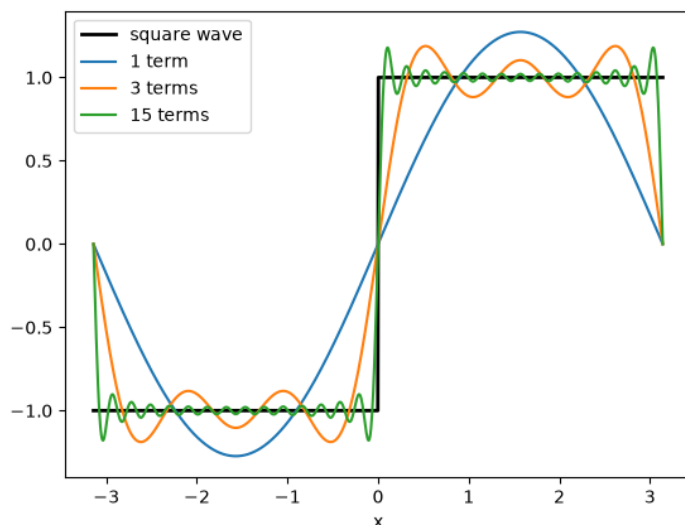


Figure 1.1: Adding more sine waves brings the sum closer and closer to a square wave.

One sine wave is a rough sketch. Three already have the flat top and the steep sides. Fifteen hug the square wave almost everywhere. Where did the numbers $\frac{4}{\pi}$, $\frac{1}{3}$, $\frac{1}{5}$, ... come from? That is what the rest of this section is about.

1.1.1 Fourier Series

A function f is *periodic* with period 2π if $f(x + 2\pi) = f(x)$ for every x . Its *Fourier series* is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

The numbers a_n and b_n are the *Fourier coefficients*. They say how much of each frequency is in the signal: b_3 , for example, is the amount of the wave $\sin 3x$, which wiggles three times as fast as $\sin x$.

1.1.2 The Key Fact: Orthogonality

Finding the coefficients rests on one fact about integrals. For positive whole numbers m and n ,

$$\int_{-\pi}^{\pi} \sin mx \sin nx \, dx = \begin{cases} 0 & m \neq n \\ \pi & m = n \end{cases}$$

The same is true with both sines replaced by cosines, and $\int_{-\pi}^{\pi} \sin mx \cos nx \, dx = 0$ always.

To see why, subtract the two sum formulas from the Trigonometric Identities chapter, $\cos(A - B) - \cos(A + B) = 2 \sin A \sin B$. So

$$\sin mx \sin nx = \frac{1}{2} [\cos((m - n)x) - \cos((m + n)x)]$$

For any nonzero whole number k , $\int_{-\pi}^{\pi} \cos kx \, dx = \left[\frac{\sin kx}{k} \right]_{-\pi}^{\pi} = 0$, because $\sin$ is zero at every multiple of π . When $m \neq n$, both cosines have a nonzero k , so the integral is 0. When $m = n$, the first cosine is $\cos 0 = 1$, and the integral is $\frac{1}{2}(2\pi) = \pi$. Finally, $\sin mx \cos nx$ is an odd function, so its integral over a symmetric interval is 0.

Sines and cosines of different frequencies “cancel out” when you multiply them and integrate, the same way perpendicular vectors have a dot product of zero. That is why mathematicians call them *orthogonal*.

1.1.3 Finding the Coefficients

Multiply both sides of the Fourier series by $\sin mx$ and integrate from $-\pi$ to π . By orthogonality, every term on the right vanishes except one, $b_m\pi$. The same trick with $\cos mx$ picks out a_m :

Fourier Coefficients

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx, \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx$$

Each integral measures how much f “looks like” one particular wave: it is large when f rises and falls in step with that wave and zero when they have nothing in common.

Example: Find the Fourier series of the square wave $f(x) = -1$ for $-\pi < x < 0$ and $f(x) = 1$ for $0 < x < \pi$.

Solution: The square wave is an odd function, and $\cos nx$ is even, so $f(x) \cos nx$ is odd

and every $a_n = 0$. For b_n , the two halves contribute equally:

$$b_n = \frac{1}{\pi} \left(\int_{-\pi}^0 -\sin nx \, dx + \int_0^{\pi} \sin nx \, dx \right) = \frac{2}{\pi} \int_0^{\pi} \sin nx \, dx = \frac{2}{\pi} \cdot \frac{1 - \cos n\pi}{n}$$

Since $\cos n\pi$ is -1 for odd n and 1 for even n , $b_n = \frac{4}{n\pi}$ when n is odd and $b_n = 0$ when n is even. That is exactly the recipe in the program above.

1.1.4 A Surprise Bonus

Set $x = \frac{\pi}{2}$, where the square wave equals 1. Since $\sin \frac{\pi}{2} = 1$, $\sin \frac{3\pi}{2} = -1$, $\sin \frac{5\pi}{2} = 1$, and so on,

$$1 = \frac{4}{\pi} \left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots \right) \implies \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

That is the series for $\arctan 1$ from the Taylor and Maclaurin Series chapter, reached by a completely different road.

1.1.5 The Gibbs Phenomenon

Look closely at figure 1.1: next to each jump, the partial sums overshoot. You might expect the overshoot to fade as you add terms. It does not. It gets narrower, but its peak stays near 1.179, about 9% of the jump, no matter how many terms you use. This is the *Gibbs phenomenon*. Engineers hear it as “ringing” when a sharp-edged signal passes through equipment that cuts off high frequencies.

Exercise 1 Orthogonality by Hand

Use the identity $\sin A \sin B = \frac{1}{2}[\cos(A - B) - \cos(A + B)]$ to show that $\int_{-\pi}^{\pi} \sin 2x \sin 3x \, dx = 0$.

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Answer on Page 49

Exercise 2 The Sawtooth Wave

The sawtooth wave is $f(x) = x$ for $-\pi < x < \pi$, repeated every 2π . Find its

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Fourier coefficients. (Hint: integrate by parts.)

Answer on Page 49

Exercise 3 Three Terms

Evaluate the first three nonzero terms of the square wave's series at $x = \frac{\pi}{2}$. Is the result above or below 1? Use figure 1.1 to explain.

Working Space

Answer on Page 49

1.2 Frequencies, Harmonics, and Timbre

Real signals do not have a period of 2π . A note with frequency f_1 repeats every $T = \frac{1}{f_1}$ seconds, and its Fourier series becomes

$$s(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(2\pi n f_1 t) + b_n \sin(2\pi n f_1 t))$$

Every term is a pure tone at a whole-number multiple of the *fundamental frequency* f_1 : the *harmonics* $f_1, 2f_1, 3f_1, \dots$. These are the overtones from the Sound chapter, and the coefficients are the recipe. An instrument's *timbre*, the quality that makes a violin sound like a violin, is the pattern of sizes of its Fourier coefficients.

A square wave contains only odd harmonics, which gives it a hollow, reedy sound, a bit like a clarinet. A sawtooth contains all of them and sounds bright and buzzy, which is why it is a favorite starting point for synthesizers.

1.2.1 The Complex Form

With Euler's formula from the previous chapter, cos and sin combine into one exponential, and the series becomes shorter:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}, \quad c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx$$

Each e^{inx} is a point spinning around the unit circle n times per period, and c_n is a complex number whose size is the amplitude and whose angle is the phase (how far the wave is shifted). Engineers and physicists almost always use this form.

Exercise 4 Harmonics of a Square Wave

An electronic keyboard plays a square-wave tone at 220 Hz. List the frequencies and relative amplitudes of its first four nonzero harmonics.

Working Space

Answer on Page 49

1.3 Sampled Signals and the FFT

A microphone does not hand a computer a formula. Digital audio stores *samples*: the height of the wave measured thousands of times per second. The *discrete Fourier transform* (DFT) does for a list of samples what the integrals did for a function: it finds the amplitude of each frequency. A clever way of computing it, the *fast Fourier transform* (FFT), made it practical; it is one of the most important algorithms ever written.

Here is a one-second signal made of a 440 Hz note (the A above middle C) and two overtones. The FFT pulls the recipe back out:

```

1 import numpy as np
2 import matplotlib.pyplot as plt
3
4 rate = 8000                                # samples per second
5 t = np.arange(0, 1, 1 / rate)              # one second of time
6
7 # A 440 Hz note with two overtones
8 signal = (1.0 * np.sin(2 * np.pi * 440 * t)
```

```

9         + 0.5 * np.sin(2 * np.pi * 880 * t)
10        + 0.25 * np.sin(2 * np.pi * 1320 * t))
11
12 spectrum = np.abs(np.fft.rfft(signal)) / (len(t) / 2) # amplitudes
13 freqs = np.fft.rfftfreq(len(t), 1 / rate) # in Hz
14
15 for f, a in zip(freqs, spectrum):
16     if a > 0.1:
17         print(f"{f:.0f} Hz: amplitude {a:.2f}")
18
19 fig, (top, bottom) = plt.subplots(2, 1)
20 top.plot(t[:160] * 1000, signal[:160])
21 top.set_xlabel("time (ms)")
22 bottom.plot(freqs, spectrum)
23 bottom.set_xlim(0, 2000)
24 bottom.set_xlabel("frequency (Hz)")
25 fig.tight_layout()
26 fig.savefig("spectrum.png")

```

440 Hz: amplitude 1.00
 880 Hz: amplitude 0.50
 1320 Hz: amplitude 0.25

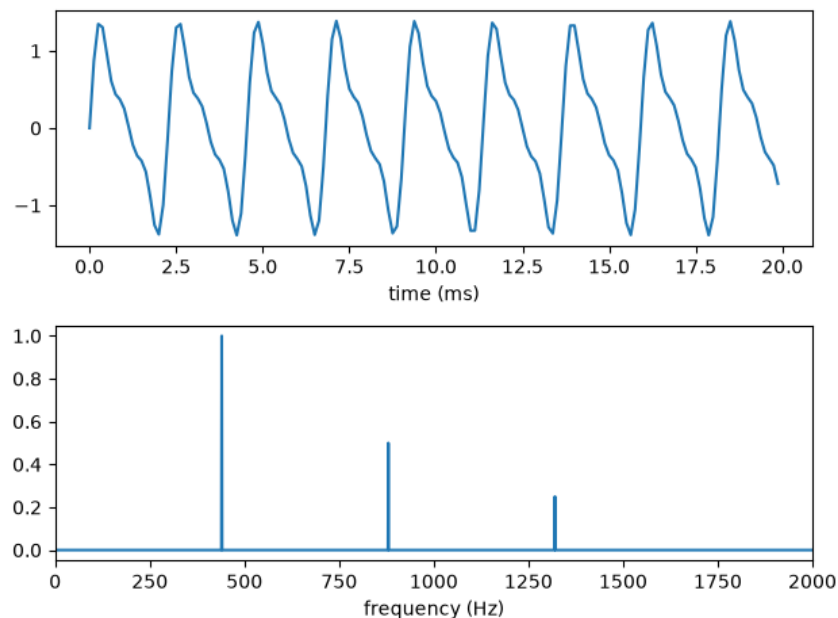


Figure 1.2: Top: 20 ms of the signal, a lopsided wave. Bottom: its *spectrum*, three spikes at the three frequencies inside it.

The top picture is hard to read; the bottom one tells you everything. Viewing a signal as a *spectrum*, amplitude against frequency, is often far more useful than viewing it against

time.

1.3.1 How Fast Must You Sample?

To capture a wave, you need more than two samples per cycle. A wave sampled too slowly is mistaken for a slower one, an effect called *aliasing* (it is why car wheels sometimes seem to spin backward in videos). So a sampling rate of r samples per second can only represent frequencies below $\frac{r}{2}$, the *Nyquist frequency*. CDs sample at 44,100 Hz, so they can hold frequencies up to 22,050 Hz, just above the roughly 20,000 Hz limit of human hearing.

Exercise 5 Sampling Rates

A phone records audio at 48,000 samples per second. What is the highest frequency it can capture? Could it record the 40,000 Hz call of a bat?

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1.4 Filters and Speaker Crossovers

Once a signal is a sum of frequencies, changing its sound is simple: turn some frequencies up and others down. A device that does this is a *filter*. A *low-pass* filter lets low frequencies through and weakens high ones; a *high-pass* filter does the opposite. The bass and treble knobs on a stereo are filters, and so is a graphic equalizer, which has a slider for each band of frequencies.

1.4.1 Why Speakers Need Crossovers

No single speaker cone is good at everything. To make deep bass, a speaker must push a lot of air, so it needs a big, heavy cone, called a *woofer*. But a heavy cone cannot shake back and forth 10,000 times a second. High notes need a small, light cone, a *tweeter*, and feeding a tweeter strong bass can tear it apart. So most speakers contain both, plus a *crossover*: a pair of filters that sends the low frequencies to the woofer and the high frequencies to the tweeter.

The simplest crossover uses just two parts, using the reactances from the Pendulums and Alternating Current chapter:

- A **capacitor** in series with the tweeter. Its reactance, $X_C = \frac{1}{2\pi fC}$, is huge at low frequencies, so it blocks the bass and lets the treble through: a high-pass filter.
- An **inductor** in series with the woofer. Its reactance, $X_L = 2\pi fL$, is huge at high frequencies, so it blocks the treble: a low-pass filter.

The handoff happens at the *crossover frequency* f_c , where the part's reactance equals the speaker's resistance R :

$$f_c = \frac{1}{2\pi RC} \quad (\text{tweeter}), \quad f_c = \frac{R}{2\pi L} \quad (\text{woofer})$$

For these first-order filters, the fraction of the signal each speaker receives at frequency f is

$$\text{woofer: } \frac{1}{\sqrt{1 + (f/f_c)^2}}, \quad \text{tweeter: } \frac{f/f_c}{\sqrt{1 + (f/f_c)^2}}$$

Example: Design a crossover at 2500 Hz for 8 Ω speakers.

Solution: For the tweeter, $C = \frac{1}{2\pi(8)(2500)} \approx 7.96 \times 10^{-6}$ F, about 8 μ F. For the woofer, $L = \frac{8}{2\pi(2500)} \approx 0.51 \times 10^{-3}$ H, about 0.51 mH. Check the capacitor: at 100 Hz its reactance is about 199 Ω , far more than the tweeter's 8 Ω , so bass is blocked; at 10,000 Hz it is only about 2 Ω .

```

1 import numpy as np
2 import matplotlib.pyplot as plt
3
4 fc = 2500                                # crossover frequency, Hz
5 f = np.logspace(1, np.log10(20000), 500) # 10 Hz to 20 kHz
6
7 woofer = 1 / np.sqrt(1 + (f / fc) ** 2)   # low-pass
8 tweeter = (f / fc) / np.sqrt(1 + (f / fc) ** 2) # high-pass
9
10 for test in [100, 2500, 10000]:
11     w = 1 / np.sqrt(1 + (test / fc) ** 2)
12     tw = (test / fc) / np.sqrt(1 + (test / fc) ** 2)
13     print(f"{test} Hz: woofer {w:.2f}, tweeter {tw:.2f}")
14
15 plt.semilogx(f, woofer, label="woofer (low-pass)")
16 plt.semilogx(f, tweeter, label="tweeter (high-pass)")
17 plt.axvline(fc, color="gray", linestyle="--")
18 plt.xlabel("frequency (Hz)")
19 plt.ylabel("fraction of the signal passed")
20 plt.legend()
21 plt.savefig("crossover.png")

```

100 Hz: woofer 1.00, tweeter 0.04
 2500 Hz: woofer 0.71, tweeter 0.71
 10000 Hz: woofer 0.24, tweeter 0.97

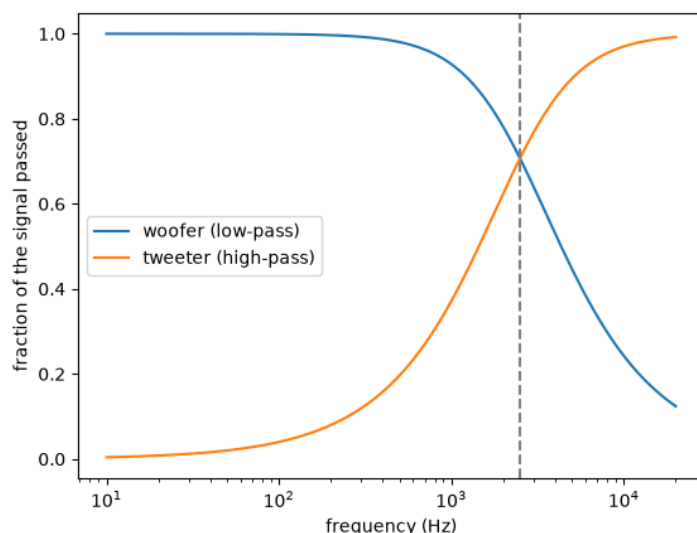


Figure 1.3: A first-order crossover at 2500 Hz. Bass goes to the woofer, treble to the tweeter, and both share the frequencies near f_c .

At the crossover frequency, each speaker gets $\frac{1}{\sqrt{2}} \approx 0.71$ of the signal, and therefore half the power (power goes as the square). In fact, at *every* frequency, the squares of the two fractions add to exactly 1: $\frac{1}{1+u^2} + \frac{u^2}{1+u^2} = 1$ with $u = f/f_c$. The total power reaching the two speakers stays the same; the crossover only decides who gets it. Real speakers often use steeper, higher-order crossovers, and many powered speakers now do the job digitally, with an FFT-style filter running on a chip.

Exercise 6 Design a Crossover

Design a first-order crossover at 3000 Hz for $4\ \Omega$ speakers. What capacitor goes with the tweeter, and what inductor goes with the woofer?

Working Space

Answer on Page 50

Exercise 7 **Filtering a Square Wave**

A 440 Hz square-wave tone passes through a first-order low-pass filter with $f_c = 1000$ Hz. What fraction of each of the first four harmonics gets through? How will the tone sound different?

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Answer on Page 50

Exercise 8 **Add a Tone**

Change `spectrum.py` to add a fourth tone at 1760 Hz with amplitude 0.3. What does the program print now?

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Answer on Page 50

Population Proportion Statistics

Let us say that you are trying to get a candidate elected. The candidate asks you, “What proportion of the voting population is going to vote for me?” So you ask a random sample of 12 voters, and 9 say they will vote for your candidate. Can you promise a win? How sure can you be? This chapter turns those 9 yeses into an honest answer.

2.1 Sample Probabilities from Population Proportion

To address these sorts of questions (and there are a lot of them), we start with the opposite question: If we knew what the proportion was in the entire population, what sort of results should we expect in a random sample of just 12?

For example, let us say that 62% of the entire population plans to vote for your candidate. You ask 12 people, “Will you vote for my candidate?” How many will say “Yes”? You do not know — it depends on the sample. For example, there is some chance that you will just happen to choose all 12 from the 38% of the population that does not plan to vote for your candidate.

We can compute the probability of each outcome using the binomial distribution. Let r be the probability that a random person will say, “I plan to vote for your candidate.” Let n be the number of people you ask. The probability that exactly k people will say, “I plan to vote for your candidate” is given by:

$$p(k) = \binom{n}{k} r^k (1 - r)^{n-k}$$

Note that even though most people support your candidate, there is some chance that no one you ask will say that they will vote for your candidate.

Using $r = 0.62$, we can compute the probability of each outcome:

k	p(k)
0	0.000009
1	0.000177
2	0.001593
3	0.008663
4	0.031801
5	0.083017
6	0.158024
7	0.220996
8	0.225358
9	0.163418
10	0.079989
11	0.023729
12	0.003226

Looking at this, the most likely outcome is that 8 people will say “Yes.” Even so, there is less than a 1 in 4 chance of that outcome. It is very unlikely that fewer than 2 people will say “Yes.” Here is a bar chart of the data:

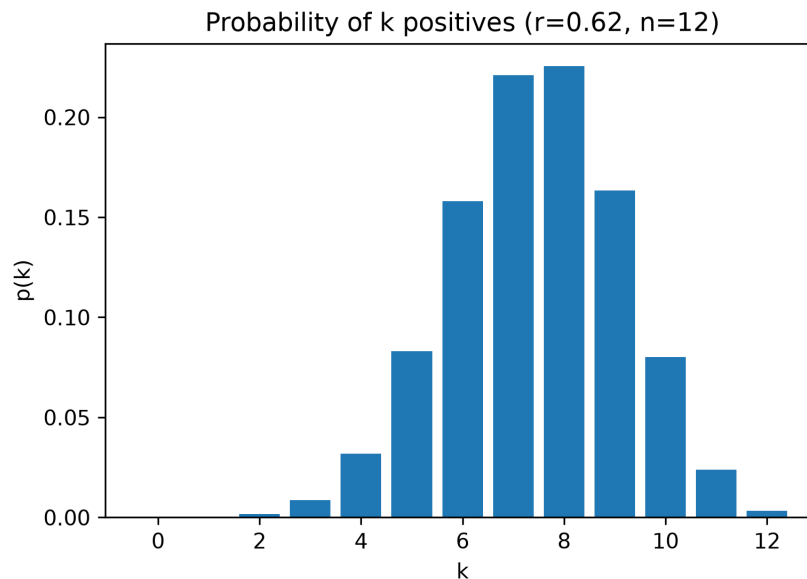


Figure 2.1: Binomial distribution of k positive votes.

In this section, we knew the proportion of the population (r) and used it to find the probability of each possible number of positives in a random sample (k). Now we go the other way: you know k , and you want to know which values of r are plausible.

2.2 Population Proportion from Sample

You ask 12 people if they will vote for your candidate. 9 say “Yes.”

Next, you do a thought experiment: “If only 10% of the population were going to vote for my candidate, what is the probability that I would see this outcome?”

$$p(9) = \binom{n}{k} r^k (1-r)^{n-k} = \binom{12}{9} (0.1)^9 (0.9)^{12-9} = 0.00000016$$

This outcome would be quite unusual. What if 70% of the population were going to vote for your candidate? What is the probability that you would see this outcome?

$$p(9) = \binom{n}{k} r^k (1-r)^{n-k} = \binom{12}{9} (0.7)^9 (0.3)^{12-9} \approx 0.2397$$

In this case, the observed outcome would be a lot less unusual.

So, you decide to plot out the likelihood of this outcome for every possible value of r :

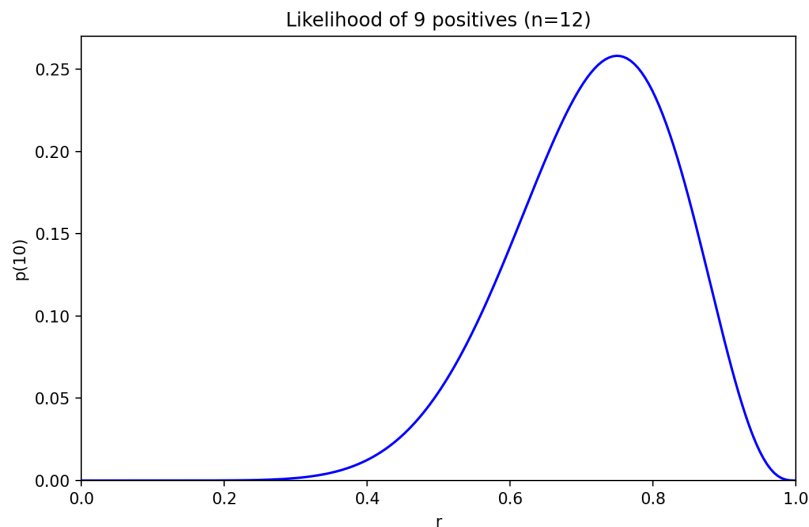


Figure 2.2: Likelihood of 9 positives.

This looks very similar to a probability distribution, but *it is not*. The area under the curve is about 0.077 (exactly $1/13$), not 1.0. This is called a *likelihood*.

It still tells us something. The value of r that makes the observed data most likely, called the maximum likelihood estimator, is $9/12 = 0.75$.

2.3 From Likelihood to Probability Density Function

How can we make this likelihood into a probability density function? We use Bayes' Law for continuous probability:

$$p(r|k) = \frac{p(k|r)p(r)}{\int_{r=0}^1 p(k|r)p(r)dr}$$

In other words, given that we had k positive responses, what is the probability that the proportion of the population that will vote for your candidate is r ? The numerator of the fraction is the likelihood scaled up or down by our prior belief about the value of r . What is the denominator of the fraction? For this to be a probability distribution, we need it to integrate to 1. This is taken care of by the denominator.

Let us say we have no prior belief about the value of r . That is, $p(r)$ is the continuous uniform distribution between 0 and 1; thus, $p(r) = 1$ for all possible values of r . Our formula becomes:

$$p(r|k) = \frac{p(k|r)}{\int_{r=0}^1 p(k|r)dr}$$

That is the likelihood scaled up so that it integrates to 1. If we plot this, we get:

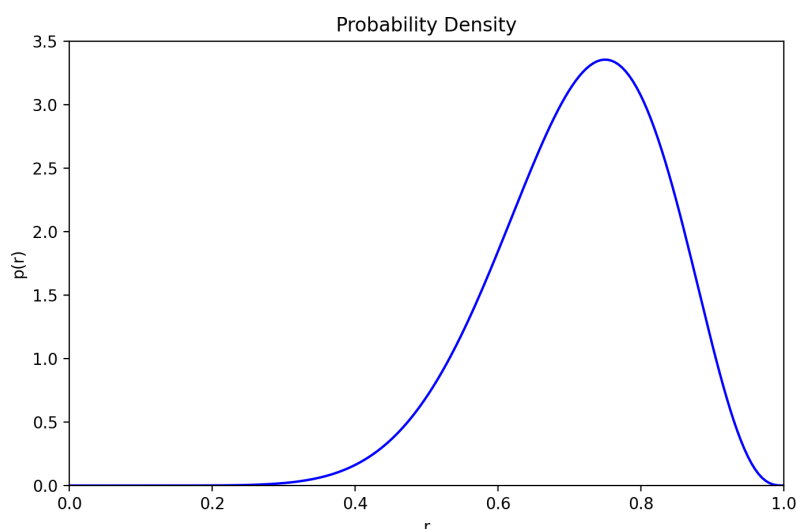


Figure 2.3: PDF of sampled likelihood function.

Here, then, is your report to your candidate: "I asked 12 voters if they were going to vote

for you. 9 said yes. Using a uniform prior, here is what I believe about your support in the general population.” You also include this graph.

What happens to this graph if you ask 120 voters and 90 say yes?

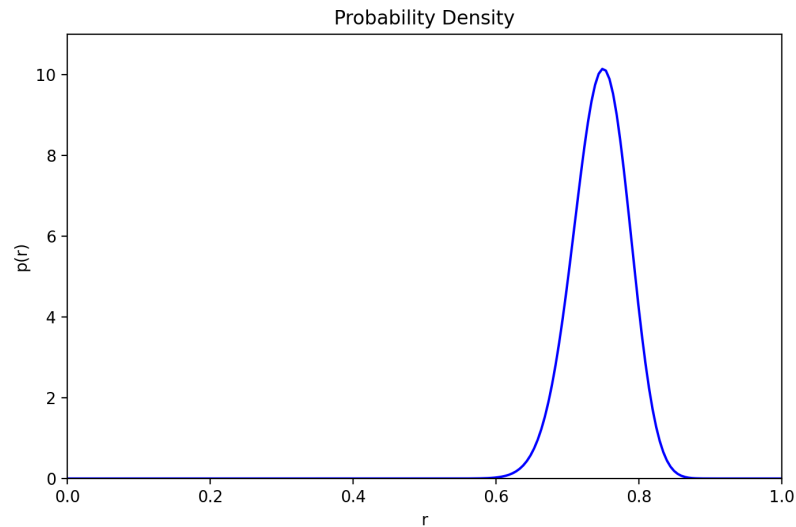


Figure 2.4: Bayes shifted for 90 positive votes from 120 voters.

The MLE (0.75) is the same, but because of the much larger sample size, you are more confident when you say “It is probably close to 75%.”

2.4 Beta Distribution

This probability distribution that you discovered is actually pretty common. It is known as the *beta distribution*.

The beta distribution has two parameters a and b that determine its shape. If you get k positives out of n , then use $a = k + 1$ and $b = n - k + 1$.

When you make your report to your candidate, they will look at your probability distribution with quiet awe and ask “Based on your sample of 12 people, what is the probability that at least 50% of the population will vote for me?” So you would fill in the region and say, “This area represents that probability.”

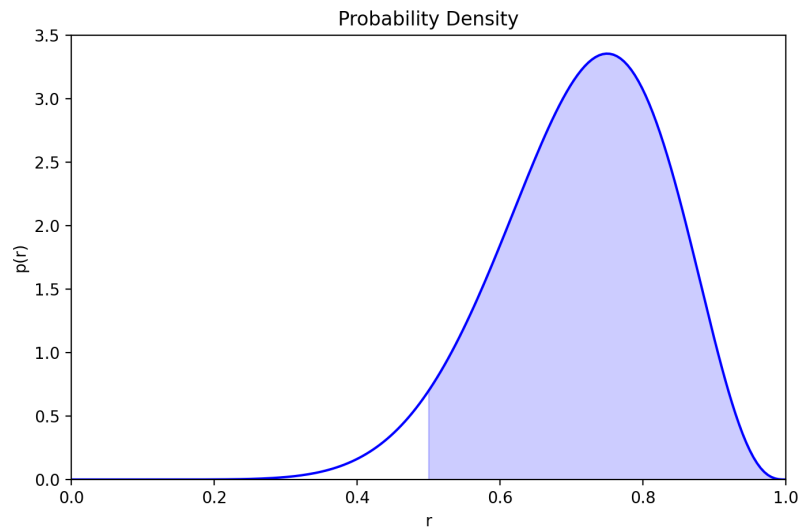


Figure 2.5: PDF for the beta distribution at least 50% positives.

Once again, there will be a long silence, followed by a simple question: “Can you give me a number?” Here is the Python code:

```

1  from scipy.stats import beta
2
3  # Constants
4  K = 9
5  N = 12
6
7  # What is the probability  $r \leq 0.5$ ?
8  p_less = beta.cdf(0.5, K + 1, N - K + 1)
9
10 # What is the probability  $r > 0.5$ ?
11 p_more = 1.0 - p_less
12 print(f"I am {p_more * 100.0:.2f}% sure you have majority support.")

```

This will give you:

```

1  I am 95.39% sure you have majority support.

```

Exercise 9

Your candidate’s team asks 20 voters, and 15 say they will vote for the candidate. Using a uniform prior, what is

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the probability that at least half of the population will vote for the candidate?
Give your answer as a percent.

Answer on Page 50

Change of Variables

You are making ice spheres for a party by freezing water in small balloons. The radius of the spheres is normally distributed with a mean of 0.7 cm and a standard deviation of 0.08 cm. But the question you care about is how much *ice* you have, which depends on mass, not radius. If radius is normal, is mass normal too? (Spoiler: no.) This chapter shows how to convert a probability distribution from one variable to another.

Here are the probability density and cumulative distribution for the radius:

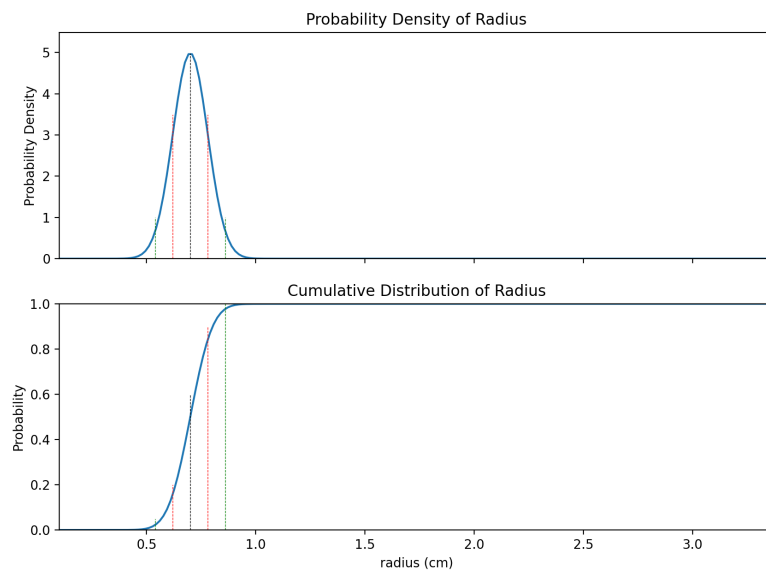


Figure 3.1: Ice sphere PDF and cumulative distribution function.

This includes lines indicating the mean and two standard deviations on each side.

Now, what is the cumulative distribution for the *mass* of the spheres? A cubic centimeter of ice has a mass of about 0.92 g; to keep the numbers simple, we will round that to 1 g. Then the mass of a sphere is:

$$m = \frac{4}{3}\pi r^3$$

So, for example, a sphere with the mean radius of 0.7 cm has a mass in grams of $\frac{4\pi(0.7^3)}{3} \approx 1.44$ g.

Thinking about the graph of the cumulative distribution, if half the balloons have a radius less than 0.7 cm (the median), then half of the balloons have a mass less than 1.44 g. For each point on the cumulative graph, we can use the radius of that point to compute the corresponding mass — the CDF gets stretched out:

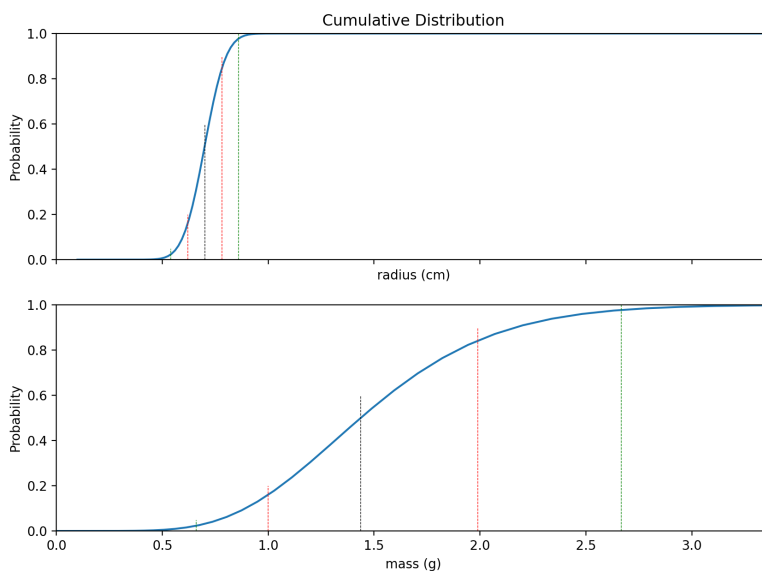


Figure 3.2: Cumulative distribution of both the radius and mass of the ice spheres.

If F is the original cumulative distribution function, and g is the function that maps the new variable (mass, in this case) to the old one (radius), then the new cumulative distribution function H is given by

$$H(m) = F(g(m))$$

In this case, F is the cumulative function for the normal distribution, with mean 0.7 and standard deviation of 0.08. g maps the mass to the radius:

$$g(m) = \left(\sqrt[3]{\frac{3}{4\pi}} \right) m^{\frac{1}{3}}$$

3.1 Making a Probability Density Function

Now, we know how to calculate a new cumulative distribution function using the new variable. However, we usually want a probability density.

Here is the CDF and the PDF of the mass of the ice spheres:

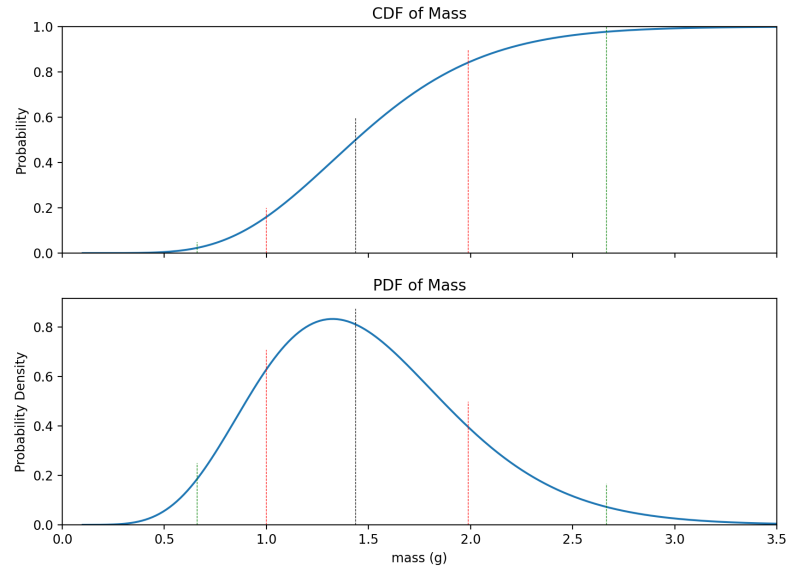


Figure 3.3: CDF and PDF of the masses.

Reminder: The probability density function is the derivative of the cumulative distribution function. We know the CDF is

$$H(m) = F(g(m))$$

By the chain rule:

$$H'(m) = F'(g(m))g'(m)$$

The function F is the cumulative distribution for the normal distribution with mean 0.7 and standard deviation of 0.08. So, we know its derivative:

$$F'(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

where $\mu = 0.7$ and $\sigma = 0.08$.

We have already said that

$$g(m) = \left(\sqrt[3]{\frac{3}{4\pi}} \right) m^{\frac{1}{3}}$$

which is easy to differentiate:

$$g'(m) = \left(\frac{1}{3} \right) \left(\sqrt[3]{\frac{3}{4\pi}} \right) m^{-\frac{2}{3}}$$

Here is the code that generated that plot:

```
1 import numpy as np
2 from scipy.stats import norm
3 import matplotlib.pyplot as plt
4
5 # Constants
6 MEAN_RADIUS = 0.7
7 STD_RADIUS = 0.08
8
9 # Range to plot
10 MIN_MASS = 0.1
11 MAX_MASS = 3.5
12
13 # Number of points to plot
14 N = 200
15
16 # Needed for radius_for_mass and d_radius_for_mass
17 C = np.power(3 / (4 * np.pi), 1/3)
18
19 # In these three functions, x can
20 # be a number or a numpy array
21
22 def mass_for_radius(x):
23     return 4 * np.pi * np.power(x, 3) / 3
24
25 def radius_for_mass(x):
26     return C * np.power(x, 1/3)
27
28 # Derivative of radius_for_mass()
29 def d_radius_for_mass(x):
30     return (C/3) * np.power(x, -2/3)
31
32 # Compute mean and 2 standard deviations in each direction
33 m_mean = mass_for_radius(MEAN_RADIUS)
```

```

34 m_minus_std = mass_for_radius(MEAN_RADIUS - STD_RADIUS)
35 m_plus_std = mass_for_radius(MEAN_RADIUS + STD_RADIUS)
36 m_minus_2std = mass_for_radius(MEAN_RADIUS - 2 * STD_RADIUS)
37 m_plus_2std = mass_for_radius(MEAN_RADIUS + 2 * STD_RADIUS)
38
39 # Make N possible values for mass
40 m_values = np.linspace(MIN_MASS, MAX_MASS, N)
41
42 # Compute g(m) for each of these masses
43 # That is: What is the radius for each of these masses?
44 r_values = radius_for_mass(m_values)
45
46 # Compute F(g(m)) for each of these masses
47 # That is: What is the cumulative distribution for each those radii?
48 cdf_values = norm.cdf(r_values, loc=MEAN_RADIUS, scale=STD_RADIUS)
49
50 # Compute g'(m) for each of these masses
51 dg_values = d_radius_for_mass(m_values)
52
53 # What is F'(g(m))g'(m)?
54 pdf_values = norm.pdf(r_values, loc=MEAN_RADIUS, scale=STD_RADIUS) * dg_values
55
56 # Sanity check: It should integrate to a little less than 1.0
57 dx = (MAX_MASS - MIN_MASS)/N
58 area_under_curve = pdf_values.sum() * dx
59 print(f"Integral from {MIN_MASS:.2f} to {MAX_MASS:.2f}: {area_under_curve:.3f}")
60
61 # Make a figure with two axes
62 fig, axs = plt.subplots(nrows=2, sharex=True, figsize=(10, 7), dpi=200)
63
64 # Draw the CDF on the first axis
65 axs[0].set_title("CDF of Mass")
66 axs[0].set_ylim(bottom=0.0, top=1.0)
67 axs[0].set_xlim(left=0.0, right=MAX_MASS)
68 axs[0].set_ylabel("Probability")
69 axs[0].plot(m_values, cdf_values)
70
71 # Add lines for mean, mean-std, and mean+std
72 axs[0].vlines(m_minus_2std, 0, 0.05, "g", linestyle="dashed", lw=0.5)
73 axs[0].vlines(m_minus_std, 0, 0.2, "r", linestyle="dashed", lw=0.5)
74 axs[0].vlines(m_mean, 0, 0.6, "k", linestyle="dashed", lw=0.5)
75 axs[0].vlines(m_plus_std, 0, 0.9, "r", linestyle="dashed", lw=0.5)
76 axs[0].vlines(m_plus_2std, 0, 1.0, "g", linestyle="dashed", lw=0.5)
77
78 # How high does the pdf go?
79 max_density = pdf_values.max()
80
81 # Draw the PDF on the second axis
82 axs[1].set_title("PDF of Mass")
83 axs[1].set_ylim(bottom=0.0, top=max_density * 1.1)
84 axs[1].set_xlim(left=0.0, right=MAX_MASS)
85 axs[1].set_xlabel("mass (g)")
86 axs[1].set_ylabel("Probability Density")

```

```

87 axs[1].plot(m_values, pdf_values)
88
89 # Add lines for mean, mean-std, and mean+std
90 axs[1].vlines(m_minus_2std, 0, max_density * .3, "g", linestyle="dashed",lw=0.5)
91 axs[1].vlines(m_minus_std, 0, max_density * .85, "r", linestyle="dashed",lw=0.5)
92 axs[1].vlines(m_mean, 0, max_density * 1.05, "k", linestyle="dashed",lw=0.5)
93 axs[1].vlines(m_plus_std, 0, max_density * .6, "r", linestyle="dashed",lw=0.5)
94 axs[1].vlines(m_plus_2std, 0, max_density * .2, "g", linestyle="dashed",lw=0.5)
95 fig.savefig("pdf.png")

```

It prints Integral from 0.10 to 3.50: 0.994. The area is a little less than 1 because a sliver of probability lies outside the plotted range. Notice in the figure that the mass density is lopsided, with a longer tail to the right: cubing stretches large radii much more than small ones.

3.2 Decreasing Conversions

The last case (mass and radius) is relatively straightforward, because the function g is always increasing. What if we have a change of variables where g is decreasing? For example, $V = IR$ so $\frac{V}{R} = I$.

Let us say that you work at a lightbulb factory and you sample the lightbulbs to see what their resistance is. You find the resistances of the lightbulbs are normally distributed with a mean of 24 ohms and a standard deviation of 3 ohms. The voltage will be exactly 12 volts. What is the PDF of the currents that will pass through the lightbulbs?

$$I = \frac{12}{R}$$

so

$$g(i) = \frac{12}{i}$$

is the function that converts a current i into a resistance $R = g(i)$. Taking the derivative with respect to i , we get:

$$g'(i) = -\frac{12}{i^2}$$

Because g is decreasing, a larger current means a smaller resistance. So the cumulative

distribution for the current is

$$H(i) = P(I \leq i) = P(R \geq g(i)) = 1 - F(g(i))$$

where F is the cumulative distribution for the resistance. Taking the derivative with the chain rule gives

$$H'(i) = -F'(g(i)) g'(i) = F'(g(i)) |g'(i)|$$

The minus sign disappears when we use $|g'(i)|$, because a probability density cannot be negative.

For the lightbulbs, F' is the normal density with mean 24 and standard deviation 3. So the density of the current is

$$h(i) = \frac{1}{3\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{12/i-24}{3}\right)^2} \cdot \frac{12}{i^2}$$

Here is the code to compute this density:

```

1  from scipy.stats import norm
2
3  VOLTAGE = 12
4  MEAN_R = 24
5  STD_R = 3
6
7  def pdf_current(i):
8      r = VOLTAGE / i
9      return norm.pdf(r, loc=MEAN_R, scale=STD_R) * VOLTAGE / i**2
10
11 print(f"{pdf_current(0.5):.3f}")

```

This prints 6.383. The density peaks near 0.49 A, a little below the current at the mean resistance ($12/24 = 0.5$ A), because of the $\frac{12}{i^2}$ factor.

Example: What is the probability that a bulb carries more than 0.6 A?

Solution: A current above 0.6 A means $\frac{12}{R} > 0.6$, which is the same as $R < 20$. So

$$P(I > 0.6) = P(R < 20) = \Phi\left(\frac{20-24}{3}\right) = \Phi\left(-\frac{4}{3}\right) \approx 0.091$$

About 9.1% of the bulbs carry more than 0.6 A. (We can also get this from the code above: `norm.cdf(20, loc=24, scale=3)` gives 0.091.)

Exercise 10

Using the lightbulb example, find the probability that a bulb carries less than 0.4 A.

Working Space

Answer on Page 50

Exercise 11

In the ice sphere example, what fraction of the spheres have a mass greater than 2 g? (Hint: convert the mass back to a radius first.)

Working Space

Answer on Page 51

Double Integrals over Rectangular Regions

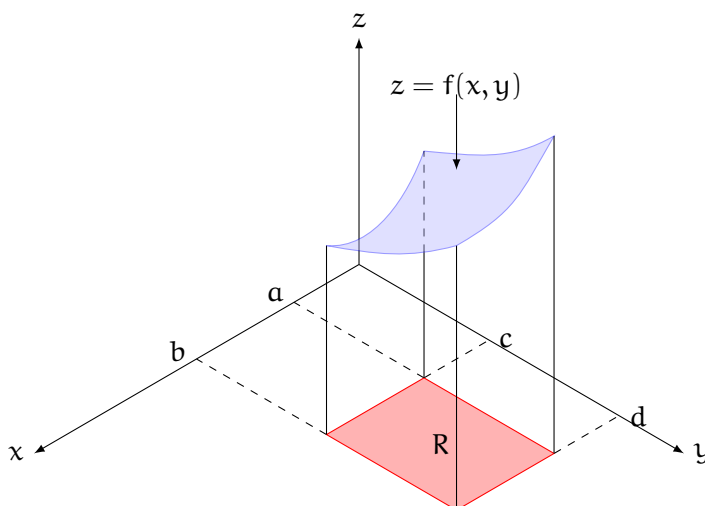
How much snow is sitting on a rectangular roof after a storm? If the snow were the same depth everywhere, you would multiply area by depth. But drifts pile up in some places and thin out in others, so the depth is a function $f(x, y)$ of position. To get the total, you need to add up depth over a two-dimensional region. That is a **double integral**.

The idea is the same one you used for area under a curve: chop the region into tiny pieces, approximate each piece, add them up, and take a limit. In one variable the pieces were thin rectangles. Now they are thin columns. In this chapter we integrate over rectangles; later chapters handle curved regions, polar coordinates, and three dimensions.

4.1 Double Integrals

4.1.1 Over Rectangular Regions

Suppose there is some function, $z = f(x, y)$, that is defined over the rectangular region, R , defined by $R = [a, b] \times [c, d] = \{(x, y) | a \leq x \leq b, c \leq y \leq d\}$, and f is such that $f \geq 0$ for all $(x, y) \in R$. Think of R as a rectangular garden and $f(x, y)$ as the height of the ground above each point. So, the graph of f is a surface that lies above the rectangular region, R (see figure 4.1).

Figure 4.1: The graph of f over the region R

Let us call the solid that fills the space between the xy -plane and the surface $z = f(x, y)$ S . Formally, this is written as

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid 0 \leq z \leq f(x, y), (x, y) \in R\}$$

How can we find the volume of the solid, S ? We will apply what we learned about Riemann sums and definite integrals in two dimensions to this three-dimensional problem.

First, we divide R into rectangular subregions. We do this by dividing the interval $[a, b]$ into m subintervals with width $\Delta x = (b - a)/m$ and the interval $[c, d]$ into n subintervals with width $\Delta y = (d - c)/n$. Drawing lines through these divisions parallel to the x - and y -axes, we create a field of subrectangles, each with area $\Delta A = \Delta x \Delta y$ (see figure 4.2).

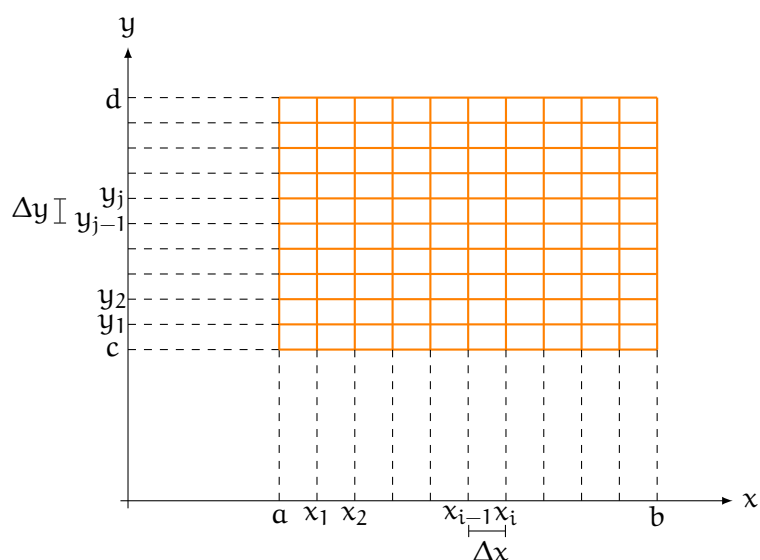


Figure 4.2: The region, R , on the xy -plane divided into subrectangles

Each subrectangle is defined by:

$$R_{ij} = \{(x, y) | x_{i-1} \leq x \leq x_i, y_{j-1} \leq y \leq y_j\}$$

In each subrectangle, pick a sample point (x_{ij}^*, y_{ij}^*) . We can approximate the volume between the xy -plane and $z = f(x, y)$ over the subrectangle as a column with base area ΔA and height $f(x_{ij}^*, y_{ij}^*)$ (see figure 4.3) and the volume of the column is given by:

$$V_{ij} = f(x_{ij}^*, y_{ij}^*) \Delta A$$

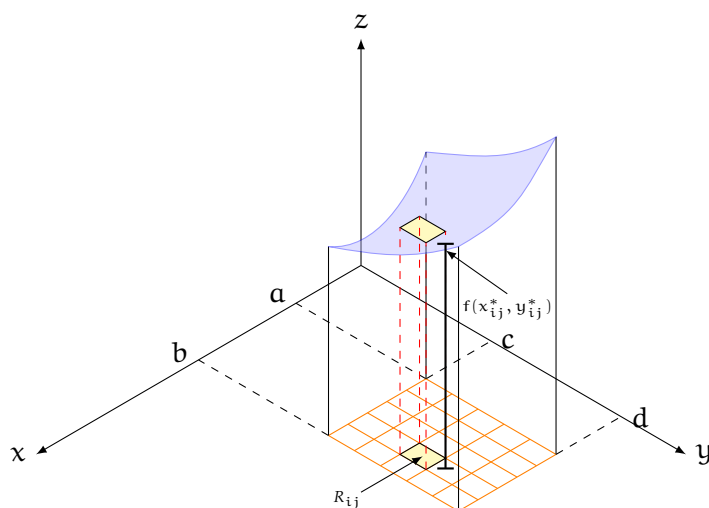


Figure 4.3: A single column with base ΔA and height $f(x_{ij}^*, y_{ij}^*)$

Therefore, the approximate volume of the solid, S , that lies between the region, R , and $z = f(x, y)$ is the sum of all the columns over i and j :

$$V_S \approx \sum_{i=1}^m \sum_{j=1}^n f(x_{ij}^*, y_{ij}^*) \Delta A$$

Just like with the area under a curve, we get the true volume by taking the limit as m and n go to ∞ , which becomes a **double integral**:

Volume of a Solid over a Region

$$V_S = \lim_{m, n \rightarrow \infty} \sum_{i=1}^m \sum_{j=1}^n f(x_{ij}^*, y_{ij}^*) \Delta A = \iint_R f(x, y) \, dA$$

4.2 Iterated Integrals

To be able to evaluate the double integral as outlined above, we must first discuss iterated integrals. Iterated integrals happen when you evaluate two single integrals, one inside the other. Consider some function, $g(x, y)$. We could integrate that function from $x = q$ to $x = r$:

$$\int_q^r g(x, y) \, dx$$

Notice that we are integrating with respect to x , so y terms will be treated as constants (this reverses a partial derivative in x). Let us call the result of this first integral $A(y)$:

$$A(y) = \int_q^r g(x, y) \, dx$$

We can then integrate the resulting function, $A(y)$, from $y = s$ to $y = t$:

$$\int_s^t A(y) \, dy = \int_s^t \left[\int_q^r g(x, y) \, dx \right] dy$$

This is called an **iterated integral**. When evaluating iterated integrals, we work from the inside out. You can also write it without the brackets:

$$\int_s^t \int_q^r g(x, y) \, dx \, dy$$

Example: Evaluate the iterated integral $\int_0^3 \int_1^2 xy^2 \, dy \, dx$.

Solution: We can re-write this to more explicitly show the inner and outer integrals:

$$\int_0^3 \left[\int_1^2 xy^2 dy \right] dx$$

As you can see, the inner integral is with respect to y . Let us isolate and evaluate the inner integral:

$$\begin{aligned} \int_1^2 xy^2 dy &= x \int_1^2 y^2 dy = x \left[\frac{1}{3} y^3 \right]_{y=1}^{y=2} \\ &= \frac{x}{3} [2^3 - 1^3] = \frac{x}{3} [8 - 1] = \frac{7x}{3} \end{aligned}$$

We were able to move x outside the integral because when we are integrating with respect to a specific variable (in this case, y), other variables are treated as constants. Now, we can substitute $\int_1^2 xy^2 dy = \frac{7x}{3}$ into the iterated integral:

$$\begin{aligned} \int_0^3 \left[\int_1^2 xy^2 dy \right] dx &= \int_0^3 \left[\frac{7x}{3} \right] dx \\ &= \frac{7}{3} \left[\frac{1}{2} x^2 \right]_{x=0}^{x=3} = \frac{7}{6} [3^2 - 0^2] = \frac{7 \cdot 9}{6} = \frac{21}{2} \end{aligned}$$

Exercise 12 Order of Evaluating Iterated Integrals

Show that $\int_0^3 \int_1^2 xy^2 dy dx = \int_1^2 \int_0^3 xy^2 dx dy$.

Working Space

Answer on Page 51

Exercise 13 Evaluating Iterated Integrals

Evaluate the following iterated integrals.

Working Space

- $\int_0^1 \int_1^2 (x + e^{-y}) dx dy$
- $\int_{-3}^3 \int_0^{\pi/2} (2y + y^2 \cos x) dx dy$
- $\int_0^3 \int_0^{\pi/2} t^2 \sin^3 \theta d\theta dt$

4.3 Fubini's Theorem for Double Integrals

Fubini's Theorem states that for a function, f , that is continuous over the rectangular region, R , the double integral of f over the region $R = \{(x, y) | a \leq x \leq b, c \leq y \leq d\}$ is equal to the iterated integral of f with respect to x and y . This is expressed mathematically below:

Fubini's Theorem

If f is continuous on the rectangle $R = \{(x, y) | a \leq x \leq b, c \leq y \leq d\}$, then

$$\iint_R f(x, y) \, dA = \int_a^b \int_c^d f(x, y) \, dy \, dx = \int_c^d \int_a^b f(x, y) \, dx \, dy$$

Example: Back to the snowy roof. The roof covers $R = [0, 10] \times [0, 6]$ (in meters), and wind piles the snow so that its depth is $f(x, y) = 0.2 + 0.05x$ meters. How much snow is on the roof?

Solution: By Fubini's Theorem, integrate in either order. Doing y first:

$$\int_0^{10} \int_0^6 (0.2 + 0.05x) \, dy \, dx = \int_0^{10} 6(0.2 + 0.05x) \, dx = \left[1.2x + 0.15x^2 \right]_0^{10} = 12 + 15 = 27$$

There are 27 cubic meters of snow. (At roughly 100 kg per cubic meter for fresh snow, that is about 2.7 metric tons.)

Exercise 14 Applying Fubini's Theorem

Rewrite the following double integrals as iterated integrals.

Working Space

- $\iint_R \frac{xy^2}{x^2+1} \, dA$, $R = \{(x, y) | 0 \leq x \leq 1, -3 \leq y \leq 3\}$
- $\iint_R \frac{\sec \theta}{\sqrt{1+t^2}} \, dA$, $R = \{(\theta, t) | 0 \leq \theta \leq \frac{\pi}{4}, 0 \leq t \leq 1\}$

Exercise 15

Evaluate the double integral.

Working Space

1. $\iint_R \frac{xy^2}{x^2+1} dA, R = \{(x, y) \mid 0 \leq x \leq 2, -3 \leq y \leq 3\}$
2. $\iint_R \frac{\tan \theta}{\sqrt{1-t^2}} dA, R = \{(\theta, t) \mid 0 \leq \theta \leq \pi/3, 0 \leq t \leq \frac{1}{2}\}$
3. $\iint_R x \sin(x+y) dA, R = [0, \pi/6] \times [0, \pi/3]$

Answer on Page 52

Double Integrals Over Non-Rectangular Regions

Real regions are rarely rectangles. A pizza is round, a lake has a wiggly shoreline, and a soccer field's penalty arc is curved. To integrate over those shapes, we need double integrals whose limits are curves instead of constants.

5.1 Extending the Function by Zero

Suppose we want the integral of $f(x, y)$ over a region D that fits inside a rectangle R (see figure 5.1). We can then define a new function:

$$F(x, y) = \begin{cases} f(x, y) & \text{if } (x, y) \text{ is in } D \\ 0 & \text{if } (x, y) \text{ is in } R \text{ but not in } D \end{cases}$$

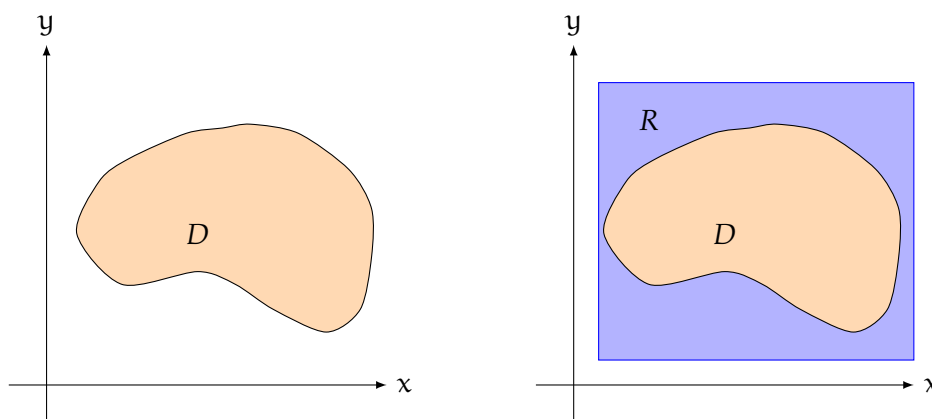


Figure 5.1: We can find a rectangular region, R , that completely encloses D

Then, we can see that:

$$\iint_D f(x, y) \, dA = \iint_R F(x, y) \, dA$$

This makes sense intuitively, since integrating over F outside of D does not contribute anything to the integral, and the integral of F inside D is equal to the integral of f inside

D . In general, there are two types of regions for D . A region is **type I** if it lies between two continuous functions of x :

$$D = \{(x, y) \mid a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}$$

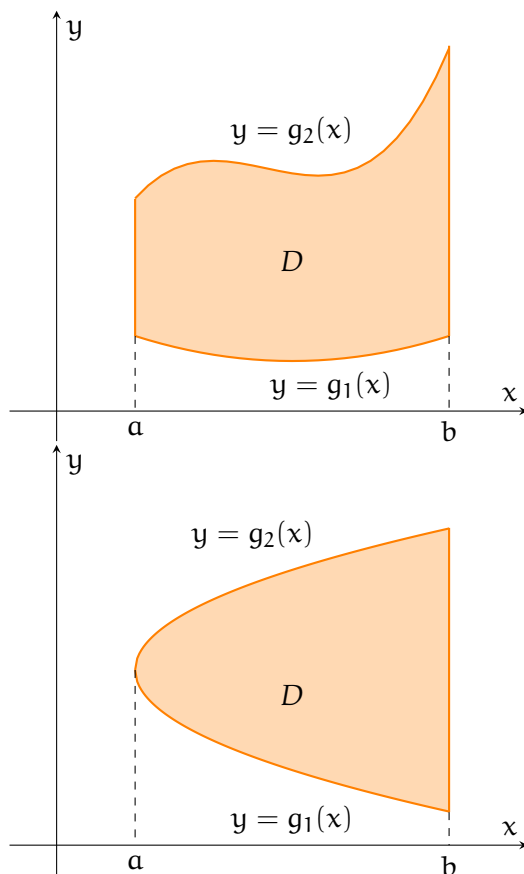


Figure 5.2: Two examples of type I domains

5.2 Integrating over Type I Regions

Some type I regions are shown in figure 5.2. To evaluate $\iint_D f(x, y) \, dA$, we begin by choosing a rectangle $R = [a, b] \times [c, d]$ such that D is completely contained in R . We again define $F(x, y)$ such that $F(x, y) = f(x, y)$ on D and $F = 0$ outside of D . Then, by Fubini's theorem:

$$\iint_D f(x, y) \, dA = \iint_R F(x, y) \, dA = \int_a^b \int_c^d F(x, y) \, dy \, dx$$

Since $F(x, y) = 0$ when $y \leq g_1(x)$ or $y \geq g_2(x)$, we know that:

$$\int_c^d F(x, y) dy = \int_{g_1(x)}^{g_2(x)} F(x, y) dy = \int_{g_1(x)}^{g_2(x)} f(x, y) dy$$

Substituting this into the iterated integral above, we see that for a type I region $D = \{(x, y) \mid a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}$,

$$\iint_D f(x, y) dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y) dy dx$$

Another way to visualize the double integral over a type I region is shown in figure 5.3. For any value of $x \in [a, b]$, we know that $g_1(x) \leq y \leq g_2(x)$. The inner integral represents moving along one blue line from $y = g_1(x)$ to $y = g_2(x)$ and integrating with respect to y . Next, for the outer integral, we integrate with respect to x , which is represented by moving the line from $x = a$ to $x = b$.

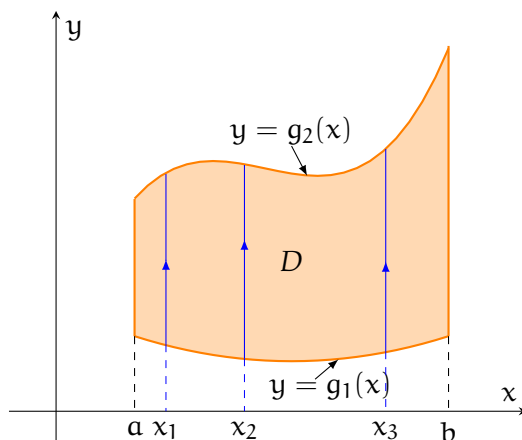


Figure 5.3: On type I domains, for a given value of x , $g_1(x) \leq y \leq g_2(x)$

5.3 Type II Regions

A **type II** region is a region such that we can define the limits of x in terms of y (see figure 5.4). In other words, a type II region can be defined as:

$$D = \{(x, y) \mid c \leq y \leq d, h_1(y) \leq x \leq h_2(y)\}$$

In a similar manner to above, we can show that:

$$\iint_D f(x, y) dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y) dx dy$$

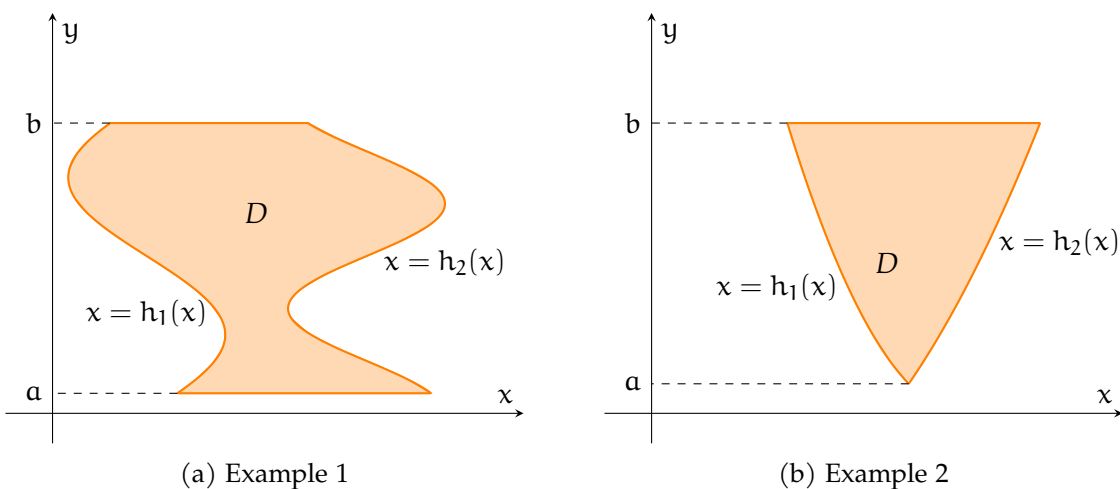
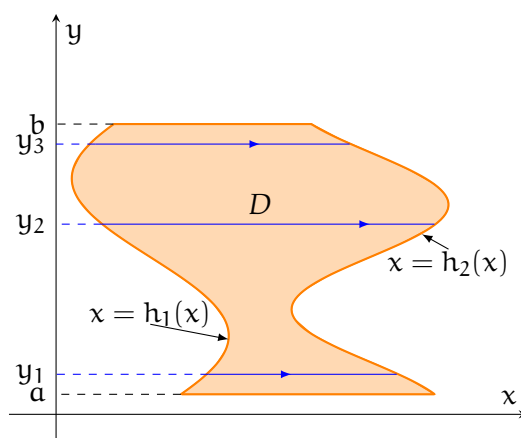


Figure 5.4: Two examples of type II domains

You can annotate type II regions with horizontal lines to show that, for a given y value, all x values in the region are contained in $h_1(y) \leq x \leq h_2(y)$ (see figure 5.5).

Figure 5.5: On type II domains, for a given value of y , $h_1(y) \leq x \leq h_2(y)$

5.4 Determining Region Type

Many regions can be described as either type I or type II. Consider the region between the curves $y = \frac{3}{2}(x-1)$ and $y = \frac{1}{2}(x-1)^2$ (see figure 5.6).

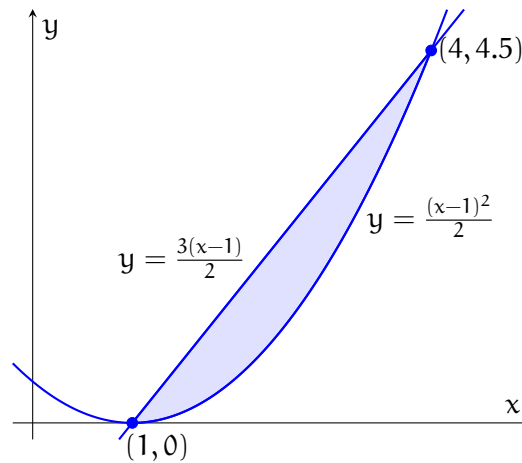
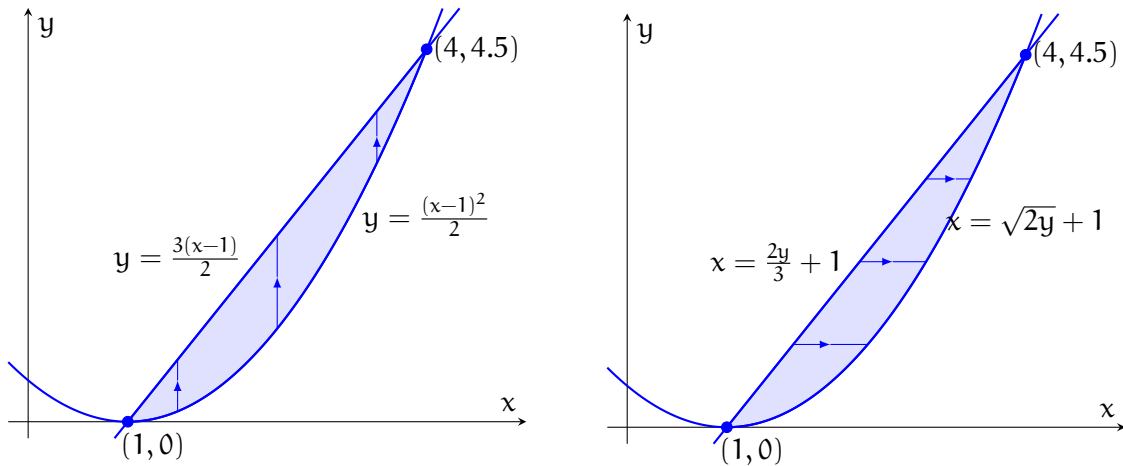


Figure 5.6: The region that lies between $y = \frac{(x-1)^2}{2}$ and $y = \frac{3(x-1)}{2}$ can be classified as type I or type II

We could classify this as a type I (see figure 5.7a) or a type II domain (see figure 5.7b).



(a) Type I region: $\frac{(x-1)^2}{2} \leq y \leq \frac{3(x-1)}{2}$

(b) Type II region: $\frac{2y}{3} + 1 \leq x \leq \sqrt{2y} + 1$

Figure 5.7: Comparison of Type I and Type II descriptions of the same region.

However, not every region is both type I and type II. A region can be classified as type I if you can take a vertical line ($x = c$, where c is some number in the domain of the region) and move it across the region without any gaps. Consider the two regions shown in figure ?? . The top is type I, and if you put vertical lines, the line traverses the entire region without leaving it. On the other hand, the lower region is not type I. There are vertical lines where the line exits the lower region before traversing the entire region.

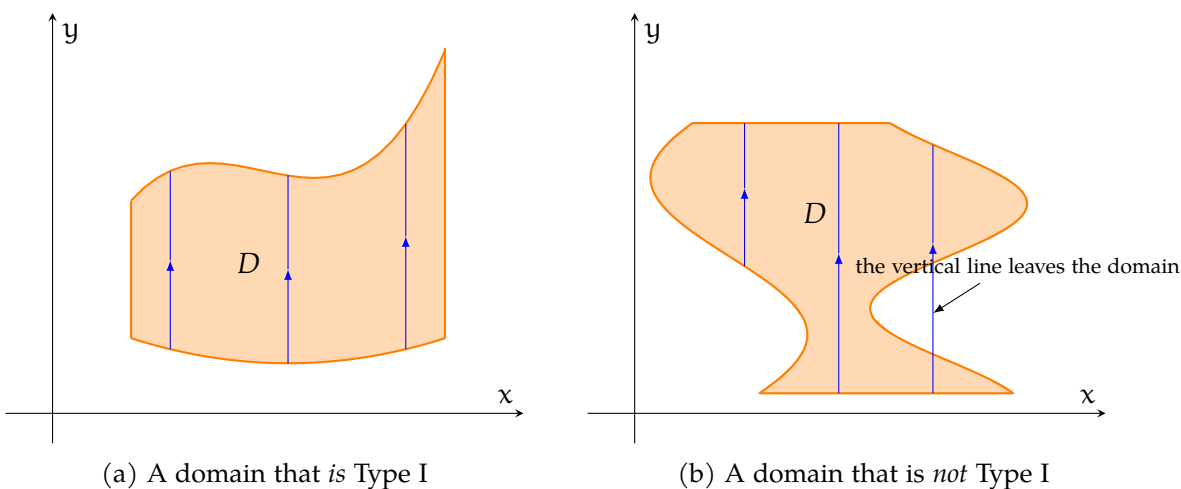


Figure 5.8: Comparing two regions: one that satisfies the Type I condition and one that does not.

To determine if a region can be classified as type II, you can use a horizontal line. If you can move a horizontal line ($y = c$, where c is in the domain of the region) across the region and the line always traverses the entire region without leaving and re-entering it, then the region is type II (see figure ??).

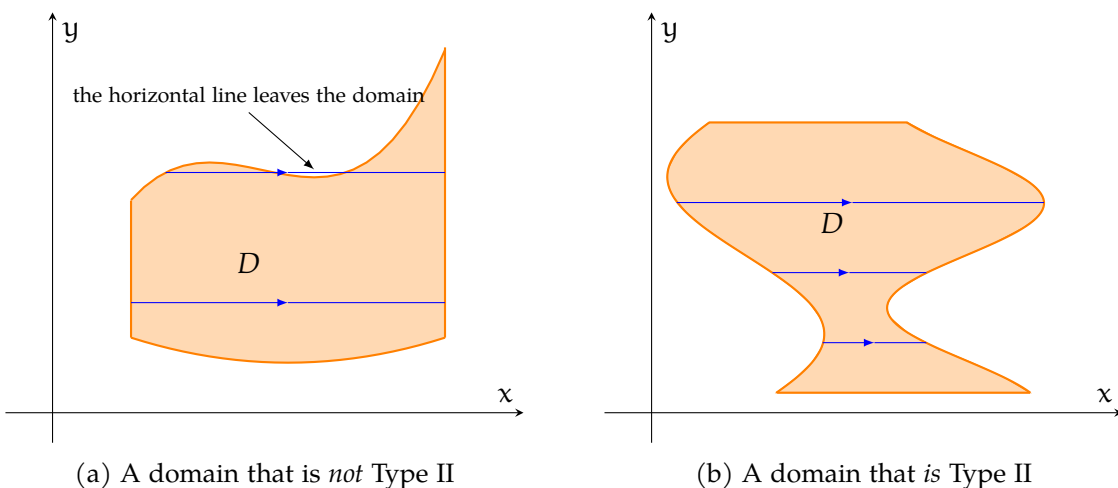


Figure 5.9: Comparing two regions: one that fails the Type II condition and one that satisfies it.

5.5 Worked Examples

Example: Evaluate $\iint_D (2x+y) \, dA$, where D is the region bounded by the parabolas $y = 3x^2$ and $y = 2 + x^2$. Region D is shown in figure 5.10.

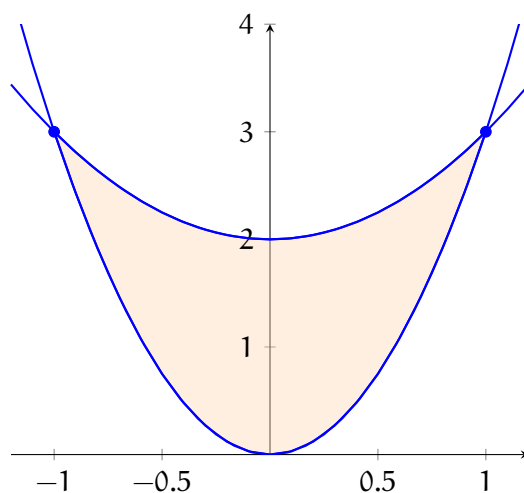


Figure 5.10: Region D is bounded above by $y = 2 + x^2$ and below by $y = 3x^2$

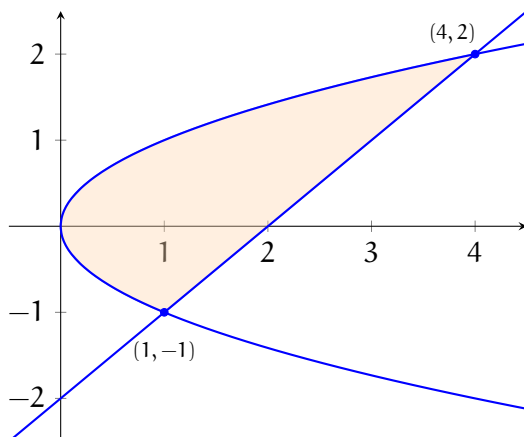
Solution: This is a type I region, since for a given x , $y \in [3x^2, 2 + x^2]$. We can define region D as $D = \{(x, y) \mid -1 \leq x \leq 1, 3x^2 \leq y \leq 2 + x^2\}$. Therefore,

$$\begin{aligned}
 \iint_D (2x + y) \, dA &= \int_{-1}^1 \int_{3x^2}^{2+x^2} (2x + y) \, dy \, dx \\
 &= \int_{-1}^1 \left[\int_{3x^2}^{2+x^2} 2x \, dy + \int_{3x^2}^{2+x^2} y \, dy \right] dx \\
 &= \int_{-1}^1 \left[2xy \Big|_{y=3x^2}^{y=2+x^2} + \frac{1}{2} y^2 \Big|_{y=3x^2}^{y=2+x^2} \right] dx \\
 &= \int_{-1}^1 \left[2x(2 + x^2 - 3x^2) + \frac{1}{2}((2 + x^2)^2 - (3x^2)^2) \right] dx \\
 &= \int_{-1}^1 [2 + 4x + 2x^2 - 4x^3 - 4x^4] \, dx \\
 &= \left[2x + 2x^2 + \frac{2}{3}x^3 - x^4 - \frac{4}{5}x^5 \right]_{x=-1}^{x=1} \\
 &= \left(2 + 2 + \frac{2}{3} - 1 - \frac{4}{5} \right) - \left(-2 + 2 - \frac{2}{3} - 1 + \frac{4}{5} \right) \\
 &= 4 + \frac{4}{3} - \frac{8}{5} = \frac{56}{15}
 \end{aligned}$$

Example: Set up integrals to evaluate $\iint_D xy \, dA$ if D is the region bounded by $y = x - 2$ and $x = y^2$ as both a type I and type II region. Which method will be easier to evaluate?

Evaluate the easier double integral.

Solution: Let us begin by visualizing D :



This region could be classified as type I and type II. To set up an integral as if D were a type I region, we need to describe y in terms of x . However, we run into a little problem. For $0 \leq x \leq 1$, the region is bounded by the parabola $x = y^2$, and for $1 \leq x \leq 4$, the region is bounded by both the parabola and the line. So, we will have to split the integral into two parts: $0 \leq x \leq 1$ and $1 \leq x \leq 4$:

$$\iint_D xy \, dA = \int_0^1 \int_{-\sqrt{x}}^{\sqrt{x}} xy \, dy \, dx + \int_1^4 \int_{x-2}^{\sqrt{x}} xy \, dy \, dx$$

To set up an integral as if D were a type II region, we need to describe x in terms of y . This time, we will not need to split the integral, because $y^2 \leq x \leq y + 2$ over the entire region:

$$\iint_D xy \, dA = \int_{-1}^2 \int_{y^2}^{y+2} xy \, dx \, dy$$

It is easier to evaluate the integral with D as a type II region, since we do not need to split the integral. Evaluating:

$$\begin{aligned} \int_{-1}^2 \int_{y^2}^{y+2} xy \, dx \, dy &= \int_{-1}^2 \frac{y}{2} \left[x^2 \right]_{x=y^2}^{x=y+2} dy \\ &= \int_{-1}^2 \frac{y}{2} \left[(y+2)^2 - (y^2)^2 \right] dy = \frac{1}{2} \int_{-1}^2 y \left[(y^2 + 4y + 4) - y^4 \right] dy \\ &= \frac{1}{2} \int_{-1}^2 (y^3 + 4y^2 + 4y - y^5) dy = \frac{1}{2} \left[\frac{1}{4}y^4 + \frac{4}{3}y^3 + \frac{4}{2}y^2 - \frac{1}{6}y^6 \right]_{y=-1}^{y=2} \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2} \left[\left(\frac{1}{4}(2)^4 + \frac{4}{3}(2)^3 + \frac{4}{2}(2)^2 - \frac{1}{6}(2)^6 \right) - \left(\frac{1}{4}(-1)^4 + \frac{4}{3}(-1)^3 + \frac{4}{2}(-1)^2 - \frac{1}{6}(-1)^6 \right) \right] \\
&= \frac{1}{2} \left[\left(4 + \frac{32}{3} + 8 - \frac{32}{3} \right) - \left(\frac{1}{4} - \frac{4}{3} + 2 - \frac{1}{6} \right) \right] = \frac{1}{2} \left(12 - \frac{3}{4} \right) = \frac{45}{8}
\end{aligned}$$

Exercise 16 Double Integrals over Non-Rectangular Regions

Evaluate the double integral.

Working Space

1. $\iint_D e^{-y^2} dA$, $D = \{(x, y) \mid 0 \leq y \leq 3, 0 \leq x \leq 2y\}$.
2. $\iint_D x \sin y dA$, D is bounded by $y = 0$, $y = x^2$, $x = 2$.
3. $\iint_D (2x - y) dA$, D is bounded by the circle with center at the origin and radius 3.

Answer on Page 54

Answers to Exercises

Answer to Exercise 1 (on page 6)

$\sin 2x \sin 3x = \frac{1}{2}[\cos(-x) - \cos 5x] = \frac{1}{2}[\cos x - \cos 5x]$. Then

$$\int_{-\pi}^{\pi} \frac{1}{2}(\cos x - \cos 5x) dx = \frac{1}{2} \left[\sin x - \frac{\sin 5x}{5} \right]_{-\pi}^{\pi} = 0$$

because $\sin$ is 0 at $\pm\pi$ and $\pm 5\pi$.

Answer to Exercise 2 (on page 6)

The function is odd, so every $a_n = 0$. By parts, with $u = x$ and $dv = \sin nx dx$,

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin nx dx = \frac{1}{\pi} \left[-\frac{x \cos nx}{n} \right]_{-\pi}^{\pi} + \frac{1}{\pi n} \int_{-\pi}^{\pi} \cos nx dx = -\frac{2 \cos n\pi}{n} + 0 = \frac{2(-1)^{n+1}}{n}$$

So $x = 2 \left(\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \dots \right)$. Unlike the square wave, the sawtooth contains every harmonic, not just the odd ones.

Answer to Exercise 3 (on page 7)

$\frac{4}{\pi} \left(1 - \frac{1}{3} + \frac{1}{5} \right) \approx 1.1035$, which is above 1. In the figure, the 3-term curve wiggles above and below the flat top, and $x = \frac{\pi}{2}$ is at one of its peaks. Adding more terms makes the wiggles smaller.

Answer to Exercise 4 (on page 8)

Only odd harmonics appear, with amplitudes proportional to $\frac{1}{n}$: 220 Hz (amplitude 1), 660 Hz ($\frac{1}{3}$), 1100 Hz ($\frac{1}{5}$), and 1540 Hz ($\frac{1}{7}$).

Answer to Exercise 5 (on page 10)

The Nyquist frequency is $\frac{48,000}{2} = 24,000$ Hz. A 40,000 Hz call is above that, so the phone cannot record it. (It would show up, if at all, as an alias at a false, lower frequency.)

Answer to Exercise 6 (on page 12)

$$C = \frac{1}{2\pi(4)(3000)} \approx 13.3 \text{ } \mu\text{F} \text{ and } L = \frac{4}{2\pi(3000)} \approx 0.212 \text{ mH.}$$

Answer to Exercise 7 (on page 13)

The harmonics are at 440, 1320, 2200, and 3080 Hz. Using $\frac{1}{\sqrt{1+(f/1000)^2}}$, the fractions are about 0.92, 0.60, 0.41, and 0.31. The fundamental passes almost untouched, but the higher harmonics are cut more and more, so the corners of the square wave round off and the tone sounds softer and more muffled, closer to a pure sine wave.

Answer to Exercise 8 (on page 13)

Add `+ 0.3 * np.sin(2 * np.pi * 1760 * t)` to the signal. The program then prints a fourth line, 1760 Hz: amplitude 0.30, after the other three.

Answer to Exercise 9 (on page 20)

With $k = 15$ and $n = 20$, the posterior is a beta distribution with $a = k + 1 = 16$ and $b = n - k + 1 = 6$. Change $K = 15$ and $N = 20$ in the Python code above. The probability that $r > 0.5$ is about 0.9867, so the answer is about 98.67%.

Answer to Exercise 10 (on page 30)

A current below 0.4 A means $\frac{12}{R} < 0.4$, which is the same as $R > 30$. So $P(I < 0.4) = P(R > 30) = 1 - \Phi\left(\frac{30-24}{3}\right) = 1 - \Phi(2) \approx 0.023$. About 2.3% of the bulbs carry less than 0.4 A.

Answer to Exercise 11 (on page 30)

A mass of 2 g means $\frac{4}{3}\pi r^3 = 2$, so $r = \sqrt[3]{\frac{3}{2\pi}} \approx 0.782$ cm. Because mass increases with radius, $P(m > 2) = P(r > 0.782) = 1 - \Phi\left(\frac{0.782-0.7}{0.08}\right) = 1 - \Phi(1.02) \approx 0.154$. About 15% of the spheres are heavier than 2 g.

Answer to Exercise 12 (on page 35)

We have already shown that $\int_0^3 \int_1^2 xy^2 dy dx = \frac{21}{2}$. We will evaluate $\int_1^2 \int_0^3 xy^2 dx dy$ and see if we get the same result.

$$\begin{aligned} \int_0^3 xy^2 dx &= y^2 \int_0^3 x dx = y^2 \left[\frac{1}{2}x^2 \right]_{x=0}^{x=3} \\ &= \frac{y^2}{2} [3^2 - 0^2] = \frac{9y^2}{2} \end{aligned}$$

Substituting this back into the iterated integral:

$$\begin{aligned} \int_1^2 \int_0^3 xy^2 dx dy &= \int_1^2 \frac{9y^2}{2} dy = \frac{9}{2} \int_1^2 y^2 dy \\ &= \frac{9}{2} \left[\frac{1}{3}y^3 \right]_{y=1}^{y=2} = \frac{9}{2} \cdot \frac{1}{3} [2^3 - 1^3] \\ &= \frac{3}{2}(8 - 1) = \frac{21}{2} \end{aligned}$$

Answer to Exercise 13 (on page 35)

1. Answer: $\frac{5}{2} - \frac{1}{e}$. Solution: $\int_0^1 \int_1^2 (x + e^{-y}) dx dy = \int_0^1 \left(\frac{1}{2}x^2 + xe^{-y} \right) \Big|_{x=1}^{x=2} dy = \int_0^1 \left(2 - \frac{1}{2} + 2e^{-y} - e^{-y} \right) dy = \int_0^1 \left(\frac{3}{2} + e^{-y} \right) dy = \left[\frac{3}{2}y - e^{-y} \right]_{y=0}^{y=1} = \left(\frac{3}{2}(1) - e^{-1} \right) - \left(\frac{3}{2}(0) - e^0 \right) = \frac{5}{2} - \frac{1}{e}$
2. Answer: 18. Solution: $\int_{-3}^3 \int_0^{\pi/2} (2y + y^2 \cos x) dx dy = \int_{-3}^3 [2xy + y^2 \sin x]_{x=0}^{x=\pi/2} dy = \int_{-3}^3 [(\pi y + y^2) - (0 + 0)] dy = \int_{-3}^3 (\pi y + y^2) dy = \left[\frac{\pi}{2}y^2 + \frac{1}{3}y^3 \right]_{y=-3}^{y=3} = \left(\frac{\pi}{2}(9) + \frac{1}{3}(27) \right) - \left(\frac{\pi}{2}(9) + \frac{1}{3}(-27) \right) = 9 - (-9) = 18$
3. Answer: 6. Solution: $\int_0^3 \int_0^{\pi/2} t^2 \sin^3 \theta d\theta dt = \left(\int_0^3 t^2 dt \right) \times \left(\int_0^{\pi/2} \sin^3 \theta d\theta \right) = \left[\frac{1}{3}t^3 \right]_{t=0}^{t=3} \times \left(\int_0^{\pi/2} \sin \theta \sin^2 \theta d\theta \right) = 9 \int_0^{\pi/2} \sin \theta (1 - \cos^2 \theta) d\theta = 9 \left[\int_0^{\pi/2} \sin \theta d\theta - \int_0^{\pi/2} \sin \theta \cos^2 \theta d\theta \right] = 9 \left[(-\cos \theta) \Big|_{\theta=0}^{\theta=\pi/2} + \left(\frac{1}{3} \cos^3 \theta \right) \Big|_{\theta=0}^{\theta=\pi/2} \right] = 9 \left[-(-\cos 0) + \left(-\frac{1}{3} \cos^3 0 \right) \right] = 9 \left(1 - \frac{1}{3} \right) = 9 \left(\frac{2}{3} \right) = 6$

Answer to Exercise 14 (on page 36)

1. $\int_0^1 \int_{-3}^3 \frac{xy^2}{x^2+1} dy dx$ OR $\int_{-3}^3 \int_0^1 \frac{xy^2}{x^2+1} dx dy$
2. $\int_0^{\pi/4} \int_0^1 \frac{\sec \theta}{\sqrt{1+t^2}} dt d\theta$ OR $\int_0^1 \int_0^{\pi/4} \frac{\sec \theta}{\sqrt{1+t^2}} d\theta dt$

Answer to Exercise 15 (on page 37)

1. $\iint_R \frac{xy^2}{x^2+1} dA$, $R = \{(x, y) | 0 \leq x \leq 2, -3 \leq y \leq 3\} = \int_0^2 \int_{-3}^3 \frac{xy^2}{x^2+1} dy dx = \int_0^2 \frac{x}{x^2+1} dx \cdot \int_{-3}^3 y^2 dy$. To evaluate the integral with respect to x , we use the u -substitution $u = x^2+1$, $(x)dx = \frac{1}{2}du$: $\int_0^2 \frac{x}{x^2+1} dx \cdot \int_{-3}^3 y^2 dy = \int_{x=0}^{x=2} \frac{1}{2} \frac{1}{u} du \cdot \int_{-3}^3 y^2 dy = \frac{1}{2} \ln |u| \Big|_{x=0}^{x=2} \cdot \frac{1}{3} [y^3]_{y=-3}^{y=3} = \frac{1}{2} [\ln(2^2+1) - \ln(0^2+1)] \cdot \frac{1}{3} [3^3 - (-3)^3] = \frac{1}{2} \ln 5 \cdot \frac{1}{3} (27 - (-27)) = \frac{\ln 5}{2} \frac{54}{3} = 9 \ln 5$
2. $\iint_R \frac{\tan \theta}{\sqrt{1-t^2}} dA$, $R = \{(\theta, t) | 0 \leq \theta \leq \pi/3, 0 \leq t \leq \frac{1}{2}\} = \int_0^{\pi/3} \int_0^{1/2} \frac{\tan \theta}{\sqrt{1-t^2}} dt d\theta = \left[\int_0^{\pi/3} \tan \theta d\theta \right] \cdot \left[\int_0^{1/2} \frac{1}{\sqrt{1-t^2}} dt \right]$. Recall that $\frac{d}{dt} \arcsin t = \frac{1}{\sqrt{1-t^2}}$. Applying FTC, then $\left[\int_0^{\pi/3} \tan \theta d\theta \right] \cdot \left[\int_0^{1/2} \frac{1}{\sqrt{1-t^2}} dt \right] = \left[\int_0^{\pi/3} \tan \theta d\theta \right] \cdot [\arcsin t]_{t=0}^{t=1/2} = \left[\int_0^{\pi/3} \tan \theta d\theta \right] \cdot [\arcsin \frac{1}{2} - \arcsin 0] = \left[\int_0^{\pi/3} \tan \theta d\theta \right] \cdot \left[\frac{\pi}{6} \right] = \frac{\pi}{6} \int_0^{\pi/3} \frac{\sin \theta}{\cos \theta} d\theta$. To evaluate this final integral, we use the u -substitution $u = \cos \theta$ and $-du = \sin \theta d\theta$: $\frac{\pi}{6} \int_0^{\pi/3} \frac{\sin \theta}{\cos \theta} d\theta = -\frac{\pi}{6} \int_{\theta=0}^{\theta=\pi/3} \frac{1}{u} du = -\frac{\pi}{6} \ln u \Big|_{\theta=0}^{\theta=\pi/3} = -\frac{\pi}{6} [\ln(\cos \theta)]_{\theta=0}^{\theta=\pi/3} = \frac{\pi}{6} [\ln(\cos 0) - \ln(\cos \frac{\pi}{3})] = \frac{\pi}{6} [\ln 1 - \ln \frac{1}{2}] = \frac{\pi}{6} \ln \frac{1}{1/2} = \frac{\pi}{6} \ln 2$
3. $\iint_R x \sin(x+y) dA$, $R = [0, \pi/6] \times [0, \pi/3] = \int_0^{\pi/6} \int_0^{\pi/3} x \sin(x+y) dy dx$. Recall the sum formula for sine:

$$\sin(x+y) = \sin x \cos y + \cos x \sin y$$

We can substitute this into our iterated integral:

$$\begin{aligned} \int_0^{\pi/6} \int_0^{\pi/3} x \sin(x+y) dy dx &= \int_0^{\pi/6} \int_0^{\pi/3} x [\sin x \cos y + \cos x \sin y] dy dx \\ &= \int_0^{\pi/6} \left[\int_0^{\pi/3} x \sin x \cos y dy + \int_0^{\pi/3} x \cos x \sin y dy \right] dx \end{aligned}$$

Let us designate $\int_0^{\pi/3} x \sin x \cos y dy$ as integral **A**, and $\int_0^{\pi/3} x \cos x \sin y dy$ as integral

B. First, we will evaluate integral **A**:

$$\begin{aligned}
 \int_0^{\pi/3} x \sin x \cos y \, dy &= x \sin x \int_0^{\pi/3} \cos y \, dy \\
 &= x \sin x [\sin y]_{y=0}^{y=\pi/3} = x \sin x \left[\sin \frac{\pi}{3} - \sin 0 \right] \\
 &= x \sin x \left(\frac{\sqrt{3}}{2} \right) = \frac{x\sqrt{3}}{2} \sin x
 \end{aligned}$$

Next, we evaluate integral **B**:

$$\begin{aligned}
 \int_0^{\pi/3} x \cos x \sin y \, dy &= x \cos x \int_0^{\pi/3} \sin y \, dy \\
 &= x \cos x [-\cos y]_{y=0}^{y=\pi/3} = x \cos x \left[-\cos \frac{\pi}{3} - (-\cos 0) \right] \\
 &= x \cos x \left[-\frac{1}{2} - (-1) \right] = \frac{x}{2} \cos x
 \end{aligned}$$

Substituting back in for integrals **A** and **B**:

$$\begin{aligned}
 \int_0^{\pi/6} \left[\int_0^{\pi/3} x \sin x \cos y \, dy + \int_0^{\pi/3} x \cos x \sin y \, dy \right] dx &= \int_0^{\pi/6} \left[\frac{x\sqrt{3}}{2} \sin x + \frac{x}{2} \cos x \right] dx \\
 &= \frac{\sqrt{3}}{2} \int_0^{\pi/6} x \sin x \, dx + \frac{1}{2} \int_0^{\pi/6} x \cos x \, dx
 \end{aligned}$$

Again, we will designate $\int_0^{\pi/6} x \sin x \, dx$ as integral **C** and $\int_0^{\pi/6} x \cos x \, dx$ as integral **D**. We start by using integration by parts to evaluate integral **C**:

Let $u = x$ and $dv = \sin x \, dx$. Then, $v = -\cos x$ and $du = dx$. Therefore:

$$\begin{aligned}
 \int_0^{\pi/6} x \sin x \, dx &= [x(-\cos x)]_{x=0}^{x=\pi/6} - \int_0^{\pi/6} (-\cos x) \, dx \\
 &= \left[\frac{\pi}{6} \left(-\cos \frac{\pi}{6} \right) \right] - [0(-\cos 0)] + \sin x \Big|_{x=0}^{x=\pi/6} \\
 &= -\frac{\pi}{6} \cdot \frac{\sqrt{3}}{2} - 0 + \sin \frac{\pi}{6} - \sin 0 = \frac{1}{2} - \frac{\pi\sqrt{3}}{12} = \frac{6 - \pi\sqrt{3}}{12}
 \end{aligned}$$

Next, we will use integration by parts to evaluate integral **D**. Let $u = x$ and $dv = \cos x \, dx$. Then $du = dx$ and $v = \sin x$ and therefore:

$$\begin{aligned}
 \int_0^{\pi/6} x \cos x \, dx &= [x \sin x]_{x=0}^{x=\pi/6} - \int_0^{\pi/6} \sin x \, dx \\
 &= \left[\frac{\pi}{6} \sin \frac{\pi}{6} - 0 \sin 0 \right] - (-\cos x) \Big|_{x=0}^{x=\pi/6} = \frac{\pi}{6} \cdot \frac{1}{2} + \cos \frac{\pi}{6} - \cos 0
 \end{aligned}$$

$$= \frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1 = \frac{\pi + 6\sqrt{3} - 12}{12}$$

Substituting back in for integrals **C** and **D**:

$$\begin{aligned} \frac{\sqrt{3}}{2} \int_0^{\pi/6} x \sin x \, dx + \frac{1}{2} \int_0^{\pi/6} x \cos x \, dx &= \frac{\sqrt{3}}{2} \left(\frac{6 - \pi\sqrt{3}}{12} \right) + \frac{1}{2} \left(\frac{\pi + 6\sqrt{3} - 12}{12} \right) \\ &= \frac{6\sqrt{3} - 3\pi + \pi + 6\sqrt{3} - 12}{24} = \frac{6\sqrt{3} - 6 - \pi}{12} \end{aligned}$$

Answer to Exercise 16 (on page 47)

1. $\iint_D e^{-y^2} \, dA = \int_0^3 \int_0^{2y} e^{-y^2} \, dx \, dy = \int_0^3 \left[e^{-y^2} x \Big|_{x=0}^{x=2y} \right] \, dy = \int_0^3 2ye^{-y^2} \, dy = -e^{-y^2} \Big|_{y=0}^{y=3} = 1 - e^{-9} \approx 0.9999$. The order matters here: e^{-y^2} has no elementary antiderivative in y , so integrating x first is the only way to finish by hand.
2. $\iint_D x \sin y \, dA = \int_0^2 \int_0^{x^2} x \sin y \, dy \, dx = \int_0^2 x \int_0^{x^2} \sin y \, dy \, dx = \int_0^2 x [-\cos y]_{y=0}^{y=x^2} \, dx$
 $= \int_0^2 x (\cos 0 - \cos x^2) \, dx = \int_0^2 (x - x \cos x^2) \, dx = \left[\frac{1}{2}x^2 - \frac{1}{2} \sin x^2 \right]_{x=0}^{x=2} = \frac{1}{2}(2)^2 - \frac{1}{2}(\sin 2^2 - \sin 0)$
 $= 2 - \frac{1}{2}(\sin 4 - 0) = 2 - \frac{\sin 4}{2} \approx 2.378$
3. We can describe the region as $D = \{(x, y) \mid -3 \leq x \leq 3, -\sqrt{9-x^2} \leq y \leq \sqrt{9-x^2}\}$.
Therefore, $\iint_D (2x - y) \, dA = \int_{-3}^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} (2x - y) \, dy \, dx = \int_{-3}^3 \left[2xy - \frac{1}{2}y^2 \right]_{y=-\sqrt{9-x^2}}^{y=\sqrt{9-x^2}} \, dx$
 $= \int_{-3}^3 \left[2x \left(\sqrt{9-x^2} + \sqrt{9-x^2} \right) - \frac{1}{2} (9-x^2 - (9-x^2)) \right] \, dx = \int_{-3}^3 4x\sqrt{9-x^2} \, dx$. Let $u = 9-x^2$, then $du = -2x \, dx$ and $4x \, dx = -2 \, du$. Substituting, $\int_{-3}^3 4x\sqrt{9-x^2} \, dx = \int_{x=-3}^{x=3} -2\sqrt{u} \, du$
 $= -2 \cdot \frac{2}{3} u^{3/2} \Big|_{x=-3}^{x=3} = -\frac{4}{3} \left[(9-x^2)^{3/2} \right]_{x=-3}^{x=3} = 0$. You could also see this without any work: the disk is symmetric, $2x$ is positive on the right half and negative on the left, and y likewise cancels top against bottom.



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