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Series

To walk to the end of a hallway, you must first walk half of it. Then half of what is left. Then half of *that*. You take infinitely many steps, yet you reach the end. That means $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$ must add up to exactly 1. Adding infinitely many numbers and getting a finite answer is the subject of this chapter.

You have done it before without noticing. Every infinite decimal, such as the Golden Ratio $\phi = 1.618033988\cdots$, is an infinite sum:

$$\phi = 1 + \frac{6}{10} + \frac{1}{10^2} + \frac{8}{10^3} + \frac{0}{10^4} + \frac{3}{10^5} + \cdots$$

You might recall from the chapter on Riemann Sums that we can write the sum of many terms with sigma notation:

$$\sum_{i=1}^n a_i$$

where i is the index and n is the number of terms. When the sum goes on forever, $\sum_{i=1}^{\infty} a_i$, we call it a *series*.

1.1 Partial Sums

A *partial sum* adds up only the first n terms of a series. For the series $\sum_{i=1}^{\infty} a_i$, the partial sums are:

$$s_1 = a_1$$

$$s_2 = a_1 + a_2$$

$$s_3 = a_1 + a_2 + a_3$$

...

$$s_n = a_1 + a_2 + \cdots + a_n = \sum_{i=1}^n a_i$$

Example: A series is given by $\sum_{i=1}^{\infty} (-\frac{3}{4})^i$. What is the value of the partial sum s_4 ?

Solution: s_4 is the sum of the first 4 terms:

$$(-\frac{3}{4})^1 + (-\frac{3}{4})^2 + (-\frac{3}{4})^3 + (-\frac{3}{4})^4$$

$$= \frac{-3}{4} + \frac{9}{16} + \frac{-27}{64} + \frac{81}{256} = \frac{-75}{256}$$

1.2 Reindexing

Sometimes it is necessary to re-index series. This means changing what n the series starts at. In general,

$$\sum_{n=i}^{\infty} a_n = \sum_{n=i+1}^{\infty} a_{n-1} \text{ and } \sum_{n=i}^{\infty} a_n = \sum_{n=i-1}^{\infty} a_{n+1}$$

In other words, to increase the index by 1, you need to replace n with $(n - 1)$ and to decrease the index by 1, you need to replace n with $(n + 1)$. Let us visualize why this is true (see figure 1.1). Notice that for each series, the terms are the same. This is similar to shifting functions: to move the function to the left on the x -axis, you plot $f(x + 1)$, and to move it to the right, $f(x - 1)$.

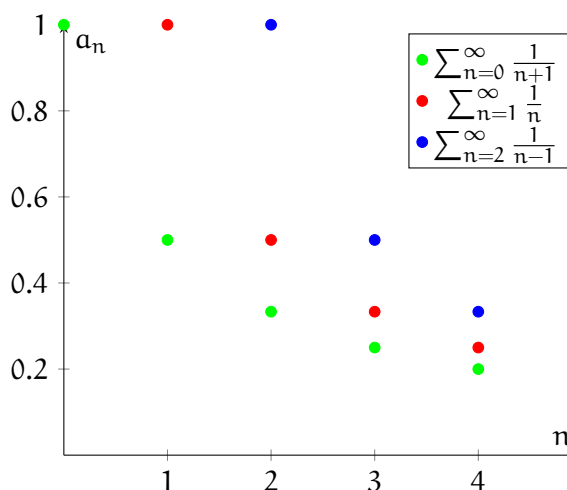


Figure 1.1: $\sum_{n=0}^{\infty} \frac{1}{n+1} = \sum_{n=1}^{\infty} a_n = \sum_{n=2}^{\infty} \frac{1}{n-1} = 1 + \frac{1}{2} + \frac{1}{3} + \dots$

We can also prove each reindexing rule mathematically. Recall that

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \dots$$

We also know that

$$\sum_{n=2}^{\infty} a_{n-1} = a_{2-1} + a_{3-1} + a_{4-1} + \dots = a_1 + a_2 + a_3 + \dots$$

Therefore, $\sum_{n=1}^{\infty} a_n = \sum_{n=2}^{\infty} a_{n-1}$.

Similarly,

$$\sum_{n=0}^{\infty} a_{n+1} = a_{0+1} + a_{1+1} + a_{2+1} + \cdots = a_1 + a_2 + a_3 + \cdots = \sum_{n=1}^{\infty} a_n$$

Example: Reindex the series $\sum_{n=3}^{\infty} \frac{n+1}{n^2-2}$ to begin with $n = 1$.

Solution: We are decreasing the index, so we will use $\sum_{n=i-1}^{\infty} a_{n+1} = \sum_{n=i}^{\infty} a_n$. We will apply this rule twice, to decrease the index from 3 to 1:

$$\begin{aligned} \sum_{n=2}^{\infty} \frac{(n+1)+1}{(n+1)^2-2} &= \sum_{n=2}^{\infty} \frac{n+2}{(n+1)^2-2} \\ \sum_{n=1}^{\infty} \frac{(n+1)+2}{[(n+1)+1]^2-2} &= \sum_{n=1}^{\infty} \frac{n+3}{(n+2)^2-2} \end{aligned}$$

It is easier and faster to be able to reindex a series by more than one step at a time. Using the example above, we can write an even more general rule for reindexing:

$$\sum_{n=i}^{\infty} a_n = \sum_{n=i+j}^{\infty} a_{n-j}$$

where i and j are integers. (Then, to decrease the index, you would choose a j such that $j < 0$.)

1.3 Convergent and Divergent Series

Just like sequences, series can also be convergent or divergent. Consider the series $\sum_{i=1}^{\infty} i$. Given what you already know about the meaning of "convergent" and "divergent", guess whether $\sum_{i=1}^{\infty} i$ is convergent or divergent.

Let us determine the first few partial sums of the series (shown graphically in figure 1.2):

n	Terms	Partial Sum
1	1	1
2	1+2	3
3	1+2+3	6
4	1+2+3+4	10

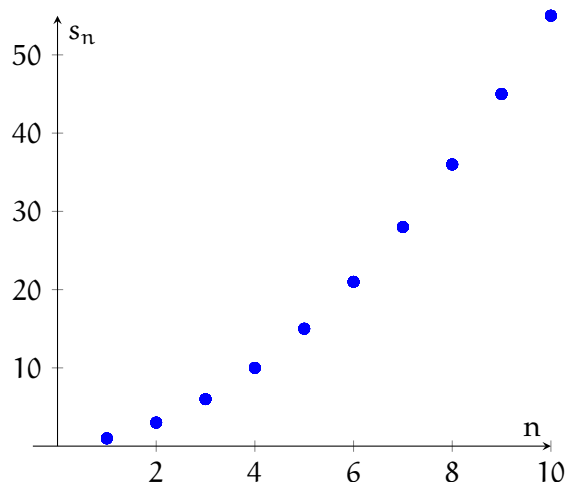


Figure 1.2: For the divergent series $\sum_{i=1}^n i$, the value of the partial sum increases to infinity as n increases

As you can see, as n increases, the value of the partial sum increases without approaching a particular value. We can also see that the value of the first n terms summed together is $\frac{n(n+1)}{2}$. This means that as n approaches ∞ , the sum also approaches ∞ and the series is divergent.

For a series to stay finite, its terms had better shrink. Take the series $\sum_{i=1}^{\infty} \frac{1}{2^i}$. As i increases, $\frac{1}{2^i}$ decreases. Let us look at the first few partial sums of this series (shown graphically in figure 1.3):

n	Terms	Partial Sum
1	$\frac{1}{2}$	$\frac{1}{2}$
2	$\frac{1}{2} + \frac{1}{4}$	$\frac{3}{4}$
3	$\frac{1}{2} + \frac{1}{4} + \frac{1}{8}$	$\frac{7}{8}$
4	$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}$	$\frac{15}{16}$

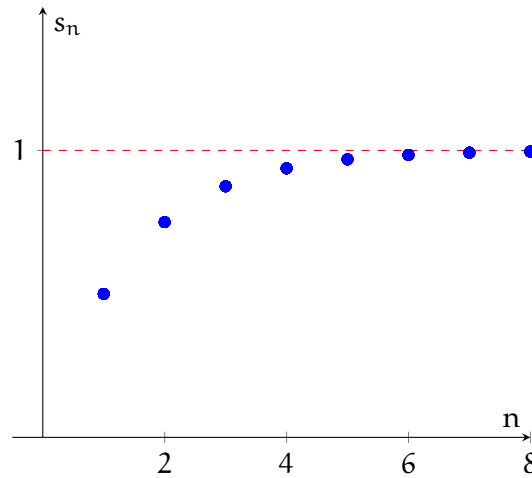


Figure 1.3: For the convergent series $\sum_{i=1}^n \frac{1}{2^i}$, the value of the partial sum approaches 1 as n increases

Do you see the pattern? The n^{th} partial sum is equal to $\frac{2^n - 1}{2^n} = 1 - \frac{1}{2^n}$. And as n approaches ∞ , the partial sum approaches 1. The series $\sum_{i=1}^{\infty} \frac{1}{2^i}$ is convergent.

Let us define the sequence $\{s_n\}$, where s_n is the n^{th} partial sum of a series:

$$s_n = \sum_{i=1}^n a_i$$

If the sequence $\{s_n\}$ converges to s , then the series $\sum_{i=1}^{\infty} a_i$ is also convergent, and its sum is s . If the sequence $\{s_n\}$ is divergent, then the series $\sum_{i=1}^{\infty} a_i$ is also divergent.

Example: Is the harmonic series, $\sum_{n=1}^{\infty} \frac{1}{n}$ convergent or divergent?

Solution: You may think that the series is convergent, since $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$. Let us see if we can confirm this. We begin by looking at the partial sums s_2 , s_4 , s_8 , and s_{16} :

$$s_2 = 1 + \frac{1}{2}$$

$$s_4 = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) = 1 + \frac{2}{2}$$

$$s_8 = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) = 1 + \frac{3}{2}$$

$$s_{16} = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \left(\frac{1}{9} + \cdots + \frac{1}{16}\right) >$$

$$1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \left(\frac{1}{16} + \cdots + \frac{1}{16}\right) = 1 + \frac{4}{2}$$

Notice that, in general, $s_{2^n} > 1 + \frac{n}{2}$ for $n \geq 1$. Taking the limit as $n \rightarrow \infty$, we see that $\lim_{n \rightarrow \infty} s_{2^n} > \lim_{n \rightarrow \infty} 1 + \frac{n}{2} = \infty$. Therefore, s_{2^n} also approaches ∞ as n gets larger and the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

This example shows a very important point: A series whose terms decrease to zero as n gets large is not necessarily convergent. What we can say, though, is that if the limit of the terms a_n as n approaches infinity does not exist or is not zero, then the series is divergent. This is called the **Test for Divergence**, and we will explore it further in the next chapter.

1.3.1 Properties of Convergent Series

We just saw that if $\lim_{n \rightarrow \infty} a_n \neq 0$ then the series $\sum_{n=1}^{\infty} a_n$ diverges. The contrapositive statement gives a property of convergent series:

$$\text{If the series } \sum_{n=1}^{\infty} a_n \text{ is convergent, then } \lim_{n \rightarrow \infty} a_n = 0$$

If a series is made of other convergent series, it may be convergent. Recall, if a series is convergent, this means the $\lim_{n \rightarrow \infty} \sum_{i=1}^n a_i = L$. By the properties of limits, we can also say that the series multiplied by a constant is convergent:

$$\sum_{n=1}^{\infty} c a_n = c \cdot L = c \sum_{n=1}^{\infty} a_n$$

Suppose there is another convergent series such that $\lim_{n \rightarrow \infty} \sum_{i=1}^n b_i = M$. In this case, the sum of those series is also convergent. That is:

$$\sum_{n=1}^{\infty} (a_n + b_n) = L + M = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$$

Similarly, the difference of the series is convergent:

$$\sum_{n=1}^{\infty} (a_n - b_n) = L - M = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$$

1.4 Geometric Series

A geometric series is the sum of a geometric sequence, and has the form:

$$\sum_{n=1}^{\infty} ar^n \text{ or } \sum_{n=1}^{\infty} ar^{n-1}$$

Where a is some constant and r is the common ratio. For $\sum_{n=1}^{\infty} ar^{n-1}$, a is also the first term.

Example: Write the series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots$ in sigma notation.

Solution: We see that the first term is $a = 1$ and the common ratio is $\frac{1}{2}$, so we can write the series:

$$\sum_{n=1}^{\infty} 1\left(\frac{1}{2}\right)^{n-1} = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1}$$

Geometric series appear in everyday situations. Suppose a ball dropped from 1 m bounces back to half of its previous height each time. The total distance it travels is $1 + 2\left(\frac{1}{2} + \frac{1}{4} + \dots\right) = 1 + 2(1) = 3$ m. Each bounce adds the up and down trips, and the geometric series gives the total.

When are geometric series convergent? First, let us consider the case where $r = 1$. If this is true, then $s_n = a + a + a + \dots + a = na$. As n approaches ∞ , the sum will approach $\pm\infty$ (depending on whether a is positive or negative), and the series is divergent.

When $r \neq 1$, we can write s_n and rs_n :

$$s_n = a + ar + ar^2 + \dots + ar^{n-1}$$

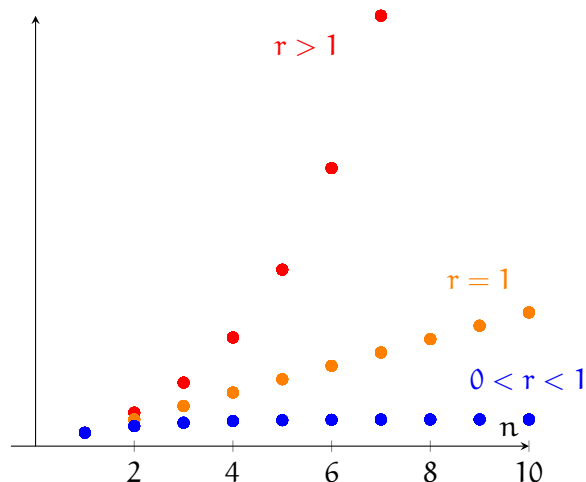
$$rs_n = ar + ar^2 + ar^3 + \dots + ar^n$$

Subtracting rs_n from s_n , we get:

$$\begin{aligned} s_n - rs_n &= (a + ar + ar^2 + \dots + ar^{n-1}) - (ar + ar^2 + ar^3 + \dots + ar^{n-1} + ar^n) \\ &= a - ar^n \end{aligned}$$

Solving for s_n , we find:

$$s_n = \frac{a(1 - r^n)}{1 - r}$$

Figure 1.4: Geometric series are divergent if $r \geq 1$

We take the limit as $n \rightarrow \infty$ to determine for what values of r the series converges:

$$\begin{aligned} \lim_{n \rightarrow \infty} s_n &= \lim_{n \rightarrow \infty} \frac{a(1 - r^n)}{1 - r} \\ &= \lim_{n \rightarrow \infty} \left[\frac{a}{1 - r} - \frac{ar^n}{1 - r} \right] = \frac{a}{1 - r} - \left(\frac{a}{1 - r} \right) \lim_{n \rightarrow \infty} r^n \end{aligned}$$

This introduces the question: When is $\lim_{n \rightarrow \infty} r^n$ convergent? From the sequences chapter, we know this limit converges if $|r| < 1$ (that is, $-1 < r < 1$). If this is true, then $\lim_{n \rightarrow \infty} r^n = 0$ and

$$\lim_{n \rightarrow \infty} s_n = \frac{a}{1 - r}$$

(see figures 1.4 and 1.5 for a visual)

Example: Find the sum of the geometric series given by $2 - \frac{2}{3} + \frac{2}{9} - \frac{2}{27} + \cdots$.

Solution: The first term is $a = 2$, and the common ratio is $r = -\frac{1}{3}$. Since $|r| < 1$, we know that the series converges. We can calculate the value of the sum using the geometric series formula:

$$\begin{aligned} \sum_{i=1}^{\infty} a(r)^{i-1} &= \frac{a}{1 - r} \\ \sum_{i=1}^{\infty} 2\left(-\frac{1}{3}\right)^{i-1} &= \frac{2}{1 - (-\frac{1}{3})} = \frac{2}{\frac{4}{3}} = \frac{6}{4} = 1.5 \end{aligned}$$

We can confirm this graphically (see figure 1.6). You can also write out the first several partial sums. You should find the sums approach 1.5 as n increases.

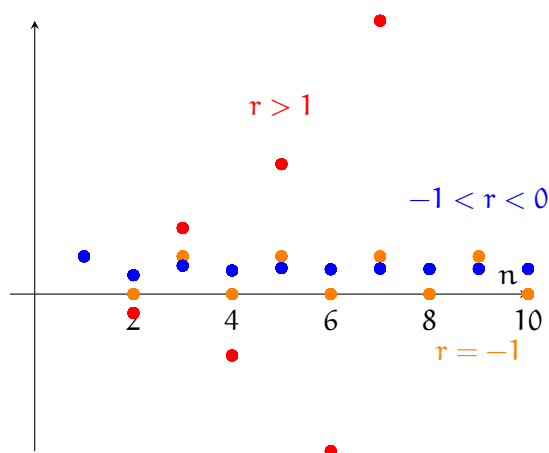


Figure 1.5: Geometric series are divergent if $r \leq -1$. Notice that for $r = -1$, the partial sums alternate between the initial term and zero.

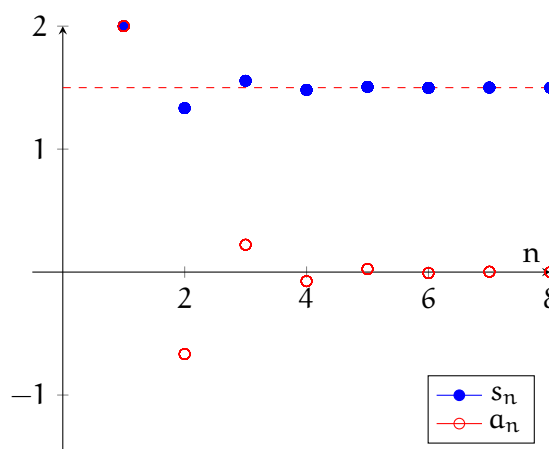


Figure 1.6: The n^{th} term and partial sums of $\sum_{i=1}^n 2\left(\frac{-1}{3}\right)^{i-1}$

Example: What is the value of $\sum_{n=1}^{\infty} 2^{2n} 5^{1-n}$

Solution: The key here is to re-write the series in the form $\sum_{n=1}^{\infty} ar^{n-1}$ so we can use the fact that convergent geometric series sum to $\frac{a}{1-r}$.

$$\begin{aligned} \sum_{n=1}^{\infty} 2^{2n} 5^{1-n} &= \sum_{n=1}^{\infty} (2^2)^n \left(\frac{1}{5}\right)^{n-1} \\ &= \sum_{n=1}^{\infty} 4 \cdot (4)^{n-1} \left(\frac{1}{5}\right)^{n-1} = \sum_{n=1}^{\infty} 4 \cdot \left(\frac{4}{5}\right)^{n-1} \end{aligned}$$

Which is in the form $\sum_{n=1}^{\infty} ar^{n-1}$ with $a = 4$ and $r = \frac{4}{5}$. Since $|r| < 1$, the series converges

to

$$\frac{a}{1-r} = \frac{4}{1-\frac{4}{5}} = \frac{4}{\frac{1}{5}} = 20$$

Exercise 1

Determine whether the geometric series is convergent or divergent. If it is convergent, find its sum.

1. $3 - 4 + \frac{16}{3} - \frac{64}{9} + \cdots$

2. $2 + 0.5 + 0.125 + 0.03125 + \cdots$

3. $\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{4^n}$

4. $\sum_{n=1}^{\infty} \frac{e^{2n}}{6^{n-1}}$

Working Space

Answer on Page 65

Exercise 2

Find a value of c such that $\sum_{n=0}^{\infty} (1 + c)^{-n} = \frac{5}{3}$.

Working Space

Answer on Page 65

Exercise 3

For what values of p does the series $\sum_{n=1}^{\infty} \left(\frac{p}{2}\right)^n$ converge?

Working Space

Answer on Page 66

1.5 p-series

A p-series takes the form $\sum_{n=1}^{\infty} \frac{1}{n^p}$ and converges if $p > 1$ and diverges if $p \leq 1$. We will not prove this here, since it requires the application of a test you will learn about in the next chapter.

Example: Write the series $1 + \frac{1}{\sqrt[3]{2}} + \frac{1}{\sqrt[3]{3}} + \frac{1}{\sqrt[3]{4}} + \cdots$. Is it convergent or divergent?

Solution: We see that $a_n = \frac{1}{\sqrt[3]{n}}$, so the infinite series is

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}}$$

We see that this is a p-series with $p = \frac{1}{3}$. Since $p < 1$, the series is divergent.

Exercise 4

Euler found that the exact sum of the p-series where $p = 2$ is:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

And that the exact sum of the p-series where $p = 4$ is:

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$$

Use this and the properties of convergent series to find the sum of each of the following series:

1. $\sum_{n=1}^{\infty} \frac{n^2+1}{n^4}$
2. $\sum_{n=2}^{\infty} \frac{1}{n^2}$
3. $\sum_{n=3}^{\infty} \frac{1}{(n+1)^2}$
4. $\sum_{n=1}^{\infty} \left(\frac{3}{n}\right)^4$
5. $\sum_{n=1}^{\infty} \left(\frac{4}{n^2} + \frac{3}{n^4}\right)$

Working Space

Answer on Page 66

Exercise 5

For what values of k does the series

$$\sum_{n=1}^{\infty} \frac{1}{n^{2k}} \text{ converge?}$$

Working Space

Answer on Page 66

1.6 Alternating Series

An alternating series is one in which the terms alternate between positive and negative . Here is an example:

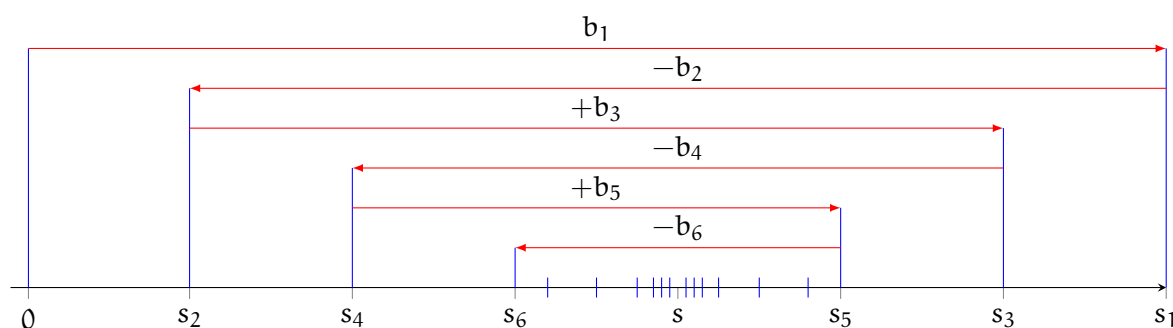
$$-\frac{1}{2} + \frac{2}{3} - \frac{3}{4} + \frac{4}{5} - \frac{5}{6} + \cdots = \sum_{n=1}^{\infty} (-1)^n \frac{n}{n+1}$$

Alternating series are generally of the form

$$a_n = (-1)^n b_n \text{ or } a_n = (-1)^{n-1} b_n$$

Where b_n is positive (and therefore, $|a_n| = b_n$).

An alternating series is convergent if (i) $b_{n+1} \leq b_n$ and (ii) $\lim_{n \rightarrow \infty} b_n = 0$. In other words, if the sizes of the terms shrink toward zero, the series converges. Each step backtracks a little less than the step before it, so the partial sums zero in on a single value. This is called the **Alternating Series Test**.


 Figure 1.7: As n increases, s_n approaches s

Example: Is the alternating harmonic series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ convergent?

Solution: We check both conditions of the Alternating Series Test. First, the terms shrink:

$$\left| \frac{(-1)^{n-1+1}}{n+1} \right| < \left| \frac{(-1)^{n-1}}{n} \right|$$

$$\frac{1}{n+1} < \frac{1}{n}$$

Second, $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$. Both conditions hold, so $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ is convergent.

Exercise 6

Test the following alternating series for convergence:

1. $\sum_{n=1}^{\infty} \frac{(-1)^n 3n}{4n-1}$
2. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^3+1}$
3. $\sum_{n=1}^{\infty} (-1)^{n-1} e^{2/n}$

Working Space

Answer on Page 66

Convergence Tests for Series

Does $\frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \dots$ add up to something finite? What about $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \dots$? The terms look almost the same, yet one sum is finite and the other grows forever. Most series do not have a formula for their partial sums, so we need tests that tell us which kind we have without adding everything up. This chapter is your toolbox.

2.1 Test for Divergence

Recall from the previous chapter that if the terms of a series do not approach zero as n approaches infinity, then the series is divergent. This is the Test for Divergence, and there are two possible outcomes. For a series $\sum_{n=1}^{\infty} a_n$:

If $\lim_{n \rightarrow \infty} a_n \neq 0$, then the series diverges

If $\lim_{n \rightarrow \infty} a_n = 0$, then the test is inconclusive

It is important to remember that the Test for Divergence cannot tell us conclusively that a series converges. Rather, it only identifies series that are divergent.

Example: Apply the Test for Divergence to the series $\sum_{n=1}^{\infty} \sqrt{n}$ and $\sum_{n=1}^{\infty} \frac{1}{n}$

Solution: $\lim_{n \rightarrow \infty} \sqrt{n} = \infty \neq 0$. Therefore, the series $\sum_{n=1}^{\infty} \sqrt{n}$ is divergent. $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$. Therefore, the series $\sum_{n=1}^{\infty} \frac{1}{n}$ may be divergent or convergent. This is the harmonic series, which we proved to be divergent in the previous chapter. This is a good example that demonstrates that just because $\lim_{n \rightarrow \infty} a_n = 0$ does not mean the series is convergent.

2.2 The Integral Test

We were able to determine the exact value of some infinite series because it was possible to write the n^{th} partial sum, s_n , in terms of n . For example, we determined that the n^{th} partial sum of $\sum_{i=1}^n \frac{1}{2^i}$ is $s_n = 1 - \frac{1}{2^n}$. However, it is not always possible to do this. How can we estimate the value of an infinite series in cases where we cannot explicitly write s_n in terms of n ?

Consider the series $\sum_{i=1}^{\infty} \frac{1}{i^2}$. The first few terms are:

$$\sum_{i=1}^{\infty} \frac{1}{i^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots$$

The terms are decreasing, but is the series convergent? Let us plot this series on an xy -plane (see figure 2.1).

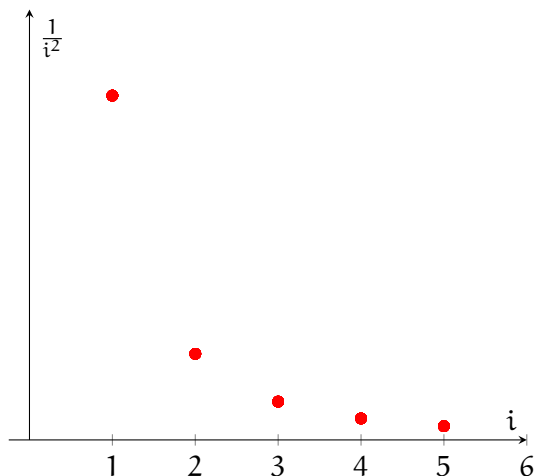


Figure 2.1: The first 5 terms of $\sum_{i=1}^{\infty} \frac{1}{i^2}$

We can overlay the function $y = \frac{1}{x^2}$ (figure 2.2). We can draw rectangles of width 1 and height $\frac{1}{x^2}$ (see figure 2.3). The area of the first n rectangles is equal to the n^{th} partial sum.

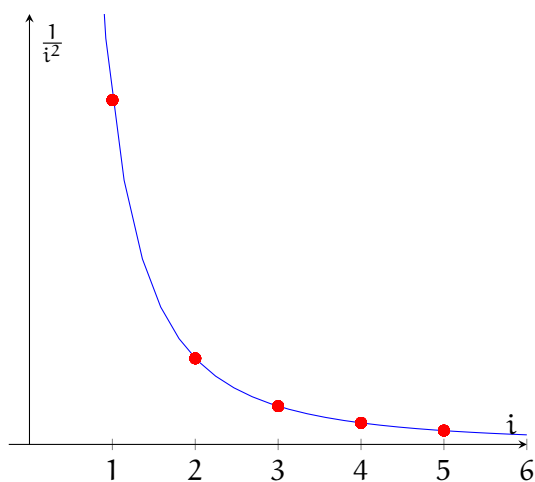


Figure 2.2: The first 5 terms of $\sum_{i=1}^{\infty} \frac{1}{i^2}$ lie on the curve $y = \frac{1}{x^2}$

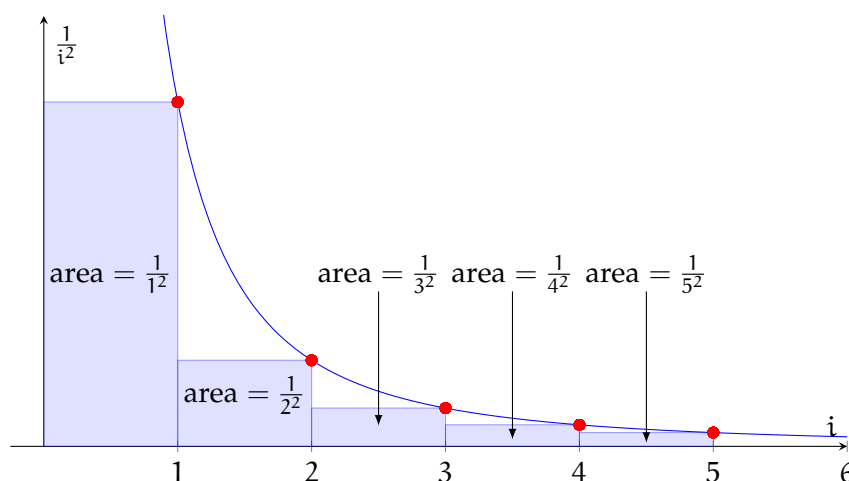


Figure 2.3: The partial sum $\sum_{i=1}^{n=5} \frac{1}{i^2}$ is equal to the area of the rectangles

This should remind you of a Riemann sum. Since the total area of the rectangles is less than the area under the curve, we can state:

$$\sum_{i=2}^{\infty} \frac{1}{i^2} < \int_1^{\infty} \frac{1}{x^2} dx$$

The first rectangle has area 1. Adding it to the bound gives:

$$\sum_{i=1}^{\infty} \frac{1}{i^2} < 1 + \int_1^{\infty} \frac{1}{x^2} dx$$

We can evaluate this integral:

$$\begin{aligned} \int_1^{\infty} \frac{1}{x^2} dx &= \lim_{t \rightarrow \infty} \left[\int_1^t \frac{1}{x^2} dx \right] \\ &= \lim_{t \rightarrow \infty} \left. \frac{-1}{x} \right|_{x=1}^t = \lim_{t \rightarrow \infty} \left(\frac{-1}{t} \right) - \frac{-1}{1} = 0 - (-1) = 1 \end{aligned}$$

Therefore:

$$\sum_{i=1}^{\infty} \frac{1}{i^2} < 1 + 1 = 2$$

This means the partial sums of $\sum_{i=1}^{\infty} \frac{1}{i^2}$ are bounded above by 2. They are also increasing (each term is positive), so they are monotonic and bounded, and the series converges.

Let us look at a divergent example: $\sum_{i=1}^{\infty} \frac{1}{\sqrt{i}}$. Again, we will make a visual, but this time we will draw rectangles that lie above the curve $y = \frac{1}{\sqrt{x}}$ (see figure 2.4). In this case, $\sum_{i=1}^{\infty} \frac{1}{\sqrt{i}} > \int_1^{\infty} \frac{1}{\sqrt{x}} dx$. Let us evaluate the integral:

$$\begin{aligned} \int_1^{\infty} \frac{1}{\sqrt{x}} dx &= \lim_{t \rightarrow \infty} \left[\int_1^t \frac{1}{\sqrt{x}} dx \right] \\ &= \lim_{t \rightarrow \infty} [2\sqrt{x}]_{x=1}^t = \lim_{t \rightarrow \infty} (2\sqrt{t}) - 2\sqrt{1} = \infty - 2 \rightarrow \text{divergent} \end{aligned}$$

Since the integral diverges to infinity and the series is greater than the integral, the series must also diverge to infinity. Its terms shrink toward zero, and yet the series diverges.

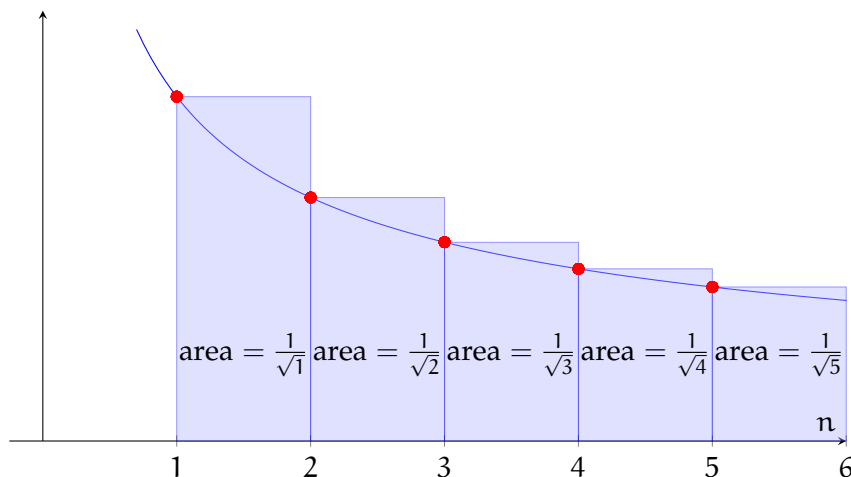


Figure 2.4: $\sum_{i=1}^{\infty} \frac{1}{\sqrt{i}} > \int_1^{\infty} \frac{1}{\sqrt{x}} dx$

The Integral Test

This leads us to the **Integral Test**. If f is a continuous, positive, decreasing function on the interval $x \in [1, \infty)$ and $a_n = f(n)$, then the series $\sum_{n=1}^{\infty} a_n$ converges if and only if $\int_1^{\infty} f(x) dx$ is convergent. If $\int_1^{\infty} f(x) dx$ is divergent, then the series is also divergent.

Example: Is the series $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ convergent or divergent?

Solution: To apply the integral test, we define $f(x) = \frac{1}{x^2+1}$, which is a positive, decreasing function on the interval $x \in [1, \infty)$.

$$\int_1^{\infty} \frac{1}{x^2+1} dx = \lim_{t \rightarrow \infty} \int_1^t \frac{1}{x^2+1} dx$$

$$= \lim_{t \rightarrow \infty} [\arctan x]_{x=1}^t = \lim_{t \rightarrow \infty} (\arctan t) - \arctan 1 = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

Because the integral $\int_1^{\infty} \frac{1}{x^2+1} dx$ converges, so does the series $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$.

Exercise 7

Use the integral test to determine if the following series are convergent or divergent.

1. $\sum_{n=1}^{\infty} 2n^{-3}$

2. $\sum_{n=1}^{\infty} \frac{5}{3n-1}$

3. $\sum_{n=1}^{\infty} \frac{n}{3n^2+1}$

Working Space

Answer on Page 67

Exercise 8

Apply the Integral Test to show that p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ are convergent only when $p > 1$ (hint: consider the cases $p \leq 0$, $0 < p < 1$, $p = 1$ and $p > 1$).

Working Space

Answer on Page 67

2.2.1 Using Integrals to Estimate the Value of a Series

Recall that $\sum_{i=1}^{\infty} a_i = a_1 + a_2 + a_3 + \cdots = s$ and that the n^{th} partial sum, often represented as s_n , is $s_n = a_1 + a_2 + \cdots + a_{n-1} + a_n$. We can then define the n^{th} remainder $R_n = s - s_n$. Expanding s and s_n , we see that:

$$R_n = [a_1 + a_2 + \cdots + a_{n-1} + a_n + a_{n+1} + \cdots] - [a_1 + a_2 + \cdots + a_{n-1} + a_n]$$

The terms a_1 through a_n cancel, so only the tail is left.

$$R_n = a_{n+1} + a_{n+2} + a_{n+3} + \cdots$$

Just like the integral test, suppose there is some continuous, positive, decreasing function, such that $a_n = f(n)$. We can then represent R_n as the right Riemann sum with width $\Delta x = 1$ from $x = n$ to ∞ . Since the rectangles are below the curve (see figure 2.5), we can state that $R_n \leq \int_n^\infty f(x) dx$.

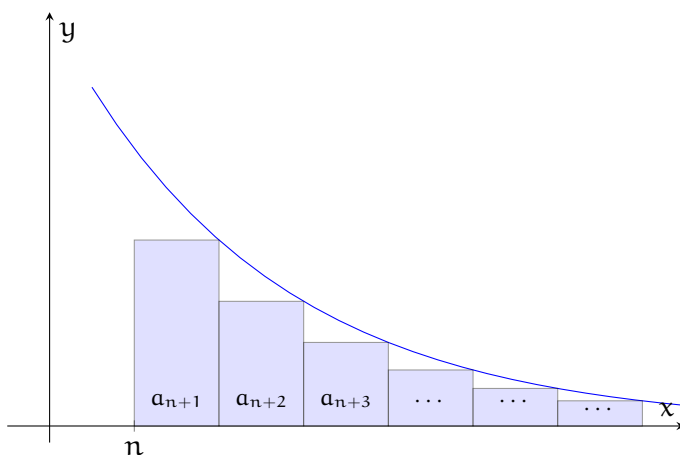


Figure 2.5: $R_n \leq \int_n^\infty f(x) dx$

Similarly, we can represent R_n as the left Riemann sum with width $\Delta x = 1$ from $x = n + 1$ to ∞ . This time, the rectangles are above the curve (see figure 2.6), and we can state that $R_n \geq \int_{n+1}^\infty f(x) dx$. Putting this all together, we have an estimate for the remainder, R_n , from the integral test:

Suppose there is a function such that $f(k) = a_k$, where f is a continuous, positive, decreasing function for $x \geq n$ and $\sum a_n$ is convergent. Then, $\int_{n+1}^\infty f(x) dx \leq R_n \leq \int_n^\infty f(x) dx$, where R_n is $s - s_n$.

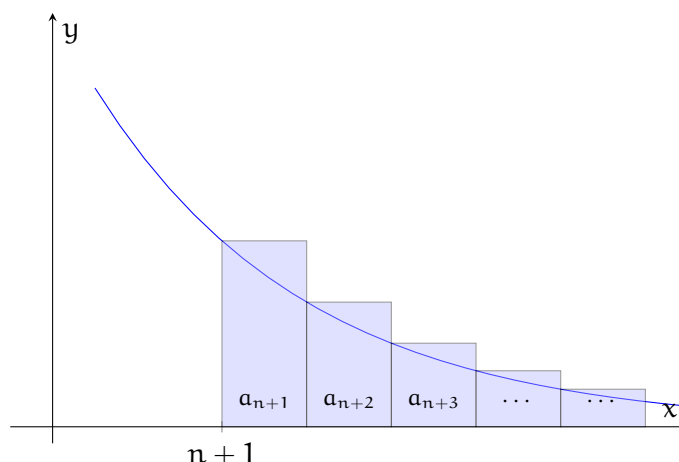


Figure 2.6: $R_n \geq \int_{n+1}^{\infty} f(x) \, dx$

Example: Approximate the sum of the series $\sum_{n=1}^{\infty} \frac{3}{n^3}$ by finding the 10th partial sum. Estimate the error of this approximation.

Solution: Using a calculator, you can find the 10th partial sum:

$$\sum_{n=1}^{10} \frac{3}{n^3} = \frac{3}{1^3} + \frac{3}{2^3} + \frac{3}{3^3} + \cdots + \frac{3}{10^3} \approx 3.593 = s_{10}$$

Recall that the remainder, R_{10} , is the difference between the actual sum, s , and the partial sum, s_{10} . Using the integral test to estimate the remainder, we can state that:

$$R_{10} \leq \int_{10}^{\infty} \frac{3}{x^3} \, dx = \frac{3}{2(10)^2} = \frac{3}{200} = 0.015$$

Therefore, the size of the error is at most 0.015.

Example: How many terms are required for the error to be less than 0.0001 for the sum presented above?

Solution: We are looking for an n such that $R_n \leq 0.0001$. Recalling that $R_n \leq \int_n^{\infty} \frac{3}{x^3} \, dx$, we need to find an n such that $\int_n^{\infty} \frac{3}{x^3} \, dx \leq 0.0001$.

$$\begin{aligned} \int_n^{\infty} \frac{3}{x^3} \, dx &\leq 0.0001 \\ \left. \frac{-3}{2x^2} \right|_{x=n}^{\infty} &\leq 0.0001 \\ \lim_{x \rightarrow \infty} \frac{-3}{2x^2} - \frac{-3}{2n^2} &\leq 0.0001 \end{aligned}$$

$$0 + \frac{3}{2n^2} = \frac{3}{2n^2} \leq 0.0001$$

$$15000 \leq n^2$$

$$122.5 \leq n \rightarrow n = 123$$

Therefore, $s - s_{123} \leq 0.0001$ and the partial sum $\sum_{n=1}^{123} \frac{3}{n^3}$ is less than 0.0001 from the value of the infinite sum $\sum_{n=1}^{\infty} \frac{3}{n^3}$.

Exercise 9

Working Space

- Find the partial sum s_{10} of the series $\sum_{n=1}^{\infty} \frac{1}{n^4}$.
- Estimate the error from using s_{10} as an approximation of the series.
- Use $s_n + \int_{n+1}^{\infty} \frac{1}{x^4} dx \leq s \leq s_n + \int_n^{\infty} \frac{1}{x^4} dx$ to give an improved estimate of the sum.
- The actual value of $\sum_{n=1}^{\infty} \frac{1}{n^4}$ is $\frac{\pi^4}{90}$. Compare your estimate with the actual value.
- Find a value of n such that s_n is within 0.00001 of the sum.

Answer on Page 68

2.3 Comparison Tests

In comparison tests, we compare a series to a known convergent or divergent series. Take the series $\sum_{n=1}^{\infty} \frac{1}{3^n+3}$. This is similar to $\sum_{n=1}^{\infty} \frac{1}{3^n}$, which is a geometric series that converges to $\frac{1}{2}$. Notice that:

$$\frac{1}{3^n+3} < \frac{1}{3^n}$$

This implies that

$$\sum_{n=1}^{\infty} \frac{1}{3^n+3} < \sum_{n=1}^{\infty} \frac{1}{3^n}$$

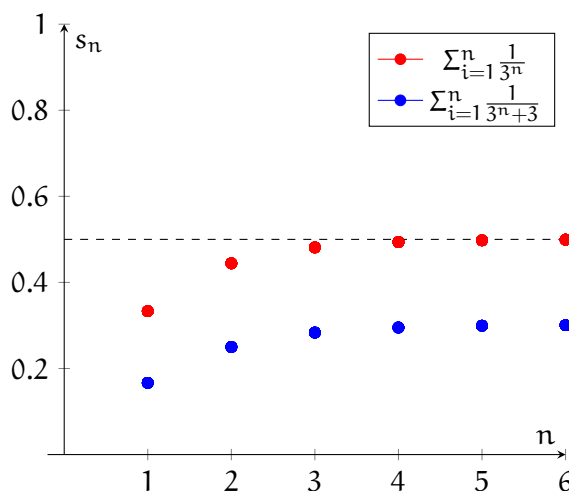


Figure 2.7: $\sum_{i=1}^n \frac{1}{3^i+3} < \sum_{i=1}^n \frac{1}{3^i}$ for all n

Since $\sum_{n=1}^{\infty} \frac{1}{3^n}$ is convergent, it follows that $\sum_{n=1}^{\infty} \frac{1}{3^n+3}$ is also convergent (see figure 2.7). As you can see, since $\sum_{n=1}^{\infty} \frac{1}{3^n}$ approaches $\frac{1}{2}$, $\sum_{n=1}^{\infty} \frac{1}{3^n+3}$ must be $\leq \frac{1}{2}$ and therefore convergent.

2.3.1 The Direct Comparison Test

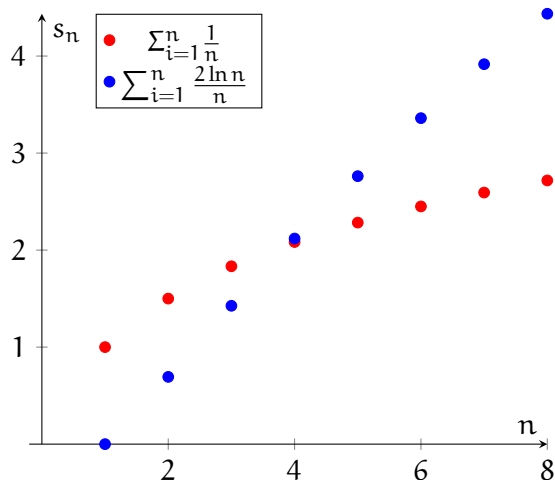
For the **Direct Comparison Test**, we compare the terms a_n to b_n directly. Take $\sum a_n$ and $\sum b_n$ to be series with positive terms. Then,

1. If $a_n \leq b_n$ and $\sum b_n$ is convergent, then $\sum a_n$ is also convergent.
2. If $a_n \geq b_n$ and $\sum b_n$ is divergent, then $\sum a_n$ is also divergent.

We already discussed above why the first part is true. The second part follows a similar argument: If every a_n is at least b_n and $\sum b_n$ grows without bound, then $\sum a_n$, which is even bigger, must grow without bound too. Consider the series $\sum_{n=1}^{\infty} \frac{2 \ln n}{n}$. For $n \geq 2$, $2 \ln n > 1$, so $\frac{2 \ln n}{n} > \frac{1}{n}$. (The first term does not matter: dropping finitely many terms never changes whether a series converges.) We recognize the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent. Therefore, $\sum_{n=1}^{\infty} \frac{2 \ln n}{n}$ is also divergent (see figure 2.8).

2.3.2 The Limit Comparison Test

Consider the series $\sum_{n=1}^{\infty} \frac{1}{2^{n-1}}$. We may want to compare this to the convergent series $\sum_{n=1}^{\infty} \frac{1}{2^n}$. The direct comparison test is not helpful here, since $\frac{1}{2^{n-1}} > \frac{1}{2^n}$, so $\sum_{n=1}^{\infty} \frac{1}{2^n}$ does not put a cap on $\sum_{n=1}^{\infty} \frac{1}{2^{n-1}}$ like our earlier example (see figure 2.7). In a case such

Figure 2.8: $\sum_{i=1}^n \frac{2 \ln i}{i} > \sum_{i=1}^n \frac{1}{i}$ for $n \geq 4$

as this, we can use the **Limit Comparison Test**, which states that if $\sum a_n$ and $\sum b_n$ are series with positive terms and $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = c > 0$, then either both series converge or both series diverge.

Let us apply this to the series $\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$. We know that $\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges, since it is a geometric series with $r < 1$.

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{\frac{1}{2^n - 1}}{\frac{1}{2^n}} &= \lim_{n \rightarrow \infty} \frac{1}{2^n - 1} \cdot \frac{2^n}{1} \\ &= \lim_{n \rightarrow \infty} \frac{2^n}{2^n - 1} = \lim_{n \rightarrow \infty} \frac{1}{1 - 1/2^n} = \frac{1}{1 - 0} = 1 > 0 \end{aligned}$$

Therefore, by the Limit Comparison Test, $\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$ converges.

In general, comparison tests are most useful for series resembling geometric or p-series. When choosing a p-series to compare the unknown series to, choose p such that the order of your p series is the same as the order of the unknown series.

Example: What p-series should one compare the series $\sum_{n=1}^{\infty} \frac{\sqrt{n^3+1}}{3n^3+4n^2+2}$ to?

Solution: We can determine the order of $\frac{\sqrt{n^3+1}}{3n^3+4n^2+2}$ by looking at the highest-order terms in the numerator and denominator:

$$\frac{\sqrt{n^3}}{n^3} = \frac{n^{3/2}}{n^3} = \frac{1}{n^{3/2}}$$

So, we should compare $\sum_{n=1}^{\infty} \frac{\sqrt{n^3+1}}{3n^3+4n^2+2}$ to the convergent p-series $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$.

Example: Is $\sum_{n=1}^{\infty} \frac{\sqrt{n^3+1}}{3n^3+4n^2+2}$ convergent or divergent?

Solution: We have already determined that we should compare this series to $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$. To apply the limit test, we need to evaluate

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{\frac{\sqrt{n^3+1}}{3n^3+4n^2+2}}{\frac{1}{n^{3/2}}} \\ &= \lim_{n \rightarrow \infty} \frac{n^{3/2}\sqrt{n^3+1}}{3n^3+4n^2+2} \\ &= \lim_{n \rightarrow \infty} \frac{\sqrt{n^6+n^3}}{3n^3+4n^2+2} = \frac{1}{3} > 0 \end{aligned}$$

Therefore, by the Limit Comparison Test, $\sum_{n=1}^{\infty} \frac{\sqrt{n^3+1}}{3n^3+4n^2+2}$ is convergent because the p-series $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ is convergent.

Exercise 10

Use the Comparison Test or the Limit Comparison Test to determine if the following series are convergent or divergent.

1. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+1}}$
2. $\sum_{n=1}^{\infty} \frac{9^n}{3+10^n}$
3. $\sum_{n=1}^{\infty} \frac{n \sin^2 n}{1+n^3}$

Working Space

Answer on Page 68

Ratio, Root, and Strategy Tests

The tests in the last chapter work best on series that look like p-series. But what about $\frac{8^n}{n!}$, where a power races a factorial? Or $\left(\frac{2n+3}{3n+2}\right)^n$, where the whole term is raised to the n th power? This chapter adds two tests built for exactly those series, then gives you a game plan for choosing the right test quickly.

3.1 Ratio and Root Tests for Convergence

3.1.1 Absolute Convergence

Suppose there is a series $\sum_{n=1}^{\infty} a_n$, then there is a corresponding series $\sum_{n=1}^{\infty} |a_n| = |a_1| + |a_2| + |a_3| + \cdots$. If $\sum_{n=1}^{\infty} |a_n|$ is convergent, then the series $\sum_{n=1}^{\infty} a_n$ is called **absolutely convergent**.

Example: Consider the alternating series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \cdots$$

Is this series absolutely convergent?

Solution: We examine the corresponding series where we take the absolute value of each term:

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n^2} \right| = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

We can identify $\sum_{n=1}^{\infty} \frac{1}{n^2}$ as a convergent p-series. Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent, we can state that $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$ is absolutely convergent.

Example: Is the convergent series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ absolutely convergent?

Solution: We consider the sum of the absolute values of the terms:

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}$$

You should recognize this as the harmonic series, which is divergent. When a series is

convergent but the corresponding series of absolute values is not, we call it **conditionally convergent**.

We will not prove the theorem here, but it is useful to know that if a series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, then it is convergent. You can prove this yourself using the Comparison Test.

Exercise 11

Is the series given by

$$\frac{\cos 1}{1^2} + \frac{\cos 2}{2^2} + \frac{\cos 3}{3^2} + \cdots$$

convergent or divergent?

Working Space

Answer on Page 69

Exercise 12

Determine whether each of the following series is absolutely or conditionally convergent.

1. $\sum_{n=1}^{\infty} \frac{(-1)^n}{3n+2}$

2. $\sum_{n=1}^{\infty} \frac{\sin n}{4^n}$

3. $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{2n}{n^2+4}$

Working Space

Answer on Page 69

3.1.2 The Ratio Test

The ratio test compares the $(n + 1)^{\text{th}}$ term of a series to the n^{th} term and takes the limit as $n \rightarrow \infty$ of the absolute value of this ratio:

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = L$$

There are three possible outcomes of the ratio test:

1. If $L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).
2. If $L = 1$, then the ratio test is inconclusive and we cannot draw any conclusions about whether $\sum_{n=1}^{\infty} a_n$ is convergent or divergent.
3. If $L > 1$ or $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

Example: Apply the ratio test to determine if $\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{3^n}$ is convergent or divergent.

Solution:

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{\frac{(-1)^{n+1}(n+1)^3}{3^{n+1}}}{\frac{(-1)^n n^3}{3^n}} \right| &= \lim_{n \rightarrow \infty} \frac{(n+1)^3}{3^{n+1}} \cdot \frac{3^n}{n^3} \\ &= \lim_{n \rightarrow \infty} \frac{(n+1)^3}{n^3} \cdot \frac{3^n}{3 \cdot 3^n} \\ &= \lim_{n \rightarrow \infty} \left(\frac{n+1}{n} \right)^3 \cdot \frac{1}{3} = \frac{1}{3} \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n} \right)^3 = \frac{1}{3} \cdot 1 = \frac{1}{3} \end{aligned}$$

Since $L < 1$, the series $\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{3^n}$ is absolutely convergent.

The ratio test is most useful for series that contain factorials, constants raised to the n^{th} power, or other products.

Exercise 13

[This question was originally presented as a multiple-choice, no-calculator problem on the 2012 AP Calculus BC exam.]
Which of the following series are convergent?

1. $\sum_{n=1}^{\infty} \frac{8^n}{n!}$
2. $\sum_{n=1}^{\infty} \frac{n!}{n^{100}}$

Working Space

3. $\sum_{n=1}^{\infty} \frac{n+1}{(n)(n+2)(n+3)}$

Answer on Page 70

Exercise 14

[This question was originally presented as a multiple-choice, calculator-allowed problem on the 2012 AP Calculus BC exam.] If the series $\sum_{n=1}^{\infty} a_n$ converges and $a_n > 0$ for all n , which of the following statements must be true? Explain why.

Working Space

1. $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = 0$
2. $|a_n| < 1$ for all n
3. $\sum_{n=1}^{\infty} a_n = 0$
4. $\sum_{n=1}^{\infty} na_n$ diverges
5. $\sum_{n=1}^{\infty} \frac{a_n}{n}$ converges

Answer on Page 70

3.1.3 Root Test

The root test examines the behavior of the n^{th} root of a_n as $n \rightarrow \infty$. Similar to the ratio test, there are three possible outcomes:

1. If $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, and therefore convergent.
2. If $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = L > 1$ or $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.
3. If $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = L = 1$, then the Root Test is inconclusive.

The root test is best when there is a term or terms raised to the n^{th} power.

Example: Is the series $\sum_{n=1}^{\infty} \left(\frac{2n+3}{3n+2}\right)^n$ convergent or divergent?

Solution: Since a_n consists of terms raised to the n^{th} power, we will apply the root test for convergence:

$$\lim_{n \rightarrow \infty} \sqrt[n]{\left|\left(\frac{2n+3}{3n+2}\right)^n\right|} = \lim_{n \rightarrow \infty} \frac{2n+3}{3n+2} = \frac{2}{3} < 1$$

Therefore, by the root test, the series $\sum_{n=1}^{\infty} \left(\frac{2n+3}{3n+2}\right)^n$ is convergent.

Exercise 15

Use the Root Test to determine whether the following series are convergent or divergent.

1. $\sum_{n=3}^{\infty} \left(\frac{3n^2+1}{n^2-4}\right)^n$
2. $\sum_{n=2}^{\infty} \frac{(-1)^n}{(\ln n)^n}$
3. $\sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^{n^2}$

Working Space

Answer on Page 71

3.2 Strategies for Testing Series

When testing series for convergence, we want to choose a test based on the form of the series. While you may be tempted to try each test one-by-one until you find an answer, this quickly becomes cumbersome and time-consuming. Additionally, if you plan to take an AP Calculus exam, you need to be able to choose a test quickly. Here are some tips:

1. Check if the series is a p-series ($\sum_{n=1}^{\infty} \frac{1}{n^p}$). If so, then if $p > 1$, the series converges. Otherwise, the series diverges.
2. If the series is not a p-series, check to see if you can write it as a geometric series ($\sum_{n=1}^{\infty} ar^{n-1}$ or $\sum_{n=1}^{\infty} ar^n$). Recall that geometric series are convergent if $|r| < 1$ and divergent otherwise.
3. If the series cannot be written as a p-series or geometric series, but has a similar form,

consider the comparison tests (the Direct Comparison Test and the Limit Comparison Test). When choosing a p-series to compare your series to, follow the guidelines outlined in the Comparison Tests section above.

4. If you can see at a glance that $\lim_{n \rightarrow \infty} a_n \neq 0$, then apply the Test for Divergence to show the series is divergent. Remember: $\lim_{n \rightarrow \infty} a_n \neq 0$ implies the series $\sum_{n=1}^{\infty} a_n$ is divergent, but $\lim_{n \rightarrow \infty} a_n = 0$ **does not** necessarily imply the series $\sum_{n=1}^{\infty} a_n$ is convergent.
5. If the series is alternating (has $(-1)^n$ or $(-1)^{n-1}$ in the term), the Alternating Series test may provide an answer.
6. The Ratio Test is excellent for series with factorials, other products, or constants to the n^{th} power. Remember that the Ratio Test will be inconclusive for p-series, rational functions of n , and algebraic functions of n .
7. If a_n is of the form $(b_n)^n$, use the Root Test.
8. If $a_n = f(n)$ where $f(n)$ is continuous, positive, and decreasing and you can evaluate $\int_1^{\infty} f(x) dx$, use the Integral Test.

You do not need to treat this as a checklist, where you check for every condition. Rather, you should use this as a guide to quickly determine the convergence test most likely to be useful.

Exercise 16

Choose an appropriate test to determine if the series is convergent or divergent. Apply the test and classify the series as convergent or divergent.

Working Space

1. $\sum_{n=1}^{\infty} \frac{e^n}{n^2}$
2. $\sum_{n=1}^{\infty} \frac{3^n n^2}{n!}$
3. $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$
4. $\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2}$
5. $\sum_{n=1}^{\infty} \left(\sqrt[n]{2} - 1\right)^n$

Answer on Page 71

Power Series

When your calculator shows $\sin(0.3)$ or $e^{1.7}$, it is not looking anything up. It is adding up a polynomial with enough terms to get every digit right. Polynomials are the only functions a chip really knows how to evaluate: just multiply and add. This chapter shows how to turn other functions into “infinitely long polynomials.”

Consider the function $f(x) = \frac{1}{1-x}$. This looks like the sum of a convergent geometric series, $\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$. If we let $a = 1$ and $r = x$, then we see that $\sum_{n=1}^{\infty} x^{n-1} = \frac{1}{1-x}$. We can reindex to begin at $n = 0$ and see that:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

This is a **power series**. We use power series in place of functions to evaluate integrals with no explicit antiderivative, to solve differential equations, and to compute functions on computers. Consider $f(x) = \frac{1}{1-x^2}$. What is $\int f(x) dx$? A u -substitution does not work, and it is not the derivative of an inverse trigonometric function. (Partial fractions would work here, but for many functions nothing works.) Another approach: write $f(x)$ as a power series and integrate term by term, since we know $\int x^n dx = \frac{1}{n+1}x^{n+1} + C$. We will come back to this integral at the end of the chapter.

4.1 Power Series

Power series are series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots$$

for some fixed x . Depending on x , the series may converge or diverge. For example, the power series $\sum_{n=0}^{\infty} x^n$ converges for $-1 < x < 1$ and diverges for all other values of x . This is because $\sum_{n=0}^{\infty} x^n$ is essentially a geometric series with $r = x$, which we already know converges for $|r| < 1$.

The form given above is for a power series centered on 0, but a power series can be centered

on any value, a . In that case, it looks like this:

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

which we call is a *power series in $(x-a)$* , or a *power series centered at a* , or a *power series about a* .

Example: Find a power series representation for $f(x) = \frac{2x-4}{x^2-4x+3}$.

Solution: Since we know $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, we will use partial fractions to decompose the function into two fractions. (Try it yourself.) We find that:

$$\frac{2x-4}{x^2-4x+3} = \frac{1}{x-1} + \frac{1}{x-3}$$

Noting that $\frac{1}{x-1} = (-1) \cdot \frac{1}{1-x}$, we can say that:

$$\frac{1}{x-1} = (-1) \cdot \sum_{n=0}^{\infty} x^n$$

Now, let us look at $\frac{1}{x-3}$. We can show that:

$$\frac{1}{x-3} = \frac{\frac{1}{3}}{\frac{x}{3}-1} = \frac{1}{3} \frac{1}{\frac{x}{3}-1} = \frac{-1}{3} \frac{1}{1-\frac{x}{3}}$$

Substituting $\frac{x}{3}$ for x into $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ we see that:

$$\frac{1}{1-\frac{x}{3}} = \sum_{n=0}^{\infty} \left(\frac{x}{3}\right)^n$$

Therefore:

$$\frac{1}{x-3} = \left(\frac{-1}{3}\right) \sum_{n=0}^{\infty} \left(\frac{x}{3}\right)^n$$

Adding the terms, we see that:

$$\frac{1}{x-1} + \frac{1}{x-3} = (-1) \sum_{n=0}^{\infty} x^n + \left(\frac{-1}{3}\right) \sum_{n=0}^{\infty} \left(\frac{x}{3}\right)^n = - \sum_{n=0}^{\infty} \left(1 + \frac{1}{3^{n+1}}\right) x^n$$

The first series needs $|x| < 1$ and the second needs $|x| < 3$, so the combined series represents f when $|x| < 1$.

4.2 Power Series Convergence

Sometimes, you will be asked to find one or more values of x for which a power series converges. To do this, choose a test to apply, then find x such that the test is passed.

Example: For what values of x is the series $\sum_{n=0}^{\infty} n!x^n$ convergent?

Solution: We will apply the Ratio Test (since there is a factorial in the series) and find x such that $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$.

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(n+1)!x^{n+1}}{n!x^n} \right| \\ &= \lim_{n \rightarrow \infty} \frac{(n+1)n!x \cdot x^n}{n!x^n} = \lim_{n \rightarrow \infty} \frac{(n+1) \cdot x}{1} \end{aligned}$$

which converges to 0 when $x = 0$ and diverges for all other values of x . Therefore, $\sum_{n=0}^{\infty} n!x^n$ converges if $x = 0$.

Example: For what values of x does the series $\sum_{n=1}^{\infty} \frac{(x-4)^n}{2n}$ converge?

Solution: We will use the Ratio Test again. We are looking for an x such that:

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &< 1 \\ \lim_{n \rightarrow \infty} \left| \frac{\frac{(x-4)^{n+1}}{2(n+1)}}{\frac{(x-4)^n}{2n}} \right| &< 1 \\ \lim_{n \rightarrow \infty} \left| \frac{(x-4)(x-4)^n(2n)}{(x-4)^n(2n+2)} \right| &< 1 \\ \lim_{n \rightarrow \infty} \left| \frac{(x-4)(2n)}{2n+2} \right| &< 1 \\ \lim_{n \rightarrow \infty} \left| \frac{x-4}{1 + \frac{1}{n}} \right| &< 1 \\ |x-4| &< 1 \end{aligned}$$

Which is true when

$$\begin{aligned} -1 &< x-4 < 1 \\ 3 &< x < 5 \end{aligned}$$

We are not done yet, though! We know the series converges for $3 < x < 5$ and diverges for $x < 3$ and $x > 5$. What about when $x = 3$ and $x = 5$? (These are the cases where the Ratio Test is indeterminate, because $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$.) We need to test each case.

Substituting $x = 3$ into the series yields:

$$\sum_{n=1}^{\infty} \frac{(3-4)^n}{2n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{2n}$$

This is an alternating series, so we apply the alternating series test. First, we check that $|a_{n+1}| < |a_n|$:

$$\frac{1}{2(n+1)} < \frac{1}{2n}$$

Which is true for all $n > 0$. Next, we check if $\lim_{n \rightarrow \infty} |a_n| = 0$:

$$\lim_{n \rightarrow \infty} \frac{1}{2n} = 0$$

Therefore, $\sum_{n=1}^{\infty} \frac{(-1)^n}{2n}$ is convergent, and the power series converges for $x = 3$. Next, we test $x = 5$:

$$\sum_{n=1}^{\infty} \frac{(5-4)^n}{2n} = \sum_{n=1}^{\infty} \frac{1}{2n}$$

This is half of the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$, which diverges. So the series diverges for $x = 5$. Our final answer to the original question is that the series is convergent for $3 \leq x < 5$.

4.2.1 Radius of Convergence

There are three possible outcomes when testing a power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ for convergence:

1. The series only converges for $x = a$
2. The series converges for all x
3. The series converges if $|x-a| < R$ and diverges for $|x-a| > R$, where R is some positive number

We call R the **radius of convergence**. If we rearrange $|x-a| < R$, we can see why this is called a radius (see figure 4.1):

$$a - R < x < a + R$$

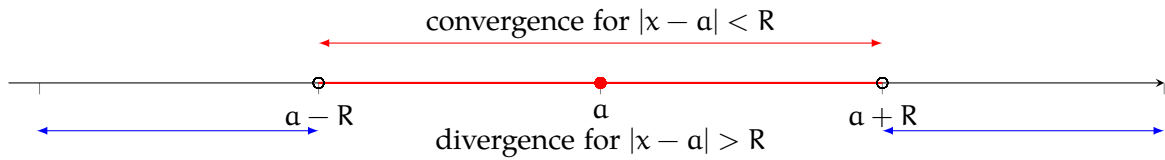


Figure 4.1: R is called the radius of convergence because it is half the width of the window of convergence

When $x = a \pm R$, the series could be convergent or divergent. You will need to test the endpoints of the window of convergence to determine if the interval is open or closed. Thus, there are four possibilities for the interval of convergence:

1. $(a - R, a + R)$
2. $[a - R, a + R)$
3. $(a - R, a + R]$
4. $[a - R, a + R]$

It is important to check each endpoint to determine the interval of convergence individually.

In the example of $\sum_{n=1}^{\infty} \frac{(x-4)^n}{2^n}$ (shown above), $a = 4$ and $R = 1$, and we found that the power series is convergent for $x \in [3, 5)$.

Example: For what values of x is the Bessel function $J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$ convergent?

Solution: Because there is a factorial, we will apply the Ratio Test:

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &< 1 \\ \lim_{n \rightarrow \infty} \left| \frac{\frac{(-1)^{n+1} x^{2(n+1)}}{2^{2(n+1)}((n+1)!)^2}}{\frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}} \right| &< 1 \\ \lim_{n \rightarrow \infty} \frac{x^{2n} x^2 2^{2n} n! n!}{2^{2n} 2^2 (n+1)! (n+1)! x^{2n}} &< 1 \\ \lim_{n \rightarrow \infty} \frac{x^2 n! n!}{2^2 (n+1) n! (n+1) n!} &< 1 \\ \lim_{n \rightarrow \infty} \frac{x^2}{4(n+1)^2} &= 0 < 1 \end{aligned}$$

Because $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = 0$ for all x , the Bessel function $J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$ is conver-

gent for all real values of x , and the interval of convergence is $(-\infty, \infty)$.

Example: Find the radius and interval of convergence for the series $\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$.

Solution: Again, we apply the ratio test to find values of x such that $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$:

$$\lim_{n \rightarrow \infty} \left| \frac{(-3)^{n+1} x^{n+1}}{\sqrt{n+1+1}} \cdot \frac{\sqrt{n+1}}{(-3)^n x^n} \right| < 1$$

$$\lim_{n \rightarrow \infty} \left| \frac{(-3)x}{\sqrt{n+2}} \cdot \frac{\sqrt{n+1}}{1} \right| < 1$$

$$\lim_{n \rightarrow \infty} \left| (-3)x \sqrt{\frac{n+2}{n+1}} \right| < 1$$

$$3|x| \lim_{n \rightarrow \infty} \sqrt{\frac{n+2}{n+1}} < 1$$

$$3|x| \lim_{n \rightarrow \infty} \sqrt{\frac{1+2/n}{1+1/n}} = 3|x|(1) < 1$$

$$3|x| < 1$$

$$|x| < \frac{1}{3}$$

Therefore, the radius of convergence is $\frac{1}{3}$. We need to test the endpoints, $x = \frac{-1}{3}$ and $x = \frac{1}{3}$, to determine the interval of convergence. First, we will test if $\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$ when $x = \frac{-1}{3}$:

$$\sum_{n=0}^{\infty} \frac{(-3)^n \left(\frac{-1}{3}\right)^n}{\sqrt{n+1}} = \sum_{n=0}^{\infty} \frac{1^n}{\sqrt{n+1}} = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}}$$

This is a p -series such that $p < 1$, so it is divergent, and our original series does not converge for $x = \frac{-1}{3}$. Next, we test $x = \frac{1}{3}$:

$$\sum_{n=0}^{\infty} \frac{(-3)^n \left(\frac{1}{3}\right)^n}{\sqrt{n+1}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$$

which is an alternating series that converges by the alternating series test. Therefore, the interval of convergence for $\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$ is $x \in \left(\frac{-1}{3}, \frac{1}{3}\right]$.

Exercise 17

[This problem was originally presented as a no-calculator, multiple-choice ques-

Working Space

tion on the 2012 AP Calculus BC exam.]
What is the radius of convergence of the series $\sum_{n=0}^{\infty} \frac{(x-4)^{2n}}{3^n}$?

Answer on Page 72

Exercise 18

[This problem was originally presented as a no-calculator, multiple-choice question on the 2012 AP Calculus BC exam.]
A power series is given by $\frac{x}{3} - \frac{x^3}{5} + \frac{x^5}{7} - \frac{x^7}{9} + \dots$. Write the series in sigma notation and use the Ratio Test to determine the interval of convergence.

Working Space

Answer on Page 72

4.3 Calculus with Power Series

You can integrate and differentiate power series. Let $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$. Recall that $f(x)$ is just a very long polynomial and that the derivative of a polynomial x^n is $n \cdot x^{n-1}$. We can then state that:

$$\frac{d}{dx}f(x) = \frac{d}{dx} \left[c_0 + c_1(x-a)^1 + c_2(x-a)^2 + c_3(x-a)^3 + \dots \right]$$

$$f'(x) = 0 + c_1 + 2c_2(x-a)^1 + 3c_3(x-a)^2 + \dots$$

$$f'(x) = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$$

which is true when x is in the interval of convergence for the series.

Similarly, we know $\int x^n dx = \frac{1}{n+1}x^{n+1}$. We can then say that:

$$\begin{aligned}\int f(x) dx &= \int \left[c_0 + c_1(x-a)^1 + c_2(x-a)^2 + \cdots \right] dx \\ \int f(x) dx &= C + c_0(x-a) + \frac{c_1}{2}(x-a)^2 + \frac{c_2}{3}(x-a)^3 + \cdots \\ \int f(x) dx &= C + \sum_{n=0}^{\infty} \frac{c_n}{n+1} (x-a)^{n+1}\end{aligned}$$

Where C is the integration constant. Again, this is true when x is in the interval of convergence for the series.

Example: Express $\frac{1}{(1-x)^2}$ as a power series by differentiating $\frac{1}{1-x}$.

Solution: Recall that $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ when $|x| < 1$.

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

Differentiating both sides:

$$\begin{aligned}\frac{d}{dx} \left[\frac{1}{1-x} \right] &= \frac{d}{dx} \sum_{n=0}^{\infty} x^n \\ (-1) \cdot \frac{1}{(1-x)^2} \cdot \frac{d}{dx}(1-x) &= \sum_{n=1}^{\infty} nx^{n-1} \\ \frac{1}{(1-x)^2} &= \sum_{n=1}^{\infty} nx^{n-1}\end{aligned}$$

Reindexing to begin at $n = 0$

$$\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} (n+1)x^n$$

Because $\sum_{n=0}^{\infty} x^n$ has a radius of convergence of 1, so does $\sum_{n=0}^{\infty} (n+1)x^n$. We can confirm our series makes sense by plotting the partial sums for $n = 3, 5$, and 7 with the original function (see figure 4.2).

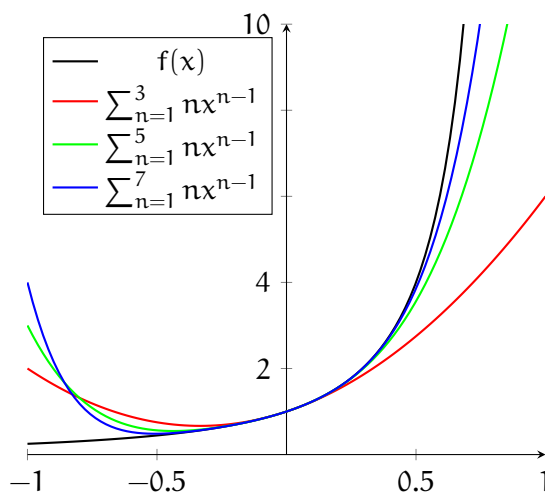


Figure 4.2: Partial sums of $\sum_{n=1}^{\infty} nx^{n-1}$ hug $f(x) = \frac{1}{(1-x)^2}$ more closely as terms are added, for $|x| < 1$

Example: Find a power series representing $\ln(1+x)$.

Solution: We know that $\frac{d}{dx} \ln(1+x) = \frac{1}{1+x}$. Replacing x with $-x$, we see that:

$$\frac{1}{1-(-x)} = \frac{1}{1+x} = \sum_{n=0}^{\infty} (-x)^n$$

which converges for $|x| < 1$. We can then integrate both sides:

$$\int \frac{1}{1+x} dx = \int \left[\sum_{n=0}^{\infty} (-x)^n \right] dx$$

$$\ln(1+x) = \int \left(1 - x + x^2 - x^3 + \cdots \right) dx$$

$$\ln(1+x) = C + x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots + (-1)^{n-1} \frac{x^n}{n}$$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} + C$$

when $|x| < 1$. To find C , substitute $x = 0$ and solve:

$$\ln(1+0) = C + \sum_{n=1}^{\infty} (-1)^{n-1} \frac{0^n}{n} = C + 0$$

$$C = \ln 1 = 0$$

So, our final answer is $\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$.

Exercise 19

Find a power series representation for $f(x) = \arctan x$.

Working Space

Answer on Page 73

Example: Back to the opening question. Find $\int \frac{1}{1-x^2} dx$ as a power series.

Solution: Replace x with x^2 in the geometric series:

$$\frac{1}{1-x^2} = \sum_{n=0}^{\infty} x^{2n} = 1 + x^2 + x^4 + \cdots \quad (|x| < 1)$$

Integrate term by term:

$$\int \frac{1}{1-x^2} dx = C + x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots = C + \sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1}$$

No partial fractions needed.

Exercise 20

Use the series for $\ln(1+x)$ to estimate $\ln(1.1)$ with three terms. How close is it to the calculator value?

Working Space

Answer on Page 73

Taylor and Maclaurin Series

In the last chapter, we built power series by tinkering with the geometric series. That trick only works for a few functions. What if you want a series for $\sin x$ or e^x ? It turns out a function's derivatives at a single point tell you everything you need. Knowing how a function starts, how fast it changes, how fast *that* changes, and so on, is enough to rebuild it.

Suppose f is given by a power series:

$$f(x) = c_0 + c_1(x - a) + c_2(x - a)^2 + c_3(x - a)^3 + \dots$$

Notice that substituting $x = a$ yields:

$$f(a) = c_0$$

Suppose we differentiate both sides, which gives:

$$f'(x) = c_1 + 2c_2(x - a) + 3c_3(x - a)^2 + \dots$$

Again, substituting $x = a$:

$$f'(a) = c_1$$

Continuing, we see that

$$f''(x) = 2c_2 + 2 \cdot 3c_3(x - a) + 3 \cdot 4c_4(x - a)^2 + \dots$$

And

$$f''(a) = 2c_2$$

Once more, we see that:

$$f'''(x) = 2 \cdot 3c_3 + 2 \cdot 3 \cdot 4c_4(x - a) + \dots$$

And substituting $x = a$ gives

$$f'''(a) = 2 \cdot 3c_3 = 3!c_3$$

Do you see the pattern? It is

$$f^{(n)}(a) = 2 \cdot 3 \cdot 4 \cdot \dots \cdot nc_n = n!c_n$$

Which also means that

$$c_n = \frac{f^{(n)}(a)}{n!}$$

(recall the convention that $0! = 1$, so this is valid for $n = 0$ as well).

If we substitute for c_n into our definition of a power series, we get a **Taylor series**:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x - a)^n$$

This is called the *Taylor series of f centered at a* . If $a = 0$, we call this special case a **Maclaurin series**:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

Example: Find the Maclaurin series for $f(x) = e^x$ and its radius of convergence.

Solution: We know that if $f(x) = e^x$, then $f'(x) = e^x$, and $f''(x) = e^x, \dots, f^{(n)}(x) = e^x$. Since $e^0 = 1$, we can say that $f^{(n)}(0) = 1$ for all n . Substituting this into our equation for Maclaurin series:

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{1}{n!} x^n \\ &= \sum_{n=0}^{\infty} \frac{x^n}{n!} \end{aligned}$$

Taking $a_n = \frac{x^n}{n!}$ and applying the Ratio test, we can find the radius of convergence:

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &< 1 \\ \lim_{n \rightarrow \infty} \left| \frac{x^{n+1}}{(n+1)n!} \cdot \frac{n!}{x^n} \right| &< 1 \\ \lim_{n \rightarrow \infty} \frac{|x|}{n+1} &= 0 < 1 \end{aligned}$$

The limit converges to 0 for all x , so the radius of convergence is ∞ .

We can confirm this graphically. But first, a quick definition. The n^{th} -degree **Taylor polynomial** keeps the terms of the Taylor series up through $(x - a)^n$:

$$T_n(x) = \sum_{i=0}^n \frac{f^{(i)}(a)}{i!} (x - a)^i$$

In the case of $f(x) = e^x$ and $a = 0$, the n^{th} degree Taylor polynomial is given by:

$$T_n(x) = \sum_{i=0}^n \frac{x^i}{i!}$$

As expected, as the degree of the polynomial increases, the closer it is to e^x (see figure 5.1).

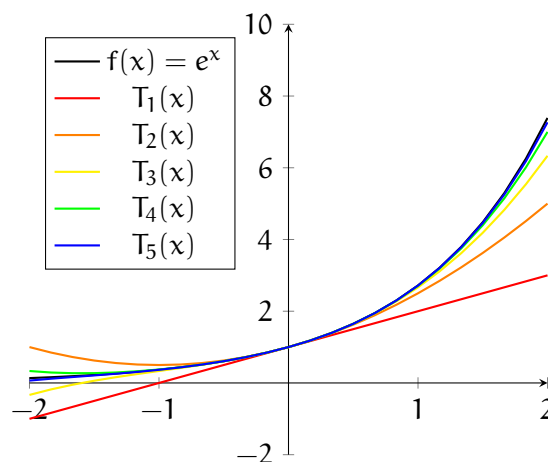


Figure 5.1: As n increases, $T_n(x)$ approaches $f(x)$

Be sure to write this Taylor series down: you will see it on the AP Calculus exam and in your college mathematics classes!

5.1 Maclaurin series of trigonometric functions

Let us write a Maclaurin series for $\sin x$. We begin by determining c_n for the first few n :

n	$f^{(n)}(0)$	c_n
0	$\sin(0) = 0$	$0/0!$
1	$\cos(0) = 1$	$1/1!$
2	$-\sin(0) = 0$	$0/2!$
3	$-\cos(0) = -1$	$-1/3!$
4	$\sin(0) = 0$	$0/4!$

As you can see, $f^{(n)}(0)$ will cycle between 0, 1, and -1 as n continues to increase. Using this, we can begin to write the first few terms of a Maclaurin series for $f(x) = \sin x$:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \frac{f(0)}{0!} x^0 + \frac{f'(0)}{1!} x^1 + \frac{f''(0)}{2!} x^2 + \frac{f'''(0)}{3!} x^3 + \frac{f^{(IV)}(0)}{4!} x^4 + \cdots + \frac{f^{(n)}(0)}{n!} x^n$$

$$\begin{aligned}
&= \frac{0}{0!} + \frac{1}{1!}x + \frac{0}{2!}x^2 + \frac{-1}{3!}x^3 + \frac{0}{4!}x^4 + \frac{1}{5!}x^5 + \cdots \\
&= x + \frac{-1}{3!}x^3 + \frac{1}{5!}x^5 + \frac{-1}{7!}x^7 + \cdots + \frac{(-1)^n}{(2n+1)!}x^{2n+1}
\end{aligned}$$

Therefore, $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$. This is also a series you will see again and again.

Exercise 21

Derive a Maclaurin series for $\cos x$. Use it and the Maclaurin series for $\sin x$ to find the first four nonzero terms of the Maclaurin series for $\tan x$. Confirm your series for $\tan x$ by plotting.

Working Space

Answer on Page 73

Another important trigonometric Maclaurin series is

$$\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

which converges for $|x| < 1$.

5.2 Other common Maclaurin series

We have already seen that:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

which converges for $|x| < 1$. And also:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Which converges for all real x .

In the previous chapter, we also derived a Maclaurin series for $f(x) = \ln(1+x)$:

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$$

Which converges for $|x| < 1$.

Example: Find a Maclaurin series for $f(x) = (1+x)^k$, where k is any real number, and determine its radius of convergence.

Solution: We begin by finding the first few derivatives and evaluating at $x = 0$:

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$(1+x)^k$	1
1	$k(1+x)^{k-1}$	k
2	$k(k-1)(1+x)^{k-2}$	$k(k-1)$
3	$k(k-1)(k-2)(1+x)^{k-3}$	$k(k-1)(k-2)$
$\dots$	$\dots$	$\dots$
n	$k(k-1)\dots(k-n+1)(1+x)^{k-n}$	$k(k-1)(k-2)\dots(k-n+1)$

Therefore, $c_n = \frac{k(k-1)\dots(k-n+1)}{n!}$ and we can write the Maclaurin series:

$$(1+x)^k = \sum_{n=0}^{\infty} \frac{k(k-1)(k-2)\dots(k-n+1)}{n!} x^n$$

This is called the **binomial series**. Another notation you can use is:

$$\binom{k}{n} = \frac{k(k-1)(k-2)\dots(k-n+1)}{n!}$$

To determine the radius of convergence, we apply the ratio test:

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$$

$$\lim_{n \rightarrow \infty} \left| \frac{k(k-1)(k-2)\dots(k-n+1)(k-n)x^{n+1}}{(n+1)n!} \cdot \frac{n!}{k(k-1)(k-2)\dots(k-n+1)x^n} \right| < 1$$

$$\lim_{n \rightarrow \infty} \left| \frac{(k-n)x}{n+1} \right| = \lim_{n \rightarrow \infty} \frac{|k-n||x|}{n+1} < 1$$

$$\lim_{n \rightarrow \infty} \frac{\left| 1 - \frac{k}{n} \right|}{1 + \frac{1}{n}} |x| = |x| < 1$$

Therefore, the series converges for $|x| < 1$ and the radius of convergence is 1.

In the Counting chapter, you saw that the binomial coefficient $\binom{n}{k}$ makes sense for any real n . The binomial series is the binomial theorem for those exponents.

Example: Expand $\sqrt{1+5x} = (1+5x)^{1/2}$.

Solution: First compute the coefficients $\binom{1/2}{n}$ for the binomial series of $(1+x)^{1/2}$:

$$\begin{aligned}\binom{1/2}{0} &= 1 \\ \binom{1/2}{1} &= \frac{1}{2} \\ \binom{1/2}{2} &= \frac{\frac{1}{2}(\frac{1}{2}-1)}{2!} = \frac{\frac{1}{2} \cdot (-\frac{1}{2})}{2} = -\frac{1}{8} \\ \binom{1/2}{3} &= \frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})}{3!} = \frac{3/8}{6} = \frac{1}{16}\end{aligned}$$

So,

$$(1+x)^{1/2} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \dots$$

Now apply this to $\sqrt{1+5x}$. This is just $(1+x)^{1/2}$ with x replaced by $5x$, so we substitute $5x$ everywhere:

$$\begin{aligned}\sqrt{1+5x} &= 1 + \frac{1}{2}(5x) - \frac{1}{8}(5x)^2 + \frac{1}{16}(5x)^3 - \dots \\ &= 1 + \frac{5}{2}x - \frac{25}{8}x^2 + \frac{125}{16}x^3 - \dots\end{aligned}$$

This expansion is only valid for $|5x| < 1$, i.e. $|x| < \frac{1}{5}$.

Exercise 22 Fractional Binomial Expansion

Working Space

1. Find the first four terms in the expansion of $(1+x)^{-1}$, in ascending powers of x . For what values of x is this expansion valid?
2. Find the first three terms in the expansion of $\sqrt{1-2x}$, in ascending powers of x .

Answer on Page 74

Example: Find the sum of the series $\frac{1}{1 \cdot 2} - \frac{1}{2 \cdot 2^2} + \frac{1}{3 \cdot 2^3} - \frac{1}{4 \cdot 2^4} + \cdots$.

Solution: We can write this series in sigma notation:

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n \cdot 2^n} = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} \left(\frac{1}{2}\right)^n$$

Notice this is the series for $\ln(1+x)$ with $x = \frac{1}{2}$. Therefore,

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} \left(\frac{1}{2}\right)^n = \ln\left(1 + \frac{1}{2}\right) = \ln \frac{3}{2}$$

5.3 How Good Is the Approximation?

A Taylor polynomial is only useful if you know how far off it might be. The **Lagrange error bound** answers that. If $|f^{(n+1)}(t)| \leq M$ for every t between a and x , then

$$|f(x) - T_n(x)| \leq \frac{M}{(n+1)!} |x - a|^{n+1}$$

In words: the error looks like the next term of the series, using the largest value the next derivative can take.

Example: Use T_2 for e^x at $a = 0$ to estimate $e^{0.1}$, and bound the error.

Solution: $T_2(0.1) = 1 + 0.1 + \frac{0.01}{2} = 1.105$. The third derivative is e^t , which is at most $e^{0.1} < 3$ on $[0, 0.1]$. So

$$|e^{0.1} - 1.105| \leq \frac{3}{3!} (0.1)^3 = 0.0005$$

The true value is $1.1051709\dots$, so the actual error is about 0.00017 , well within the bound.

Example: Find T_2 for $f(x) = \sqrt{x}$ centered at $a = 4$, and use it to estimate $\sqrt{4.2}$.

Solution: $f(4) = 2$, $f'(x) = \frac{1}{2}x^{-1/2}$ so $f'(4) = \frac{1}{4}$, and $f''(x) = -\frac{1}{4}x^{-3/2}$ so $f''(4) = -\frac{1}{32}$. Then

$$T_2(x) = 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2$$

and $T_2(4.2) = 2 + 0.05 - 0.000625 = 2.049375$. A calculator gives $\sqrt{4.2} \approx 2.049390$. Centering at 4, where we know the square root exactly, is what makes this work.

Exercise 23

[The problem was originally presented

Working Space

as a no-calculator, multiple-choice question on the 2012 AP Calculus BC exam.]

The Maclaurin series for the function f is given by $f(x) = \sum_{n=0}^{\infty} \left(-\frac{x}{4}\right)^n$. What is the value of $f(3)$?

Answer on Page 75

Exercise 24

[This problem was originally presented as a calculator-allowed, multiple-choice question on the 2012 AP Calculus BC exam.] Let f be a function having derivatives of all orders for $x > 0$ such that $f(3) = 2$, $f'(3) = -1$, $f''(3) = 6$, and $f'''(3) = 12$. Write a 3rd-degree Taylor polynomial for f about $x = 3$ and estimate the value of $f(3.4)$

Working Space

Answer on Page 75

Exercise 25

[This problem was originally presented as a calculator-allowed, free-response question on the 2012 AP Calculus BC exam.] The function f is twice differentiable for $x > 0$ with $f(1) = 15$, $f'(1) = 8$, and $f''(1) = 20$. Write the second-degree Taylor polynomial about $x = 1$ and use it to estimate $f(1.4)$.

Working Space

Answer on Page 75

Exercise 26

[The problem was originally presented as a no-calculator, multiple-choice question on the 2012 AP Calculus BC exam.]

For $x > 0$, the power series $1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots + (-1)^n \frac{x^{2n}}{(2n+1)!}$ converges to which of the following? (a) $\cos x$

(b) $\sin x$

(c) $\frac{\sin x}{x}$

(d) $e^x - e^{x^2}$

(e) $1 + e^x - e^{x^2}$

Working Space

Answer on Page 75

Exercise 27

Use $T_3(x) = x - \frac{x^3}{6}$ to estimate $\sin(0.5)$.

Use the Lagrange error bound to show your estimate is within 0.003 of the true value.

Working Space

Answer on Page 75

Euler's Formula and the Complex Exponential

Mathematicians love to argue about which equation is the most beautiful. The usual winner is

$$e^{i\pi} + 1 = 0$$

It connects the five most important numbers in mathematics: 0, 1, e , π , and i . And it should make you suspicious. What could it possibly mean to raise e to an *imaginary* power? You cannot multiply e by itself i times.

In this chapter, you will see that the question has a beautiful answer, and that the answer explains things you have already met: why multiplying complex numbers rotates them, where the trig sum formulas come from, and why springs and circuits oscillate the way they do.

6.1 What Could $e^{i\theta}$ Mean?

For whole-number exponents, e^3 means $e \cdot e \cdot e$. Fractions and negative exponents get their meaning from the exponent rules, like $e^{a+b} = e^a e^b$. None of that tells us what e^i should be.

The Taylor and Maclaurin Series chapter gives us a way in. For every real x ,

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

The right side uses only adding, multiplying, and dividing, and those all work for complex numbers. So we *define* e^z , for any complex number z , by the same series:

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

This is the only sensible choice: it agrees with the old e^x for real numbers, and it still obeys the rule $e^{a+b} = e^a e^b$ for complex a and b .

6.2 Deriving Euler's Formula

Put $z = i\theta$, where θ is a real number. The powers of i repeat every four steps ($i, -1, -i, 1, \dots$, as you saw in the Complex Numbers chapter), so

$$\begin{aligned} e^{i\theta} &= 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \dots \\ &= 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} - \dots \end{aligned}$$

Gather the terms without i and the terms with i :

$$e^{i\theta} = \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \dots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots\right)$$

Look closely at the two series in parentheses. They are exactly the Maclaurin series for $\cos \theta$ and $\sin \theta$.

Euler's Formula

For every real number θ ,

$$e^{i\theta} = \cos \theta + i \sin \theta$$

Leonhard Euler published this in 1748. It says that the exponential function and the trigonometric functions, which look completely unrelated, are the same function seen from two directions.

6.2.1 Watching the Series Converge

You can watch the series for $e^{i\pi}$ home in on its value. Each term is the previous one multiplied by $\frac{i\pi}{n}$: a quarter turn and a stretch. So the partial sums turn corners as they go:

```
1 import math
2
3 z = 1j * math.pi          # the exponent i*pi
4 total = 0
5 for n in range(12):
6     total += z**n / math.factorial(n)
7     print(n, round(total.real, 4), round(total.imag, 4))
```

```
0 1.0 0.0
1 1.0 3.1416
2 -3.9348 3.1416
3 -3.9348 -2.0261
```

```

4 0.1239 -2.0261
5 0.1239 0.524
6 -1.2114 0.524
7 -1.2114 -0.0752
8 -0.976 -0.0752
9 -0.976 0.0069
10 -1.0018 0.0069
11 -1.0018 -0.0004

```

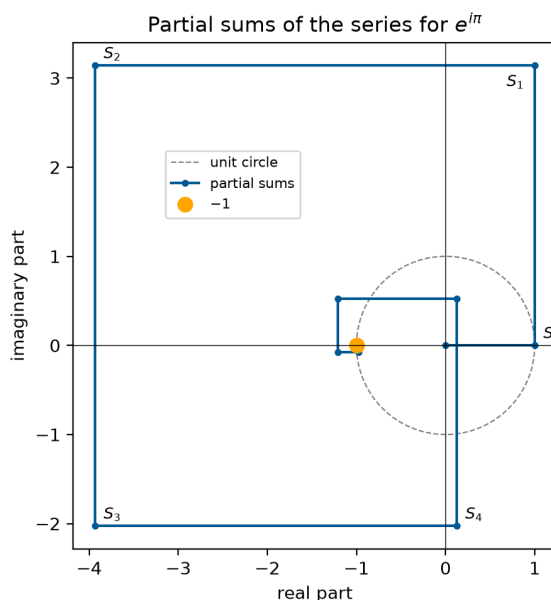


Figure 6.1: Each partial sum turns a corner, and the path spirals in on -1 .

The path spirals in on $-1 + 0i$. So $e^{i\pi} = -1$, which is Euler's identity, $e^{i\pi} + 1 = 0$.

6.3 A Point on the Unit Circle

In the Trigonometric Functions chapter, $(\cos \theta, \sin \theta)$ was the point on the unit circle at angle θ . In the complex plane, the point (a, b) is the number $a + bi$. So Euler's formula says:

$e^{i\theta}$ is the point on the unit circle at angle θ .

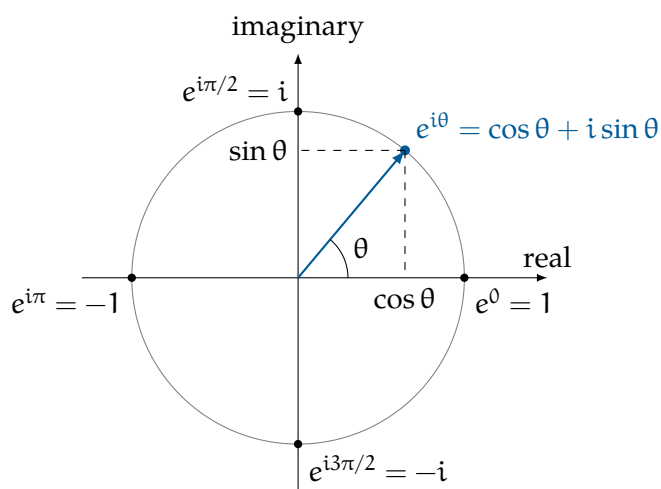


Figure 6.2: As θ increases, $e^{i\theta}$ travels counterclockwise around the unit circle.

As θ goes from 0 to 2π , $e^{i\theta}$ walks once around the circle. Some landmarks:

$$e^0 = 1, \quad e^{i\pi/2} = i, \quad e^{i\pi} = -1, \quad e^{i3\pi/2} = -i, \quad e^{2\pi i} = 1$$

Euler's identity is no longer mysterious: walk halfway around the unit circle, and you land on -1 . Because every point of the circle is at distance 1 from 0,

$$|e^{i\theta}| = \sqrt{\cos^2 \theta + \sin^2 \theta} = 1$$

Exercise 28 Around the Circle

Write each number in the form $a + bi$.

1. $e^{i\pi/3}$
2. $e^{i\pi/4}$
3. $e^{i3\pi/2}$
4. $e^{10\pi i}$

Working Space

Answer on Page 76

6.4 Polar Form

Every point in the plane can be described by its distance r from the origin and its angle θ , as in the Polar Coordinates chapter. For a complex number, that means

$$z = a + bi = r(\cos \theta + i \sin \theta) = re^{i\theta}$$

This is the *polar form* of z . The distance $r = |z| = \sqrt{a^2 + b^2}$ is the *modulus*, and the angle θ is the *argument*. As with polar coordinates, find θ from $\tan \theta = \frac{b}{a}$, and check which quadrant the point is in.

For example, $1 + i$ is at distance $\sqrt{2}$ and angle $\frac{\pi}{4}$, so $1 + i = \sqrt{2}e^{i\pi/4}$. And $-2 = 2e^{i\pi}$.

6.4.1 Multiplying: Stretch and Rotate

The Complex Numbers chapter claimed that multiplying by a complex number rotates and stretches. Polar form shows exactly how. Using the exponent rule,

$$(r_1 e^{i\theta_1}) (r_2 e^{i\theta_2}) = r_1 r_2 e^{i(\theta_1 + \theta_2)}$$

To multiply complex numbers, multiply their lengths and add their angles.

Multiplying by $i = e^{i\pi/2}$ keeps the length (it is 1) and adds 90° to the angle: a quarter turn, just as you saw before.

Example: Multiply $(1 + i)(\sqrt{3} + i)$ using polar form.

Solution: $1 + i = \sqrt{2}e^{i\pi/4}$ and $\sqrt{3} + i = 2e^{i\pi/6}$. So the product has length $2\sqrt{2} \approx 2.83$ and angle $\frac{\pi}{4} + \frac{\pi}{6} = \frac{5\pi}{12}$, which is 75° . Check by multiplying the ordinary way: $(1 + i)(\sqrt{3} + i) = (\sqrt{3} - 1) + (\sqrt{3} + 1)i \approx 0.732 + 2.732i$, whose length is $\sqrt{0.536 + 7.464} = \sqrt{8} = 2\sqrt{2}$.

6.4.2 Powers: De Moivre's Formula

Multiplying a number by itself n times multiplies its length n times and adds its angle n times:

$$(re^{i\theta})^n = r^n e^{in\theta}$$

This is *De Moivre's formula*. It turns big powers into easy ones.

Example: Find $(1 + i)^{10}$.

Solution: $(1 + i)^{10} = (\sqrt{2}e^{i\pi/4})^{10} = (\sqrt{2})^{10} e^{i10\pi/4} = 32e^{i5\pi/2}$. An angle of $\frac{5\pi}{2}$ is one full

turn plus $\frac{\pi}{2}$, so $(1 + i)^{10} = 32i$. Doing this by expanding would take a long time.

6.4.3 Roots of Unity

How many numbers satisfy $z^3 = 1$? The real number 1 does, but the Fundamental Theorem of Algebra promises three solutions. Write $z = e^{i\theta}$. Then $z^3 = e^{i3\theta}$, which equals 1 whenever 3θ is a whole number of turns: $3\theta = 0, 2\pi, 4\pi$. So the three *cube roots of unity* are

$$1, \quad e^{i2\pi/3} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i, \quad e^{i4\pi/3} = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$$

In general, the n solutions of $z^n = 1$ are $e^{i2\pi k/n}$ for $k = 0, 1, \dots, n-1$. They sit evenly spaced around the unit circle, at the corners of a regular n -sided polygon.

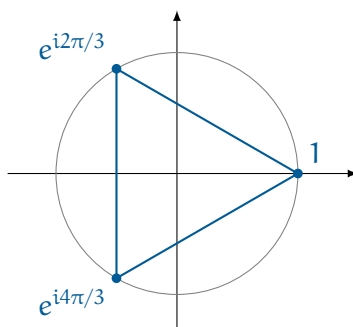


Figure 6.3: The three cube roots of 1 form an equilateral triangle.

Exercise 29 Polar Form

Working Space

1. Write $-3 + 3i$ in polar form.
2. Use polar form to find $(1 - \sqrt{3}i)^6$.
3. Find all three cube roots of 8.

Answer on Page 76

6.5 Trigonometry for Free

In the Trigonometric Identities chapter, the sum formulas took a careful geometric proof. Euler's formula gives them in two lines. Start with $e^{i(a+b)} = e^{ia}e^{ib}$ and expand both sides:

$$\begin{aligned}\cos(a+b) + i \sin(a+b) &= (\cos a + i \sin a)(\cos b + i \sin b) \\ &= (\cos a \cos b - \sin a \sin b) + i(\sin a \cos b + \cos a \sin b)\end{aligned}$$

Two complex numbers are equal only when their real parts match and their imaginary parts match:

$$\cos(a+b) = \cos a \cos b - \sin a \sin b, \quad \sin(a+b) = \sin a \cos b + \cos a \sin b$$

You never need to memorize these again; you can rebuild them whenever you want.

Euler's formula also runs backward. Since $e^{-i\theta} = \cos \theta - i \sin \theta$, adding and subtracting gives

$$\cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \quad \sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

so every trig function is secretly made of exponentials.

Exercise 30 Double Angles

Use $e^{i2\theta} = (e^{i\theta})^2$ to find formulas for $\cos 2\theta$ and $\sin 2\theta$.

Working Space

Answer on Page 76

6.6 Spinning and Oscillating

Now let the angle grow with time: $z(t) = e^{i\omega t}$ is a point going around the unit circle with angular speed ω . Differentiating each part,

$$\frac{d}{dt}e^{i\omega t} = \frac{d}{dt}(\cos \omega t + i \sin \omega t) = -\omega \sin \omega t + i\omega \cos \omega t = i\omega (\cos \omega t + i \sin \omega t) = i\omega e^{i\omega t}$$

So the usual rule $\frac{d}{dt}e^{kt} = ke^{kt}$ still works with $k = i\omega$. And it has a picture: the velocity is the position multiplied by $i\omega$, that is, rotated 90° and stretched by ω . The velocity of circular motion is always tangent to the circle, just as you saw in the Uniform Circular Motion and Centripetal Force chapter.

Differentiate again: $\frac{d^2}{dt^2} e^{i\omega t} = (i\omega)^2 e^{i\omega t} = -\omega^2 e^{i\omega t}$. That is the equation of a spring, $x'' = -\omega^2 x$. Its real part, $\cos \omega t$, is the spring's back-and-forth motion: a mass on a spring is the shadow of a point going around a circle.

6.6.1 Damped Oscillations

In the Differential Equations chapter, a damped spring had characteristic roots $r = \alpha \pm i\beta$, and the table claimed the motion was $e^{\alpha t}(C_1 \cos \beta t + C_2 \sin \beta t)$. Now you can see why. The guess e^{rt} with $r = \alpha + i\beta$ gives

$$e^{(\alpha+i\beta)t} = e^{\alpha t} e^{i\beta t} = e^{\alpha t} (\cos \beta t + i \sin \beta t)$$

The factor $e^{\alpha t}$ (with $\alpha < 0$) shrinks the swing, and the $e^{i\beta t}$ part spins, producing the oscillation. A complex root is just an exponential that rotates while it decays.

6.6.2 Alternating Current

Electrical engineers use this idea every day. They write an AC voltage $V_0 \cos \omega t$ as the real part of $V_0 e^{i\omega t}$, a rotating arrow called a *phasor*. Since differentiating just multiplies by $i\omega$, the calculus in a circuit with capacitors and inductors turns into ordinary algebra with complex numbers.

6.7 Complex Exponentials in Python

Python writes i as `1j`. The `cmath` library has complex versions of the math functions, plus tools for polar form:

```

1 import cmath, math
2
3 print(cmath.exp(1j * math.pi))      # e^(i pi)
4 print(cmath.polar(1 + 1j))          # (r, theta) for 1 + i
5 print(cmath.rect(2, math.pi / 3))   # 2 e^(i pi/3) back to a + bi
6 print((1 + 1j) ** 10)               # De Moivre says 32i

```

```

(-1+1.2246467991473532e-16j)
(1.4142135623730951, 0.7853981633974483)
(1.0000000000000002+1.7320508075688772j)
32j

```

The first answer should be exactly -1 . The tiny imaginary part, about 10^{-16} , is rounding error: the computer's value of π is very close to π , but not exact.

Exercise 31 The Strangest Power

Use $i = e^{i\pi/2}$ to compute i^i . Is the answer real or imaginary?

Working Space

Answer on Page 76

Exercise 32 Conjugates on the Circle

Show that $e^{-i\theta}$ is the complex conjugate of $e^{i\theta}$, and that their product is 1. What does this mean geometrically?

Working Space

Answer on Page 76

Exercise 33 Check Euler with Python

Write a short Python program that checks Euler's formula for $\theta = 0, 0.5, 1, 1.5, \dots, 3$ by printing the difference between `cmath.exp(1j * theta)` and `math.cos(theta) + 1j * math.sin(theta)`.

Working Space

Answer on Page 77

Answers to Exercises

Answer to Exercise 1 (on page 12)

1. We need to identify a and r . If we use the form $\sum_{n=1}^{\infty} ar^{n-1}$, then $a = 3$. To find the common ratio, we can evaluate $\frac{a_{n+1}}{a_n} = \frac{-4}{3}$. We can then write the series as $\sum_{n=1}^{\infty} 3 \left(\frac{-4}{3}\right)^{n-1}$. In this case, $r = \frac{-4}{3}$ and $|r| \geq 1$, and therefore the series is divergent.
2. Following the process outlined above, we see that $a = 2$ and $r = \frac{1}{4}$. Therefore, the series is $\sum_{n=1}^{\infty} 2 \left(\frac{1}{4}\right)^{n-1}$. Since $|r| < 1$, the series converges to $\frac{a}{1-r} = \frac{2}{1-1/4} = \frac{2 \cdot 4}{3} = \frac{8}{3}$.
3. We need to rewrite the series into a standard form in order to identify a and r :

$$\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{4^n} = \sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{4(4)^{n-1}} = \sum_{n=1}^{\infty} \frac{1}{4} \left(\frac{-3}{4}\right)^{n-1}$$

So $r = \frac{-3}{4}$ and $|r| < 1$. Therefore, the series converges to $\frac{1/4}{1-(-3/4)} = \frac{1}{4} \cdot \frac{4}{7} = \frac{1}{7}$.

4. We need to rewrite the series into a standard form in order to identify a and r :

$$\sum_{n=1}^{\infty} \frac{e^{2n}}{6^{n-1}} = \sum_{n=1}^{\infty} \frac{(e^2)^n}{6^{n-1}} = \sum_{n=1}^{\infty} \frac{(e^2)(e^2)^{n-1}}{6^{n-1}} = \sum_{n=1}^{\infty} e^2 \left(\frac{e^2}{6}\right)^{n-1}$$

Therefore, $r = \frac{e^2}{6} \approx 1.232$. Since $|r| > 1$, the series diverges.

Answer to Exercise 2 (on page 12)

We want to rewrite this as a geometric series of the form $\sum_{n=i}^{\infty} ar^{n-1}$, so we can use the fact that the sum of a convergent geometric series is $\frac{a}{1-r}$. $\sum_{n=0}^{\infty} (1+c)^{-n} = \sum_{n=0}^{\infty} \left(\frac{1}{1+c}\right)^n = \sum_{n=1}^{\infty} \left(\frac{1}{1+c}\right)^{n-1}$. This is a geometric series with $a = 1$ and $r = \frac{1}{1+c}$. So, the value of the series is $\frac{1}{1-\frac{1}{1+c}} = \frac{1}{\frac{c}{1+c}} = \frac{1+c}{c}$. Setting this equal to $\frac{5}{3}$ and solving for c , we find that $c = \frac{3}{2}$.

Answer to Exercise 3 (on page 12)

$-2 < p < 2$. Let us rewrite this geometric series into standard form: $\sum_{n=1}^{\infty} \left(\frac{p}{2}\right)^n = \sum_{n=1}^{\infty} \frac{p}{2} \left(\frac{p}{2}\right)^{n-1}$ which means $a = \frac{p}{2}$ and $r = \frac{p}{2}$. We know that geometric series converge if $|r| < 1$, so we set up an inequality and solve for p :

$$\begin{aligned} \left|\frac{p}{2}\right| &< 1 \\ -1 &< \frac{p}{2} < 1 \\ -2 &< p < 2 \end{aligned}$$

Answer to Exercise 4 (on page 13)

1. Separating the terms, we see that $\sum_{n=1}^{\infty} \frac{n^2+1}{n^4} = \sum_{n=1}^{\infty} \left(\frac{n^2}{n^4} + \frac{1}{n^4}\right) = \sum_{n=1}^{\infty} \frac{1}{n^2} + \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^2}{6} + \frac{\pi^4}{90}$
2. Notice that this series starts at $n = 2$. By the properties of series, we know that $\sum_{n=1}^{\infty} a_n = a_1 + \sum_{n=2}^{\infty} a_n$. Therefore, $\sum_{n=2}^{\infty} \frac{1}{n^2} = \sum_{n=1}^{\infty} \left(\frac{1}{n^2}\right) - \frac{1}{1^2} = \frac{\pi^2}{6} - 1$
3. We can begin by reindexing this series: $\sum_{n=3}^{\infty} \frac{1}{(n+1)^2} = \sum_{n=4}^{\infty} \frac{1}{n^2}$. Similar to the previous problem, we also know that $\sum_{n=4}^{\infty} \frac{1}{n^2} = \sum_{n=1}^{\infty} \left(\frac{1}{n^2}\right) - \left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2}\right) = \frac{\pi^2}{6} - \frac{49}{36}$
4. We can rewrite this series as $\sum_{n=1}^{\infty} \left(\frac{3}{n}\right)^4 = \sum_{n=1}^{\infty} (3^4) \frac{1}{n^4} = 81 \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{81\pi^4}{90} = \frac{9\pi^4}{10}$
5. We can re-write the series as $\sum_{n=1}^{\infty} \left(\frac{4}{n^2} + \frac{3}{n^4}\right) = \sum_{n=1}^{\infty} \frac{4}{n^2} + \sum_{n=1}^{\infty} \frac{3}{n^4} = 4 \sum_{n=1}^{\infty} \frac{1}{n^2} + 3 \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{4\pi^2}{6} + \frac{3\pi^4}{90} = \frac{2\pi^2}{3} + \frac{\pi^4}{30}$

Answer to Exercise 5 (on page 14)

This is a p -series where $p = 2k$. We know that p -series converge for $p > 1$: $2k > 1 \rightarrow k > \frac{1}{2}$.

Answer to Exercise 6 (on page 15)

1. First check whether the terms shrink: $\left|\frac{(-1)^{n+1}3(n+1)}{4(n+1)-1}\right| < \left|\frac{(-1)^n3n}{4n-1}\right|$ if $\frac{3n+3}{4n+4-1} < \frac{3n}{4n-1}$ and if $\frac{3n+3}{4n+3} < \frac{3n}{4n-1}$ if $(3n+3)(4n-1) < (3n)(4n+3)$ if $12n^2 + 12n - 3n - 3 < 12n^2 + 9n$ if

$-3 < 0$ which is true. The terms b_n decrease, but $\lim_{n \rightarrow \infty} \frac{3n}{4n-1} = \frac{3}{4} \neq 0$. So the terms do not approach 0, and $\sum_{n=1}^{\infty} \frac{(-1)^n 3n}{4n-1}$ is divergent.

2. First check whether the terms shrink: $\left| (-1)^{n+1+1} \frac{(n+1)^2}{(n+1)^3+1} \right| < \left| (-1)^{n+1} \frac{n^2}{n^3+1} \right|$, which is true if $\frac{(n+1)^2}{(n+1)^3+1} < \frac{n^2}{n^3+1}$ if $(n+1)^2(n^3+1) < (n^2)((n+1)^3+1)$ if $(n^2+2n+1)(n^3+1) < (n^2)(n^3+3n^2+3n+1+1)$ if $n^5+2n^4+n^3+n^2+2n+1 < n^5+3n^4+3n^3+2n^2$, which is the same as $0 < n^4+2n^3+n^2-2n-1$. This is true for all $n \geq 1$. Also, $\lim_{n \rightarrow \infty} \frac{n^2}{n^3+1} = 0$. Therefore, $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^3+1}$ is convergent.
3. First check whether the terms shrink: $\left| (-1)^{n-1+1} e^{2/(n+1)} \right| < \left| (-1)^{n-1} e^{2/n} \right|$, which is true if $e^{2/(n+1)} < e^{2/n}$, which is true if $\frac{2}{n+1} < \frac{2}{n}$ which is true for all $n \geq 1$. Although the terms decrease, $e^{2/n} \rightarrow e^0 = 1 \neq 0$. So the terms do not approach 0, and $\sum_{n=1}^{\infty} (-1)^{n-1} e^{2/n}$ is divergent.

Answer to Exercise 7 (on page 21)

1. The function $2x^{-3}$ is positive and decreasing for $x \in [1, \infty)$. $\int_1^{\infty} 2x^{-3} dx = \lim_{t \rightarrow \infty} \int_1^t 2x^{-3} dx = \lim_{t \rightarrow \infty} [-x^{-2}]_{x=1}^t = \lim_{t \rightarrow \infty} (-t^{-2}) - (-1^{-2}) = 0 + 1 = 1$. Since the integral $\int_1^{\infty} 2x^{-3} dx$ converges, the series $\sum_{n=1}^{\infty} 2n^{-3}$ is also convergent.
2. The function $\frac{5}{3x-1}$ is positive and decreasing for $x \in [1, \infty)$. $\int_1^{\infty} \frac{5}{3x-1} dx = \lim_{t \rightarrow \infty} \int_1^t \frac{5}{3x-1} dx$. Using u-substitution to evaluate the integral, we set $u = 3x-1$ and find that $du = 3dx \rightarrow dx = \frac{du}{3}$. Substituting, $\int_1^t \frac{5}{3x-1} dx = \int_{x=1}^{x=t} \frac{5}{3} \frac{1}{u} du$. Evaluating the integral, $\int_{x=1}^{x=t} \frac{5}{3} \frac{1}{u} du = \frac{5}{3} \ln u \Big|_{x=1}^{x=t} = \frac{5}{3} \ln 3x - 1 \Big|_1^t$. Substituting this back into the limit, $\int_1^{\infty} \frac{5}{3x-1} dx = \lim_{t \rightarrow \infty} \frac{5}{3} \ln 3x - 1 \Big|_1^t = \lim_{t \rightarrow \infty} \left[\frac{5}{3} \ln 3t - 1 \right] - \frac{5}{3} \ln 2 = \infty$. Therefore, the integral $\int_1^{\infty} \frac{5}{3x-1} dx$ is divergent and so is the series $\sum_{n=1}^{\infty} \frac{5}{3n-1}$.
3. The function $\frac{x}{3x^2+1}$ is positive and decreasing for $x \in [1, \infty)$. $\int_1^{\infty} \frac{x}{3x^2+1} dx = \lim_{t \rightarrow \infty} \int_1^t \frac{x}{3x^2+1} dx$. Applying the substitution $u = 3x^2+1$ and $\frac{du}{6} = x dx$, we see that $\lim_{t \rightarrow \infty} \int_1^t \frac{x}{3x^2+1} dx = \lim_{t \rightarrow \infty} \int_{x=1}^{x=t} \frac{1}{6u} du = \lim_{t \rightarrow \infty} \frac{1}{6} \ln u \Big|_{x=1}^{x=t} = \lim_{t \rightarrow \infty} \frac{1}{6} \ln 3x^2 + 1 \Big|_1^t = \lim_{t \rightarrow \infty} \left[\frac{1}{6} \ln 3t^2 + 1 \right] - \frac{1}{6} \ln 4 = \infty$. Therefore, the integral $\int_1^{\infty} \frac{x}{3x^2+1} dx$ is divergent, and so is the series $\sum_{n=1}^{\infty} \frac{n}{3n^2+1}$.

Answer to Exercise 8 (on page 21)

1. If $p \leq 0$, then $\lim_{n \rightarrow \infty} \frac{1}{n^p} \neq 0$, and the series fails the Test for Divergence. Therefore, a p-series is divergent if $p \leq 0$.
2. If $p > 0$, then $f(x) = \frac{1}{x^p}$ is continuous, positive, and decreasing on the interval $x \in$

$[1, \infty)$, and we can apply the integral test. So, we want to know, when is $\int_1^\infty \frac{1}{x^p} dx$ convergent? When $p = 1$, $\int_1^\infty \frac{1}{x^p} dx = \ln x \Big|_{x=1}^{x=\infty} = \lim_{t \rightarrow \infty} \ln t - \ln 1 = \infty$ and the integral and p -series are both divergent.

- What about when $0 < p < 1$? In this case, the integral $\int_1^\infty \frac{1}{x^p} dx = \lim_{t \rightarrow \infty} \int_1^t x^{-p} dx = \lim_{t \rightarrow \infty} \frac{1}{1-p} x^{1-p} \Big|_{x=1}^{x=t} = \lim_{t \rightarrow \infty} \frac{1}{1-p} \frac{1}{t^{p-1}} = \left(\frac{1}{1-p} \right) \left[\lim_{t \rightarrow \infty} \left(\frac{1}{t^{p-1}} \right) - 1 \right]$. When $0 < p < 1$, then $1-p > 0$ is positive and $\lim_{t \rightarrow \infty} \frac{1}{t^{p-1}} = \lim_{t \rightarrow \infty} t^{1-p} = \infty$ and the integral diverges. Therefore, p -series are divergent for $0 < p < 1$.
- When $p > 1$, then $\int_1^\infty \frac{1}{x^p} dx = \left(\frac{1}{1-p} \right) \left[\lim_{t \rightarrow \infty} \left(\frac{1}{t^{p-1}} \right) - 1 \right]$. When $p > 1$, $p-1 > 0$ and $\lim_{t \rightarrow \infty} \frac{1}{t^{p-1}} = 0$. Therefore, $\int_1^\infty \frac{1}{x^p} dx$ converges to $\frac{1}{p-1}$ when $p > 1$, and therefore the p -series is convergent when $p > 1$.

Answer to Exercise 9 (on page 24)

- $s_{10} = \frac{1}{1^4} + \frac{1}{2^4} + \cdots + \frac{1}{10^4} \approx 1.082037$.
- $R_{10} \leq \int_{10}^\infty \frac{1}{x^4} dx = \frac{-1}{3x^3} \Big|_{x=10}^\infty = \lim_{x \rightarrow \infty} \frac{-1}{3x^3} - \frac{-1}{3 \cdot 10^3} = \frac{1}{3000} = 0.000333$. Therefore, the error is less than 0.000333.
- Given $s_{10} \approx 1.082037$, we can say that $1.082037 + \int_{11}^\infty \frac{1}{x^4} dx \leq s \leq 1.082037 + \int_{10}^\infty \frac{1}{x^4} dx$. Using a calculator to evaluate each integral, we see that: $1.082037 + 0.000250 \leq s \leq 1.082037 + 0.000333$ and therefore the sum is between 1.082287 and 1.082370.
- Writing the actual value as a decimal, $\frac{\pi^4}{90} \approx 1.082323$, which is in the estimate window from the previous part.
- We are looking for an n such that $\int_n^\infty \frac{1}{x^4} dx \leq 0.00001$. $\lim_{x \rightarrow \infty} \frac{-1}{3x^3} - \frac{-1}{3n^3} = \frac{1}{3n^3} \leq 0.00001$. $100,000 \leq 3n^3$. $33,333.33 \leq n^3$. $32.183 \leq n$. Since n must be an integer, $n = 33$ gives $R_n \leq 0.00001$.

Answer to Exercise 10 (on page 27)

- This is similar to $\sum_{n=1}^\infty \frac{1}{n}$, which is divergent. Unfortunately, $\frac{1}{n} > \frac{1}{\sqrt{n^2+1}}$, so we cannot use the direct comparison test. We will try the limit comparison test:

$$\lim_{n \rightarrow \infty} \left(\frac{\frac{1}{\sqrt{n^2+1}}}{\frac{1}{n}} \right) = \lim_{n \rightarrow \infty} \left(\frac{1}{\sqrt{n^2+1}} \cdot \frac{n}{1} \right) = \lim_{n \rightarrow \infty} \frac{n}{\sqrt{n^2+1}} = \lim_{n \rightarrow \infty} \frac{1}{\sqrt{1+1/n^2}} = \frac{1}{1+0} = 1 > 0$$

Therefore, since $\sum_{n=1}^\infty \frac{1}{n}$ diverges, so does $\sum_{n=1}^\infty \frac{1}{\sqrt{n^2+1}}$.

2. This series is similar to the convergent geometric series $\sum_{n=1}^{\infty} \left(\frac{9}{10}\right)^n$. Given that:

$$\frac{9^n}{3+10^n} < \frac{9^n}{10^n} = \left(\frac{9}{10}\right)^n$$

Since $\frac{9^n}{3+10^n} < \left(\frac{9}{10}\right)^n$ and $\sum_{n=1}^{\infty} \left(\frac{9}{10}\right)^n$ is convergent, by the direct comparison test, $\sum_{n=1}^{\infty} \frac{9^n}{3+10^n}$ is also convergent.

3. We can compare this to the convergent p-series $\sum_{n=1}^{\infty} \frac{1}{n^2}$. Noting that $\sin^2 n \leq 1$:

$$\frac{n \sin^2 n}{1+n^3} < \frac{n \sin^2 n}{n^3} \leq \frac{n}{n^3} = \frac{1}{n^2}$$

Because $\frac{n \sin^2 n}{1+n^3} \leq \frac{1}{n^2}$ for all $n \geq 1$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent, we can state by the direct comparison test that $\sum_{n=1}^{\infty} \frac{n \sin^2 n}{1+n^3}$ is also convergent.

Answer to Exercise 11 (on page 30)

We can write the series as $\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$. Since $n^2 > 0$, $\sum_{n=1}^{\infty} \left| \frac{\cos n}{n^2} \right| = \sum_{n=1}^{\infty} \frac{|\cos n|}{n^2}$. Additionally, $|\cos n| \leq 1$ for all n , and therefore $\frac{|\cos n|}{n^2} \leq \frac{1}{n^2}$. We know the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent. And since we have shown that $\sum_{n=1}^{\infty} \frac{|\cos n|}{n^2} \leq \sum_{n=1}^{\infty} \frac{1}{n^2}$, by the comparison test $\sum_{n=1}^{\infty} \frac{|\cos n|}{n^2}$ is convergent. Therefore, $\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$ is absolutely convergent and therefore convergent.

Answer to Exercise 12 (on page 30)

- Conditionally Convergent. $\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{3n+2} \right| = \sum_{n=1}^{\infty} \frac{1}{3n+2}$ Applying the integral test to this sum: $\int_1^{\infty} \frac{1}{3x+2} dx = \lim_{t \rightarrow \infty} \int_1^t \frac{1}{3x+2} dx = \lim_{t \rightarrow \infty} \left[\frac{1}{3} \ln(3x+2) \right]_{x=1}^t = \lim_{t \rightarrow \infty} \frac{1}{3} \ln(3t+2) - \frac{1}{3} \ln 5 = \infty$. Since $\int_1^{\infty} \frac{1}{3x+2} dx$ is divergent, $\sum_{n=1}^{\infty} \frac{1}{3n+2}$ is divergent. The original series still converges by the Alternating Series Test, because $\frac{1}{3n+2}$ decreases to 0. So $\sum_{n=1}^{\infty} \frac{(-1)^n}{3n+2}$ is conditionally convergent.
- Absolutely Convergent. $\sum_{n=1}^{\infty} \left| \frac{\sin n}{4^n} \right| \leq \sum_{n=1}^{\infty} \frac{1}{4^n}$. The series $\sum_{n=1}^{\infty} \frac{1}{4^n}$ is geometric with $r = \frac{1}{4}$, so it converges. And since $\sum_{n=1}^{\infty} \left| \frac{\sin n}{4^n} \right| \leq \sum_{n=1}^{\infty} \frac{1}{4^n}$, $\sum_{n=1}^{\infty} \left| \frac{\sin n}{4^n} \right|$ is also convergent, which shows that $\sum_{n=1}^{\infty} \frac{\sin n}{4^n}$ is absolutely convergent.
- Conditionally Convergent. We are asking if the series $\sum_{n=1}^{\infty} \left| (-1)^{n-1} \frac{2n}{n^2+4} \right|$ is convergent. $\sum_{n=1}^{\infty} \left| (-1)^{n-1} \frac{2n}{n^2+4} \right| = \sum_{n=1}^{\infty} \frac{2n}{n^2+4}$ We will apply the Limit Comparison test and compare this series to the known, divergent series $\sum_{n=1}^{\infty} \frac{1}{n}$. $\lim_{n \rightarrow \infty} \frac{\frac{2n}{n^2+4}}{\frac{1}{n}} = \lim_{n \rightarrow \infty} \frac{2n^2}{n^2+4} =$

$2 > 0$. Therefore, by the Limit Comparison test, $\sum_{n=1}^{\infty} \left| (-1)^{n-1} \frac{2n}{n^2+4} \right|$ is divergent. The terms $\frac{2n}{n^2+4}$ go to 0 and decrease from $n = 2$ on (the derivative of $\frac{2x}{x^2+4}$ is negative for $x > 2$), so the Alternating Series Test shows the original series converges. Therefore $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{2n}{n^2+4}$ is conditionally convergent.

Answer to Exercise 13 (on page 31)

Series 1 and 3 converge.

1. We apply the ratio test: $\lim_{n \rightarrow \infty} \left| \frac{8^{n+1}}{\frac{(n+1)!}{8^n}} \right| = \lim_{n \rightarrow \infty} \frac{8 \cdot 8^n}{(n+1)(n!)} \cdot \frac{n!}{8^n} = \lim_{n \rightarrow \infty} \frac{8}{n+1} = 0$.

Therefore, the series converges.

2. We apply the ratio test: $\lim_{n \rightarrow \infty} \left| \frac{\frac{(n+1)!}{(n+1)^{100}}}{\frac{n!}{n^{100}}} \right| = \lim_{n \rightarrow \infty} \frac{(n+1)n!}{(n+1)^{100}} \cdot \frac{n^{100}}{n!} = \lim_{n \rightarrow \infty} \left(\frac{n}{n+1} \right)^{100}$.
 $(n+1) = \lim_{n \rightarrow \infty} \frac{n^{100}}{(n+1)^{99}} = \infty$. Therefore, the series diverges.

3. We apply the comparison test: $\frac{n+1}{(n)(n+2)(n+3)} = \frac{n}{(n)(n+2)(n+3)} + \frac{1}{(n)(n+2)(n+3)} = \frac{1}{(n+2)(n+3)} + \frac{1}{(n)(n+2)(n+3)} = \frac{1}{n^2+5n+6} + \frac{1}{n^3+5n^2+6n} \leq \frac{1}{n^2} + \frac{1}{n^3}$. The series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ and $\sum_{n=1}^{\infty} \frac{1}{n^3}$ are both convergent, because they are p-series with $p > 1$. Having established that $\sum_{n=1}^{\infty} \frac{n+1}{(n)(n+2)(n+3)} \leq \sum_{n=1}^{\infty} \frac{1}{n^2} + \frac{1}{n^3}$ and that $\sum_{n=1}^{\infty} \frac{1}{n^2} + \frac{1}{n^3}$ converges, by the comparison test we can state that $\sum_{n=1}^{\infty} \frac{n+1}{(n)(n+2)(n+3)}$ converges.

Answer to Exercise 14 (on page 32)

1. This is not necessarily true. For the convergent series $\sum_{n=1}^{\infty} \frac{1}{2^n}$, the ratio is always $\frac{1}{2}$. For $\sum_{n=1}^{\infty} \frac{1}{n^2}$, the ratio approaches 1.
2. This is not necessarily true. Consider the geometric series $\sum_{n=1}^{\infty} 2\left(\frac{1}{2}\right)^{n-1}$. This series is convergent because the common ratio is less than one, but the first term is $2\left(\frac{1}{2}\right)^0 = 2 > 1$.
3. This is not necessarily true. Again, consider the geometric series $\sum_{n=1}^{\infty} 2\left(\frac{1}{2}\right)^{n-1}$, which converges to $4 \neq 0$.
4. This is not necessarily true. Consider the p-series $\sum_{n=1}^{\infty} \frac{1}{n^4}$. Then the series $\sum_{n=1}^{\infty} n \frac{1}{n^4} = \sum_{n=1}^{\infty} \frac{1}{n^3}$ is convergent.
5. This must be true. Since $a_n > 0$ and $n \geq 1$, $\frac{a_n}{n} \leq a_n$. Since $\sum_{n=1}^{\infty} a_n$ converges, so must $\sum_{n=1}^{\infty} \frac{a_n}{n}$ by the comparison test.

Answer to Exercise 15 (on page 33)

1. $\lim_{n \rightarrow \infty} \sqrt[n]{\left| \left(\frac{3n^2+1}{n^2-4} \right)^n \right|} = \lim_{n \rightarrow \infty} \frac{3n^2+1}{n^2-4} = 3 > 1$. Therefore, the series $\sum_{n=3}^{\infty} \left(\frac{3n^2+1}{n^2-4} \right)^n$ is divergent.
2. $\lim_{n \rightarrow \infty} \sqrt[n]{\left| \frac{(-1)^n}{(\ln n)^n} \right|} = \lim_{n \rightarrow \infty} \sqrt[n]{\frac{1}{(\ln n)^n}} = \lim_{n \rightarrow \infty} \frac{1}{\ln n} = \frac{1}{\infty} = 0 < 1$. Therefore, the series $\sum_{n=2}^{\infty} \frac{(-1)^n}{(\ln n)^n}$ is convergent.
3. $\lim_{n \rightarrow \infty} \sqrt[n]{\left| \left(1 + \frac{1}{n} \right)^n \right|} = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n} \right)^n = e > 1$. Therefore, $\sum_{n=1}^{\infty} \left(1 + \frac{1}{n} \right)^n$ is divergent.

Answer to Exercise 16 (on page 34)

1. Divergent. Since there is a constant to the n^{th} power and an algebraic function of n , we will try the Ratio Test. $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \frac{e^{n+1}}{(n+1)^2} \cdot \frac{n^2}{e^n} = \lim_{n \rightarrow \infty} \frac{e^n \cdot e}{e^n} \cdot \left(\frac{n}{n+1} \right)^2 = \lim_{n \rightarrow \infty} e \cdot \left(\frac{n}{n+1} \right)^2 = e \cdot 1^2 = e > 1$. Therefore, $\sum_{n=1}^{\infty} \frac{e^n}{n^2}$ is divergent.
2. Convergent. Since there is a factorial, we will try the Ratio Test. $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \frac{3^{n+1}(n+1)^2}{(n+1)!} \cdot \frac{n!}{3^n n^2} = \lim_{n \rightarrow \infty} \frac{3 \cdot 3^n}{3^n} \cdot \frac{n!}{(n+1)n!} \cdot \left(\frac{n+1}{n} \right)^2 = \lim_{n \rightarrow \infty} \frac{3(n+1)^2}{(n+1)n^2} = \lim_{n \rightarrow \infty} \frac{3(n+1)}{n^2} = 0 < 1$. Therefore, the series $\sum_{n=1}^{\infty} \frac{3^n n^2}{n!}$ is convergent.
3. Divergent. Since $\int_2^{\infty} \frac{1}{x\sqrt{\ln x}} dx$ can be integrated, we will apply the integral test. $\int_2^{\infty} \frac{1}{x\sqrt{\ln x}} dx = \lim_{t \rightarrow \infty} \int_2^t \frac{1}{x\sqrt{\ln x}} dx$. Setting $u = \ln x$, then $du = \frac{dx}{x}$ and $\frac{1}{x\sqrt{\ln x}} dx = \frac{1}{\sqrt{u}} du$. Then we can say that $\int_2^{\infty} \frac{1}{x\sqrt{\ln x}} dx = \lim_{t \rightarrow \infty} \int_{x=2}^{x=t} \frac{1}{\sqrt{u}} du = \lim_{t \rightarrow \infty} 2\sqrt{u} \Big|_{x=2}^{x=t} = \lim_{t \rightarrow \infty} 2\sqrt{\ln x} \Big|_2^t = 2 \lim_{t \rightarrow \infty} \sqrt{\ln t} - 2\sqrt{\ln 2} = \infty$. Since the integral diverges, so does the series.
4. Convergent. Since this series has terms to the n^{th} power, we will try the Root Test. $\lim_{n \rightarrow \infty} \sqrt[n]{\left| \left(\frac{n}{n+1} \right)^{n^2} \right|} = \lim_{n \rightarrow \infty} \sqrt[n]{\left(\frac{n}{n+1} \right)^{n^2}} = \lim_{n \rightarrow \infty} \left(\frac{n}{n+1} \right)^{\frac{n^2}{n}} = \lim_{n \rightarrow \infty} \left(\frac{n}{n+1} \right)^n = \lim_{n \rightarrow \infty} \left(\frac{1}{1+\frac{1}{n}} \right)^n = \frac{1}{\lim_{n \rightarrow \infty} \left(1+\frac{1}{n} \right)^n} = \frac{1}{e} < 1$. Therefore, by the root test, the series is convergent.
5. Convergent. This series also has terms raised to the n^{th} power, we will try the Root Test again. $\lim_{n \rightarrow \infty} \sqrt[n]{\left| \left(\sqrt[3]{2} - 1 \right)^n \right|} = \lim_{n \rightarrow \infty} \sqrt[n]{\left(\sqrt[3]{2} - 1 \right)^n} = \lim_{n \rightarrow \infty} \left(\sqrt[3]{2} - 1 \right)^{n/n} = \lim_{n \rightarrow \infty} \left(\sqrt[3]{2} - 1 \right) = \lim_{n \rightarrow \infty} 2^{1/n} - 1 = 1 - 1 = 0 < 1$. Therefore, the series converges.

Answer to Exercise 17 (on page 40)

Since this sum has terms to the n^{th} power, we will apply the Root Test, which states a series is convergent if $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} < 1$.

$$\lim_{n \rightarrow \infty} \sqrt[n]{\left| \frac{(x-4)^{2n}}{3^n} \right|} < 1$$

$$\lim_{n \rightarrow \infty} \left| \frac{(x-4)^{2n/n}}{3^{n/n}} \right| < 1$$

$$\lim_{n \rightarrow \infty} \frac{|(x-4)^2|}{3} < 1$$

$$(x-4)^2 < 3$$

$$|x-4| < \sqrt{3}$$

Therefore, the radius of convergence is $\sqrt{3}$.

Answer to Exercise 18 (on page 41)

We see that the series is alternating, so we know it involves $(-1)^n$ (we will begin indexing at $n = 0$). The powers of x are given by x^{2n+1} and the denominators are given by $2n+3$. Therefore, the sum in sigma notation is $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+3}$. Applying the ratio test, the series is convergent when:

$$\lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+1} x^{2(n+1)+1}}{2(n+1)+3} \cdot \frac{2n+3}{(-1)^n x^{2n+1}} \right| < 1$$

$$\lim_{n \rightarrow \infty} \left| \frac{x^{2n+3}}{2n+5} \cdot \frac{2n+3}{x^{2n+1}} \right| < 1$$

$$|x^2| \lim_{n \rightarrow \infty} \frac{2n+3}{2n+5} < 1$$

$$|x^2| < 1$$

$$|x| < 1$$

So, we know that the series is convergent on the open interval $x \in (-1, 1)$. We check the endpoints, $x = -1, 1$ for convergence.

$$\sum_{n=0}^{\infty} \frac{(-1)^n (-1)^{2n+1}}{2n+3} = \sum_{n=0}^{\infty} \frac{(-1)^{3n+1}}{2n+3}$$

When $x = -1$, the series is an alternating series such that $|a_{n+1}| < |a_n|$ and $\lim_{n \rightarrow \infty} a_n = 0$. Therefore, the series converges for $x = -1$.

$$\sum_{n=0}^{\infty} (-1)^n \frac{(1)^{2n+1}}{2n+3} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+3}$$

which is also an alternating series such that $|a_{n+1}| < |a_n|$ and $\lim_{n \rightarrow \infty} a_n = 0$. Therefore, the series converges for $x = 1$ and the interval of convergence is $x \in [-1, 1]$, which can also be written as $-1 \leq x \leq 1$.

Answer to Exercise 19 (on page 44)

Recall that $\frac{d}{dx} \arctan x = \frac{1}{1+x^2}$. Replacing x with $-x^2$, we see that $\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} [-x^2]^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}$. Then we can also say that $\arctan x = \int \frac{1}{1+x^2} dx = \int \left[\sum_{n=0}^{\infty} (-1)^n x^{2n} \right] dx$. Evaluating the integral, $\int \left[\sum_{n=0}^{\infty} (-1)^n x^{2n} \right] dx = \sum_{n=0}^{\infty} (-1)^n \int x^{2n} dx = C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$. Knowing that $\arctan 0 = 0$, we find that $C = 0$ and $\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$.

Answer to Exercise 20 (on page 44)

With $x = 0.1$: $0.1 - \frac{0.01}{2} + \frac{0.001}{3} \approx 0.0953333$. The calculator gives $\ln(1.1) \approx 0.0953102$, so the error is about 0.000023. The next term, $\frac{0.0001}{4} = 0.000025$, bounds the error, just as the Alternating Series Test promises.

Answer to Exercise 21 (on page 48)

Part I: Derive a Maclaurin series for $\cos x$. There are two methods you might have used: You could have used the same method as we used for $\sin x$ above, or you could have applied the fact that $\frac{d}{dx} \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} n c_n x^{n-1}$. We will show the second method:

$$\begin{aligned} \cos x &= \frac{d}{dx} \sin x \\ \cos x &= \frac{d}{dx} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \\ \cos x &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(2n)!} (2n+1) x^{2n} \\ \cos x &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} \end{aligned}$$

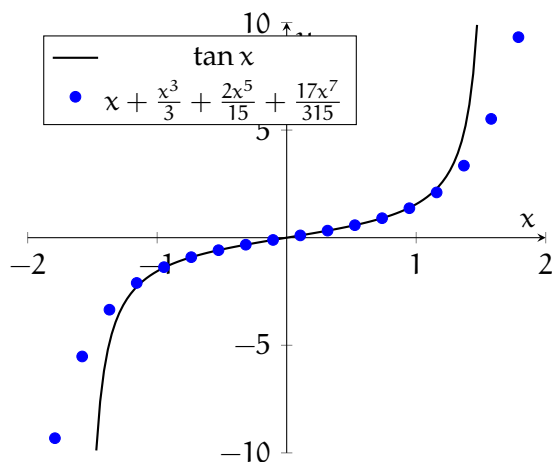
This is also an important Maclaurin series you will see again. Part II: Find the first four nonzero terms of the Maclaurin series for $\tan x$ using the series for $\cos x$ and $\sin x$. We may be tempted to use the same process of explicitly calculating the derivatives of $\tan x$ at $x = 0$. However, the derivative of $\tan x$ is $\sec^2 x$, and the derivatives only get more complicated from there. We will not have a nice, cyclical result, like we did with $\sin x$. Instead, we will use trigonometric identities to help us. Recall that $\tan x = \frac{\sin x}{\cos x}$. Therefore, we can say that:

$$\tan x = \frac{\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}}{\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}}$$

We may be tempted to divide the terms directly and have lots of beautiful cancellations. However, this would be incorrect (because $\frac{a+b+c+\dots}{A+B+C+\dots} \neq \frac{a}{A} + \frac{b}{B} + \frac{c}{C} + \dots$). Instead, we keep the first four terms of each series and perform long division of polynomials.

$$\tan x = \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}}{1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}}$$

When we evaluate this, we find the first four terms of the result are $\tan x \approx x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315}$. A plot is shown below:



As you can see, the series is an excellent estimate near $x = 0$.

Answer to Exercise 22 (on page 50)

1. $(1+x)^{-1} = 1 - x + x^2 - x^3 + \dots$, valid for $|x| < 1$
2. $\sqrt{1-2x} = 1 - x - \frac{1}{2}x^2 - \dots$

Answer to Exercise 23 (on page 51)

Substituting $x = 3$, the series is $\sum_{n=0}^{\infty} \left(-\frac{3}{4}\right)^n$, which is an alternating geometric series. Reindexing the series to begin at $n = 1$, this is equivalent to $\sum_{n=1}^{\infty} \left(-\frac{3}{4}\right)^{n-1}$, which is in the standard form $\sum_{n=1}^{\infty} ar^{n-1}$ with $a = 1$ and $r = -\frac{3}{4}$. Then the value of the series is $\frac{a}{1-r} = \frac{1}{1-\left(-\frac{3}{4}\right)} = \frac{1}{1+\frac{3}{4}} = \frac{1}{\frac{7}{4}} = \frac{4}{7}$.

Answer to Exercise 24 (on page 52)

A third-degree Taylor polynomial about $x = 3$ is given by $f(3) + \frac{f'(3)}{1!}(x-3) + \frac{f''(3)}{2!}(x-3)^2 + \frac{f'''(3)}{3!}(x-3)^3$. Substituting what we know, the 3rd-degree Taylor polynomial is $f(x) \approx 2 + \frac{-1}{1}(x-3) + \frac{6}{2}(x-3)^2 + \frac{12}{6}(x-3)^3 = 2 - (x-3) + 3(x-3)^2 + 2(x-3)^3$. Substituting $x = 3.4$, we find that $f(3.4) \approx 2 - (3.4-3) + 3(3.4-3)^2 + 2(3.4-3)^3 = 2 - 0.4 + 3(0.16) + 2(0.064) = 2 - 0.4 + 0.48 + 0.128 = 2.208$.

Answer to Exercise 25 (on page 52)

A second-degree Taylor polynomial about $x = 1$ is given by $f(x) \approx f(1) + \frac{f'(1)}{1!}(x-1) + \frac{f''(1)}{2!}(x-1)^2 = 15 + 8(x-1) + 10(x-1)^2$. Substituting $x = 1.4$, we find that $f(1.4) \approx 15 + 8(1.4-1) + 10(1.4-1)^2 = 19.8$.

Answer to Exercise 26 (on page 53)

(c). Since $\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$, then $\frac{\sin x}{x} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{x(2n+1)!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n+1)!}$.

Answer to Exercise 27 (on page 53)

$T_3(0.5) = 0.5 - \frac{0.125}{6} \approx 0.479167$. Every derivative of $\sin x$ is $\pm \sin$ or $\pm \cos$, so $M = 1$ works:

$$|\sin(0.5) - T_3(0.5)| \leq \frac{1}{4!}(0.5)^4 \approx 0.0026 < 0.003$$

(The true value is 0.479426, so the actual error is only about 0.00026.)

Answer to Exercise 28 (on page 58)

1. $\cos \frac{\pi}{3} + i \sin \frac{\pi}{3} = \frac{1}{2} + \frac{\sqrt{3}}{2}i$
2. $\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$
3. $\cos \frac{3\pi}{2} + i \sin \frac{3\pi}{2} = -i$
4. 10π is five full turns, so $e^{10\pi i} = 1$.

Answer to Exercise 29 (on page 60)

1. $r = \sqrt{9+9} = 3\sqrt{2}$. The point is in the second quadrant at angle $\frac{3\pi}{4}$ (135°), so $-3+3i = 3\sqrt{2}e^{i3\pi/4}$.
2. $1 - \sqrt{3}i = 2e^{-i\pi/3}$, so $(1 - \sqrt{3}i)^6 = 2^6e^{-i2\pi} = 64$.
3. Each cube root has length $\sqrt[3]{8} = 2$, and the angles are spaced by $\frac{2\pi}{3}$: 2 , $2e^{i2\pi/3} = -1 + \sqrt{3}i$, and $2e^{i4\pi/3} = -1 - \sqrt{3}i$.

Answer to Exercise 30 (on page 61)

$\cos 2\theta + i \sin 2\theta = (\cos \theta + i \sin \theta)^2 = \cos^2 \theta - \sin^2 \theta + 2i \sin \theta \cos \theta$. Matching parts: $\cos 2\theta = \cos^2 \theta - \sin^2 \theta$ and $\sin 2\theta = 2 \sin \theta \cos \theta$.

Answer to Exercise 31 (on page 63)

$i^i = (e^{i\pi/2})^i = e^{i \cdot i\pi/2} = e^{-\pi/2} \approx 0.2079$. An imaginary number raised to an imaginary power is a real number! (Because i is also $e^{i(\pi/2+2\pi k)}$ for any whole number k , i^i has other values too, such as $e^{-5\pi/2}$; $e^{-\pi/2}$ is the standard one, and it is what Python's `1j ** 1j` returns.)

Answer to Exercise 32 (on page 63)

$e^{-i\theta} = \cos(-\theta) + i \sin(-\theta) = \cos \theta - i \sin \theta$, which is the conjugate of $\cos \theta + i \sin \theta$. Their product is $e^{i\theta}e^{-i\theta} = e^0 = 1$ (or, directly, $\cos^2 \theta + \sin^2 \theta = 1$). Geometrically, $e^{-i\theta}$ is the reflection of $e^{i\theta}$ across the real axis, and rotating by θ and then by $-\theta$ brings you back to

1.

Answer to Exercise 33 (on page 63)

```
1 import cmath, math
2
3 for k in range(7):
4     theta = 0.5 * k
5     lhs = cmath.exp(1j * theta)
6     rhs = math.cos(theta) + 1j * math.sin(theta)
7     print(theta, abs(lhs - rhs))
```

Every difference is 0 or a rounding error around 10^{-16} .



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