Contents

Chapter 1

Definite Integrals

Integrals are a fundamental concept in calculus. They are used to calculate areas, volumes, and many other things. A definite integral calculates the net area between the function and the x-axis over a given interval.

Recall that you can use a Riemann sum to estimate the area under a function, and that as we increase the number of subintervals, the estimated area approaches the actual area. In sigma notation we can express a Riemann sum as

$$\sum_{i=1}^{n} f(x_i)\Delta x$$

1.1 Definition

The definite integral of a function $f(x)$ over an interval $[a, b]$ is defined as the limit of a Riemann sum as n approaches ∞:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x \tag{1.1}$$

where x_i^* is a sample point in the i^{th} subinterval of a partition of $[a, b]$, $\Delta x = \frac{b-a}{n}$ is the width of each subinterval, and the limit is taken as the number of subintervals n approaches infinity.

Exercise 1

Express

$$\lim_{n\to\infty}\sum_{i=1}^{n}(x_i^3 + x_i \sin x_i)\Delta x$$

as an integral on the interval $[0, \pi]$.

Working Space

Answer on Page 45

1.2 Positive and Negative Areas

What if the function dips below the x-axis? We consider that area negative. In other words, it represents a *decrease* as opposed to an increase. Consider an oscillating object where $v(t) = \sin \pi x$ (figure 1.1). From $t = 0$ to $t = 1$, the velocity is positive, which means the object is moving *away from* the starting position. This is a positive displacement. From $t = 1$ to $t = 2$, the velocity is negative. What does this tell you about the direction the object is moving and its displacement during this time period? A negative velocity means the object is moving *back towards* the starting position.

In general, areas above the x-axis are positive, while areas below the x-axis are negative.

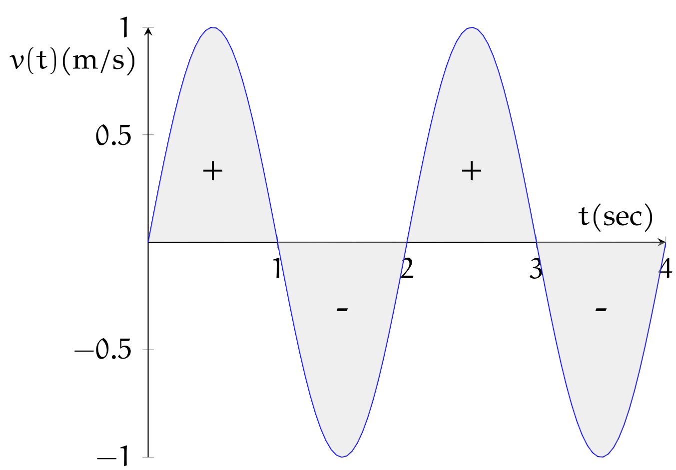


Figure 1.1: velocity of an oscillating object

1.3 Properties of Integrals

There are several important properties of integrals that will help us evaluate more complex integrals in the future. The following examples apply when $f(x)$ is continuous or has a finite number of jump discontinuities on the interval $a \leq x \leq b$:

1.3.1 What happens when $a = b$?

What if the endpoints of the integral are the same? Let's consider $\int_a^b x^2\,dx$, and take the limit as $b \to a$ (shown in figure 1.2). As you can see, as b approaches a, the calculated area decreases. Intuitively, we can guess that when $b = a$, then the width of the area (Δx) is zero, and therefore the area is also zero. Let's prove this formally.

Recall that $\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$, where $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x$. To evaluate the integral when $b = a$, we will take the limit of the limit:

$$\lim_{b\to a} \lim_{n\to\infty} f(x_i)\frac{b-a}{n}$$

This can be rewritten as

$$\lim_{b\to a}(b-a) \lim_{n\to\infty} \frac{f(x_i)}{n}$$

We know that $\lim_{b\to a}(b-a) = (a-a) = 0$, and therefore

$$\int_a^a f(x)\,dx = 0 \cdot \lim_{n\to\infty} \frac{f(x_i)}{n} = 0$$

This is true for any function.

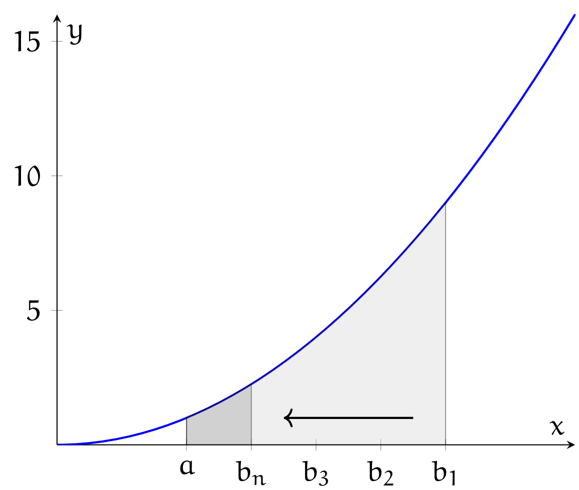


Figure 1.2: As b gets closer to a, the area represented by the integral decreases

1.3.2 The integral of a constant

When the function we are integrating is a constant (that is, it takes the form $f(x) = C$), the area is simply $(b - a) \cdot C$. This is shown graphically in figure 1.3.

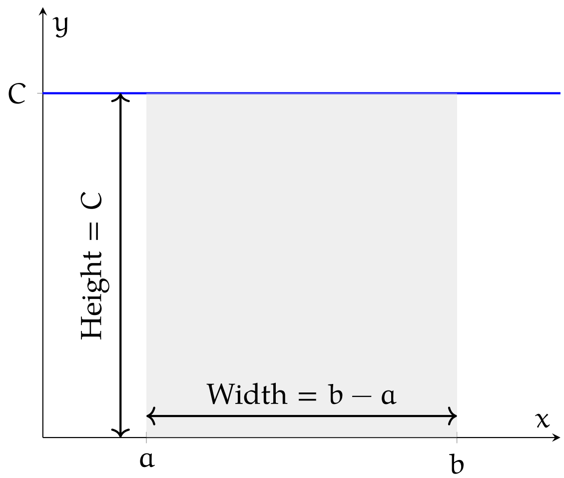


Figure 1.3: $\int_a^b f(x)\,dx = (b - a) \cdot C$

Since $f(x) = C$ is a horizontal line, the area under $f(x)$ is simply a rectangle. As you can see in figure 1.3, the width of the rectangle is $b - a$ and the height is C. To find the area of a rectangle, we multiply the width by the height, and therefore $\int_a^b C\,dx = (b - a) \cdot C$.

1.3.3 The integral of a function multiplied by a constant

How is $\int_a^b f(x)\,dx$ related to $\int_a^b C \cdot f(x)\,dx$? Intuitively, we know that multiplying a function by a constant, C, vertically stretches the graph by a factor of C. In turn, the area under the curve increases by a factor of C. Imagine a simple shape, like a triangle. If we keep the base of the triangle the same (analogous to the integral being over the same interval) and make the triangle three times taller (analogous to multiplying the function we're integrating by a factor of $C = 3$), then we would expect the total area of the triangle to be 3 times greater. Therefore, $\int_a^b C \cdot f(x)\,dx = C \int_a^b f(x)\,dx$.

1.3.4 Integrals of sums and differences of functions

If a function can be described as a sum of two other functions, then the integral of the original function is the same as the sum of the integrals of the two other functions. Concretely, we say $\int_a^b f(x) + g(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$. Figure 1.4 shows $f(x) = x + 2$, $g(x) = 4x^3 - 12x^2 + 10x$, and $f(x) + g(x)$. As you can see, the area under $f(x) + g(x)$ is equal to the area under $f(x)$ (the red area) plus the area under $g(x)$ (the diagonal lined area).

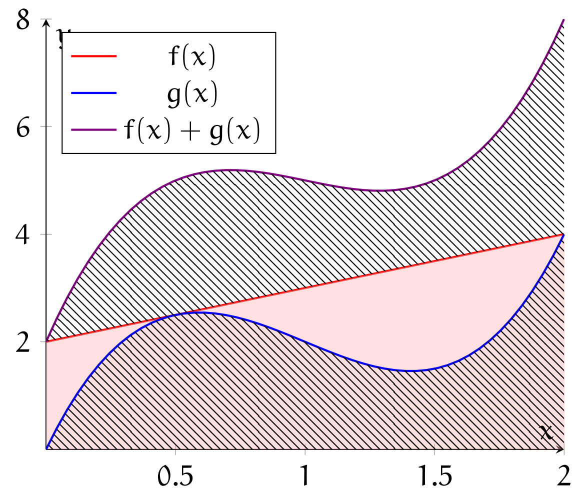


Figure 1.4: The intergral of $f(x) + g(x)$ is equal to the integral of $f(x)$ plus the integral of $g(x)$

Mathematically, we can prove this by recalling that the limit of a sum is the sum of the limits:

$$\begin{aligned}\int_a^b f(x) + g(x)\,dx &= \lim_{n\to\infty}\sum_{i=1}^{n}[f(x_i) + g(x_i)]\Delta x \\ &= \lim_{n\to\infty}[\sum_{i=1}^{n} f(x_i)\Delta x + \sum_{i=1}^{n} g(x_i)\Delta x] \\ &= \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x + \lim_{n\to\infty}\sum_{i=1}^{n} g(x_i)\Delta x \\ &= \int_a^b f(x)\,dx + \int_a^b g(x)\,dx\end{aligned}$$

Similar to the addition property, the integral of the difference between two function is equal to the difference of the integrals of two functions.

$$\int_a^b [f(x) - g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

. This is more difficult to visualize than addition, but we can easily prove it by applying the constant multiple and addition properties. Let's define $f(x) - g(x) = f(x) + (-g(x))$:

$$\int_a^b f(x) - g(x)\,dx = \int_a^b f(x) + (-g(x))\,dx$$

By the addition property,

$$= \int_a^b f(x)\,dx + \int_a^b -g(x)\,dx = \int_a^b f(x)\,dx + \int_a^b (-1)\cdot g(x)\,dx$$

By the constant multiple property:

$$= \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

1.3.5 Integrals of adjacent areas

If c is some x-value between a and b, then $\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$. This is shown graphically in figure 1.5. The total area from $x = a$ to $x = b$ is equal to the red area (the integral from a to c) plus the blue area (the integral from c to b).

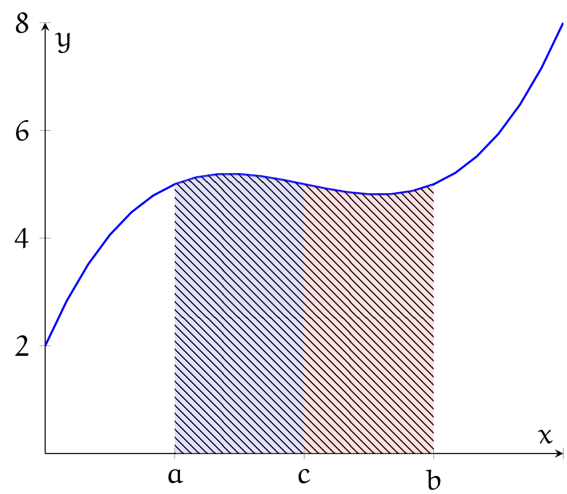


Figure 1.5: $\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$

1.3.6 Estimating the value of an integral

Suppose we need to know the area under a complex function. We can estimate a range for the value of the integral if we can bookend the function over the interval we are interested. Suppose there is some value m such that $f(x) \geq m$, and some other value M such that $f(x) \leq M$ on the interval we are interested in (see figure 1.6). The total area under $f(x)$ is the light blue plus the darker blue. The total area under $y = M$ is the darker blue, plus the light blue, plus the white area. The darker blue area under the curve has total area $m \cdot (b - a)$ and the rectangle under $y = M$ has total area $M \cdot (b - a)$ (since these are both integrals of a constant, which we learned about above). The actual area under our function is more than just the dark blue area, but less than the total area under $y = M$. Therefore, $m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$ if $m \leq f(x)$ and $M \geq f(x)$ on the interval $x \in [a, b]$.

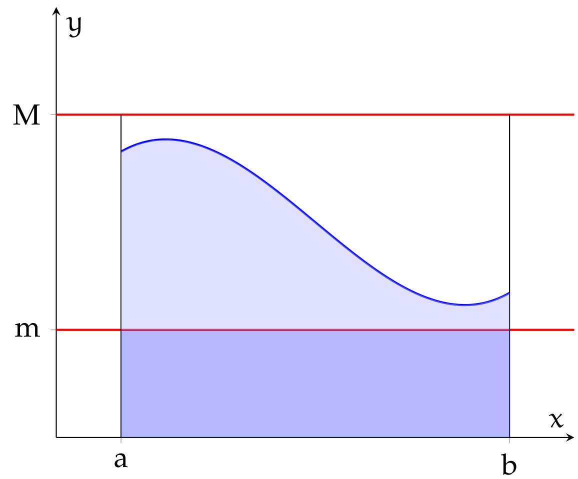


Figure 1.6: $m \leq f(x) \leq M$

1.3.7 Other Properties of Integrals

If $f(x) \geq 0$ over the for $a \leq x \leq b$, then $\int_a^b f(x)\,dx > 0$. We can make an intuitive, geometric argument to support this claim. Recall that areas above the x-axis are considered positive. If $f(x) \geq 0$, then all the area of the integral lies above the x-axisj therefore, the total area must be positive.

Similarly, if $f(x) \geq g(x)$ on the interval $a \leq x \leq b$, then $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$ (see figure 1.7). The entire area under $g(x)$ is contained in the area under $f(x)$. Therefore, $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$

Lastly, we see what happens when we switch a and b. While it is unusual to integrate from right to left (that is, in a case where $a > b$), this property will be useful. Recall that

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\frac{(b-a)}{n}$$

What is $\int_b^a f(x)\,dx$? Substituting, we see that

$$\int_b^a f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\frac{(a-b)}{n}$$

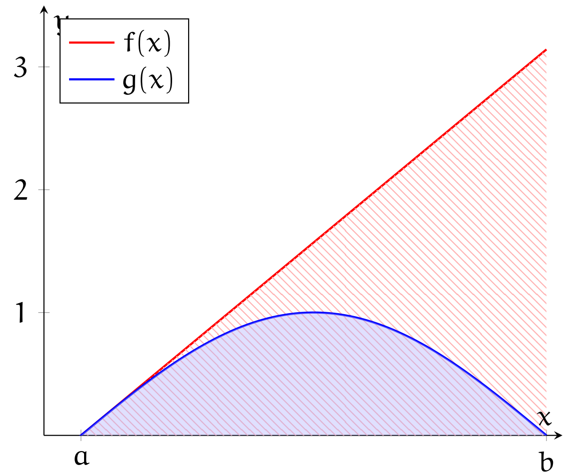


Figure 1.7: $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$

Noting that $(a-b) = -(b-a)$ we see:

$$\begin{aligned}\int_b^a f(x)\,dx &= \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\frac{-(b-a)}{n}\\ &= (-1)\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\frac{(b-a)}{n}\\ &= (-1)\int_a^b f(x)\,dx\end{aligned}$$

Therefore, it is true that $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$.

1.4 Applications in Physics

We have already seen that the area under a velocity function is displacement, and the area under an acceleration function is change in velocity (Riemann Sums). We can use integrals to determine the change in position of an object over a given time frame. If we *also* know the object's starting position, then we can state the object's ending position. Consider the graph of an object's velocity in figure 1.8:

We can determine the net displacement of the object from $t = 0$ to $t = 9$ by evaluating $\int_0^9 v(t)\,dt$. Since the definite integral is equal to the area under the curve, we need to find the total area. As the function consists of straight lines, we will leave the explicit calculation of the area as an exercise for the student. You should find that the total positive area (above the x-axis) is 10 meters, and the total negative area (below the x-axis) is 8 meters. Therefore, the object's displacement over the specified time interval is $10 - 8 = 2$ meters.

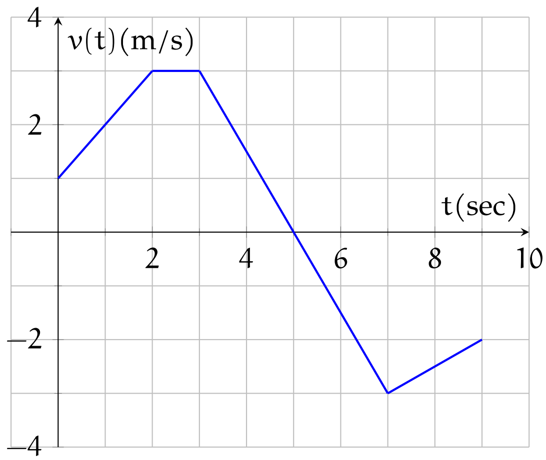


Figure 1.8: Velocity of an object from $t = 0$ to $t = 9$

When you push on something to move it, you are applying a force over a distance (assuming you are strong enough to move it!). The integral of force as a function of distance is the *work* done on that object. Work is the change in kinetic energy (KE) of an object. Mathematically, this is

$$\int_a^b F(x)\,dx = \Delta KE = \frac{1}{2}m(v_f^2 - v_i^2)$$

.

If you integrate the force as a function of time, that is *impulse*. Impulse is the change in momentum (p) of the object. Mathematically, this is

$$\int_a^b F(t)\,dt = \Delta p = m(v_f - v_i)$$

.

Example problem: You push a 3 kg box with force $F(x) = 0.5x$, where x is measured in meters and F is measured in Newtons. If the box was initially at rest, what is its speed when it reaches the 2 meter mark? (Hint: $KE = \frac{1}{2}mv^2$.)

Solution: Change in kinetic energy is the area under a force-distance curve. We can plot the force applied to the box from d=0 to d=2 (see figure 1.9):

Given that the box's initial velocity is $0\frac{m}{s}$, we know that the initial kinetic energy (KE) is 0J. This implies that $KE_f = \Delta KE$. We can find ΔKE from the shaded area:

$$\Delta KE = \frac{1}{2}(2m)(1N) = 1J = KE_f$$

Solving for the final velocity:

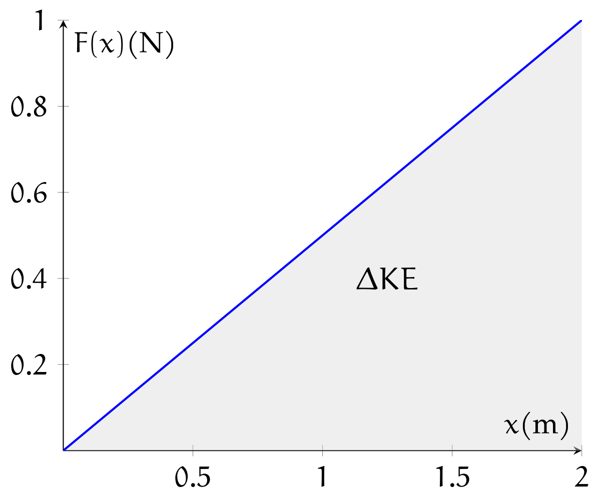


Figure 1.9: Force applied to a box over a distance; the shaded area represents the change in kinetic energy.

$$KE_f = 1\text{J} = \frac{1}{2}(3\text{kg})(v^2)$$
$$2\text{J} = (3\text{kg})(v^2)$$
$$\frac{2}{3}\frac{\text{m}^2}{\text{s}^2} = v^2$$
$$v = \sqrt{\frac{2}{3}\frac{\text{m}^2}{\text{s}^2}} \approx 0.816\frac{\text{m}}{\text{s}}$$

1.5 Practice Exercises

Exercise 2

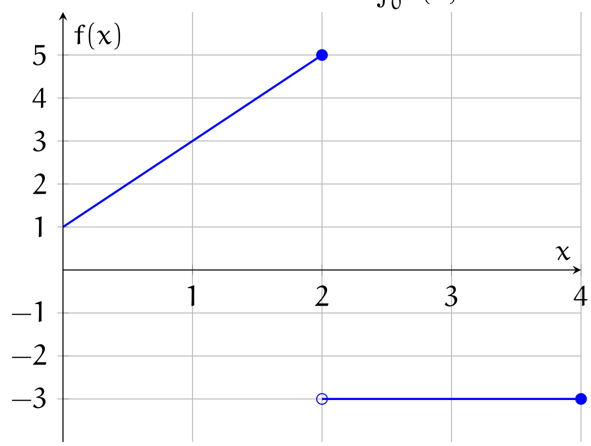
Given that $\int_0^1 x^2\,dx = \frac{1}{3}$, use the properties of integrals to evaluate $\int_0^1 (5-6x^2)\,dx$.

Working Space

Answer on Page 45

Exercise 3

This question was originally presented as a multiple-choice problem on the 2012 AP Calculus BC exam. The graph of f is shown. What is the value of $\int_0^4 f(x)\,dx$?

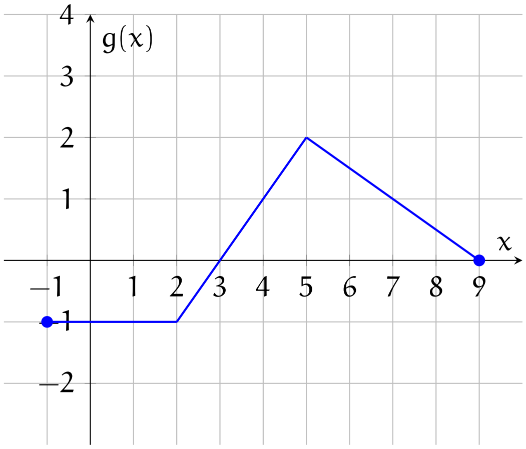


Working Space

Answer on *Page 45*

Exercise 4

This question was originally presented as a multiple-choice problem on the 2012 AP Calculus BC exam. The graph of the piecewise function $g(x)$ is shown. What is the value of $\int_{-1}^{9} 3g(x) + 2\, dx$?



Working Space

Answer on Page 46

Exercise 5

[This question was originally presented as a calculator-allowed, multiple- choice question on the 2012 AP Calculus BC exam.] If $f'(x) > 0$ for all real numbers and $\int_4^7 f(x)\,dx = 0$, which of the following could be a table of values for the function f?

(A)

x	$f(x)$
4	-4
5	-3
7	0

(B)

x	$f(x)$
4	-4
5	-2
7	5

(C)

x	$f(x)$
4	-4
5	6
7	3

(D)

x	$f(x)$
4	0
5	0
7	0

(E)

x	$f(x)$
4	0
5	4
7	6

Working Space

Answer on Page 46

Chapter 2

The Fundamental Theorem of Calculus

The Fundamental Theorem of Calculus (FTC) is a theorem that connects the concept of differentiating a function with the concept of integrating a function. This theorem is divided into two parts:

2.1 First Part

The first part of the Fundamental Theorem of Calculus states that if f is a continuous real-valued function defined on a closed interval $[a, b]$ and F is the function defined, for all x in $[a, b]$, by:

$$F(x) = \int_a^x f(t)\,dt \tag{2.1}$$

Then, F is uniformly continuous and differentiable on the open interval (a, b), and $F'(x) = f(x)$ for all x in (a, b). (That is $F(x)$ is the antiderivative of $f(x)$.)

2.2 Second Part

The second part of the Fundamental Theorem of Calculus states that if f is a real-valued function defined on a closed interval $[a, b]$ that admits an antiderivative F on $[a, b]$, and f is integrable on $[a, b]$ (it need not be continuous), then

$$\int_a^b f(t)\,dt = F(b) - F(a). \tag{2.2}$$

We will also use shorthand as follows:

$$\int_a^b f(t)\,dt = F(t)\big|_a^b \tag{2.3}$$

Which means "$F(t)$ evaluated from $t = a$ to $t = b$".

2.3 FTC and Definite Integrals

Let f be a function that is continuous on the interval $x \in [a, b]$ and $g(x)$ is given by:

$$g(x) = \int_a^x f(t)\,dt$$

So, g is continuous on $[a, b]$ and differentiable on (a, b). Additionally,

$$g'(x) = f(x)$$

Proof: Let x and $x + h$ be in (a, b). Then,

$$g(x+h) - g(x) = \int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt$$

Recall from the chapter on definite integrals that we can split the first integral, rewriting it as:

$$g(x+h) - g(x) = \left[\int_a^x f(t)\,dt + \int_x^{x+h} f(t)\,dt\right] - \int_a^x f(t)\,dt$$

$$g(x+h) - g(x) = \int_x^{x+h} f(t)\,dt$$

And for $h \neq 0$:

$$\frac{g(x+h) - g(x)}{h} = \frac{1}{h}\int_x^{x+h} f(t)\,dt$$

Since f is continuous, there is some u in (a, b), such that $f(u) = m$, where m is the minimum value of f on the interval (a, b). Similarly, there is also some v, such that $f(v) = M$, where M is the maximum value (see figure ??). We can then state the true inequality that:

$$mh \leq \int_x^{x+h} f(t)\,dt \leq Mh$$

Therefore, (assuming $h > 0$):

$$f(u) \leq \frac{1}{h}\int_x^{x+h} f(t)\,dt \leq f(v)$$

Substituting the equation above for the integral, we see that:

$$f(u) \leq \frac{g(x+h) - g(x)}{h} \leq f(v)$$

If we let h approach zero, then the window that u and v are in collapses and u and v both approach x. Therefore,

$$\lim_{h \to 0} f(u) = \lim_{u \to x} f(u) = f(x)$$

Recall also that

$$\lim_{h \to 0} \frac{g(x+h) - g(x)}{h} = g'(x)$$

Taking the limit as $h \to 0$ of the whole, inequality becomes the Squeeze Theorem:

$$\lim_{h \to 0} f(u) \leq \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} \leq \lim_{h \to 0} f(v)$$

$$f(x) \leq g'(x) \leq f(x)$$

Therefore, if $g(x) = \int_a^x f(t)\,dt$, then $g'(x) = f(x)$. Notice it doesn't matter what a is!

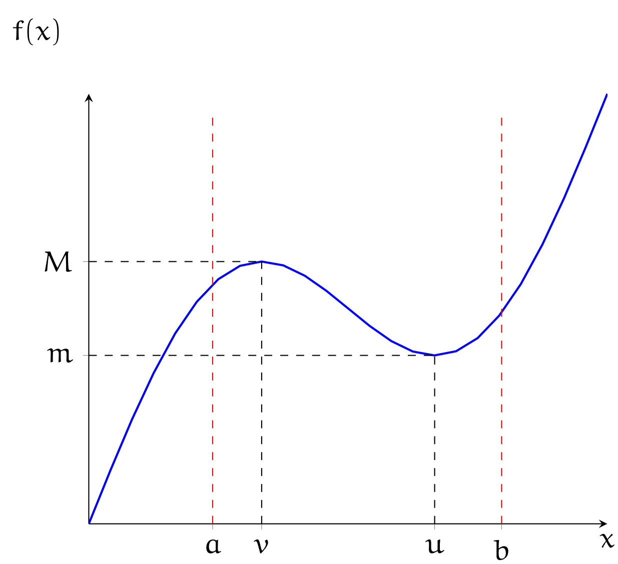


Figure 2.1: $f(v) = M$, the maximum value, and $f(u) = m$, the minimum value on the interval $x \in [a, b]$

Exercise 6

[This question was originally presented as a no-calculator, multiple-choice problem on the 2012 AP Calculus BC Exam.] Let g be a continuously differentiable function with $g(1) = 6$ and $g'(1) = 3$. What is the value of $\lim_{x\to 1} \frac{\int_1^x g(t)\,dt}{g(x)-6}$?
(A) 0
(B) $\frac{1}{2}$
(C) 1
(D) 2
(E) The limit does not exist

Working Space

Answer on Page 46

2.4 The Meaning of the FTC

What the Fundamental Theorem of Calculus is really saying is that differentiation and integration are opposite processes. Mathematically, we can say

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

. This may seem clunky, but many useful functions are defined this way. Consider the Fresnel function: $S(x) = \int_0^x \sin\frac{\pi t^2}{2}\,dt$. Originally used in optics, this equation is also used by civil engineers to design road and railway curves. According to FTC, then, $S'(x) = \sin\frac{\pi t^2}{2}$.

We can also apply the Chain Rule when taking derivatives of integrals. Let $f(x) =$

$\int_1^{x^4} \sec t \, dt$. What is $f'(x)$? First, let us define $u = x^4$. By the Chain Rule,

$$\frac{d}{dx}\int_0^{x^4} \sec t \, dt = \frac{d}{dx}\int_0^{u} \sec t \, dt$$
$$= \frac{d}{du}[\int_0^{u} \sec t \, dt]\frac{du}{dx}$$
$$= \sec u \frac{du}{dx}$$

Noting that $\frac{du}{dx} = \frac{d}{dx}x^4 = 4x^3$,

$$f'(x) = \sec x^4 (4x^3)$$

2.4.1 FTC Practice

Exercise 7

Use the Fundamental Theorem of Calculus to find the derivative of the function.

1. $g(x) = \int_0^x \sqrt{t + t^3} \, dt$
2. $F(x) = \int_x^0 \sqrt{1 + \sec t} \, dt$
3. $h(x) = \int_1^{e^x} \ln t \, dt$
4. $y = \int_{\sqrt{x}}^{\frac{\pi}{4}} \theta \tan \theta \, d\theta$

Working Space

Answer on Page 47

2.5 Using Antiderivatives to Evaluate Definite Integrals

In everyday English, the FTC states that the integral from a to b of a function is the antiderivative of that function evaluated from a to b. In the previous chapter, the integrals presented were of linear functions where the area under the curve could be equally calculated by hand. The FTC connects integrals to antiderivatives, allowing us to evaluate more complex integrals. Consider the following example:

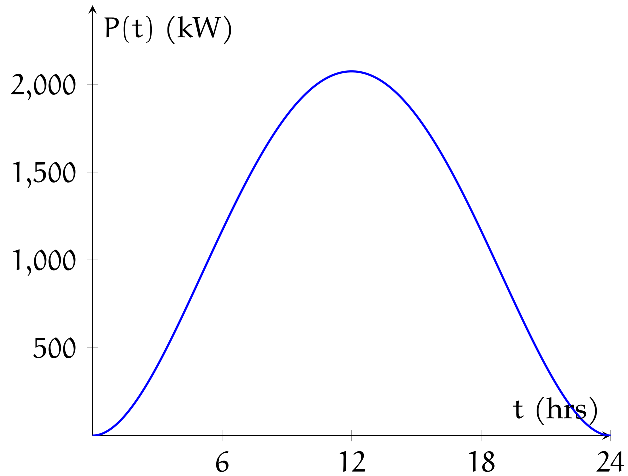


Figure 2.2: Power consumption of a household in a day

The power consumption of a household can be modeled as $P(t) = \frac{1}{10}t^2(t-24)^2$ from $t = 0$ to $t = 24$, where P is measured in watts and t is measured in hours ($t = 0$ is midnight). The total energy the household uses is given by $\int_0^{24} P(t)\, dt$. As you can see from the graph (see figure 2.2), we cannot simply use our geometry skills to determine the area under the curve.

To determine the total energy use, we need to evaluate $\int_0^{24} \frac{1}{10}t^2(t-24)^2\, dt$. First, we expand the polynomial:

$$\begin{aligned} E_{tot} &= \frac{1}{10}\int_0^{24} t^2(t^2 - 48t + 576)\, dt = \frac{1}{10}\int_0^{24} t^4 - 48t^3 + 576t^2\, dt \\ &= \frac{1}{10}\int_0^{24} t^4\, dt - \frac{24}{5}\int_0^{24} t^3\, dt + \frac{288}{5}\int_0^{24} t^2\, dt \end{aligned}$$

Using the power rule to determine the antiderivatives of t^4, t^3, and t^2, we see:

$$= \frac{1}{10}[\frac{1}{5}t^5]|_0^{24} - \frac{24}{5}[\frac{1}{4}t^4]|_0^{24} + \frac{288}{5}[\frac{1}{3}t^3]|_0^{24} = 26542.1Whr = 26.5421kWhr$$

2.5.1 Definite Integrals Practice

Exercise 8

Evaluate the following integrals:

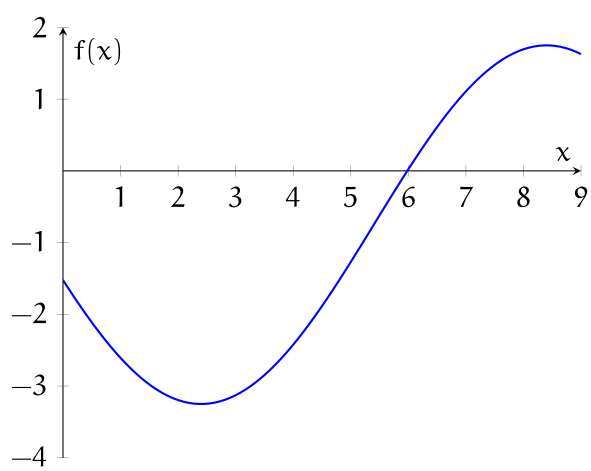
1. $\int_1^4 t^{-3/2}\,dt$

Working Space

Answer on Page 47

Exercise 9

[This question was originally presented as a multiple-choice, no-calculator problem on the 2012 Calculus BC exam.] The graph of a differentiable function f is shown in the graph. $h(x) = \int_0^x f(t)\,dt$. Rank the relative values of $h(6)$, $h'(6)$, and $h''(6)$ from lowest to highest.

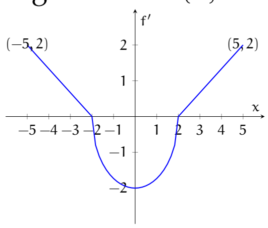


Working Space

Answer on Page 47

Exercise 10

The graph of f' is shown in the graph and consists of a semi-circle and two line segments. If $f(2) = 1$, then what is $f(-5)$?



Working Space

Answer on Page 47

Exercise 11

[This question was originally presented as a calculator-allowed, multiple- choice question on the 2012 AP Calculus BC exam.] A particle moves along a line so that its acceleration for $t \geq 0$ is given by $a(t) = \frac{t+3}{\sqrt{t^3+1}}$. If the particle's velocity at $t = 0$ is 5, what is the velocity of the particle at $t = 3$?

Working Space

Answer on Page 47

2.6 Average Value of a Function

The average value of a function, f, on an interval $[a, b]$ is given by:

Average Value of a Function

$$f_{avg} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

Example: Find the average value of $f(x) = 3 + x^2$ on the interval $[-2, 1]$.

Solution: Taking $a = -2$ and $b = 1$, we have:

$$f_{avg} = \frac{1}{1-(-2)} \int_{-2}^{1} \left[3 + x^2\right] \, dx$$

$$f_{avg} = \frac{1}{3} \int_{-2}^{1} \left[3 + x^2\right] \, dx$$

$$f_{avg} = \frac{1}{3} \left[3x + \frac{1}{3}x^3\right]_{x=-2}^{x=1}$$

$$f_{avg} = \frac{1}{3} \left[\left(3 \cdot 1 + \frac{1}{3} \cdot 1^3\right) - \left(3 \cdot (-2) + \frac{1}{3} \cdot (-2)^3\right)\right]$$

$$f_{avg} = \frac{1}{3} \left[\left(3 + \frac{1}{3}\right) - \left(-6 - \frac{8}{3}\right)\right]$$

$$f_{avg} = \frac{1}{3} \left[\frac{10}{3} + 6 + \frac{8}{3}\right] = \frac{1}{3}(12) = 4$$

Therefore, the average value of $f(x) = 3 + x^2$ on the interval $[-2, 1]$ is 4.

Exercise 12

[This question was originally presented as a multiple-choice, calculator- allowed problem on the 2012 AP Calculus BC Exam.] What is the average value of $y = \sqrt{\cos x}$ on the interval $0 \le x \le \frac{\pi}{2}$?

Working Space

Answer on Page 48

Chapter 3

Arc Lengths

3.1 Determining the Arc Length of a Curve

Another application of integrals is finding the length of a curve. In real life, we could do this by laying a piece of string up against the curve, then straightening out the string and measuring its length with a ruler (you may have done this in elementary school when you were first learning about the relationship between the radius and circumference of a circle). Archimedes estimated the circumference of a circle by inscribing a circle with polygons of increasing numbers of sides. (Archimedes' proof that π is between $3\frac{10}{71}$ and $3\frac{1}{7}$ is more complicated, but we will not dive into that here.) As we increase the number of sides of the inscribed polygon, the perimeter of the polygon (the sum of the lengths of the sides) gets closer to the circumference of the circle (see figure 3.1). Now, it's easy to find the length of a polygon: just add up the length of the line segments! Using this, we can find the length of a curve by approximating it as many short lines and adding up the lengths of those lines.

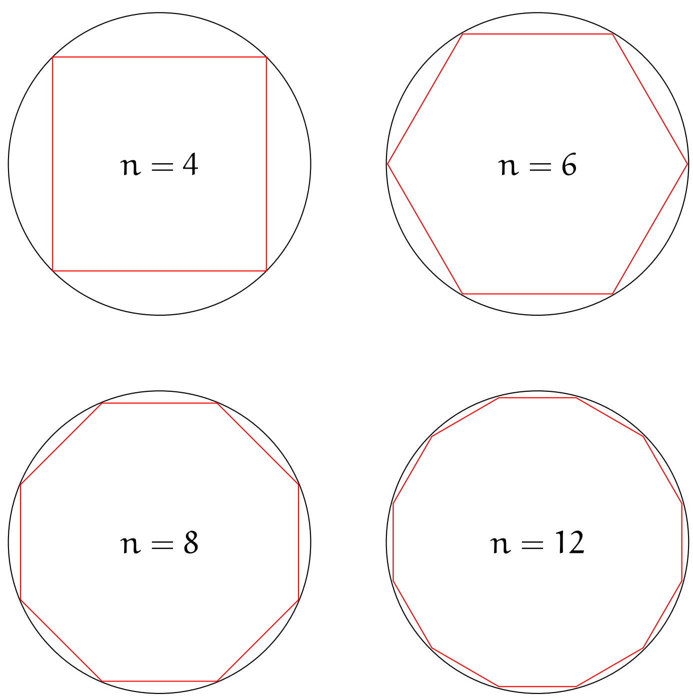


Figure 3.1: As n increases, the perimeter of the inscribed polygon approaches the circumference of the circle

We can choose n points along the graph of $f(x)$ and connect each point with a straight line (this is shown in figure 3.2). If we add up the length of the lines, we get an estimate of the length of the curve. We represent the length of the line between the i^{th} point, P_i and the previous point, P_{i-1} as $|P_{i-1}P_i|$ (recall that the absolute value sign can be used to signify the length of something). Therefore, the sum of the lengths of the lines approximating the curve is:

$$\sum_{i=1}^{n} |P_{i-1}P_i|$$

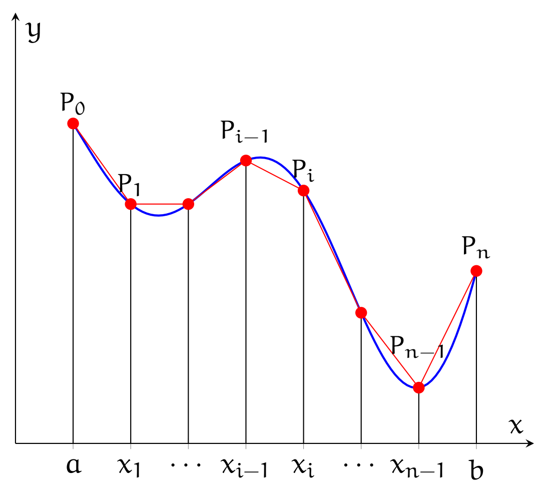


Figure 3.2: Polygon approximation of $f(x)$

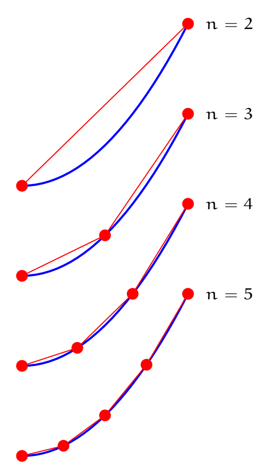


Figure 3.3: As the number of points increases, the total length of the lines segments approaches the true length of the curve

The more points we choose, the closer the lines lay to the actual curve (see figure 3.3), and the closer our estimate is to the true length. So, to find the true length, we will want to take n to ∞. Therefore, the actual curve length is the limit as $n \to \infty$ of that sum:

$$L = \lim_{n\to\infty} \sum_{i=1}^{n} |P_{i-1}P_i|$$

The length of each segment can be found using the Pythgorean theorem. Recall that the distance between two points on the xy-plane is $\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$. (For a reminder of why this is, see figure 3.4.) The coordinates of P_{i-1} are $(x_{i-1}, f(x_{i-1}))$ and the coordinates of P_i are $(x_i, f(x_i))$. Substituting this into the above sum, we see that the total length of the segments is

$$\sum_{i=1}^{n}\sqrt{(x_i-x_{i-1})^2+(f(x_i)-f(x_{i-1}))^2}$$

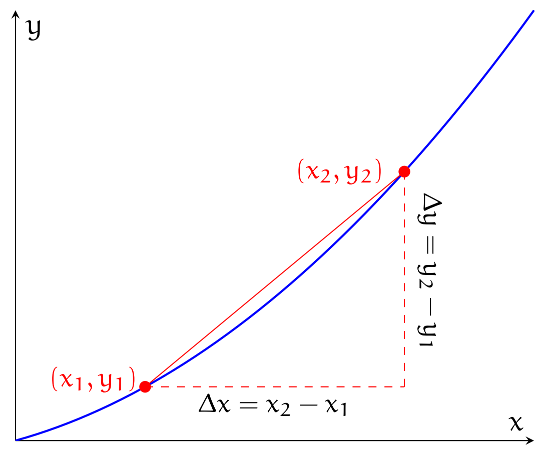


Figure 3.4: The distance between two points on the xy-plane

Recall the Mean Value Theorem, which states that there is some x_i^* such that $f(x_i) - f(x_{i-1}) = f'(x_i^*)(x_i - x_{i-1})$. Substituting this into the above sum, we get:

$$\sum_{i=1}^{n}\sqrt{(x_i-x_{i-1})^2+(f'(x_i^*)(x_i-x_{i-1}))^2}$$

Recall from the chapter on Riemann sums and the integral that we defined $\Delta x = x_i - x_{i-1}$ and we can further rewrite the sum as

$$\sum_{i=1}^{n}\sqrt{(\Delta x)^2+[f'(x_i^*)\Delta x]^2}=\sum_{i=1}^{n}\sqrt{1+[f'(x_i^*)]^2}\sqrt{(\Delta x)^2}=\sum_{i=1}^{n}\sqrt{1+[f'(x_i^*)]^2}\Delta x$$

Putting this all together, we see that that actual length of the curve is defined as

$$L=\lim_{n\to\infty}\sum_{i=1}^{n}\sqrt{1+[f'(x_i^*)]^2}\Delta x$$

.

This is the definition of the integral of $\sqrt{1+[f'(x)]^2}$ and therefore the length of some function $f(x)$ on the interval $a < x < b$ is $\int_a^b \sqrt{1+[f'(x)]^2}\,dx$. In another notation, this is equivalent to $\int_a^b \sqrt{1+(\frac{dy}{dx})^2}\,dx$.

3.2 Arc Length of Vector-valued Functions

Suppose you have a vector-valued function, $f(t) = [x(t), y(t)]$. A common example might be an artillery shell shot at an angle. For a shell shot with an initial velocity v_o at angle θ from the ground, its position can be described with the vector-valued function $f(t) = [v_o \cos\theta(t), v_o \sin\theta(t) - 4.9t^2]$. (A concrete example where $v_o = 12\frac{m}{s}$ and $\theta = 30^o$ is shown in figure 3.5.)

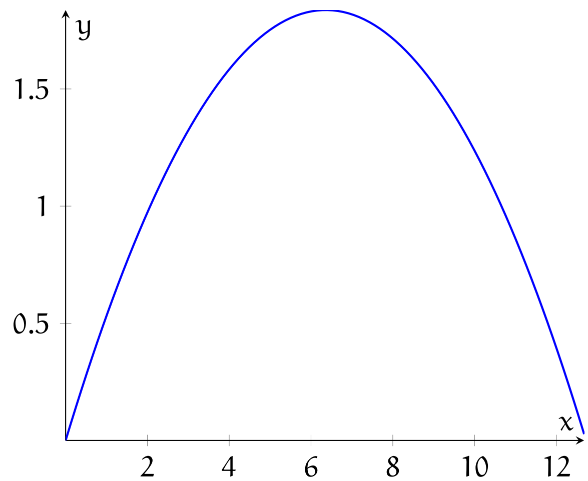


Figure 3.5: The path of an artillery shell with shot with initial velocity v_o at angle θ

How can we find the length of the flight path of the artillery shell? We can reinterpret the length integral for a vector-valued function, $f(t) = [x(t), y(t)]$.

$$L = \int_a^b \sqrt{1 + (\frac{dy}{dx})^2}\, dx = \int_a^b \sqrt{1 + \frac{(dy/dt)^2}{(dx/dt)^2}} \frac{dx}{dt}\, dt$$

Moving the $\frac{dx}{dt}$ under the square root, we see that

$$L = \int_a^b \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}\, dt$$

, which is equivalent to

$$L = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\, dx$$

.

3.3 Applications in Physics

When we take the integral of a velocity function, we get the *displacement*. For example, if you drove to school and home again, your displacement would be zero. However, the *distance* you traveled is not zero! We can use the arc length formula to find the total distance traveled. (Remember, that if $x(t)$ is the object's position, then its velocity is given by $x'(t)$.)

Suppose a block is attached to a spring on a frictionless horizontal surface. You pull on the block, initiating harmonic motion described by $v(t) = (-0.16)\sin 9t$, where v is in $\frac{m}{s}$ and t is in *sec*. (Note: We are working in radians, not degrees.) What is the block's displacement from $t = 0$ to $t = 3$? What is the total distance the block moves from $t = 0$ to $t = 3$?

To find the displacement, we integrate the velocity function over the specified interval:

$$\int_0^3 (-0.16)\sin 9t\, dt = \frac{-0.16}{9}(-\cos 9t)|_0^3$$

$$= \frac{0.16}{9}[\cos 27 - \cos 0] = \frac{(0.16)(-0.29-1)}{9} = -0.0229m$$

The position function and displacement are shown in figure 3.6. To find the total distance traveled, we need to find the length of the curve. Before we do so, take a minute to mentally predict: Will the distance be more or less than the displacement?

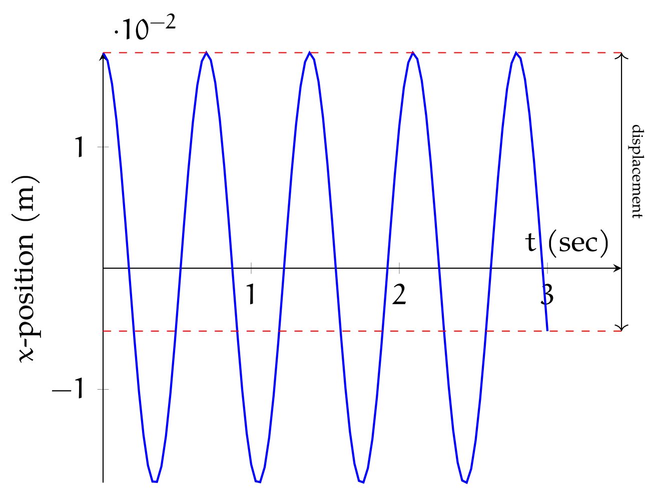


Figure 3.6: The position of the block with displacement shown. The distance traveled is the total length of the curve

Recalling that the distance traveled by an object is $\int_a^b \sqrt{1+[x'(t)]^2}\,dt = \int_a^b \sqrt{1+[v(t)]^2}\,dt$, we can write an integral to determine the total distance traveled by the block:

$$\int_0^3 \sqrt{1+[-0.16\sin 9t]^2}\,dt$$

.

Unfortunately, we do not know an antiderivative for this integral, and u-substitution won't help us. However, for definite integrals, calculators such as a TI-89 or Wolfram Alpha can easily use Riemann sums to determine the value of the integral to a high precision. Using such a tool, we find that the total distance traveled by the block is ≈ 3.019 meters. Did you predict that the distance would be greater than the displacement?

3.4 Practice

Exercise 13

Write an integral that gives the length of the requested curve.

1. $y = \ln x$ from $x = 1$ to $x = 3$
2. $y = \sin x$ from $x = 0$ to $x = \pi$
3. $y = \frac{x^3}{3} + \frac{1}{4x}$ from $x = 1$ to $x = 4$
4. $y = \ln(\cos x)$ from $x = 0$ to $x = \frac{\pi}{3}$

Working Space

Answer on Page 48

Exercise 14

The arc length function for a curve $f(x)$, where f is an increasing function, is given by $s(x) = \int_0^x \sqrt{3t+5}\, dt$. If f has y-intercept 2, find an equation for f. What point on f is three units from the y-intercept? Give your answer to the thousandths place.

Working Space

Answer on Page 48

Exercise 15

[This problem was originally presented as a no-calculator, multiple-choice question on the 2012 AP Calculus BC exam.] Write an integral that gives the length of the curve $y = \ln x$ from $x = 1$ to $x = 2$.

Working Space

Answer on Page 49

Exercise 16

An out-of-control rocket ship is spiraling out of control through space. Its velocity can be described with the vector-valued function $v(t) = [-1412 \sin t, 1412 \cos t, t]$ where v is in $\frac{m}{s}$ and t is in sec. How far does the ship travel in the first 60 seconds? In the second 60 seconds? [Hint: in three dimensions, the length of a vector-valued function is $\int_a^b \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2 + (\frac{dz}{dt})^2}\, dt$]

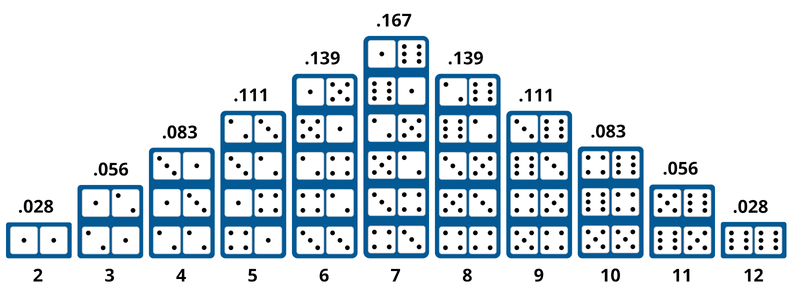
Working Space

Answer on Page 49

Chapter 4

Continuous Probability Distributions

When we talked about the probability distribution of the sum of two dice, we assigned a probability to each of the 11 possibilities:



The probabilities all added up to be 1.0. That is a way of saying "100% of the times you throw the dice, the sum will be an integer between 2 and 12."

Now we need to talk about probabilities of properties that are continuous, not discrete. For example, we might want to ask the question, "If I randomly pick a cow from all the cows in the world, what is the probability that it will weigh less than 597.34 kg?" What does a probability distribution for a continuous variable look like?

4.1 Cumulative Distribution Function

Imagine that you live in ancient times. You buy, sell, and ship cows. A lot of people come into your office and brag about their cows: "Bessie is heavier than 99% of the cows in world!" So, you need to develop some statistics on cow weights.

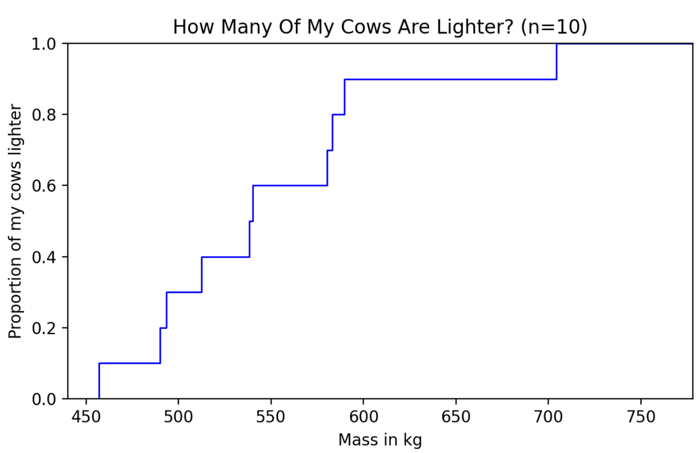
You have ten cows. You weight them:

Cow	Mass in kg
Cow 1	580.22
Cow 2	540.07
Cow 3	538.20
Cow 4	512.39
Cow 5	589.75
Cow 6	456.91
Cow 7	583.09
Cow 8	493.56
Cow 9	489.97
Cow 10	704.15

If someone comes into your shop and says, "My Bessie is an astonishing 530 kg!" it would be cool to have a list on the wall that would let you yell back, "Half my cows are heavier than that, Silly!" So, you sort the cows by weight. For each weight, you say how many of your cows are lighter than that:

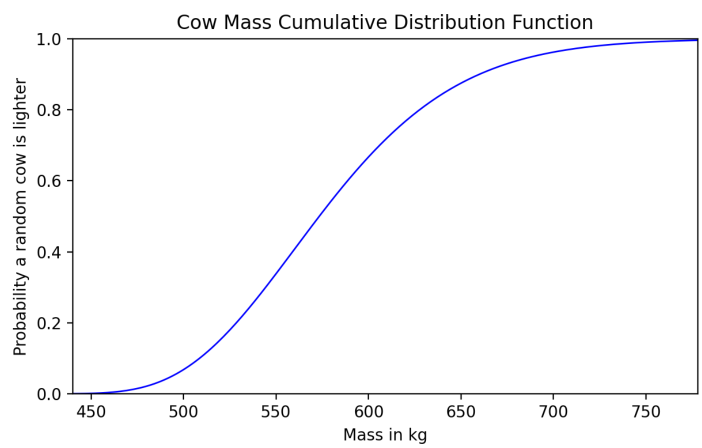
Proportion of cows lighter	Mass in kg
0.00	456.91
0.10	489.97
0.20	493.56
0.30	512.39
0.40	538.20
0.50	540.07
0.60	580.22
0.70	583.09
0.80	589.75
0.90	704.15

In fact, for easy reference, you make a plot:



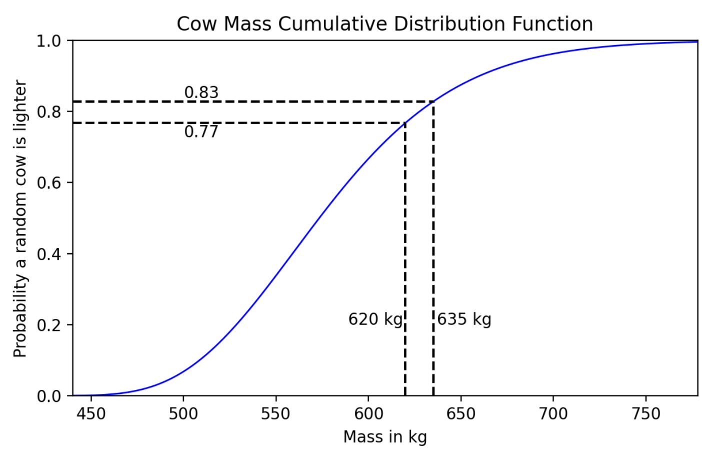
Now, for any weight, you can quickly look up what proportion of your cows are lighter (and, if you subtract that from 1, what proportion of your cows are heavier).

See how jagged that graph is? That is because you only have the data for 10 cows. However, as the years pass and you weight thousands of cows, the plot will become smoother. Because it always accumulates more cows as you move from left to right, this is known as a *cumulative distribution function*, or CDF:



A cumulative distribution function always starts at 0 and ends at 1. On that journey, it never decreases.

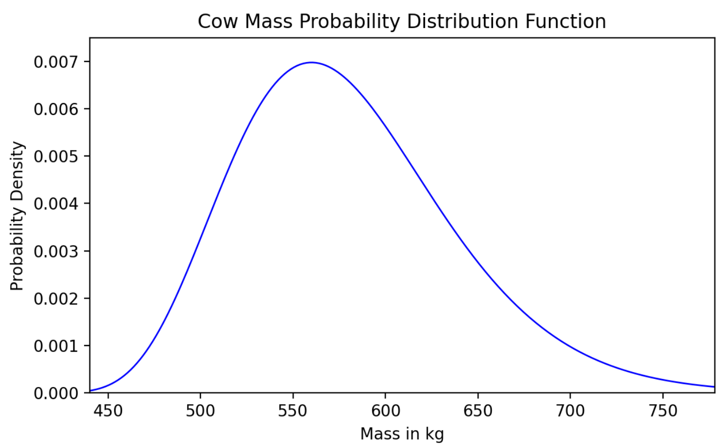
Let's say you want to know what proportion of cows weigh between 620 kg and 635 kg. Using the CDF, you could figure out that 77% of all cows weigh less than 620 kg and 83% of all cows weigh less than 635 kg. Thus 6% of all cows must weigh more than 620 kg and less than 635 kg.



4.2 Probability Density Function

The cumulative density function is handy, but some of its information can be hard to see. For example, how would you answer the question, "What is the most common weight of a cow?" You would squint at the CDF and try to determine where it was steepest.

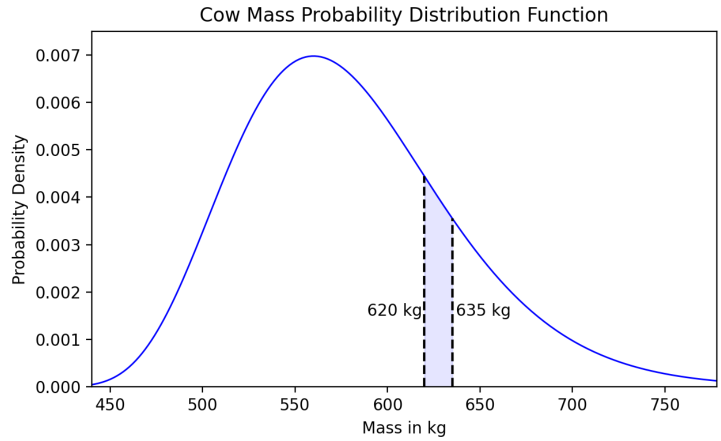
To make these sorts of questions easier to answer, we take the derivative of the CDF to get the *probability density function* (or PDF). For the cows, it would look like this:



Now you can easily see that the CDF was steepest at about 560 kg. We call the highest point on the PDF the *most likely estimator*. For example, you might say "560 kg is the most likely estimator of cow mass." Sometimes we just say "the MLE".

Note that the MLE is often different from the mean or the median. In this case, for example, the distribution is skewed right — there are more cows that are heavier than the MLE than there are cows that are lighter than the MLE. The MLE would be less than the mean or the median.

Once again, let's say you want to know what proportion of cows weight between 620 kg and 635 kg. This is more difficult with a PDF than it is with a CDF. With a PDF, you have to find the area under the curve between $x = 620$ and $x = 635$.

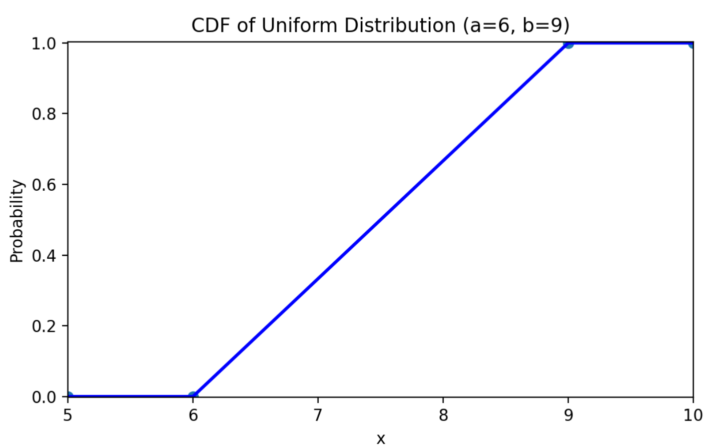


This is why it is called a ”probability density” — to get a true probability you need to multiply the density by the width of the region.

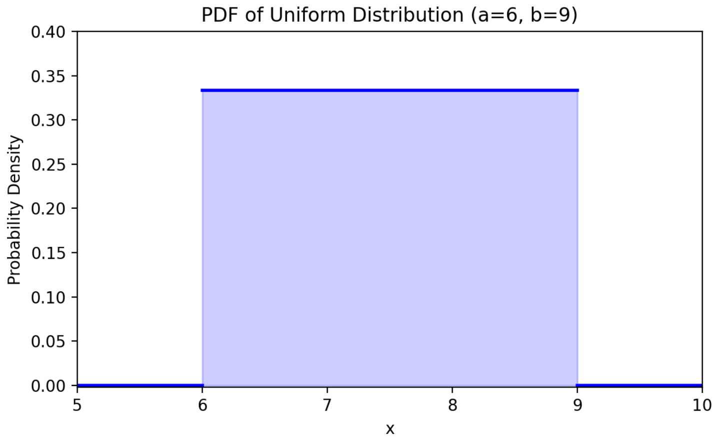
What is the area under the entire curve? If you integrated it, you would get the CDF. The CDF goes from 0 to 1.0. The area under a PDF is *always* 1.0.

4.3 The Continuous Uniform Distribution

The most simple continuous distribution is the uniform distribution between two numbers a and b. The CDF is a straight line from zero at a to 1 at b. For example, here is the CDF for the uniform distribution between 6 and 9:



That line goes from 0 to 1 over a distance of 3, so its slope is $\frac{1}{3}$ between 6 and 9 and zero everywhere else. Thus, the PDF (its derivative) looks like this:



So, we can write the probability distribution of a continuous uniform distribution between a and b as:

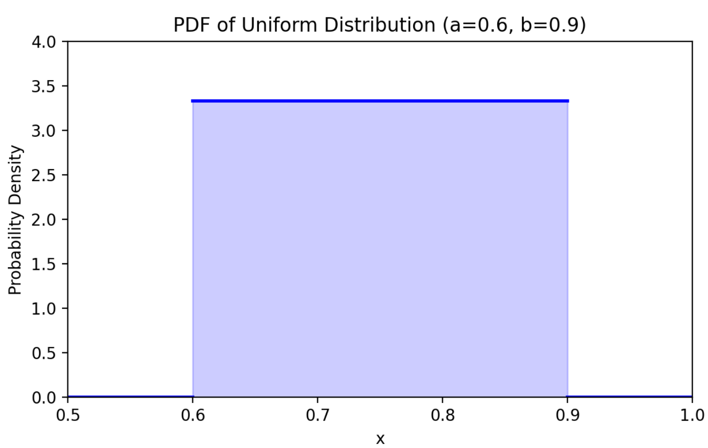
$$p(x) = \begin{cases} \frac{1}{b-a} & \text{for } a \leq x \leq b, \\ 0 & \text{for } x < a \text{ or } x > b. \end{cases}$$

Notice that if a and b are less than 1 apart, the value of $p(x)$ will be greater than 1. This is an important difference between a probability and a probability density:

- A probability will always be in the interval $[0, 1]$.
- A probability density will never be less than 0, but can be much larger than 1.

That said, the probability density will always integrate to 1.

Here is the PDF for a uniform distribution between 0.6 and 0.9:



The mean and median of a uniform distribution between a and b is its midpoint: $\frac{a+b}{2}$.

The variance (σ^2) is $\frac{(b-a)^2}{12}$.

4.4 Continuous Distributions In Python

The SciPy library has functions that let a programmer work with a large collection of different probability distributions.

For example, if you wanted to work with a continuous uniform distribution between 6 and 9, you would import the relevant functions like this:

```
from scipy.stats import uniform
```

Now, if you wanted a numpy array containing a sample of 300 numbers generated randomly from that distribution:

```
samples = uniform.rvs(loc=6,  scale=3,  size=300)
```

The `loc` argument is a. The `scale` argument is $b - a$.

If you wanted to know the value of the probability density function at 8 and 10, you could use the `pdf` function:

```
x_values = np.array([8, 10])
p_values  = uniform.pdf(x_values,  loc=6,  scale=3)
```

Now, `p_values` contains 0.33333 and 0.0.

To get the value of the cumulative distribution function at those points, you would use the `cdf` function:

```
cdf_values = uniform.cdf(x_values,  loc=6,  scale=3)
```

Now, `cdf_values` contains 0.666667 and 1.0.

The inverse of the CDF is very useful. It answers questions like “How heavy does a cow have to be to be in top 1%?”

```
bottom_top_percentiles = np.array([0.01,  0.99])
boundaries = uniform.ppf(bottom_top_percentiles,  loc=6,  scale=3)
```

Now, `boundaries` contains 6.03 and 8.97.

The SciPy library supplies these functions (`rvs`, `pdf`, `cdf`, and `ppf`) for over a hundred common continuous probability distributions.

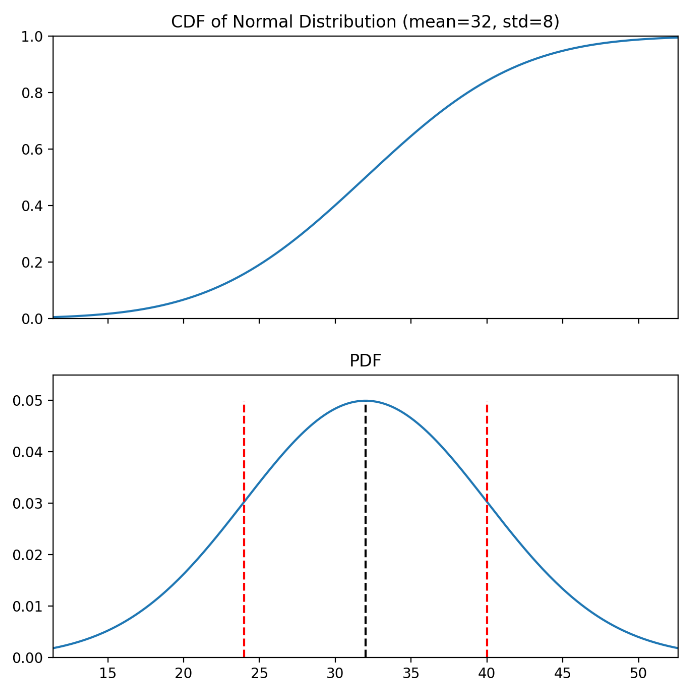
The common "bell curve" shaped distribution is called a Gaussian or Normal distribution. It is described by its mean (the midpoint of the bell) and its standard deviation. For the normal distribution, the standard deviation is the distance you have to go from the mean to reach 68% of the population. We will talk a lot more about the normal distribution in other chapters, but let's take this opportunity to plot the CDF and PDF of a normal distribution with a mean of 32 and a standard deviation of 8.

Create a file called `plot_norm.py` and add the following lines:

```
import numpy as np
from scipy.stats import norm
import matplotlib.pyplot as plt

# Constants
MEAN = 32
STD = 8

# Plottin from the 0.5 percentile to the 99.5 percentile
x_min = norm.ppf(0.005, loc=MEAN, scale=STD)
x_max = norm.ppf(0.995, loc=MEAN, scale=STD)

# Make 200 points between x_min and x_max
x_values = np.linspace(x_min, x_max, 200)

# Get CDF for each x value
cdf_values = norm.cdf(x_values, loc=MEAN, scale=STD)

# Get PDF for each x value
pdf_values = norm.pdf(x_values, loc=MEAN, scale=STD)

# What is the highest density?
max_density = norm.pdf(MEAN, loc=MEAN, scale=STD)

# Make a figure with two axes
fig, axs = plt.subplots(nrows=2, sharex=True, figsize=(8, 8), dpi=200)
axs[0].set_xlim(left=x_min, right=x_max)

# Draw the CDF on the first axis
axs[0].set_title("CDF of Normal Distribution (mean=32, std=8)")
axs[0].set_ylim(bottom=0.0, top=1.0)
axs[0].plot(x_values, cdf_values)

# Draw the PDF on the second axix
axs[1].set_title("PDF")
axs[1].set_ylim(bottom=0.0, top=max_density * 1.1)
```

```
axs[1].plot(x_values, pdf_values)

# Add lines for mean,  mean-std, and mean+std
axs[1].vlines(MEAN - STD, 0, max_density, "r", linestyle="dashed")
axs[1].vlines(MEAN + STD, 0, max_density, "r", linestyle="dashed")

# Save out the figure
fig.savefig("norm_32_8.png")
```

The resulting plot should look like this:



What do those vertical lines mean? An ornithologist might tell you, “The wingspan of adult robins are normally distributed with a mean of 32 cm and a standard deviation of 8 cm.” This means 68% of the population of adult robins would have wingspans between the two red lines.

Exercise 17 SciPy Stats

Working Space

Globally, the height of adult women is approximately normally distributed. The mean is 164.7 cm. The standard deviation is 7.1 cm.

Use Python and SciPy stats to answer these questions:

- To be in the tallest decile (the top 10%) of adult women, how tall does one need to be?
- What percentage of adult women are between 160 cm and 165 cm?

(In case you are wondering: For men the mean is 178.4 cm and the standard deviation is 7.6 cm.)

Answer on Page 49

Appendix A

Answers to Exercises

Answer to Exercise 1 (on page 4)

Following the structure shown in the formal definition of a definite integral, we can set $f(x) = x^3 + x \sin x$ and rewrite the limit of the sum as $\lim_{n\to\infty} \sum_{i=1}^{n} f(x)\Delta x = \int_0^{\pi} f(x)\,dx$. Therefore, the full definite integral would be written as $\int_0^{\pi}(x^3 + x\sin x)\,dx$.

Answer to Exercise 2 (on page 12)

By property 6, we know that

$$\int_0^1 (5 - 6x^2)\,dx = \int_0^1 5\,dx - \int_0^1 6x^2\,dx$$

By property 5, we know that

$$\int_0^1 5\,dx - \int_0^1 6x^2\,dx = \int_0^1 5\,dx - 6\int_0^1 x^2\,dx$$

By property 3, we know that

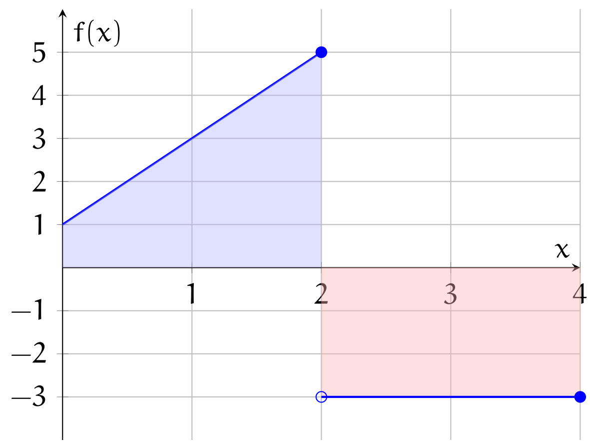
$$\int_0^1 5\,dx = 5(1 - 0) = 5$$

Putting it all together, we see that

$$\int_0^1 (5 - 6x^2)\,dx = 5 - 6(\frac{1}{3}) = 5 - 2 = 3$$

Answer to Exercise 3 (on page 13)

We can break the integral into two parts: from $x = 0$ to $x = 2$ (shaded in blue), and from $x = 2$ to $x = 4$ (shaded in red). The blue portion is a trapezoid, so it has a total area of $\frac{1}{2}(b_1 + b_2)(h) = \frac{1}{2}(1 + 5)(2) = 6$. Because it is above the x-axis, the area is positive. The red portion is a rectangle and has a total area of $2 \times 3 = 6$ and is *negative*, because it lies below the x-axis. Therefore, the total area is $6 + -6 = 0$.



Answer to Exercise 4 (on page 14)

Using the properties of integrals, we can rewrite $\int_{-1}^{9} 3g(x) + 2\,dx$ as $3\int_{-1}^{9} g(x)\,dx + 2(9 - (-1))$. From the graph, we can determine $\int_{-1}^{9} g(x)\,dx = 2.5$. Therefore, $\int_{-1}^{9} 3g(x) + 2\,dx = 3(2.5) + 2(10) = 27.5$.

Answer to Exercise 6 (on page 20)

(B). If $f'(x) > 0$ for all x, then f must be increasing for $4 < x < 7$. Since (C) decreases from $x = 5$ to $x = 7$, we can eliminate it. We can also eliminate (D), since $f(4) = f(5) = f(7)$, which implies either the slope of f is zero or changes from positive to negative. If $\int_4^7 f(x)\,dx =$, then some portion of $f(x)$ lies above the $x-axis$, while some other portion lies below (we must have positive and negative areas for the sum to be zero). This eliminates (A) and (E), since the integral of (A) would have a negative value and the integral of (E) would have a positive value. This leaves (B).

Answer to Exercise 6 (on page 20)

(D) 2. First, we try to compute the limit directly: $\lim_{x\to 1} \frac{\int_1^x g(t)\,dt}{g(x)-6} = \frac{\int_1^1 g(t)\,dt}{6-6} = \frac{0}{0}$, which is undefined. Because g is continuous and differentiable, we can apply L'Hospital's rule. $\lim_{x\to 1} \frac{\int_1^x g(t)\,dt}{g(x)-6} = \lim_{x\to 1} \frac{d}{dx}\left[\frac{\int_1^x g(t)\,dt}{g(x)-6}\right] = \lim_{x\to 1} \frac{g(x)}{g'(x)} = \frac{g(6)}{g'(6)} = \frac{6}{3} = 2$.

Answer to Exercise 7 (on page 21)

1. $g'(x) = \sqrt{x + x^3}$
2. $F(x) = -\int_0^x \sqrt{1 + \sec t}\, dt$ and therefore $F'(x) = -\sqrt{1 + \sec x}$
3. setting $u = e^x$ and noting $\frac{du}{dx} = e^x$, then $h'(x) = \frac{d}{du} \int_1^u \ln t\, dt (\frac{du}{dx})$ Taking the derivative and substituting for $\frac{du}{dx}$, we find $h'(x) = \ln u \cdot e^x = \ln(e^x) \cdot e^x = x \cdot e^x$
4. $y = -\int_{\frac{\pi}{4}}^{\sqrt{x}} \theta \tan\theta\, d\theta$. Setting $u = \sqrt{x}$ and noting that $\frac{du}{dx} = \frac{1}{2\sqrt{x}}$, we see that $y' = -\frac{d}{du}[\int_{\frac{\pi}{4}}^{u} \theta \tan\theta\, d\theta] \frac{du}{dx} = u \tan u \cdot \frac{1}{2\sqrt{x}} = -\sqrt{x} \tan\sqrt{x} \cdot \frac{1}{2\sqrt{x}} = \frac{-\sqrt{x}\tan\sqrt{x}}{2\sqrt{x}} = -\frac{1}{2}\tan\sqrt{x}$

Answer to Exercise 8 (on page 23)

1. The antiderivative of $t^{-3/2}$ is $\frac{-2}{\sqrt{t}}$. Therefore, the integral is equal to $[\frac{-2}{\sqrt{t}}]_1^4 = \frac{-2}{\sqrt{4}} - \frac{-2}{\sqrt{1}} = -1 + 2 = 1$.

Answer to Exercise 9 (on page 23)

According to FTC, $h'(x) = f(x)$ and $h''(x) = f'(x)$. Examining the graph, we see that the curve lies below the x-axis for $0 < x < 6$, which means that $h(6) = \int_0^6 f(t)\, dt < 0$. $h'(6) = f(6) = 0$ and $h''(6) = f'(6) > 0$. Therefore, $h(6) < h'(6) < h''(6)$.

Answer to Exercise 10 (on page 24)

We know that $f(2) = \int_{-5}^{2} f'(x)\, dx + f(-5)$. Examining the graph, we know that $\int_{-5}^{2} f'(x)\, dx = frac12(3)(2) - \frac{1}{2}\pi(2^2)$ (the area of the triangle above the x-axis less the area of the semicircle below the axis). Therefore, $f(-5) = f(2) - \int_{-5}^{2} f'(x)\, dx = 1 - (3 - 2\pi) = 2\pi - 2$

Answer to Exercise 11 (on page 24)

11.71. The particle's velocity will be given by its initial velocity plus the integral of its acceleration over the time period. Therefore, $v(3) = v(0) + \int_0^3 a(t)\, dt = 5 + \int_0^3 \frac{t+3}{\sqrt{t^3+1}}\, dt \approx 5 + 6.71 = 11.71$.

Answer to Exercise 12 (on page 25)

We begin by setting up the integral for average value of a function with $a = 0$, $b = \frac{\pi}{2}$, and $f(x) = \sqrt{\cos x}$:

$$\begin{aligned} &\frac{1}{\pi/2 - 0} \int_0^{\pi/2} \sqrt{\cos x}\, dx \\ &= \frac{2}{\pi} \int_0^{\pi/2} \sqrt{\cos x}\, dx \end{aligned}$$

There's not an obvious way to evaluate this integral by hand. Luckily, this question allows for the use of a calculator. Entering this integral into a calculator (such as a TI-89 or Wolfram Alpha), we find that:

$$\frac{2}{\pi} \int_0^{\pi/2} \sqrt{\cos x}\, dx \approx 0.763$$

Answer to Exercise 13 (on page 32)

1. $L = \int_1^3 \sqrt{1 + \frac{1}{x^2}}\, dx$
2. $L = \int_0^{\pi} \sqrt{1 + \cos^2 x}\, dx$
3. $L = \int_1^4 \sqrt{1 + (x^2 - \frac{1}{4x^2})^2}\, dx$
4. $L = \int_0^{\frac{\pi}{3}} \sqrt{1 + (\frac{1}{\cos x} \times - \sin x)^2}\, dx = \int_0^{\frac{\pi}{3}} \sqrt{1 + \tan^2 x}\, dx$

Answer to Exercise 14 (on page 32)

From looking at the structure of the given arc length integral, we can see that $f'(t) = \sqrt{3t + 4}$. Taking the antiderivative, we find that $f(x) = \frac{2}{9}(3x + 4)^{3/2} + C$. Substituting $f(0) = 2$, we can solve for C.

$$2 = \frac{2}{9}(3(0) + 4)^{3/2} + C$$

$$2 = \frac{2}{9}(4)^{3/2} + C$$

$$2 = \frac{2}{9}(2)^3 + C$$

$$2 = \frac{16}{9} + C$$

$$
\begin{aligned}
\frac{18}{9} &= \frac{16+9C}{9} \\
18 &= 16+9C \\
2 &= 9C \\
C &= \frac{2}{9}
\end{aligned}
$$

Therefore, $f(x) = \frac{2}{3}(3x+4)^{3/2}+\frac{2}{9}$. To find the coordinate point where $s(x) = 3$, we first note that the antiderivative of $\sqrt{3t+5}$ is $\frac{2}{9}(3t+5)^{3/2}+C$. Therefore, $s(x) = \frac{2}{9}(3x+5)^{3/2}-\frac{2}{9}(5)^{3/2}$. Setting $s(x) = 3$ and solving for x, we find that $x = 1.159$

Answer to Exercise 15 (on page 33)

Recall that arc length is given by $\int_a^b \sqrt{1+[f'(x)]^2}\,dx$. Since $f(x) = \ln x$, then $f'(x) = \frac{1}{x}$. Taking $a = 1$, $b = 2$, and $f'(x) = \frac{1}{x}$, the integral that gives the length of the curve $y = \ln x$ on the specified interval is $\int_1^2 \sqrt{1+\frac{1}{x^2}}\,dx$.

Answer to Exercise 16 (on page 33)

Since we are told the vector-valued velocity of the ship, we know that $\frac{dx}{dt} = -1412\sin t$, $\frac{dy}{dt} = 1412\cos t$, and $\frac{dz}{dt} = t$. The distance traveled in the first 60 seconds is given by $\int_0^{60} \sqrt{(-1412\sin t)^2 + (1412\cos t)^2 + t^2}\,dt$. Using a calculator, the integral evaluates to 84745 meters. The distance traveled in the second 60 seconds is given by
$\int_{60}^{120} \sqrt{(-1412\sin t)^2 + (1412\cos t)^2 + t^2}\,dt$. Using a calculator, this integral evaluates to 84898 meters.

Answer to Exercise 17 (on page 44)

```
from scipy.stats import norm

# Constants
MEAN = 164.7
STD = 7.1

# What is the cutoff for the top decile?
cutoff = norm.ppf(0.9, loc=MEAN, scale=STD);
print(f"To be in the top 10 percent, you must be at least {cutoff:.2f} cm")

# What proportion of women are between 160cm and 165cm?
shorter_than_160 = norm.cdf(160, loc=MEAN, scale=STD)
```

```
shorter_than_165 = norm.cdf(165, loc=MEAN, scale=STD)
between = shorter_than_165 - shorter_than_160
print(f"{between * 100.0:.2f}% of adult women are between 160 and 165 cm.")
```

When run, this will give you:

```
> python3 women.py
To be in the top 10 percent, you must be at least 173.80 cm
26.29% of adult women are between 160 and 165 cm.
```

Index