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Calculus with Polar Coordinates

Spirals in seashells, the petals of a rose curve, the radar sweep of an air-traffic screen: some shapes are much easier to describe by distance and angle than by x and y . Recall from the Polar Coordinates chapter that a point can be written as (r, θ) , where r is its distance from the origin and θ is its angle, in radians, from the positive x -axis (see Figure 1.1). In this chapter, you will find slopes and areas of curves written this way.

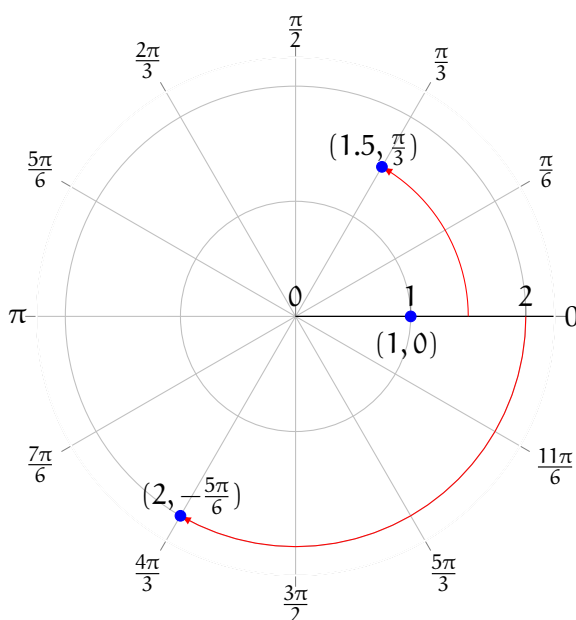
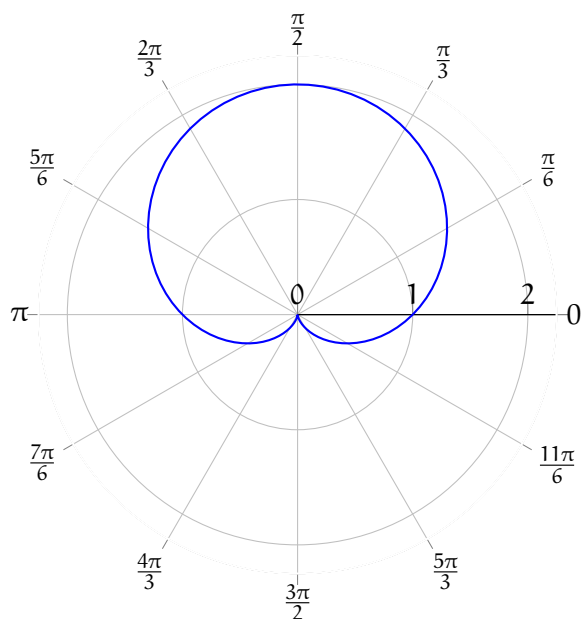


Figure 1.1: Polar coordinates give a degree of rotation, θ , and a distance from the origin, r , in the form of (r, θ)

1.1 Derivatives of Polar Functions

Consider the limaçon $r = 2 + \sin \theta$ (see Figure 1.2). What is the slope of the line tangent to the curve at $\theta = \frac{\pi}{2}$?

Figure 1.2: $r = 2 + \sin \theta$

From a visual inspection, we can guess that the slope of the tangent line is zero. Let us prove this mathematically:

First, recall that to convert polar coordinates to Cartesian coordinates, we can use the trigonometric identities:

$$x = r \cos \theta$$

$$y = r \sin \theta$$

So, we can write the parametric equation:

$$x = [2 + \sin \theta] \cos \theta$$

$$y = [2 + \sin \theta] \sin \theta$$

Recall from parametric equations that we can find $\frac{dy}{dx}$ by dividing:

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$$

Finding $\frac{dy}{d\theta}$ and $\frac{dx}{d\theta}$:

$$\frac{dy}{d\theta} = \frac{d}{d\theta} (2 \sin \theta + \sin^2 \theta) = 2 \cos \theta + 2 \sin \theta \cos \theta$$

$$\frac{dx}{d\theta} = \frac{d}{d\theta} (2 \cos \theta + \sin \theta \cos \theta) = \cos^2 \theta - \sin^2 \theta - 2 \sin \theta$$

Substituting $\theta = \frac{\pi}{2}$, we find that:

$$\frac{dy}{d\theta} = 2(0) + 2(1)(0) = 0$$

$$\frac{dx}{d\theta} = (0)^2 - (1)^2 - 2(1) = -3$$

Therefore,

$$\frac{dy}{dx} = \frac{0}{-3} = 0$$

Which is the result we expected from examining the graph of $r = 2 + \sin \theta$. So, in general for polar equations,

Tangent to a Polar Function

For a polar function, $r = f(\theta)$, the slope of a tangent line is given by:

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$$

Where $y = r \cdot \sin \theta$ and $x = r \cdot \cos \theta$.

Exercise 1

[This problem was originally presented as a no-calculator, multiple-choice question on the 2012 AP Calculus BC exam.]
What is the slope of the line tangent to the polar curve $r = 1 + 2 \sin \theta$ at $\theta = 0$?

Working Space

Answer on Page 59

Exercise 2

Find the slope of the tangent line to the given polar curve at the value of θ specified. Use this to write an equation for the tangent line in Cartesian coordinates.

1. $r = \frac{2}{3} \cos \theta, \theta = \frac{\pi}{6}$
2. $r = \frac{1}{2\theta}, \theta = \frac{\pi}{2}$
3. $r = 2 + 3 \cos \theta, \theta = \frac{2\pi}{3}$

Working Space

Answer on Page 59

1.2 Integrals of Polar Functions

Similar to Cartesian functions, an integral of a polar function tells us the area within the function. We say “within” as opposed to “under” because a polar function describes how far from the origin the graph is based on the angle. Consider the graph of $r = 2 \sin \theta$ (figure 1.3). Geometrically, we expect the area inside the curve to be $\pi r^2 = \pi$. However, this is not the result we get from directly integrating the function (we only integrate from $\theta = 0$ to $\theta = \pi$ because the circle is complete when θ reaches π):

$$\begin{aligned}\int_0^\pi 2 \sin \theta \, d\theta &= -2 \cos \theta \Big|_{\theta=0}^{\theta=\pi} \\ &= -2 [\cos \pi - \cos 0] = -2 [-1 - 1] = 4 \neq \pi\end{aligned}$$

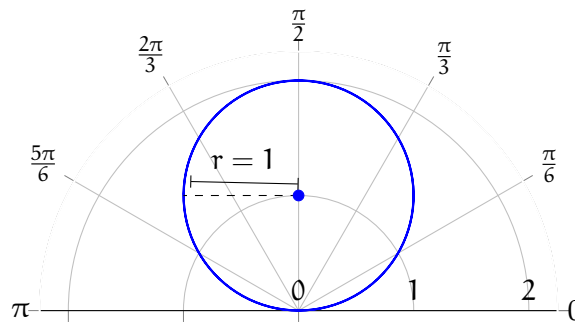


Figure 1.3: The graph of $r = 2 \sin \theta$ is a circle of radius 1 centered at $(1, \frac{\pi}{2})$

Clearly something else is happening here. We can just take the integral of a Cartesian function because the area of a rectangle is the base times the height. When integrating Cartesian functions, the base is given by the dx and the height by the function, $f(x)$. In polar coordinates, the integral sweeps across a θ interval, making a wedge, not a rectangle.

Let us consider a generic polar function, shown in figure 1.4

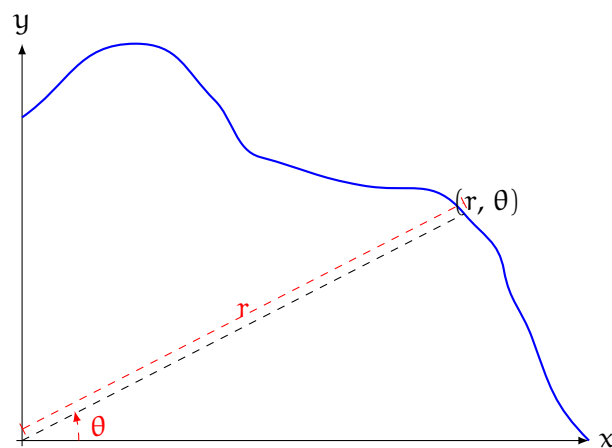


Figure 1.4: A generic polar function

Suppose we are interested in a specific region, bounded by $a \leq \theta \leq b$ (see figure 1.5).

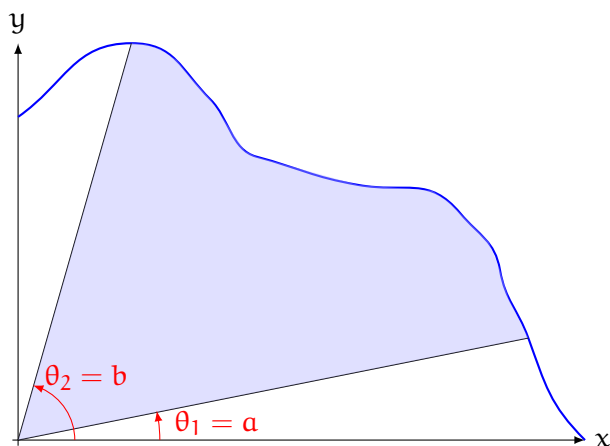


Figure 1.5: A generic polar with a region from $\theta = a$ to $\theta = b$ highlighted

We can divide the region into many small sectors. Then, each small sector has a central angle $\Delta\theta$ and a radius $r(\theta_i^*)$, where $\theta_{i-1} < \theta_i^* < \theta_i$ (see figure 1.6).

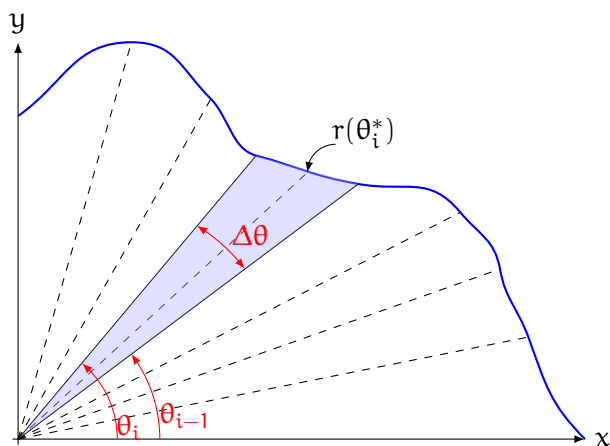


Figure 1.6: A single sector from θ_{i-1} to θ_i

What is the area of the i^{th} sector? Recall from the chapter on circles that the area of a sector with angle θ and radius r is $A = \frac{1}{2}r^2\theta$. Substituting, we see the area of the i^{th} sector is:

$$A_i = \frac{1}{2} [r(\theta_i^*)]^2 \Delta\theta$$

Therefore, the total area of the whole sector from $\theta = a$ to $\theta = b$ is the limit as the number of sectors approaches infinity of sum of the areas of all the small sectors:

$$A = \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{1}{2} [r(\theta_i^*)]^2 \Delta\theta$$

Does this look familiar? It is the definition of an integral!

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{1}{2} [r(\theta_i^*)]^2 \Delta\theta = \int_a^b \frac{1}{2} [r(\theta)]^2 d\theta$$

Area of a Polar Function

The area of a polar function is given by the integral

$$\int_a^b \frac{1}{2} r^2 d\theta$$

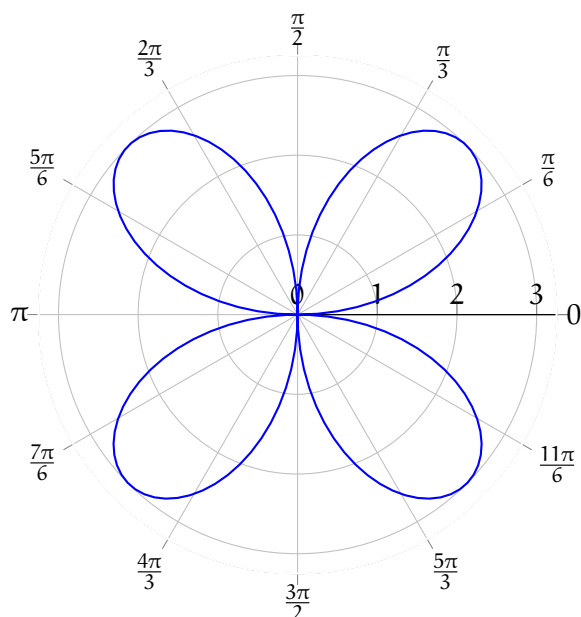
Where r is a function of θ .

We can check this with the example from the beginning of the section. Recall that the polar function $r = 2 \sin \theta$ graphs a circle with a radius of 1. Therefore, we expect the area enclosed by the graph of $r = 2 \sin \theta$ from $\theta = 0$ to $\theta = \pi$ to be π :

$$\begin{aligned} A &= \frac{1}{2} \int_0^\pi [2 \sin \theta]^2 d\theta \\ A &= 2 \int_0^\pi \sin^2 \theta d\theta = 2 \int_0^\pi \left[\frac{1 - \cos 2\theta}{2} \right] d\theta \\ A &= \int_0^\pi [1 - \cos 2\theta] d\theta = \left[\theta - \frac{1}{2} \sin 2\theta \right]_{\theta=0}^{\theta=\pi} \\ A &= [\pi - 0] - [0 - 0] = \pi \end{aligned}$$

Which is the expected result, confirming our formula for the area within a polar function.

Example: The graph of $r = 3 \sin 2\theta$ is shown below. What is the total area enclosed by the graph?

Figure 1.7: $r = 3 \sin 2\theta$

Solution: Since each lobe is symmetric to the others, we can find the area of one lobe and multiply it by four. To find the area of one lobe, we need to determine an interval for θ that defines one lobe. You can imagine each lobe being drawn out from the center and then back in. So, we will find where $r = 0$:

$$0 = 3 \sin 2\theta$$

$$\sin 2\theta = 0$$

$$2\theta = n\pi$$

$$\theta = \frac{n\pi}{2}$$

Taking the first two solutions, $\theta = 0$ and $\theta = \frac{\pi}{2}$, as our limits of integration, we see that the area of one lobe is:

$$A_{\text{lobe}} = \frac{1}{2} \int_0^{\pi/2} [3 \sin 2\theta]^2 d\theta$$

$$A_{\text{lobe}} = \frac{9}{2} \int_0^{\pi/2} \sin^2 2\theta d\theta$$

Applying the half-angle formula $\sin^2 \theta = \frac{1 - \cos 2\theta}{2}$, we see that:

$$A_{\text{lobe}} = \frac{9}{2} \int_0^{\pi/2} \frac{1 - \cos 4\theta}{2} d\theta = \frac{9}{4} \int_0^{\pi/2} 1 - \cos 4\theta d\theta$$

$$\begin{aligned}
 &= \frac{9}{4} \left[\theta - \frac{1}{4} \sin 4\theta \right]_{\theta=0}^{\theta=\pi/2} = \frac{9}{4} \left(\frac{\pi}{2} - 0 \right) - \frac{9}{4} \left(\frac{1}{4} \right) (\sin 2\pi - \sin 0) \\
 &= \frac{9\pi}{8} - \frac{9}{16} (0) = \frac{9\pi}{8}
 \end{aligned}$$

Since the area of one lobe is $\frac{9\pi}{8}$, the area of all four lobes is $\frac{9\pi}{2}$.

1.2.1 Area between polar curves

Consider the circle $r = 6 \sin \theta$ and the cardioid $r = 2 + 2 \sin \theta$. How can we find the area that lies inside the circle, but outside the cardioid (see figure 1.8)? First, let us find where these curves intersect.

$$6 \sin \theta = 2 + 2 \sin \theta$$

$$4 \sin \theta = 2$$

$$\sin \theta = \frac{1}{2}$$

$$\theta = \frac{\pi}{6}, \frac{5\pi}{6}$$

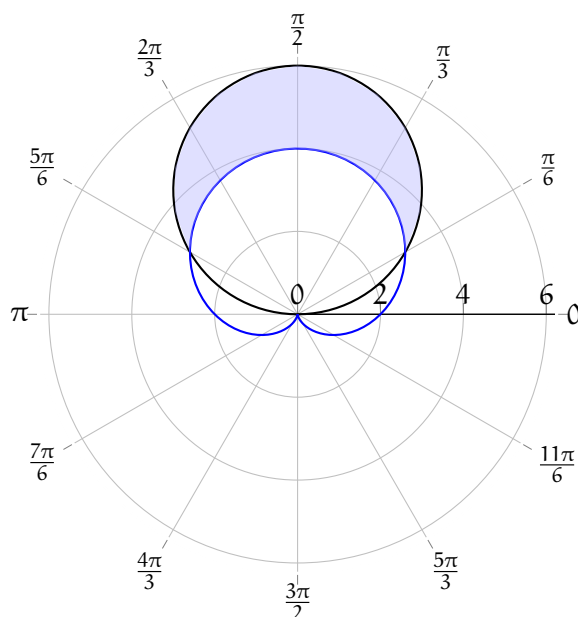


Figure 1.8: The area inside $r = 6 \sin \theta$ and outside of $r = 2 + 2 \sin \theta$ is highlighted

Recall that for Cartesian functions, to find the area between two curves, we subtract the

area under the lower curve from the total area under the higher curve. In polar coordinates, we want to subtract the area in the inner curve from the total area in the outer curve. In this case, the outer curve is $r = 6 \sin \theta$ and the inner curve is $r = 2 + 2 \sin \theta$. We have already found our limits of integration ($\frac{\pi}{6} \leq \theta \leq \frac{5\pi}{6}$), so we set up and evaluate our integral:

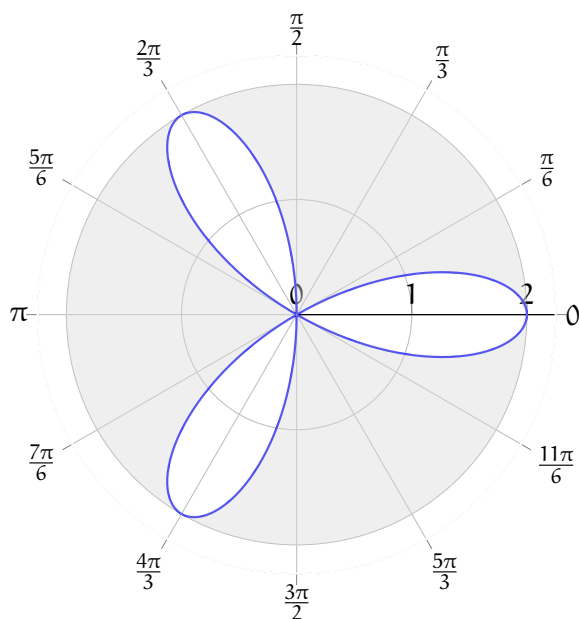
$$\begin{aligned} A_{\text{between}} &= \frac{1}{2} \int_{\pi/6}^{5\pi/6} [6 \sin \theta]^2 d\theta - \frac{1}{2} \int_{\pi/6}^{5\pi/6} [2 + 2 \sin \theta]^2 d\theta \\ &= \int_{\pi/6}^{5\pi/6} [16 \sin^2 \theta - 4 \sin \theta - 2] d\theta \\ &= \int_{\pi/6}^{5\pi/6} [8 - 8 \cos 2\theta - 4 \sin \theta - 2] d\theta \\ &= [6\theta - 4 \sin 2\theta + 4 \cos \theta]_{\theta=\pi/6}^{\theta=5\pi/6} \\ &= [5\pi - 2\sqrt{3} + 2\sqrt{3}] - [\pi - 2\sqrt{3} + 2\sqrt{3}] = 4\pi \end{aligned}$$

(Note: Because these polar functions are symmetric about the y-axis, we could have also taken the integral from $\theta = \frac{\pi}{6}$ to $\theta = \frac{\pi}{2}$ and doubled the result. We leave it as an exercise for the student to show this works.)

Exercise 3

[This question was originally presented as a multiple-choice, calculator-allowed problem on the 2012 AP Calculus BC exam.] The figure below shows the graphs of polar curves $r = 2 \cos 3\theta$ and $r = 2$. What is the sum of the areas of the shaded regions to three decimal places?

Working Space



Answer on Page 61

Exercise 4

Find the area of the region bounded by the given curve and angles.

1. $r = e^{\theta/2}$, $\pi/4 \leq \theta \leq \pi/2$
2. $r = 2 \sin \theta + \cos 2\theta$, $0 \leq \theta \leq \pi$
3. $r = 4 + 3 \sin \theta$, $-\pi/2 \leq \theta \leq \pi/2$

Working Space

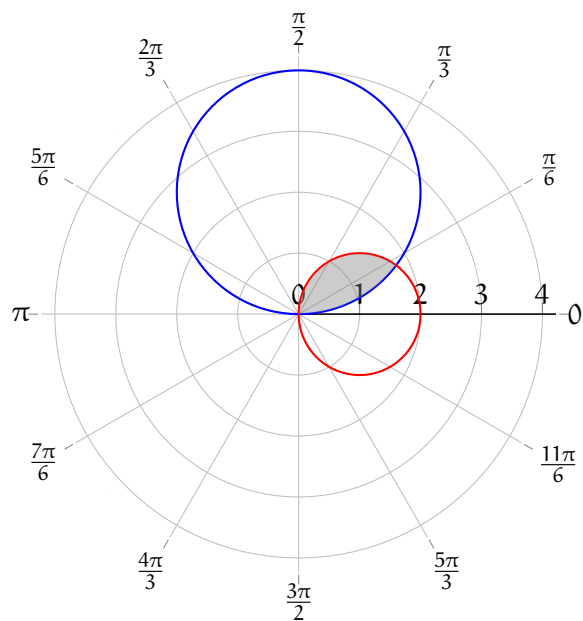
Answer on Page 61

Exercise 5

Find the area of the region that lies between the curves $r = 4 \sin \theta$ and $r =$

Working Space

$2 \cos \theta$. A graph is shown below.



Answer on Page 62

Differential Equations

Most laws of nature do not tell you where something *is*. They tell you how it *changes*. A cup of cocoa cools faster when it is hotter. A rumor spreads faster when more people at school know it. A spring pulls harder the farther you stretch it. Each of these is a **differential equation**: an equation that relates an unknown function to its own derivatives. Solving one means working backward from “how it changes” to “what it is.”

2.1 Ordinary Differential Equations

An ordinary differential equation (ODE) involves a function of a single independent variable and its derivatives. The order of an ODE is determined by the order of the highest derivative present in the equation. An example of a first-order ODE is:

$$\frac{dy}{dx} + y = x \quad (2.1)$$

Here, y is the function of the independent variable x , and $\frac{dy}{dx}$ represents its first derivative.

A real-world example of the application of differential equations is an oscillating spring (or any harmonic motion). When a spring is stretched, the restoring force (the force pulling or pushing it back to its neutral position) is proportional to the distance by which the spring has been stretched (see Figure 2.1). Mathematically, we say that

$$\text{restoring force} = -kx$$

where k is the positive spring constant (the stiffer a spring, the greater k).

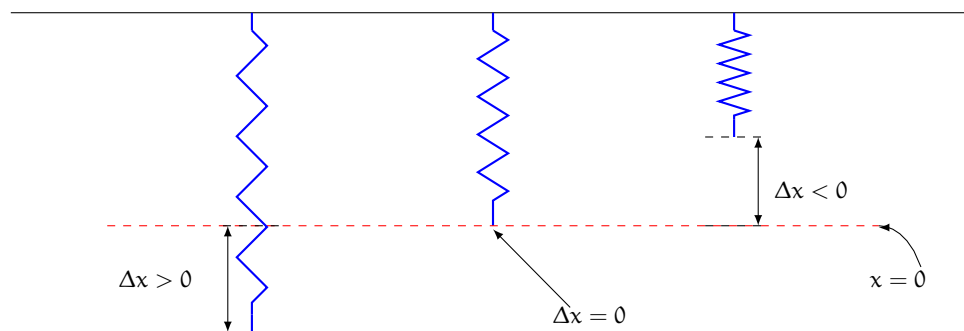


Figure 2.1: A spring can have a positive or negative displacement

Recall that Newton's Second Law tells us that force is equal to mass times acceleration, and that acceleration is the second derivative of position. We can then write the differential equation:

$$m \frac{d^2x}{dt^2} = -kx$$

This is called a **second-order differential equation**, because it involves second-order derivatives. The order of a differential equation is the same as the highest order of derivative in the equation. We can further rewrite the equation to isolate the second derivative:

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x$$

In everyday language, this is saying that the second derivative is proportional to the original function, just negative. Two trigonometric functions have this property. Can you name them before reading on?

The sine and cosine functions both have the property $\frac{d^2x}{dt^2} \propto -x(t)$ (recall that $\propto$ means "proportional to").

Example: Assuming $x(t)$ is a sine function, solve the second-order differential equation $\frac{d^2x}{dt^2} = -\frac{k}{m}x$.

Solution: Let $x(t) = \sin Ct$. Then $\frac{dx}{dt} = C \cos Ct$ and $\frac{d^2x}{dt^2} = -C^2 \sin Ct$. This implies that $C^2 = \frac{k}{m}$ and $C = \pm\sqrt{\frac{k}{m}}$. So, a solution to the differential equation $\frac{d^2x}{dt^2} = -\frac{k}{m}x$ is $x(t) = \sin \sqrt{\frac{k}{m}}t$. The same check works for $\cos \sqrt{\frac{k}{m}}t$, and in fact every solution has the form

$$x(t) = A \cos \sqrt{\frac{k}{m}}t + B \sin \sqrt{\frac{k}{m}}t$$

where the constants A and B come from where the mass starts and how fast it is moving at $t = 0$.

2.1.1 Population Growth

Another real-world application of differential equations is modeling population growth. Under ideal conditions (unlimited food, no predators, disease-free, etc.), the population of a species grows at a rate proportional to the current population size. We can identify two variables:

t = time (the independent variable)

P = the number of individuals in the population (the dependent variable)

So, what is the rate of growth? Recall that a rate is change over time. In that case, the rate of growth is given by $\frac{dP}{dt}$. If the rate of growth is proportional to the population, then we

can write a first-order differential equation:

$$\frac{dP}{dt} = kP$$

where k is a proportionality constant. This is called **natural growth** or **exponential growth**. To find a solution, we must answer the question: What function's derivative is a constant multiple of itself? Recall that we have seen that the derivative of the exponential function e^{kt} is ke^{kt} . Setting $P(t) = Ce^{kt}$ (where C is some constant), we see that the derivative is $\frac{dP}{dt} = kCe^{kt} = kP(t)$ (see figure 2.2). You can determine C from initial conditions.

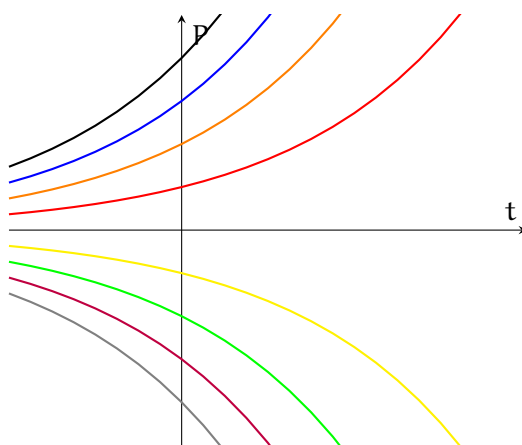


Figure 2.2: Several solutions to $\frac{dP}{dt} = kP$

Example: Suppose a population of bacteria has an initial population of 100 bacteria. If the bacteria's growth rate is given by $\frac{dP}{dt} = 2P$ (where t is in hours), how many bacteria are present after 4 hours?

Solution: We have seen that the solution to $\frac{dP}{dt} = 2P$ is $P(t) = Ce^{2t}$. We can then use the given initial condition to find C :

$$P(0) = 100 = Ce^{2 \cdot 0} = C \cdot 1 = C$$

Which means that the complete solution is:

$$P(t) = 100e^{2t}$$

To answer the question, we need to find $P(4)$:

$$P(4) = 100e^{2 \cdot 4} = 100e^8 \approx 298096$$

As stated above, this model works well for populations under specific, ideal conditions.

However, there are very few environments in which these conditions are met. Real animals suffer from disease, are hunted by predators, and have limited food supplies. Most environments have a maximum number of animals they can support, which ecologists call a **carrying capacity**. Let us call the carrying capacity of an environment M . So, the population growth can be modeled by the logistic differential equation:

$$\frac{dP}{dt} = kP \left(1 - \frac{P}{M} \right)$$

This is called a **logistic differential growth model**. Notice that if P is small, then $\frac{dP}{dt} \approx kP$. This makes sense: If the population is very small compared to the carrying capacity, the conditions are nearly ideal, and so growth should be nearly ideal too. On the other hand, if the population ever goes *above* the carrying capacity, the $\frac{dP}{dt} < 0$ and the population will decrease back below the carrying capacity (see figure 2.3). Notice that if the initial population is $P_0 = M$, then $\frac{dP}{dt} = kP(1 - 1) = 0$ and the population is stable at $P(t) = M$. We call this an **equilibrium solution**. Can you logically find the other equilibrium solution?

If there are no animals to begin with, then there are none to reproduce, and $P(t) = 0$. This is the other equilibrium solution. Notice that when the population is in equilibrium, then the rate of change is zero. Mathematically, to find equilibrium solutions, we can set $\frac{dP}{dt} = 0$ and solve for P .

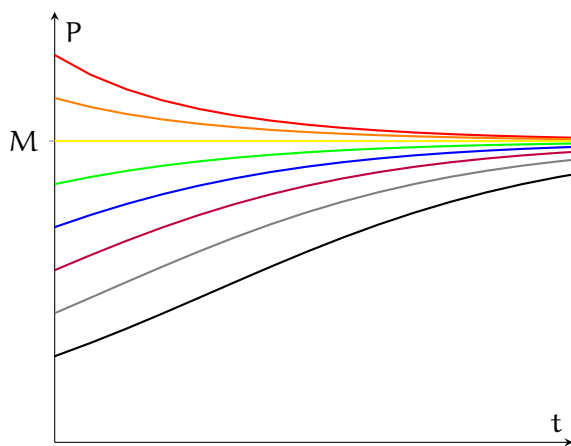


Figure 2.3: Several solutions to $\frac{dP}{dt} = kP \left(1 - \frac{P}{M} \right)$

Exercise 6

A population is modeled by the differential equation $\frac{dP}{dt} = 1.2P \left(1 - \frac{P}{4200} \right)$.

Working Space

1. What is the carrying capacity of the

environment?

2. For what values of P is the population increasing?
3. For what values of P is the population decreasing?
4. What are the equilibrium solutions?

Answer on Page 63

Exercise 7

[This problem was originally presented as a multiple-choice question on the 2012 AP Calculus BC exam.] Let k be a positive constant. Which of the following is a logistic differential equation?

- (a) $\frac{dy}{dt} = kt$
- (b) $\frac{dy}{dt} = ky$
- (c) $\frac{dy}{dt} = kt(1 - t)$
- (d) $\frac{dy}{dt} = ky(1 - t)$
- (e) $\frac{dy}{dt} = ky(1 - y)$

Working Space

Answer on Page 64

2.1.2 Separable Differential Equations

Sometimes, differential equations can be explicitly solved. A first-order differential equation is separable if $\frac{dy}{dx}$ can be written as a function of x times a function of y . Symbolically, a differential equation is separable if it takes the form

$$\frac{dy}{dx} = g(x)f(y)$$

The equations may be solvable by separating the x from the y and integrating each side.

For our generic form, we can separate the variables like this if $f(y) \neq 0$:

$$\frac{dy}{dx} \frac{1}{f(y)} = g(x)$$

$$\frac{1}{f(y)} dy = g(x) dx$$

Integrating both sides:

$$\int \frac{1}{f(y)} dy = \int g(x) dx$$

Let us look at the example $\frac{dy}{dx} = \frac{x^2}{y}$. We can separate the variables by multiplying both sides by $y dx$:

$$y dy = x^2 dx$$

Integrating both sides:

$$\int y dy = \int x^2 dx$$

$$\frac{1}{2}y^2 + C_1 = \frac{1}{3}x^3 + C_2$$

We can combine the constants by defining $C = C_2 - C_1$. Making this substitution and solving for y , we find:

$$y^2 = \frac{2}{3}x^3 + 2C$$

$$y = \pm \sqrt{\frac{2}{3}x^3 + 2C}$$

Noting that $2C$ is also a constant (which we will call K for convenience), we find the general solution is

$$y = \pm \sqrt{\frac{2}{3}x^3 + K}$$

A graph showing the solution for several values of K is in figure [2.4](#).

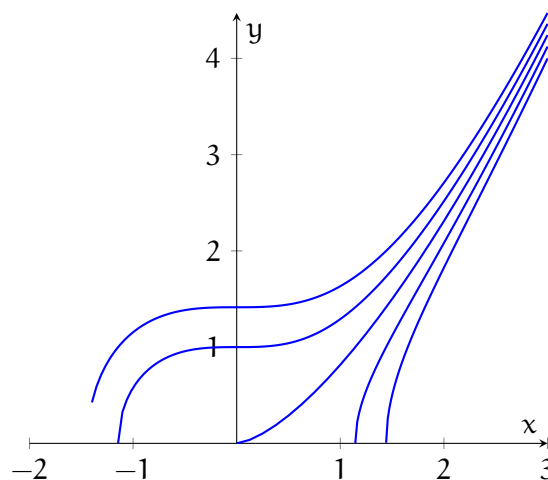


Figure 2.4: Several possible solutions to $\frac{dy}{dx} = \frac{x^2}{y}$

It is not always possible to solve for y explicitly in terms of x . The practice problem below is an example of this.

Exercise 8

Solve the differential equation $\frac{dy}{dx} = \frac{3x^2}{2y + \sin y}$.

Working Space

Answer on Page 64

Exercise 9

[This problem was originally presented as a calculator-allowed, free response question on the 2012 AP Calculus BC exam.] The rate at which a baby bird gains mass is proportional to the difference between its adult mass and its current mass. At time $t = 0$, when the bird is first weighed, its mass is 20 grams. If $B(t)$ is the mass of the bird, in grams, at time t days after it is first

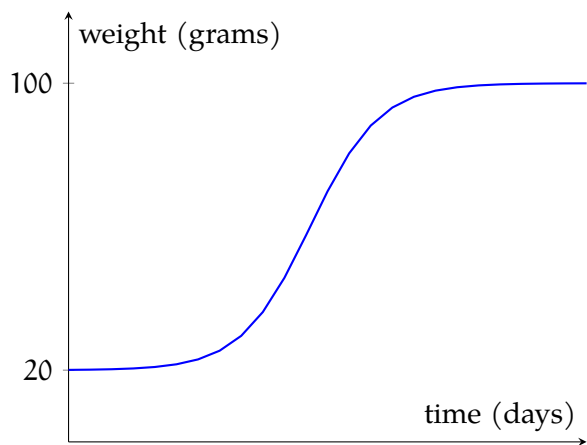
Working Space

weighed, then

$$\frac{dB}{dt} = \frac{1}{5}(100 - B)$$

Let $y = B(t)$ be the solution to the differential equation with initial condition $B(0) = 20$.

1. Is the bird gaining mass faster when it masses 40 grams or when it masses 70 grams? Explain your reasoning.
2. Find $\frac{d^2B}{dt^2}$ in terms of B . Use it to explain why the graph of B cannot resemble the graph shown below.
3. Use separation of variables to find $y = B(t)$, the particular solution to the differential equation with initial condition $B(0) = 20$.



Answer on Page 64

Exercise 10

[This problem was originally presented as a no-calculator, multiple-choice question on the 2012 AP Calculus BC exam.]
If $P(t)$ is the size of a population at time

Working Space

t , which of the following differential equations describes *linear* growth in the size of the population?

- (a) $\frac{dP}{dt} = 200$
- (b) $\frac{dP}{dt} = 200t$
- (c) $\frac{dP}{dt} = 100t^2$
- (d) $\frac{dP}{dt} = 200P$
- (e) $\frac{dP}{dt} = 100P^2$

Answer on Page 64

2.2 Mass-Spring Systems and Linear Differential Equations

In the springs and simple harmonic motion chapter, you saw that the motion of a mass attached to a spring can be modeled using differential equations. Here, we will examine this connection more closely.

2.2.1 Undamped Simple Harmonic Motion

Starting from Newton's Second Law and Hooke's Law, $F = -kx$, we get the differential equation for simple harmonic motion:

$$m \frac{d^2x}{dt^2} + kx = 0. \quad (2.2)$$

The term $m \frac{d^2x}{dt^2}$ comes directly from *Newton's Second Law*, representing the net force required to accelerate the mass. The term kx arises from *Hooke's Law*, which states that the spring exerts a restoring force proportional to displacement and directed toward equilibrium. Equation (2.2) is a **second-order linear differential equation with constant coefficients**. It describes an ideal mass-spring system that oscillates forever, with nothing slowing it down and nothing pushing it. What if friction also acts on the mass? We add another force term.

2.2.2 Introducing Damping

Real systems are often influenced by additional forces. One common example is a **damping force**, such as friction or air resistance, which opposes the motion of the mass. A

typical damping force is proportional to the velocity and can be written as

$$F_d(t) = -c \frac{dx}{dt},$$

where c is the damping coefficient. Including this force in Newton's Second Law modifies our differential equation. Instead of the undamped equation (2.2), we obtain:

$$m a(t) = -c v(t) - k x(t) \quad (2.3)$$

$$m \frac{d^2 x}{dt^2} + c \frac{dx}{dt} + kx = 0 \quad (2.4)$$

This is still a second-order linear differential equation with constant coefficients, but its solutions behave very differently. Depending on the value of c , the system may oscillate with decreasing amplitude, fail to oscillate at all, or return to equilibrium as quickly as possible. To see which, guess a solution of the form $x = e^{rt}$. Substituting gives $e^{rt}(mr^2 + cr + k) = 0$, and since e^{rt} is never zero, r must solve the **characteristic equation** $mr^2 + cr + k = 0$. The quadratic formula gives

$$r = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m}$$

Characteristic Roots	Homogeneous Solution	Damping Type
$r_1 \neq r_2$, real, both negative	$y_h = C_1 e^{r_1 t} + C_2 e^{r_2 t}$	Overdamped
r real, repeated	$y_h = (C_1 + C_2 t) e^{rt}$	Critically damped
$r = \alpha \pm i\beta$, $\alpha < 0$	$y_h = e^{\alpha t} (C_1 \cos(\beta t) + C_2 \sin(\beta t))$	Underdamped
$r = \pm i\beta$	$y_h = C_1 \cos(\beta t) + C_2 \sin(\beta t)$	Simple Harmonic Motion

Table 2.1: Classification of solutions to the damped harmonic oscillator based on characteristic roots.

The sign of $c^2 - 4mk$ decides everything: positive means overdamped (think of a screen door closer), zero means critically damped (a car's shock absorbers), and negative means underdamped (a plucked guitar string).

In general, the position of the mass is $y(t) = y_h(t) + y_p(t)$, where y_h is the homogeneous solution (the part we found by setting the right side to 0), and y_p is the particular solution, which depends on any external forcing. In this section, we focused on the homogeneous solution. If the equation (2.4) had a forcing function $F(t)$ on the right side, we would need to find a particular solution y_p to account for that.

Exercise 11 Linear Systems 1

Working Space

A 2 kg mass is attached to a spring with $k = 200 \text{ N/m}$ with no damping. Which differential equation describes the motion?

1. $x'' + 100x = 0$
2. $x'' + 10x = 0$
3. $x'' + 200x = 0$
4. $2x'' + 100x = 0$

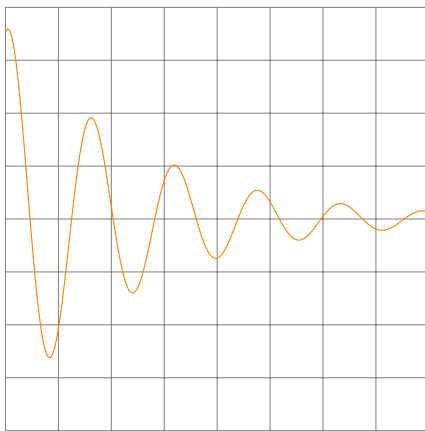
Answer on Page 65

Exercise 12 Linear Systems 2

Working Space

Answer the following questions about the graph below, which shows the position of a mass on a spring as a function of time. The equation of motion for this system is given by:

$$x(t) = 0.15e^{-0.40t} \cos(4t + 0.30)$$



- Is this system undergoing simple harmonic motion, or is it damped? If it

is damped, is it underdamped, overdamped, or critically damped?

- What would need to be changed in the equation of motion to make this system undergo simple harmonic motion instead?

Answer on Page 65

Exercise 13 Linear Systems 3

Working Space

Given that a unit (1 kg) mass obeys:

$$x'' + 16x = 0, \quad x(0) = 0.20, \quad v(0) = x'(0) = 0$$

1. Solve for $x(t)$
2. Find the acceleration, $a(t)$, and the velocity, $v(t)$, as functions of time.

Answer on Page 65

2.3 Partial Differential Equations

Everything so far has been an ordinary differential equation: one independent variable, usually time. Partial differential equations (PDEs) involve a function of several independent variables and their partial derivatives. An example of a PDE is the heat equation, a second-order PDE:

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2} \tag{2.5}$$

In this equation, $u = u(x, t)$ is a function of the two independent variables x and t , $\frac{\partial u}{\partial t}$ is

the first partial derivative of u with respect to t , and $\frac{\partial^2 u}{\partial x^2}$ is the second partial derivative of u with respect to x .

Picture a metal rod with one end in a campfire. Here $u(x, t)$ is the temperature at position x along the rod at time t . The second derivative $\frac{\partial^2 u}{\partial x^2}$ is positive where the temperature curve is concave up, meaning a spot is colder than the average of its neighbors. The equation says that such a spot warms up. Heat flows to even out the bumps, and the constant α says how quickly that happens (it is large for copper and small for wood).

Slope Fields

Separable equations are the lucky ones. Most differential equations have no formula for their solution at all. So how does a weather app predict tomorrow's temperature? It starts from what it knows right now and takes many small steps, each one following the rate of change the equation gives. You can do the same thing by hand with a **slope field** (also called a direction field): a grid of tiny tick marks showing which way a solution would head from each point. Drop a pencil anywhere, follow the ticks, and you have traced a solution.

3.1 Drawing Slope Fields

When a differential equation is in the form

$$y' = f(x, y)$$

we can use the coordinates (x, y) to determine the slope of a solution to the differential equation at that coordinate. Take $y' = x + y$ as an example. According to this differential equation, a solution that passes through the point $(1, 1)$ would have a slope of 2. We can represent this with a small tick of slope 2 at the $(1, 1)$ (see figure 3.1).

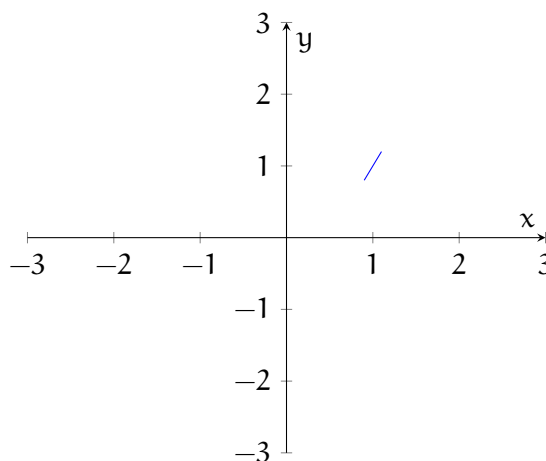


Figure 3.1: A solution to $y' = x + y$ that passes through $(1, 1)$ will have a slope of 2 at that point

Next, look for points where the slope is easy to find. Notice that $y' = 0$ when $-x = y$, so

let us go ahead and fill those ticks in (see figure 3.2):

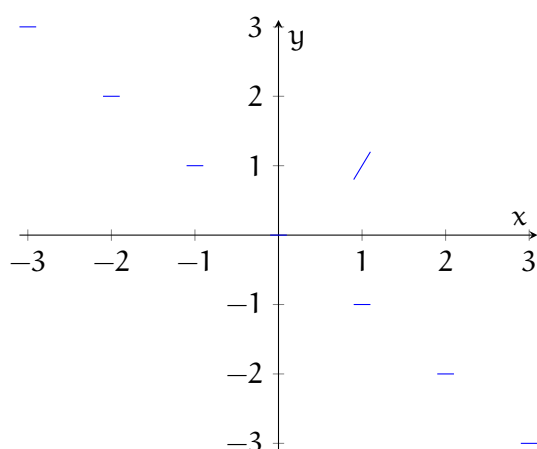


Figure 3.2: Solutions to $y' = x + y$ that lie on the line $y = -x$ will have a slope of 0.

We can repeat this process for all the coordinates shown, resulting in a slope field (see figure 3.3).

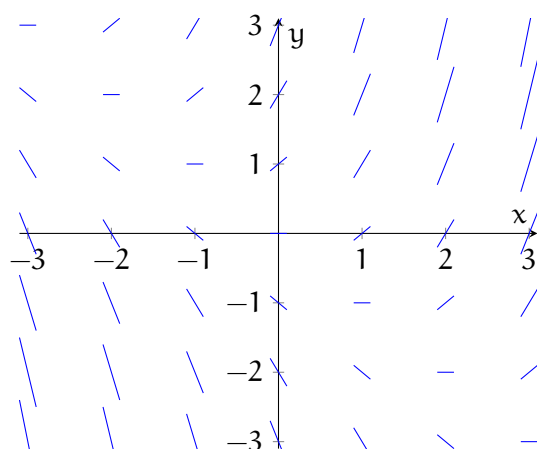


Figure 3.3: Slope field of $y' = x + y$

3.2 Sketching solutions on slope fields

If you are given an initial condition or a known point in the solution to the differential equation, you can begin sketching a curve on the slope field. Start at the given point and draw parallel to the nearby slopes. For example, suppose we know that a particular solution to $y' = x + y$ passes through the point $(1, 0)$. Begin by extending the dash at $(1, 0)$ (see figure 3.4), changing the slope of your sketched solution to be approximately parallel to the nearby slopes (see figure 3.5).

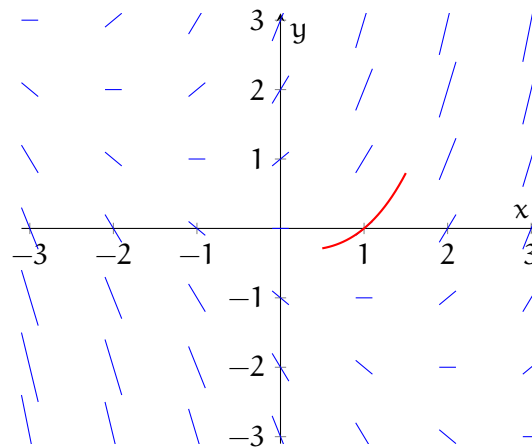


Figure 3.4: To begin sketching a solution to the differential equation, start at the point given as part of the solution

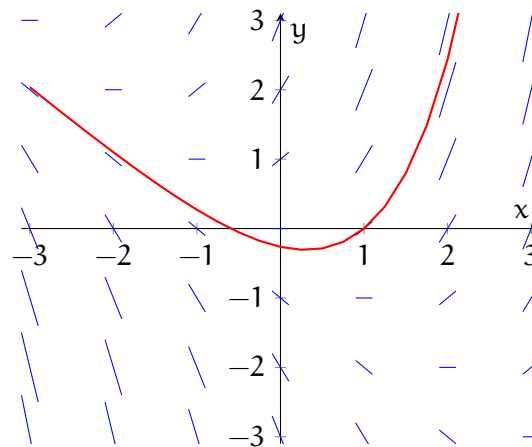


Figure 3.5: To sketch a solution to the differential equation, draw a function parallel to the nearby slopes that passes through the given point in the particular solution

While this method does not yield an exact, formulaic solution to the differential equation, it does allow us to visualize solutions and generally describe the behavior of any solutions. Sketching solutions this way is the idea behind Euler's method, which turns "follow the ticks" into arithmetic a computer can do. That is the next chapter.

3.3 Example: Application of Differential Equations to Electronics

Think back to the chapter on DC circuits. You learned that Ohm's Law relates voltage (electromotive force), current, and resistance for simple DC circuits:

$$V = IR$$

Simple resistors have a constant resistance, so once the voltage source (battery) is connected, the current is constant. There are other electronic components, such as inductors and capacitors, that behave differently. When current changes in an inductor, a voltage drop is induced across the inductor. This is described by the differential equation:

$$V = -L \frac{dI}{dt}$$

Where L is inductance, measured in henries (H), of the inductor. Consider, then, a circuit consisting of a constant-voltage battery, a fixed resistor, and an inductor (shown in figure 3.6). Since Kirchhoff's Law states that the sum of the voltage drops across each component must equal the voltage supplied by the battery, we can write a differential equation to describe the circuit:

$$V = L \frac{dI}{dt} + RI$$

Where the current, I , is a function of time, t .

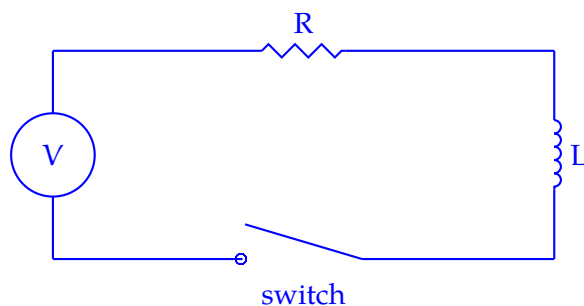


Figure 3.6: A simple circuit with a battery, resistor, inductor, and switch

Example: If the resistor is 12Ω , the inductance is $4H$, and the battery supplies a constant voltage of $60V$:

1. Draw a slope field for the differential equation describing the current in the circuit.
2. Describe the expected behavior of the current over a long period of time.
3. Identify any equilibrium solutions.
4. If the initial current at $t = 0$ is $I(0) = 0$, sketch the particular solution to the differential equation on the slope field.

Solution: Substituting the given values into the differential equation and rearranging to isolate $\frac{dI}{dt}$, we get $\frac{dI}{dt} = 15 - 3I$. Notice that the slope does not depend on time. When the slope is only dependent on the value of the function (as in this case), we call this an **autonomous differential equation**. This means that the slope is the same for all values of t at a given I . The slope field is shown in figure 3.7.

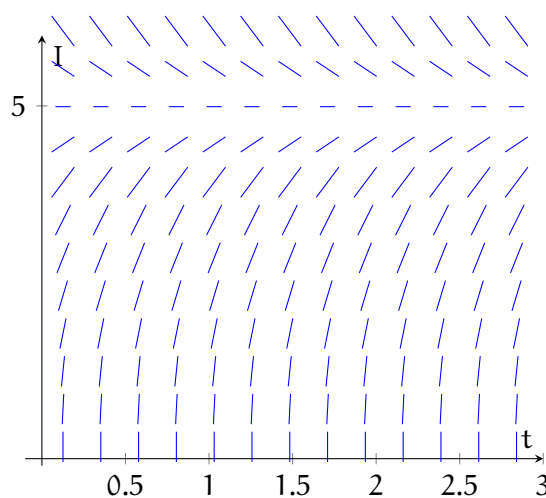


Figure 3.7: Slope field for the differential equation $\frac{dI}{dt} = 15 - 3I$

Examining the slope field, we see that the solutions tend towards $I(t) = 5$, which suggests that over an extended period of time, the current will approach 5 amperes. Similarly, if the initial current were 5 amperes, then the current would be constant at 5 amperes. Therefore, $I(t) = 5$ is an equilibrium solution. A sketch of the solution with $I(0) = 0$ is shown in figure 3.8.

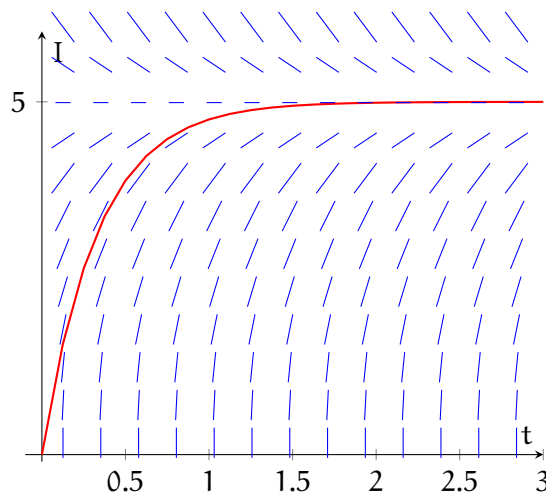


Figure 3.8: The solution with $I(0) = 0$ rises toward the equilibrium $I = 5$

3.4 Practice

Exercise 14

Sketch the slope field for the differential equation $y' = x + y^2$. Use your slope field to sketch a solution that passes through the point $(0, 0)$.

Working Space

Answer on Page 66

Exercise 15

Consider the autonomous equation $\frac{dy}{dt} = y(4 - y)$.

1. Find the equilibrium solutions.
2. For which values of y are solutions increasing? Decreasing?
3. Without drawing the whole field, describe what happens to the solution with $y(0) = 1$ and the solution with $y(0) = 6$ as t grows.

Working Space

Answer on Page 66

Exercise 16

In the slope field for $y' = x$, every tick in a vertical column has the same slope. Explain why, and name the family of curves the solutions belong to.

Working Space

Answer on Page 66

Numerical Root Finding

4.1 Why Approximate a Root?

Your calculator has a $\sqrt{}$ button, but nobody stored the answer for every number inside it. When you press it, the chip makes a guess, checks how wrong it is, and improves the guess, a few times, in a fraction of a millisecond. This chapter shows you two ways to do that.

A *root* of a function f is a value x where $f(x) = 0$. For $f(x) = x^2 - 2$, the root is $\sqrt{2}$. Many equations have no formula for their roots, such as $\cos x = x$ or $e^x = 3x$. For those we use numerical methods: we start with a guess and improve it until it is accurate enough.

4.2 The Bisection Method

Bisection uses the Intermediate Value Theorem (see the limits chapter). If f is continuous and $f(a)$ and $f(b)$ have opposite signs, then f has a root between a and b . Bisection checks the midpoint and keeps the half that still has a sign change.

For $f(x) = x^2 - 2$ on $[1, 2]$, we have $f(1) = -1$ and $f(2) = 2$. The midpoint is 1.5, and $f(1.5) = 0.25 > 0$. The sign changes between 1 and 1.5, so the new interval is $[1, 1.5]$. Each step halves the interval. To find $\sqrt{2}$ to within 10^{-6} , bisection needs 19 steps.

4.3 Newton's Method

Newton's method uses the tangent line. Start at x_0 . The tangent line at x_0 crosses the x -axis at

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

For $f(x) = x^2 - 2$, this is $x_{n+1} = x_n - \frac{x_n^2 - 2}{2x_n}$. Starting at $x_0 = 1$:

$$x_1 = 1.5, \quad x_2 = 1.41\bar{6}, \quad x_3 \approx 1.414216$$

Starting from the same point, the iterates stop changing by less than 10^{-12} after 6 steps. It works quickly when the starting guess is close, but it can fail if $f'(x_n)$ is zero or the guess is far from the root.

Example: Solve $\cos x = x$.

Solution: Rewrite it as $f(x) = \cos x - x = 0$, so $f'(x) = -\sin x - 1$. Starting at $x_0 = 1$:

$$x_1 \approx 0.750364, \quad x_2 \approx 0.739113, \quad x_3 \approx 0.739085, \quad x_4 \approx 0.739085$$

Three steps give six correct decimal places. No algebra trick would have found this number.

Newton's method can also get stuck. For $f(x) = x^3 - 2x + 2$ starting at $x_0 = 0$, you get $x_1 = 1$, then $x_2 = 0$, then $x_3 = 1$, forever. The tangent at each point aims straight at the other one. Always check that your iterates are actually settling down.

4.4 Code

```

1 def bisection(f, a, b, tol=1e-6):
2     if f(a) * f(b) > 0:
3         raise ValueError("f(a) and f(b) must have opposite signs")
4     while (b - a) / 2 > tol:
5         mid = (a + b) / 2
6         if f(a) * f(mid) <= 0:
7             b = mid
8         else:
9             a = mid
10    return (a + b) / 2
11
12 def newton(f, df, x, tol=1e-10, max_steps=50):
13     for _ in range(max_steps):
14         x_next = x - f(x) / df(x)
15         if abs(x_next - x) < tol:
16             return x_next
17         x = x_next
18     raise RuntimeError("Newton's method did not converge")
19
20 f = lambda x: x**2 - 2
21 df = lambda x: 2 * x
22 print(bisection(f, 1, 2)) # 1.4142141342163086
23 print(newton(f, df, 1.0)) # 1.4142135623730951

```

4.5 Comparing the Methods

Bisection always converges when it starts with a sign change, but it is slow. It gains about one correct binary digit per step. Newton's method usually converges much faster, but it needs the derivative, and it may diverge from a poor starting point.

Exercise 17 **One bisection step**

For $f(x) = x^2 - 2$ on $[1, 2]$, find the new interval after one bisection step.

Working Space

Answer on Page 66

Exercise 18 **One Newton step**

Starting at $x_0 = 2$, find x_1 and x_2 for $f(x) = x^2 - 2$.

Working Space

Answer on Page 67

Exercise 19 **No sign change**

Why does bisection raise an error for $f(x) = x^2 + 1$ on $[-1, 1]$?

Working Space

Answer on Page 67

Exercise 20 **How many halvings?**

You run bisection on an interval of width 1. How many steps until the interval is narrower than 0.001?

Working Space

Answer on Page 67

Euler's Method

Imagine driving at night with a speedometer that updates only every few seconds. Between readings, you assume your speed stays the same and estimate how far you have gone. Euler's method approximates a differential equation the same way: read the slope, take a step, read the slope again.

5.1 Following the Slope

Let us consider the differential equation $\frac{dy}{dx} = x + y$ with initial condition $y(0) = 1$.

This means the solution passes through the point $(0, 1)$. Additionally, the slope of the solution is $\frac{dy}{dx} = 0 + 1 = 1$ at that point. This means we can approximate the solution with the linear function $L(x) = x + 1$ (see figure 5.1). As you can see, near $(0, 1)$ the approximation is good, but as x increases, the divergence between the actual solution and the approximation grows.

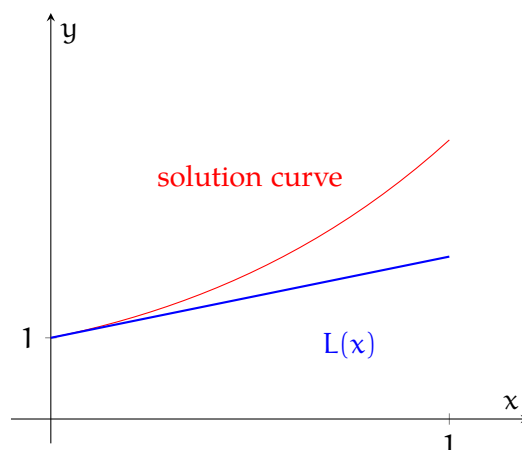


Figure 5.1: A first Euler approximation

How can we make a better approximation? Suppose we stop the first approximation at $x = 0.5$, re-evaluate $\frac{dy}{dx}$, and use that to make a second linear approximation. When $x = 0.5$, $L(x) = 0.5 + 1 = 1.5$. Taking the point $(0.5, 1.5)$, then $\frac{dy}{dx} = 0.5 + 1.5 = 2$. We can then write a second linear approximation, $L_2(x) = 2(x - 0.5) + 1.5 = 2x - 1 + 1.5 = 2x + 0.5$. As you can see (figure 5.2), this new approximation is closer than our first approximation. We call this an approximation with a step size of 0.5.

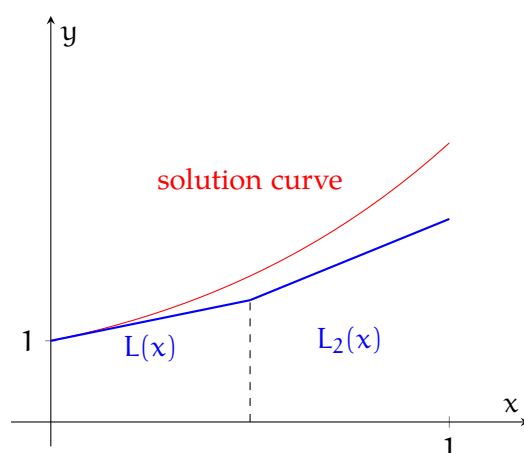


Figure 5.2: An Euler approximation with step size 0.5

We can improve this further by taking a step size of 0.25 (see figure 5.3). As the step size decreases and the step number increases, the approximation gets closer and closer to the true solution.

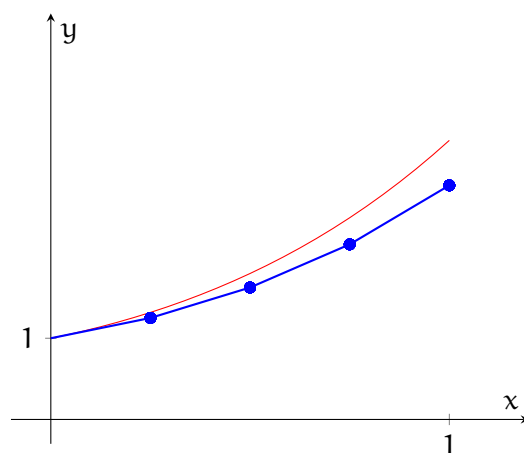


Figure 5.3: An Euler approximation with step size 0.25

5.2 The Algorithm

In general, Euler's method is a numerical process similar to sketching a solution on a slope field. You begin at the initial value and take a short step in the direction the slope field points. At the end of each step, you read the slope again and turn.

For a first-order differential equation, let $\frac{dy}{dx} = F(x, y)$ and $y(x_0) = y_0$. If we have step size h , then our successive x -values are $x_1 = x_0 + h$, $x_2 = x_1 + h$, etc. The differential equation tells us that the slope at x_0 is $F(x_0, y_0)$. So, $y_1 = y_0 + hF(x_0, y_0)$ (see figure 5.4).

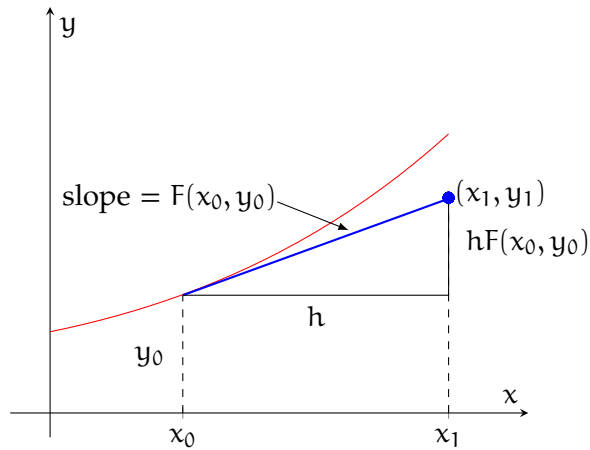


Figure 5.4: Visualization of Euler's method

Continuing, once we have found y_1 , we can then define $x_2 = x_1 + h$ and $y_2 = y_1 + hF(x_1, y_1)$. And in general, for an initial-value problem when $\frac{dy}{dx} = F(x, y)$ and $y(x_0) = y_0$, we can make an approximation with step size h where:

$$y_n = y_{n-1} + hF(x_{n-1}, y_{n-1})$$

where $n = 1, 2, 3, \dots$.

Example: Use Euler's method with a step size of 0.2 to approximate the value of $y(1)$ if $\frac{dy}{dx} = 2x + y$ and $y(0) = 1$.

Solution: We are given $h = 0.2$, $x_0 = 0$, $y_0 = 1$, and $F(x, y) = 2x + y$. This means we will need 5 steps to reach $x_5 = 1$. So, we know that:

$$y_1 = 1 + 0.2[2(0) + 1] = 1 + 0.2[1] = 1.2$$

$$y_2 = 1.2 + 0.2[2(0.2) + 1.2] = 1.2 + 0.2(1.6) = 1.52$$

$$y_3 = 1.52 + 0.2[2(0.4) + 1.52] = 1.984$$

We can continue in this manner. The results are shown in the table:

n	x_n	y_n	$F(x_n, y_n)$
0	0	1	1
1	0.2	1.2	1.6
2	0.4	1.52	2.32
3	0.6	1.984	3.184
4	0.8	2.6208	4.2208
5	1	3.46496	—

Therefore, $y(1) \approx 3.465$. The exact solution is $y = 3e^x - 2x - 2$, which gives $y(1) \approx 4.155$. Euler's method is low here because the solution curves upward, and a straight-line step falls below a curve that bends up.

5.3 Letting a Computer Take the Steps

A computer does not mind taking thousands of tiny steps:

```

1 def euler(F, x0, y0, h, steps):
2     x, y = x0, y0
3     for _ in range(steps):
4         y = y + h * F(x, y)
5         x = x + h
6     return y
7
8 F = lambda x, y: 2 * x + y
9 for steps in [5, 50, 500]:
10    print(steps, euler(F, 0, 1, 1 / steps, steps))

```

```

5 3.46496
50 4.074764087220816
500 4.146705561955173

```

Each tenfold cut in the step size shrinks the error (compared with 4.1548) by about a factor of ten.

5.4 An AP-Style Example

Example: This problem was originally presented as a no-calculator, multiple-choice question on the 2012 AP Calculus BC exam. Let $y = f(x)$ be the solution to $\frac{dy}{dx} = x - y$ with initial condition $f(1) = 3$. What is the approximation of $f(2)$ obtained using Euler's method with two steps of equal length starting at $x = 1$?

Solution: The question asks that we use Euler's method two steps. The step size should be $h = \frac{x_2 - x_0}{2} = \frac{2 - 1}{2} = \frac{1}{2}$. Taking $x_0 = 1$ and $y_0 = 3$, we find that:

$$\begin{aligned}
 y_1 &= y_0 + h[x_0 - y_0] \\
 y_1 &= 3 + \frac{1}{2}[1 - 3] \\
 y_1 &= 3 + \frac{1}{2}[-2] = 3 - 1 = 2
 \end{aligned}$$

So our intermediate point is $(x_1, y_1) = (\frac{3}{2}, 2)$. Finding y_2 :

$$y_2 = y_1 + h[x_1 - y_1]$$

$$y_2 = 2 + \frac{1}{2} \left[\frac{3}{2} - 2 \right]$$

$$y_2 = 2 + \frac{1}{2} \left[\frac{-1}{2} \right] = 2 - \frac{1}{4} = \frac{7}{4}$$

So the approximate value of $f(2)$ is $\frac{7}{4}$.

Exercise 21

Working Space

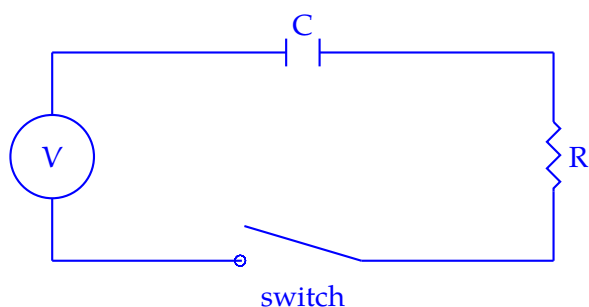
In the slope fields chapter, we discussed the behavior of inductors in electronic circuits. As you may recall, capacitors also exhibit more complex behavior than regular resistors. Consider a circuit with a resistor and capacitor (see figure below). Let the resistor have resistance R ohms and the capacitor have capacitance C farads. By Kirchhoff's law, we know that:

$$RI + \frac{Q}{C} = V$$

where Q is the charge on each side of the capacitor and $\frac{Q}{C}$ is the voltage drop across the capacitor. Recall that current is the change in charge over time. Therefore, $I = \frac{dQ}{dt}$, and we can write the differential equation:

$$R \frac{dQ}{dt} + \frac{1}{C} Q = V$$

When the switch is first closed, there is no charge (that is, $Q(0) = 0$). If the resistor is 5Ω , the battery is $60V$, and the capacitor is $0.05F$, use Euler's method with a step size of 0.1 to estimate the charge after half a second.



Answer on Page 67

Exercise 22

[This problem was originally presented as a calculator-allowed, free-response question on the 2012 AP Calculus BC exam.] The function f is twice-differentiable for $x > 0$ with $f(1) = 15$ and $f''(1) = 20$. Values of f' , the derivative of f , are given for selected values of x in the table below. Use Euler's method, starting at $x = 1$ with two steps of equal size, to approximate $f(1.4)$. Show the computations that lead to your answer.

x	1	1.1	1.2	1.3	1.4
$f'(x)$	8	10	12	13	14.5

Working Space

Answer on Page 67

Sequences in Calculus

Your phone's battery drops from 100% to 99% to 98%. Your savings account grows a little each month. A rumor reaches 1, then 3, then 9 classmates. Each of these is a list of numbers that follows a rule, which is a sequence. We introduced sequences in Chapter ?? . Now we ask the calculus question: where is the list heading? You already know about arithmetic and geometric sequences, but not all sequences can be classified as arithmetic or geometric. Take the famous Fibonacci sequence, $\{1, 1, 2, 3, 5, 8, \dots\}$, which is defined by the rule $a_n = a_{n-1} + a_{n-2}$, with $a_1 = a_2 = 1$. There is no common difference or common ratio, so the Fibonacci sequence is not arithmetic or geometric. Another example is $a_n = \sin \frac{n\pi}{6}$, which will cycle through a set of values.

Sequences have many real-world applications, including compound interest and modeling population growth. In later chapters, you will learn that the sum of all the values in a sequence is a series and how to use series to describe functions. First, we need to understand sequences well.

Some sequences are defined explicitly, like $a_n = \sin \frac{n\pi}{6}$, while others are defined recursively, like $a_n = a_{n-1} + a_{n-2}$.

Example: Write the first 5 terms for the explicitly defined sequence $a_n = \frac{n}{n+1}$.

Solution: We can construct a table to keep track of our work:

n	work	a_n
1	$\frac{1}{1+1}$	$\frac{1}{2}$
2	$\frac{2}{2+1}$	$\frac{2}{3}$
3	$\frac{3}{3+1}$	$\frac{3}{4}$
4	$\frac{4}{4+1}$	$\frac{4}{5}$
5	$\frac{5}{5+1}$	$\frac{5}{6}$

So, the first five terms are $\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \text{ and } \frac{5}{6}\}$.

Exercise 23

Write the first five terms for each sequence.

1. $a_n = \frac{2^n}{2n+1}$

Working Space

2. $a_n = \cos \frac{n\pi}{2}$
3. $a_1 = 1, a_{n+1} = 5a_n - 3$
4. $a_1 = 6, a_{n+1} = \frac{a_n}{n+1}$

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6.1 The Idea of Approaching

6.1.1 Law of Large Numbers Revisited

Recall that, in the limits chapter, we talked about the *Law of Large Numbers*. It says that as the number of trials (coin flips, in that chapter) grows, the proportion of successes (heads) gets closer to the expected value.

Let us bring back our flip sequence from that chapter

H, T, T, H, T, H, T, T, H, T

Let us turn it into a sequence:

$$a_n = \frac{\text{heads so far}}{\text{flips so far}}$$

The first few terms would be

$$a_1 = \frac{1}{1} = 1$$

$$a_2 = \frac{1}{2} = 0.5$$

$$a_3 = \frac{1}{3} \approx 0.333$$

$$a_4 = \frac{2}{4} = 0.5$$

And after all 10 flips,

$$1, \frac{1}{2}, \frac{1}{3}, \frac{2}{4}, \frac{2}{5}, \frac{3}{6}, \frac{3}{7}, \frac{3}{8}, \frac{4}{9}, \frac{4}{10}$$

or as decimals,

$$1, 0.5, 0.333, 0.5, 0.4, 0.5, 0.429, 0.375, 0.444, 0.4$$

We represent this with the equation

$$\bar{X}_n \rightarrow \mu$$

the average value of X approaches the true mean, μ .

The Law of Large Numbers says that, over the long run, this sequence tends to settle near 0.5. In calculus, this kind of behavior has a name. When the terms of a sequence approach a fixed value as n gets larger, we say the sequence *converges*. If the terms do not approach one fixed value, we say the sequence *diverges*. Let us look at this with graphs and actual equations.

The coin-flip sequence is random, so now we will look at sequences whose rules are known exactly.

6.1.2 Convergence and Divergence

You can visualize a sequence on an xy -plane or a number line. Figures 6.1 and 6.2 show visualizations of the sequence $a_n = \frac{n}{n+1}$. To visualize this on the xy -plane, we take points such that $x = n$ and $y = a_n$, where n is a positive integer. What do you notice? As n increases, a_n gets closer and closer to 1.

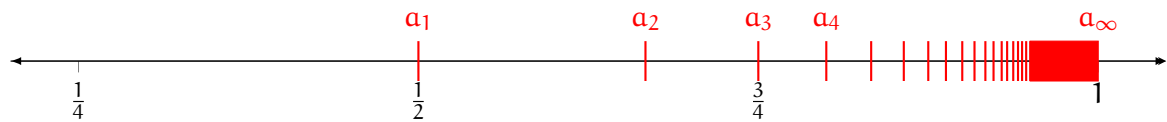


Figure 6.1: $a_n = \frac{n}{n+1}$ on a number line

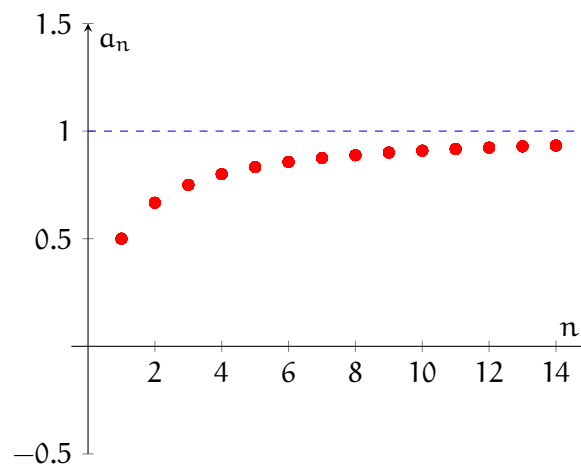


Figure 6.2: $a_n = \frac{n}{n+1}$ on an xy -plane

Because a_n approaches a specific number as $n \rightarrow \infty$, we call the sequence $a_n = \frac{n}{n+1}$

convergent. We prove a sequence is convergent by taking the limit as n approaches ∞ . If the limit exists and approaches a specific number, the sequence is convergent. If the limit does not exist or approaches $\pm\infty$, the sequence is divergent.

We can see graphically that $\lim_{n \rightarrow \infty} \frac{n}{n+1} = 1$, so that sequence is convergent. What about $b_n = \frac{n}{\sqrt{10+n}}$? Is b_n convergent or divergent?

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{n}{\sqrt{10+n}} &= \lim_{n \rightarrow \infty} \frac{n/n}{\sqrt{\frac{10}{n^2} + \frac{n}{n^2}}} \\ &= \lim_{n \rightarrow \infty} \frac{1}{\sqrt{\frac{10}{n^2} + \frac{1}{n}}} = \infty \end{aligned}$$

Therefore, the sequence $b_n = \frac{n}{\sqrt{10+n}}$ is divergent.

Here is another example of a divergent sequence: $c_n = \sin \frac{n\pi}{2}$. The graph is shown in figure 6.3. As you can see, the value of c_n oscillates between 1, 0, and -1 without approaching a specific number. This means that c_n does not approach a particular number as $n \rightarrow \infty$ and the sequence is divergent.

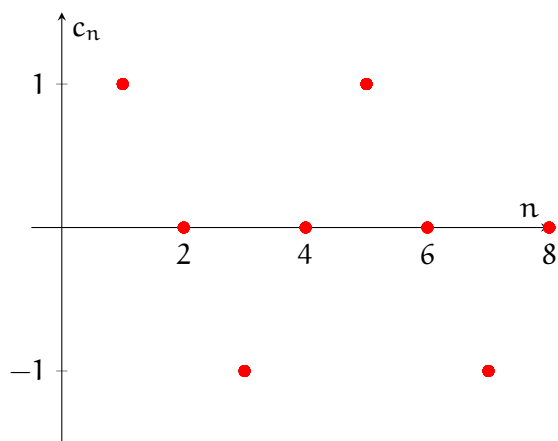


Figure 6.3: $c_n = \sin \frac{n\pi}{2}$ on an xy -plane

Exercise 24

Classify each sequence as convergent or divergent. If the sequence is convergent, find the limit as $n \rightarrow \infty$.

1. $a_n = \frac{3+5n^2}{n+n^2}$

Working Space

2. $a_n = \frac{n^4}{n^3 - 2n}$
3. $a_n = 2 + (0.86)^n$
4. $a_n = \cos \frac{n\pi}{n+1}$
5. $a_n = \sin n$

Answer on Page 68

6.2 Evaluating limits of sequences

Recall that a sequence can be considered a function where the domain is restricted to positive integers. If there is some $f(x)$ such that $a_n = f(n)$ when n is an integer, then $\lim_{n \rightarrow \infty} a_n = \lim_{x \rightarrow \infty} f(x)$ (see figure 6.4). This means that all the rules that apply to the limits of functions also apply to the limits of sequences, including the Squeeze Theorem and l'Hôpital's rule.

Example: What is $\lim_{n \rightarrow \infty} \frac{\ln n}{n}$?

Solution: First, we will try to compute the limit directly:

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{\ln n}{n} &= \\ \frac{\lim_{n \rightarrow \infty} \ln n}{\lim_{n \rightarrow \infty} n} &= \frac{\infty}{\infty} \end{aligned}$$

This is an indeterminate form, so we can use l'Hôpital's rule:

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{\ln n}{n} &= \lim_{n \rightarrow \infty} \frac{\frac{d}{dn} \ln n}{\frac{d}{dn} n} \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} = 0 \end{aligned}$$

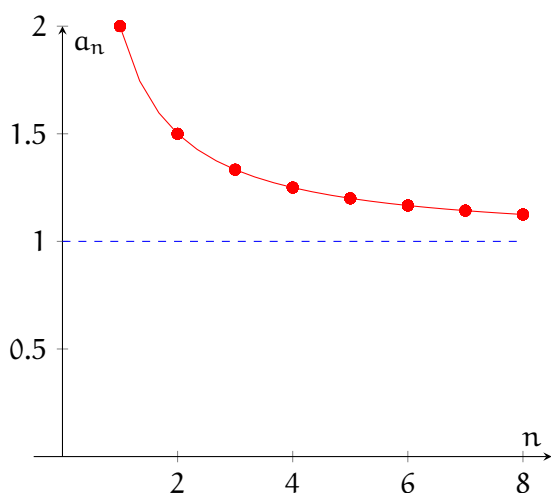


Figure 6.4: The limit of the function is the same as the limit of the sequence

Example: Is the sequence $a_n = \frac{n!}{n^n}$ convergent or divergent?

Solution: First trying to take the limit directly, we see that:

$$\lim_{n \rightarrow \infty} \frac{n!}{n^n} = \frac{\infty}{\infty}$$

which is indeterminate. Because the factorial cannot be described as a continuous function, we cannot use l'Hôpital's rule. We can examine this sequence graphically (see figure 6.5) and mathematically. We examine it mathematically by writing out a few terms to get an idea of what happens to a_n as n gets large:

$$\begin{aligned} a_1 &= \frac{1!}{1^1} = 1 \\ a_2 &= \frac{2!}{2^2} = \frac{1 \cdot 2}{2 \cdot 2} \\ a_3 &= \frac{3!}{3^3} = \frac{1 \cdot 2 \cdot 3}{3 \cdot 3 \cdot 3} \\ &\dots \\ a_n &= \frac{n!}{n^n} = \frac{1 \cdot 2 \cdot 3 \cdots n}{n \cdot n \cdot n \cdots n} \end{aligned}$$

From examining the graph in figure 6.5, we can guess that $\lim_{n \rightarrow \infty} a_n = 0$. Let us prove that mathematically. We can rewrite our expression for a_n as n gets larger:

$$a_n = \frac{n!}{n^n} = \frac{1 \cdot 2 \cdot 3 \cdots n}{n \cdot n \cdot n \cdots n} = \frac{1}{n} \left(\frac{2 \cdot 3 \cdots n}{n \cdot n \cdots n} \right)$$

Each fraction inside the parentheses is at most 1, so the product is at most 1; therefore, $0 < a_n \leq \frac{1}{n}$. Since $\lim_{n \rightarrow \infty} 0 = 0$ and $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$, by Squeeze Theorem, we know that $\lim_{n \rightarrow \infty} \frac{n!}{n^n} = 0$. Therefore, the sequence $a_n = \frac{n!}{n^n}$ is convergent.

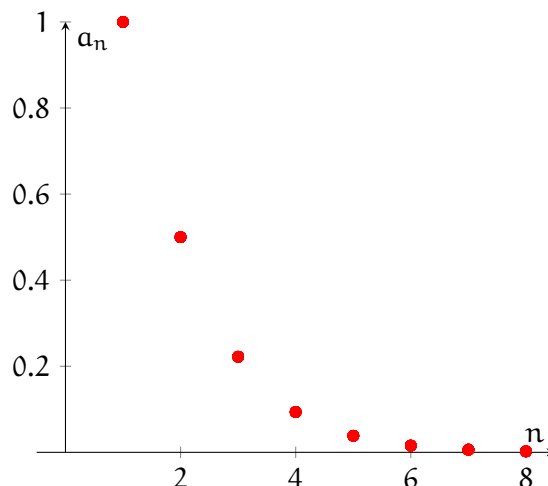


Figure 6.5: $a_n = \frac{n!}{n^n}$

6.2.1 Continuous Functions of Sequences

Some sequences are formed by applying a function to a simpler sequence, such as $\sin \frac{\pi}{n}$ or $e^{1/n}$. In these cases, we can find the limit of the inner sequence first and then evaluate the function at that limit, provided the function is continuous there.

If $\lim_{n \rightarrow \infty} a_n = L$ and the function f is continuous at L , then

$$\lim_{n \rightarrow \infty} f(a_n) = f\left(\lim_{n \rightarrow \infty} a_n\right) = f(L)$$

Example: What is $\lim_{n \rightarrow \infty} \sin \frac{\pi}{n}$?

Solution: We know that $\lim_{n \rightarrow \infty} \frac{\pi}{n} = 0$ and that the sine function is continuous at 0. Therefore,

$$\lim_{n \rightarrow \infty} \sin \frac{\pi}{n} = \sin\left(\lim_{n \rightarrow \infty} \frac{\pi}{n}\right) = \sin 0 = 0$$

6.3 Monotonic and Bounded sequences

Just like functions, sequences can be increasing or decreasing. A sequence is increasing if $a_n < a_{n+1}$ for $n \geq 1$. Similarly, a sequence is decreasing if $a_n > a_{n+1}$ for $n \geq 1$. If a sequence is strictly increasing or decreasing, it is called *monotonic*.

The sequence $a_n = \frac{1}{n+6}$ is decreasing. We prove this formally by comparing a_n to a_{n+1} :

$$\frac{1}{n+6} > \frac{1}{(n+1)+6} = \frac{1}{n+7}$$

Example: Is the sequence $a_n = \frac{n}{n^2+1}$ increasing or decreasing?

Solution: First, we find an expression for a_{n+1} :

$$a_{n+1} = \frac{n+1}{(n+1)^2+1} = \frac{n+1}{n^2+2n+2}$$

Since the degree of n is greater in the denominator, we have a guess that the sequence is decreasing. To prove this, we check if $a_n > a_{n+1}$ is true:

$$\frac{n}{n^2+1} > \frac{n+1}{n^2+2n+2}$$

We can cross-multiply, because $n > 0$ and the denominators are positive:

$$(n)(n^2+2n+2) > (n+1)(n^2+1)$$

$$n^3+2n^2+2n > n^3+n^2+n+1$$

Subtracting (n^3+n^2+n) from both sides we see that:

$$n^2+n > 1$$

This is true for all $n \geq 1$. Therefore, $a_n > a_{n+1}$ for all $n \geq 1$ and the sequence is decreasing.

A sequence is *bounded above* if there is some number M such that $a_n \leq M$ for all $n \geq 1$. A sequence is *bounded below* if there is some other number m such that $a_n \geq m$ for all $n \geq 1$. If a sequence is bounded above and below, then it is a *bounded sequence*.

Not all bounded sequences are convergent. Take our earlier example of $a_n = \sin \frac{n\pi}{6}$. This sequence is bounded, since we can say that $-1 \leq a_n \leq 1$ for all n . However, $a_n = \sin \frac{n\pi}{6}$ is divergent because $\lim_{n \rightarrow \infty} \sin \frac{n\pi}{6}$ does not exist (see figure 6.6). Additionally, not all monotonic sequences are convergent. Consider $b_n = 2^n$ (shown in figure 6.7). This is monotonically increasing (that is, $b_n > b_{n-1}$ for all n), but $\lim_{n \rightarrow \infty} 2^n = \infty$ and the sequence is divergent.

A sequence must be convergent if it is **both** monotonic and bounded. Why is this? Recall that to be bounded, a sequence must be bounded above and below, which means there is some m and some M such that $m \leq a_n \leq M$ for all n . If the sequence is increasing,

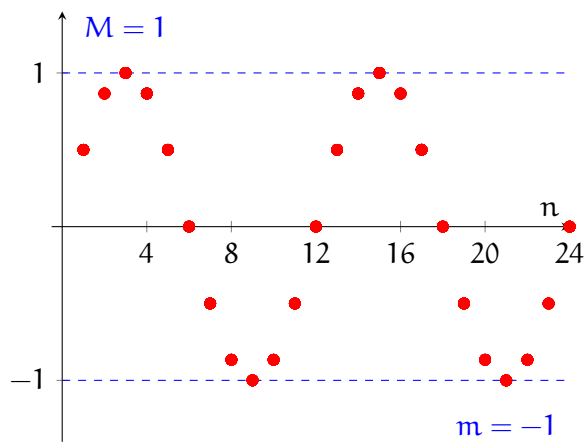


Figure 6.6: The sequence $a_n = \sin \frac{n\pi}{6}$ is bounded and divergent

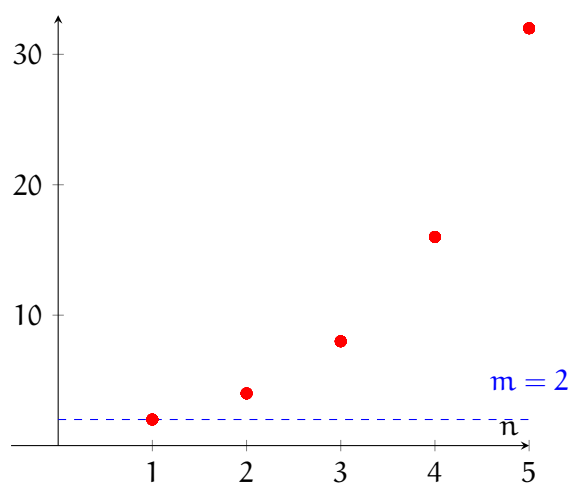


Figure 6.7: The sequence $b_n = 2^n$ is bounded below, monotonically increasing, and divergent

the terms must get close to but not exceed M . Likewise, if the sequence is decreasing, the terms must get close to, but not be less than m .

Example: Is the sequence given by $a_1 = 4$ and $a_{n+1} = \frac{1}{2}(a_n + 7)$ bounded above, below, both, or neither?

Solution: We start by calculating the first several terms:

Term	Work	Value
a_1	$a_1 = 4$	4
a_2	$= \frac{1}{2}(4 + 7)$	5.5
a_3	$= \frac{1}{2}(5.5 + 7)$	6.25
a_4	$= \frac{1}{2}(6.25 + 7)$	6.625
a_5	$= \frac{1}{2}(6.625 + 7)$	6.8125
a_6	$= \frac{1}{2}(6.8125 + 7)$	6.90625
a_7	$= \frac{1}{2}(6.90625 + 7)$	6.953125
a_8	$= \frac{1}{2}(6.953125 + 7)$	6.9765625

The sequence is increasing, so it is bounded below by the initial term, $a_1 = 4$, and we can state that $a_n \geq 4$. Examining the computed terms, we see that $a_n \rightarrow 7$ as n grows larger. We can guess that this sequence is bounded above, with $a_n \leq 7$. We can prove this by induction. Suppose that there is some k such that $a_k < 7$ (which is true for a_1 , etc.). Then,

$$\begin{aligned}
 a_k &< 7 \\
 a_k + 7 &< 14 \\
 \frac{1}{2}(a_k + 7) &< \frac{1}{2}(14) \\
 a_{k+1} &< 7
 \end{aligned}$$

Therefore, $a_n < 7$ for all n and the sequence is bounded above. Because the sequence is monotonic and bounded, we know the sequence is convergent and, therefore, that the limit of a_n as $n \rightarrow \infty$ exists.

Exercise 25

Let $a_1 = 1$ and $a_{n+1} = \sqrt{2 + a_n}$.

Working Space

1. Compute a_2 , a_3 , and a_4 to three decimal places.
2. Show that if $a_k < 2$, then $a_{k+1} < 2$.
3. The sequence is increasing. Explain why it must converge, and guess its

limit.

Answer on Page 68

6.4 Applications of Sequences

6.4.1 Compound Interest

You previously learned about compound interest and modeled the accumulation of compound interest by $P_n = P_0(1 + r)^n$, where P_0 is the principal investment, r is the yearly interest rate, and n is the number of elapsed years. This sequence describes the value of an investment accumulating interest, but most people add to their savings on a regular schedule. We can write a sequence to model the value of a savings account that the owner makes regular deposits into.

Example: Suppose you open a savings account with an initial deposit of \$3,000 and you plan to deposit an additional \$1,200 at the end of every year. If your savings account has an annual interest rate of 3.25%, how long will it take you to save \$10,000?

Solution: We can write a recursive definition for the sequence. At the end of each year, the account will gain the interest on the entirety of the previous year's balance plus \$1200:

$$P_n = P_{n-1}(1 + 0.0325) + \$1200$$

With an initial investment $P_0 = \$3000$. We can write out the first few terms to find how many years it will take to save \$10,000:

Year	Savings
0	\$3,000
1	\$4,297.50
2	\$5,637.17
3	\$7,020.38
4	\$8,448.54
5	\$9,923.12
6	\$11,445.62

Since the balance is still below \$10,000 after year 5 and passes it during year 6, it takes six years to save \$10,000.

The accumulation of interest with deposits is better described by a sequence than a func-

tion. That is because the deposits are happening at discrete times, not continuously.

Exercise 26

You invest \$1500 at 5%, compounded annually. Write an explicit formula that describes the value of your investment every year. What will your investment be worth after 10 years? Is the sequence convergent or divergent? Explain.

Working Space

Answer on Page 68

6.4.2 Population Growth

Sequences can be used to model a reproducing population that is being occasionally culled from or added to. Similar to compound interest, a population of living things (plants, animals, fungi, etc.) reproducing at a rate r can be modeled with an exponential function:

$$P_n = P_0(1 + r)^n$$

where P_0 is the initial population, r is the yearly reproductive rate, and n is the number of years elapsed.

Example: Suppose the population of deer in a national park is estimated to be 6,500. If the deer reproduce at a rate of 8% per year and wolves hunt and kill 500 deer per year, how many deer will be in the park in 5 years?

Solution: We can write a recursive sequence:

$$P_n = P_{n-1}(1 + 0.08) - 500$$

$$P_0 = 6500$$

And calculate P_5 (we round to the nearest whole number because half of a deer is not a living deer):

Year		Population
1	$6500(1.08) - 500$	6520
2	$6520(1.08) - 500$	6542
3	$6542(1.08) - 500$	6565
4	$6565(1.08) - 500$	6590
5	$6590(1.08) - 500$	6617

There will be 6617 deer in the park after 5 years.

Exercise 27

A farmer keeps a pond stocked with fish. Predators eat 5% of the fish each month, and the farmer can afford to add 10 fish at the end of every 6 months. If the farmer starts with 100 fish, how many total fish will the farmer have lost to predation after 4 years?

Working Space

Answer on Page 69

Answers to Exercises

Answer to Exercise 1 (on page 5)

Recall that for a polar function, $\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$. We also know that $x = r \cos \theta$, which equals $[1 + 2 \sin \theta] \cdot \cos \theta = \cos \theta + 2 \sin \theta \cos \theta$ in this case. We also know that $y = r \cdot \sin \theta$, which equals $[1 + 2 \sin \theta] \cdot \sin \theta = \sin \theta + 2 \sin^2 \theta$ in this case. Taking the derivative with respect to θ :

$$\begin{aligned}\frac{dy}{d\theta} &= \frac{d}{d\theta} [\sin \theta + 2 \sin^2 \theta] \\ \frac{dy}{d\theta} &= \cos \theta + 4 \sin \theta \cos \theta\end{aligned}$$

And

$$\begin{aligned}\frac{dx}{d\theta} &= \frac{d}{d\theta} [\cos \theta + 2 \sin \theta \cos \theta] \\ \frac{dx}{d\theta} &= -\sin \theta - 2 \sin^2 \theta + 2 \cos^2 \theta\end{aligned}$$

Evaluating each at $\theta = 0$:

$$\begin{aligned}\frac{dy}{d\theta} &= \cos 0 + 4 \sin 0 \cos 0 = 1 + 0 = 1 \\ \frac{dx}{d\theta} &= -\sin 0 - 2 \sin^2 0 + 2 \cos^2 0 = 0 - 0 + 2 = 2\end{aligned}$$

Therefore, $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{1}{2}$

Answer to Exercise 2 (on page 6)

1. Answer: slope $= -\frac{\sqrt{3}}{3}$ and an equation for the tangent line is $y - \frac{\sqrt{3}}{6} = -\frac{\sqrt{3}}{3} (x - \frac{1}{2})$.

$$\text{Explanation: } \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\frac{d}{d\theta} r \cdot \sin \theta}{\frac{d}{d\theta} r \cdot \cos \theta} = \frac{\frac{d}{d\theta} \left(\frac{2}{3} \cos \theta \sin \theta \right)}{\frac{d}{d\theta} \left(\frac{2}{3} \cos^2 \theta \right)} = \frac{\frac{2}{3} (\cos^2 \theta - \sin^2 \theta)}{\frac{2}{3} (-2 \cos \theta \sin \theta)} = \frac{\sin^2 \theta - \cos^2 \theta}{2 \cos \theta \sin \theta}$$

$$\text{Substituting } \theta = \frac{\pi}{6}: \frac{dy}{dx} = \frac{\sin^2 \pi/6 - \cos^2 \pi/6}{2 \cos \pi/6 \sin \pi/6} = \frac{(1/2)^2 - (\sqrt{3}/2)^2}{2(\sqrt{3}/2)(1/2)} = \frac{1/4 - 3/4}{\sqrt{3}/2} = \frac{-1/2}{\sqrt{3}/2} = \frac{-1}{\sqrt{3}} = -\frac{1}{\sqrt{3}} \cdot \frac{2}{2} = -\frac{\sqrt{3}}{3}$$

To write an equation for a line, we need a Cartesian point. First, we find r at $\theta = \frac{\pi}{6}$: $r = \frac{2}{3} \cos\left(\frac{\pi}{6}\right) = \frac{2}{3} \cdot \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{3}$. So the point the tangent passes through is the polar coordinate $\left(\frac{\sqrt{3}}{3}, \frac{\pi}{6}\right)$. We convert this to Cartesian coordinates: $x = r \cos \theta = \frac{\sqrt{3}}{3} \cdot \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{3} \cdot \frac{\sqrt{3}}{2} = \frac{3}{6} = \frac{1}{2}$ And $y = r \sin \theta = \frac{\sqrt{3}}{3} \cdot \sin\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{3} \cdot \frac{1}{2} = \frac{\sqrt{3}}{6}$

So, an equation for a line with slope $-\frac{\sqrt{3}}{3}$ that passes through Cartesian coordinate $\left(\frac{1}{2}, \frac{\sqrt{3}}{6}\right)$ is: $y - \frac{\sqrt{3}}{6} = -\frac{\sqrt{3}}{3} \left(x - \frac{1}{2}\right)$

2. Answer: slope $= \frac{2}{\pi}$ and an equation for the tangent line is $y - \frac{1}{\pi} = \frac{2}{\pi}x$

$$\text{Explanation: } \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\frac{d}{d\theta}\left(\frac{\sin \theta}{2\theta}\right)}{\frac{d}{d\theta}\left(\frac{\cos \theta}{2\theta}\right)} = \frac{\frac{\theta \cos \theta - \sin \theta}{2\theta^2}}{-\frac{\theta \sin \theta + \cos \theta}{2\theta^2}} = \frac{\sin \theta - \theta \cos \theta}{\theta \sin \theta + \cos \theta}.$$

$$\text{Substituting } \theta = \frac{\pi}{2}: \frac{dy}{dx} = \frac{\sin \frac{\pi}{2} - \left(\frac{\pi}{2}\right) \cos \frac{\pi}{2}}{\left(\frac{\pi}{2}\right) \sin \frac{\pi}{2} + \cos \frac{\pi}{2}} = \frac{1 - \left(\frac{\pi}{2}\right) \cdot 0}{\left(\frac{\pi}{2}\right) \cdot 1 + 0} = \frac{1}{\frac{\pi}{2}} = \frac{2}{\pi}$$

To write an equation for a line, we need a Cartesian point. First, we find r at $\theta = \frac{\pi}{2}$: $r = \frac{1}{2\theta} = \frac{1}{2 \cdot \frac{\pi}{2}} = \frac{1}{\pi}$. So the tangent line passes through the point with polar coordinates $\left(\frac{1}{\pi}, \frac{\pi}{2}\right)$. We convert this to Cartesian coordinates: $x = r \cdot \cos \theta = \frac{1}{\pi} \cdot \cos \frac{\pi}{2} = \frac{1}{\pi} \cdot 0 = 0$ and $y = r \cdot \sin \theta = \frac{1}{\pi} \cdot \sin \frac{\pi}{2} = \frac{1}{\pi} \cdot 1 = \frac{1}{\pi}$.

So, an equation for a line with slope $\frac{2}{\pi}$ that passes through Cartesian coordinate $(0, \frac{1}{\pi})$ is $y - \frac{1}{\pi} = \frac{2}{\pi}x$

3. Answer: slope $= -\frac{5}{\sqrt{3}}$ and an equation for the tangent line is $y - \frac{\sqrt{3}}{4} = \left(-\frac{5}{\sqrt{3}}\right) \left(x + \frac{1}{4}\right)$.

$$\text{Explanation: } \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\frac{d}{d\theta}[(2+3\cos\theta) \cdot \sin\theta]}{\frac{d}{d\theta}[(2+3\cos\theta) \cdot \cos\theta]} = \frac{\cos\theta(2+3\cos\theta) - 3\sin^2\theta}{(2+3\cos\theta) \cdot (-\sin\theta) + \cos\theta(-3\sin\theta)} = \frac{\cos\theta(2+3\cos\theta) - 3\sin^2\theta}{-2\sin\theta(1+3\cos\theta)}.$$

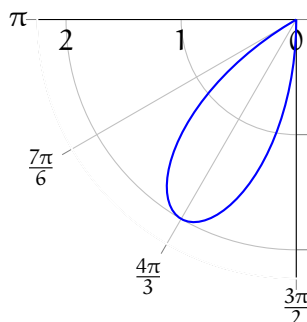
$$\text{Substituting } \theta = \frac{2\pi}{3}: \frac{dy}{dx} = \frac{\cos \frac{2\pi}{3} \left(2+3\cos \frac{2\pi}{3}\right) - 3\sin^2 \frac{2\pi}{3}}{-2\sin \frac{2\pi}{3} \left(1+3\cos \frac{2\pi}{3}\right)} = \frac{\left(-\frac{1}{2}\right) \left(2+3\left(-\frac{1}{2}\right)\right) - 3\left(\frac{\sqrt{3}}{2}\right)^2}{-2\left(\frac{\sqrt{3}}{2}\right) \left(1+3\left(-\frac{1}{2}\right)\right)} = \frac{\left(-\frac{1}{2}\right) \left(2-\frac{3}{2}\right) - 3\left(\frac{3}{4}\right)}{-\sqrt{3} \left(1-\frac{3}{2}\right)} = \frac{\left(-\frac{1}{2}\right) \left(\frac{1}{2}\right) - \frac{9}{4}}{-\sqrt{3} \left(-\frac{1}{2}\right)} = \frac{-\frac{1}{4} - \frac{9}{4}}{\frac{\sqrt{3}}{2}} = \frac{-\frac{10}{4}}{\frac{\sqrt{3}}{2}} = -\frac{10 \cdot 2}{4 \cdot \sqrt{3}} = -\frac{5}{\sqrt{3}}$$

To write an equation for a tangent line, we need a Cartesian point. First, we find r at $\theta = \frac{2\pi}{3}$: $r = 2+3\cos \frac{2\pi}{3} = 2+3\left(-\frac{1}{2}\right) = 2-\frac{3}{2} = \frac{1}{2}$. So the tangent line passes through polar coordinate $\left(\frac{1}{2}, \frac{2\pi}{3}\right)$. We convert this to Cartesian coordinates: $x = r \cos \theta = \frac{1}{2} \cos \frac{2\pi}{3} = \frac{1}{2} \left(-\frac{1}{2}\right) = -\frac{1}{4}$ and $y = r \sin \theta = \frac{1}{2} \sin \frac{2\pi}{3} = \frac{1}{2} \left(\frac{\sqrt{3}}{2}\right) = \frac{\sqrt{3}}{4}$.

So, an equation with slope $-\frac{5}{\sqrt{3}}$ that passes through the Cartesian coordinate $\left(-\frac{1}{4}, \frac{\sqrt{3}}{4}\right)$ is: $y - \frac{\sqrt{3}}{4} = \left(-\frac{5}{\sqrt{3}}\right) \left(x + \frac{1}{4}\right)$

Answer to Exercise 3 (on page 12)

We know the area of the circle is $\pi r^2 = \pi(2)^2 = 4\pi$. To find the area of the shaded regions, we need to subtract the area of the trefoil from the area of the circle. The trefoil has three equal areas. We can find the area of the leaf that is formed on the interval $\frac{\pi}{6} \leq \theta \leq \frac{\pi}{2}$ (see figure below).



The area of one leaf of the trefoil is given by $\frac{1}{2} \int_{\pi/6}^{\pi/2} [2 \cos 3\theta]^2 d\theta$. Using a calculator, the area of one leaf is ≈ 1.0472 . The area of the circle is given by $\pi r^2 = \pi(2)^2 \approx 12.5664$. The area of the shaded region is the area of the circle minus three times the area of a single leaf: $12.5664 - 3 \cdot 1.0472 = 9.4248 \approx 9.425$.

Answer to Exercise 4 (on page 13)

1. Answer: $A = \frac{1}{2} (e^{\pi/2} - e^{\pi/4}) \approx 1.309$

Explanation: The integrand is $[e^{\theta/2}]^2 = e^\theta$, so $A = \frac{1}{2} \int_{\pi/4}^{\pi/2} e^\theta d\theta = \frac{1}{2} [e^\theta]_{\pi/4}^{\pi/2} = \frac{1}{2} (e^{\pi/2} - e^{\pi/4}) \approx 1.309$

2. Answer: $A = \frac{5\pi}{4} - \frac{4}{3} \approx 2.594$

Explanation: Squaring $r = 2 \sin \theta + \cos 2\theta$ gives $4 \sin^2 \theta + 4 \sin \theta \cos 2\theta + \cos^2 2\theta$. Over $0 \leq \theta \leq \pi$, the three integrals are $\int_0^\pi 4 \sin^2 \theta d\theta = 2\pi$, $\int_0^\pi 4 \sin \theta \cos 2\theta d\theta = -\frac{8}{3}$, and $\int_0^\pi \cos^2 2\theta d\theta = \frac{\pi}{2}$. So $A = \frac{1}{2} (2\pi - \frac{8}{3} + \frac{\pi}{2}) = \frac{5\pi}{4} - \frac{4}{3}$.

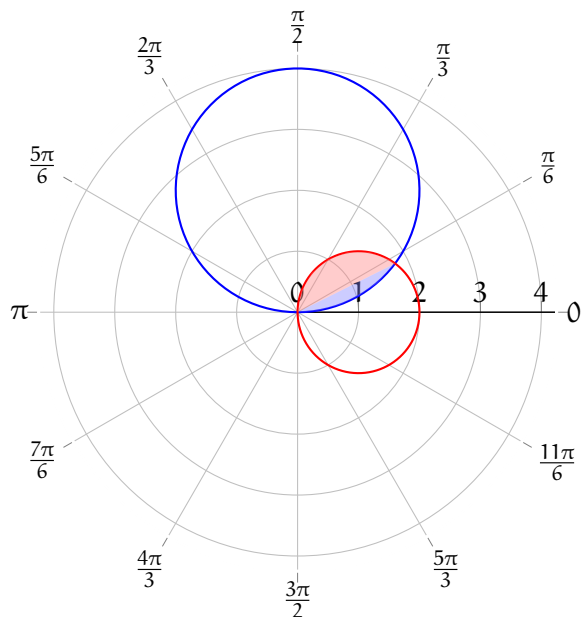
3. Answer: $A = \frac{41}{4}\pi \approx 32.201$

Explanation: $A = \frac{1}{2} \int_{-\pi/2}^{\pi/2} [4 + 3 \sin \theta]^2 d\theta = \frac{1}{2} \int_{-\pi/2}^{\pi/2} [16 + 24 \sin \theta + 9 \sin^2 \theta] d\theta = \int_{-\pi/2}^{\pi/2} 8 d\theta + 12 \int_{-\pi/2}^{\pi/2} \sin \theta d\theta + \frac{9}{2} \int_{-\pi/2}^{\pi/2} \sin^2 \theta d\theta = [8\theta]_{\theta=-\pi/2}^{\theta=\pi/2} + 12 [-\cos \theta]_{\theta=-\pi/2}^{\theta=\pi/2} + \frac{9}{2} \int_{-\pi/2}^{\pi/2} \frac{1 - \cos 2\theta}{2} d\theta = 8 \left[\left(\frac{\pi}{2} \right) - \left(-\frac{\pi}{2} \right) \right] + 12 \left[(-\cos \frac{\pi}{2}) - (-\cos -\frac{\pi}{2}) \right] + \frac{9}{4} \int_{-\pi/2}^{\pi/2} 1 d\theta - \frac{9}{4} \int_{-\pi/2}^{\pi/2} \cos 2\theta d\theta = 8\pi + 12(0 - 0) + \frac{9}{4} [\theta]_{\theta=-\pi/2}^{\theta=\pi/2} - \frac{9}{4} \left[\frac{1}{2} \sin 2\theta \right]_{\theta=-\pi/2}^{\theta=\pi/2} = 8\pi + \frac{9}{4} \left[\left(\frac{\pi}{2} \right) - \left(-\frac{\pi}{2} \right) \right] - \frac{9}{8} [\sin(2 \cdot \frac{\pi}{2}) - \sin(2 \cdot -\frac{\pi}{2})] = 8\pi + \frac{9}{4}\pi - \frac{9}{8} [\sin(\pi) - \sin(-\pi)] = \frac{41}{4}\pi - \frac{9}{8} [0 - (-0)] = \frac{41}{4}\pi \approx 32.201$

Answer to Exercise 5 (on page 13)

Answer: The area between the circles is approximately 0.96174.

Explanation: Examining the graph, we see that the region we are interested in is the area within $r = 4 \sin \theta$ from $\theta = 0$ to $\theta = \theta_i$ plus the area within $r = 2 \cos \theta$ from $\theta = \theta_i$ to $\theta = \frac{\pi}{2}$, where θ_i is the angle where the two curves intersect. Examine the graph below to see why this is true.



Setting the equations equal to each other to find θ_i :

$$4 \sin \theta_i = 2 \cos \theta_i$$

$$\frac{\sin \theta_i}{\cos \theta_i} = \tan \theta_i = \frac{2}{4}$$

$$\theta_i = \arctan 1/2 \approx 0.464$$

So, the total area between the circles is:

$$\begin{aligned}
 & \frac{1}{2} \int_0^{\theta_i} [4 \sin \theta]^2 d\theta + \frac{1}{2} \int_{\theta_i}^{\pi/2} [2 \cos \theta]^2 d\theta \\
 &= 8 \int_0^{\theta_i} \sin^2 \theta d\theta + 2 \int_{\theta_i}^{\pi/2} \cos^2 \theta d\theta \\
 &= 8 \int_0^{\theta_i} \frac{1}{2} (1 - \cos 2\theta) d\theta + 2 \int_{\theta_i}^{\pi/2} \frac{1}{2} (1 + \cos 2\theta) d\theta \\
 &= 4 \int_0^{\theta_i} (1 - \cos 2\theta) d\theta + \int_{\theta_i}^{\pi/2} (1 + \cos 2\theta) d\theta \\
 &= 4 \left[\theta - \frac{1}{2} \sin 2\theta \right]_{\theta=0}^{\theta=\theta_i} + \left[\theta + \frac{1}{2} \sin 2\theta \right]_{\theta=\theta_i}^{\theta=\pi/2} \\
 &= 4 \left[(\theta_i - 0) - \frac{1}{2} (\sin 2\theta_i - \sin 0) \right] + \left[\left(\frac{\pi}{2} - \theta_i \right) + \frac{1}{2} (\sin \pi - \sin 2\theta_i) \right] \\
 &= 4 \left[\theta_i - \frac{1}{2} \sin 2\theta_i \right] + \frac{\pi}{2} - \theta_i - \frac{1}{2} \sin 2\theta_i \\
 &= 4\theta_i - 2 \sin 2\theta_i + \frac{\pi}{2} - \theta_i - \frac{1}{2} \sin 2\theta_i = 3\theta_i - \frac{5}{2} \sin 2\theta_i + \frac{\pi}{2}
 \end{aligned}$$

Substituting $\theta_i = \arctan 1/2 \approx 0.464$:

$$= 3(0.464) - \frac{5}{2} \sin 0.927 + \frac{\pi}{2} \approx 0.96174$$

Answer to Exercise 6 (on page 18)

- 4200
- Logically, we can say that the population will increase if it is below the carrying capacity (that is, $P < 4200$), but we can also prove it mathematically: $\frac{dP}{dt} > 0 \rightarrow 1.2P \left(1 - \frac{P}{4200}\right) > 0 \rightarrow P \left(1 - \frac{P}{4200}\right) > 0$. Since we are talking about population, we can assume that $P > 0$ and continue: $1 - \frac{P}{4200} > 0 \rightarrow \frac{P}{4200} < 1 \rightarrow P < 4200$, which is the result we expected.
- Similarly, we know the population should be decreasing when P is greater than the carrying capacity of 4200.
- The equilibrium solutions can be found by setting $\frac{dP}{dt} = 0$ and solving. The solutions are $P(t) = 0$ and $P(t) = 4200$.

Answer to Exercise 7 (on page 19)

Recall that logistic differential equations are of the form $\frac{dy}{dt} = ky(1 - \frac{y}{m})$ where y is a function and t is the independent variable. (e) is the only logistic differential equation, with $m = 1$.

Answer to Exercise 8 (on page 21)

Multiply both sides by $(2y + \sin y) dx$:

$$\begin{aligned}(2y + \sin y) dy &= 3x^2 dx \\ \int 2y dy + \int \sin y dy &= \int 3x^2 dx \\ y^2 - \cos y &= x^3 + C\end{aligned}$$

There is no way to solve this for y with algebra, so we leave the solution in this implicit form.

Answer to Exercise 9 (on page 21)

1. Since $\frac{dB}{dt}$ depends only on B , we can use the given masses to find the rate of growth for each mass. $\frac{dB}{dt}(40) = \frac{1}{5}(100 - 40) = \frac{1}{5}(60) = 12$ and $\frac{dB}{dt}(70) = \frac{1}{5}(100 - 70) = \frac{1}{5}(30) = 6$. Since $\frac{dB}{dt}$ is greater when $B = 40$, the baby bird is gaining mass faster when it has a mass of 40 grams.
2. $\frac{d^2B}{dt^2} = \frac{d}{dt}(\frac{dB}{dt}) = \frac{d}{dt}[\frac{1}{5}(100 - B)] = \frac{1}{5}(-\frac{dB}{dt}) = -\frac{1}{5}[\frac{1}{5}(100 - B)] = -\frac{1}{25}(100 - B)$. For $20 < B < 100$, $\frac{d^2B}{dt^2} < 0$ and the graph of B should be concave down. The graph shown has a concave up portion, so it cannot represent $B(t)$.
3. $\frac{dB}{dt} = \frac{1}{5}(100 - B) \rightarrow \frac{dB}{100 - B} = \frac{1}{5}dt \rightarrow \int \frac{dB}{100 - B} = \int \frac{1}{5}dt \rightarrow -\ln(100 - B) = \frac{t}{5} + C \rightarrow e^{-\frac{t}{5} - C} = 100 - B \rightarrow ke^{-\frac{t}{5}} = 100 - B \rightarrow B(t) = 100 - ke^{-\frac{t}{5}}$. Setting $B(0) = 20$ to find k : $20 = 100 - ke^0 \rightarrow 20 = 100 - k \rightarrow k = 80$. So, the particular solution is $B(t) = 100 - 80e^{-\frac{t}{5}}$.

Answer to Exercise 10 (on page 22)

(a). Integrating gives $P(t) = 200t + C$, a line: the population gains 200 every unit of time. Choice (b) gives $100t^2 + C$ and (c) gives $\frac{100}{3}t^3 + C$, which are not linear. Choice (d) is exponential growth, and (e) blows up even faster.

Answer to Exercise 11 (on page 24)

The correct differential equation is $x'' + 100x = 0$. This is derived from the equation of motion for any simple harmonic oscillator: $mx'' + kx = 0$, where $m = 2 \text{ kg}$ and $k = 200 \text{ N/m}$. Dividing by m , we get $x'' + \frac{k}{m}x = 0$, which simplifies to $x'' + 100x = 0$.

Answer to Exercise 12 (on page 25)

- This system is damped. Specifically, it is underdamped because the amplitude of oscillation decreases over time.
- To make this system undergo simple harmonic motion, we would need to remove the damping term from the equation of motion. This means setting the damping coefficient c to zero in the differential equation, which would eliminate the exponential decay factor in the position function. The equation becomes $x(t) = 0.15e^{-0.40t} \cos(4t+0.30) \rightarrow x(t) = 0.15 \cos(4t + 0.30)$

Answer to Exercise 13 (on page 26)

1. Notice that the system equation is linear as it is defined with an x'' and x . The general solution to the differential equation $x'' + 16x = 0$ is of the form $x(t) = A \cos(4t) + B \sin(4t)$. This comes from the idea that the characteristic equation $r^2 + 16 = 0$ has roots $r = \pm 4i$, where the β term is 4. Using the initial conditions $x(0) = 0.20$ and $v(0) = x'(0) = 0$, we find our velocity equation to be $v(t) = x'(t) = -4A \sin(4t) + 4B \cos(4t)$. From $x(0) = A \cos 0 + B \sin 0 = A(1) + B(0) = A = 0.20$, we find that $A = 0.20$. From $v(0) = -4A \sin 0 + 4B \cos 0 = -4(0.20)(0) + 4B(1) = 4B = 0$, we find that $B = 0$. Therefore, the solution is:

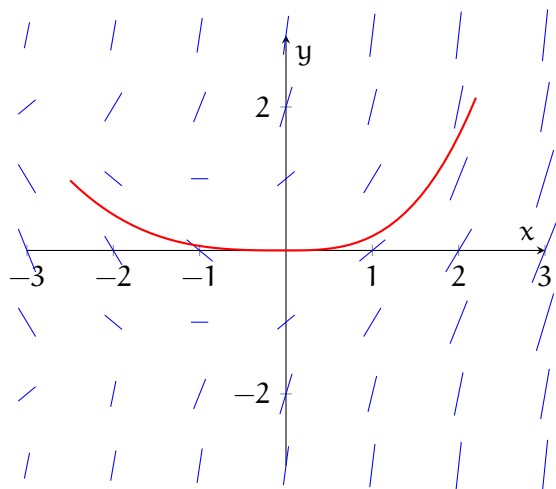
$$x(t) = 0.20 \cos(4t)$$

2. Differentiate once for velocity and again for acceleration:

$$v(t) = x'(t) = -0.80 \sin(4t)$$

$$a(t) = v'(t) = -3.20 \cos(4t)$$

Since cosine and sine functions oscillate between -1 and 1 (regardless of angular frequency $\omega = 4$), the maximum velocity is 0.80 m/s and the maximum acceleration is 3.20 m/s^2 .

Answer to Exercise 14 (on page 34)**Answer to Exercise 15 (on page 34)**

1. Setting $y(4 - y) = 0$ gives $y = 0$ and $y = 4$.
2. The slope is positive when $0 < y < 4$, so solutions increase there. It is negative when $y > 4$ or $y < 0$, so solutions decrease there.
3. Starting at 1, the solution rises and levels off toward 4. Starting at 6, it falls and levels off toward 4. Every solution that starts above 0 is drawn to $y = 4$, just like a population approaching its carrying capacity.

Answer to Exercise 16 (on page 34)

The slope depends only on x , so moving up or down a column does not change it. Each solution is a vertical shift of the others: integrating gives $y = \frac{1}{2}x^2 + C$, a family of parabolas.

Answer to Exercise 17 (on page 37)

The midpoint is 1.5 and $f(1.5) = 0.25 > 0$, so the sign change is on $[1, 1.5]$.

Answer to Exercise 18 (on page 37)

$$x_1 = 2 - \frac{2}{4} = 1.5, \text{ and } x_2 = 1.5 - \frac{0.25}{3} \approx 1.4167.$$

Answer to Exercise 19 (on page 37)

$f(-1) = 2$ and $f(1) = 2$ have the same sign, so the Intermediate Value Theorem does not guarantee a root in the interval. (The function has no real roots at all.)

Answer to Exercise 20 (on page 37)

After n steps the width is $\frac{1}{2^n}$. Since $2^{10} = 1024 > 1000$, you need 10 steps.

Answer to Exercise 21 (on page 43)

Substituting the given values, we find that $(5)\frac{dQ}{dt} + \frac{1}{0.05}Q = 60$. Solving for $\frac{dQ}{dt}$:

$$(5)\frac{dQ}{dt} + (20)Q = 60$$

$$\frac{dQ}{dt} + 4Q = 12$$

$$\frac{dQ}{dt} = 12 - 4Q$$

We also know that $Q(0) = 0$. Using Euler's method with step size $h = 0.1$, $Q(0.1) \approx Q(0) + h[12 - 4Q(0)] = 0 + 0.1[12 - 4(0)] = 1.2$. And $Q(0.2) \approx Q(0.1) + h[12 - 4Q(0.1)] = 1.2 + 0.1[12 - 4(1.2)] = 1.92$. And $Q(0.3) \approx Q(0.2) + h[12 - 4Q(0.2)] = 1.92 + 0.1[12 - 4(1.92)] = 2.352$. And $Q(0.4) \approx Q(0.3) + h[12 - 4Q(0.3)] = 2.352 + 0.1[12 - 4(2.352)] = 2.6112$. And finally, $Q(0.5) \approx Q(0.4) + h[12 - 4Q(0.4)] = 2.6112 + 0.1[12 - 4(2.6112)] = 2.76672$. Because we are finding a charge, the unit is coulombs (C), so our approximate answer is $Q(0.5) \approx 2.77C$. The exact solution is $Q(t) = 3(1 - e^{-4t})$, which gives $Q(0.5) \approx 2.59C$. Euler's method is high here because the charge curve bends downward, and the straight-line steps stay above it.

Answer to Exercise 22 (on page 44)

We are given $x_0 = 1$ and $x_2 = 1.4$. Therefore we will use step size $h = \frac{1.4-1}{2} = \frac{0.4}{2} = 0.2$. Taking $x_0 = 1$ and $y_0 = f(1) = 15$, we find y_1 : $y_1 = y_0 + h \cdot f'(x_0) = 15 + 0.2 \cdot f'(1) =$

$15 + 0.2(8) = 15 + 1.6 = 16.6$. And then $y_2 = y_1 + h \cdot f'(x_1) = 16.6 + 0.2 \cdot f'(1.2) = 16.6 + 0.2(12) = 16.6 + 2.4 = 19$. Therefore, $f(1.4) \approx 19$.

Answer to Exercise 23 (on page 45)

1. $\frac{2}{3}, \frac{4}{5}, \frac{8}{7}, \frac{16}{9}, \frac{32}{11}$
2. 0, -1, 0, 1, 0
3. 1, 2, 7, 32, 157
4. 6, 3, 1, $\frac{1}{4}, \frac{1}{20}$

Answer to Exercise 24 (on page 48)

1. convergent, 5
2. divergent
3. convergent, 2
4. convergent, -1
5. divergent

Answer to Exercise 25 (on page 54)

1. $a_2 \approx 1.732$, $a_3 \approx 1.932$, $a_4 \approx 1.983$.
2. If $a_k < 2$, then $2 + a_k < 4$, so $\sqrt{2 + a_k} < 2$.
3. By induction it is bounded above by 2 (and below by 1), and it is increasing, so it is monotonic and bounded and therefore converges. The terms approach 2. (If the limit is L , then $L = \sqrt{2 + L}$, which gives $L = 2$.)

Answer to Exercise 26 (on page 56)

Our principal is $P = 1500$ and the interest rate is $r = 0.05$. After n years, your investment will be worth $a_n = 1500(1.05)^n$. For $n = 10$, your investment will be valued at $a_{10} = 1500(1.05)^{10} \approx \$2443.34$ (that is over \$900 in interest). To determine if the sequence is

convergent or divergent, we examine the limit as $n \rightarrow \infty$:

$$\lim_{n \rightarrow \infty} 1500(1.06)^n = 1500 \cdot \lim_{n \rightarrow \infty} (1.05)^n = \infty$$

The sequence is divergent.

Answer to Exercise 27 (on page 57)

The number of fish in the pond is:

$$P_n = 0.95^6 P_{n-1} + 10$$

$$P_0 = 100$$

where n is the number of 6-month periods that have passed. The four-year period is given by $1 \leq n \leq 8$. The amount lost to predation every 6 months is given by $P_{n-1}(1 - 0.95^6)$.

n	Fish Population	Lost to Predators
0	100.0	
1	83.5	26.5
2	71.4	22.1
3	62.5	18.9
4	55.9	16.6
5	51.1	14.8
6	47.6	13.5
7	45.0	12.6
8	43.1	11.9

Adding up all the fish lost to predators, we find that over 4 years, the farmer loses about 137 fish.



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