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Chapter 1

Conditional Probability

Let's say there is a virus going around, and there is a vaccine for it that requires two shots. You are working at a school, and you are wondering how effective the vaccines are. Some students are unvaccinated, some have had one shot, and some have had two shots. One day, you test all 644 students to see who has the virus. You end up with the following table:

	V_0	V_1	V_2
T_+	88 students	36 students	96 students
T_-	92 students	76 students	256 students

Here is what each symbol means:

- V_0: student has had zero vaccination shots
- V_1: student has had one vaccination shot
- V_2: student has had both vaccination shots
- T_+: student tested positive for the virus
- T_-: student tested negative for the virus

So, for example, your data indicates that 76 students who had only one of the two shots and tested negative for the virus.

Your principal has a few questions. The first is, "If I put five randomly chosen students in a study group together, what is the probabiltiy that one of them has the virus?"

The first thing you might do is make a new table that shows what is the probability of a randomly chosen student being in any particular group. You just divide each entry by 644 (the total number of students).

	V_0	V_1	V_2
T_+	$p(V_0 \text{ AND } T_+) = 13.7\%$	$p(V_1 \text{ AND } T_+) = 5.6\%$	$p(V_2 \text{ AND } T_+) = 14.9\%$
T_-	$p(V_0 \text{ AND } T_-) = 14.3\%$	$p(V_1 \text{ AND } T_-) = 11.8\%$	$p(V_2 \text{ AND } T_-) = 39.8\%$

(In this table, we expressed the number as a percentage with a decimal point — you had to round off the numbers. If you wanted exact answers, you would have to keep each as a fraction: 36 students represents $\frac{9}{161}$ of the student body.)

1.1 Marginalization

Now we can sum across the columns and rows.

	V_0	V_1	V_2	sum
T_+	0.137	0.056	0.149	$p(T_+) = 0.342$
T_-	0.143	0.118	0.398	$p(T_+) = 0.547$
sum	$p(V_0) = 0.280$	$p(V_1) = 0.174$	$p(V_2) = 0.547$	

If a child is chosen randomly from the entire student body, there is a 34.2% that the student has tested positive for the virus, and there is 17.4% chance that the student has one shot of the vaccine.

This summing of the probabilities across one dimension is known as *marginalizing*. Marginalization is just summing across all the variables that you don't care about. If you don't care who has the virus, just the probability that a student has not received even one shot of the vaccine, you can simply marginalize all the vaccine statuses.

To answer the principal's question, the easy thing to do is find the answer of the opposite: "if I put five randomly chosen students in a study group together, what is the probabiltiy that *none* of them has tested positive for the virus?"

The chance that a randomly chosen student doesn't have the virus $(p(T_-)$ is 54.7%. This means the chance that 5 randomly chosen students don't have the virus is $0.547 \times 0.547 \times 0.547 \times 0.547 \times 0.547 = 0.0489$ Thus, the probability of the opposite is $1.0 - 0.0489 = 0.951$

The answer, then, is "If you put 5 kids in a study group together, there is a 95.1 % probability that at least one of them has the virus."

1.2 Conditional Probability

Now the principal asks you, "What if I make a group of five kids who have had both shots of the vaccine? What are the odds that one of them has tested positive for the virus?"

This involves the idea of *Conditional probability*. You want to know the odds that a student doesn't have the virus, given that the student has had both shots of the vaccine.

There is a mathematical notation for this:

$$p(T_-|V_2)$$

That is the probability that a student who has had both vaccination shots will test negative for the virus.

How would you calculate this? You would count all the students who had a positive test *and* both vaccination shots, which you would divide by the total number of students who had both vaccination shots.

$$p(T_-|V_2) = \frac{256}{96 + 256} = \frac{8}{11} \approx 72.7\%$$

If we are working from the probabilities, you can get the same result this way. Divide the probability that a randomly chosen student had a positive test *and* both vaccination shots by the probabiltiy that a student had both vaccination shots:

$$p(T_-|V_2) = \frac{p(T_- \text{ AND } V_2)}{p(T_-)} = \frac{0.398}{0.547} \approx 72.7\%$$

Notice that this is different from $p(V_2|T_-)$, which is the probability that a student has had both vaccinations, given they tested negative for the virus.

Back to the principal's question: "If you have five students who have had both vaccinations, what is the probability that all of them tested negative for the virus?" The probability that one student is virus-free is $\frac{8}{11}$, so the probability that five students are virus-free is $\frac{8}{11}^5 \approx 0.203$. So, there is a 79.6% chance that at least one of the five has the virus.

1.3 Chain Rule for Probability

You just used this equality: For any events A and B

$$p(A|B) = \frac{p(A \text{ AND } B)}{p(B)}$$

This is more commonly written like this:

$$p(A \text{ AND } B) = \frac{p(A|B)}{p(B)}$$

This is an abstract way of writing the idea, but the idea itself is pretty intuitive. The probability that you are going to buy a ticket and win the lottery is equal to the probability that you buy a ticket times the probability that you win, given that you have bought a ticket. (Here A is "win the lottery" and B is "buy a ticket".)

This is known as *The Chain Rule of Probability*. We can chain together as many events as we want: The probability that you are going to die in the car that you bought with your winnings from the lottery ticket you bought is:

$$p(W \text{ AND } X \text{ AND } Y \text{ AND } Z) = p(W|X \text{ AND } Y \text{ AND } Z)p(X|Y \text{ AND } Z)p(Y|Z)p(Z)$$

where

- W = Dying in car accident
- X = Buying a car with lottery winnings
- Y = Winning the lottery
- Z = Buying a lottery ticket

In English, the equation says:

"The probability that you will die in a car accident, buy a car with lottery winnings, win the lottery, and buy a lottery ticket is equal to the probability that you buy a lottery ticket times the probability that you win the lottery (given that you have bought a ticket) times the probability that buy a car with those lottery winnings (given that bought a ticket and won) times the probability that you crash that car (given that you have bought the car, won the lottery, and bought a ticket)."

Chapter 2

Bayes' Theorem

Let's say that you are holding two bags of marbles. You know that one bag contains 60 white marbles and 40 red marbles, and you know that the other holds 10 white marbles and 90 red marbles. You don't know which is which, and you can't see the marbles.

Your friend says, "Guess which bag is mostly red marbles." You pick one.

"What is the probability that this is the bag that is mostly red marbles?" You think to yourself "There is a 50 percent chance that this bag is mostly red marbles, and there is also a 50 percent probability that it is the mostly-white-marbles bag."

You then pick one marble from the bag: it is red. Now you must update your beliefs. It is more likely that this is the mostly-red-marbles bag. What is the probability now?

Bayes Theorem gives you the rule for updating your beliefs based on new data.

2.1 Bayes Theorem

Let's say you have two events or conditions C and D. C is "The person has a cough" and D is "The person is waiting to see a doctor."

Using the chain rule of probability, we now have two ways to calculate $p(C \text{ AND } D)$:

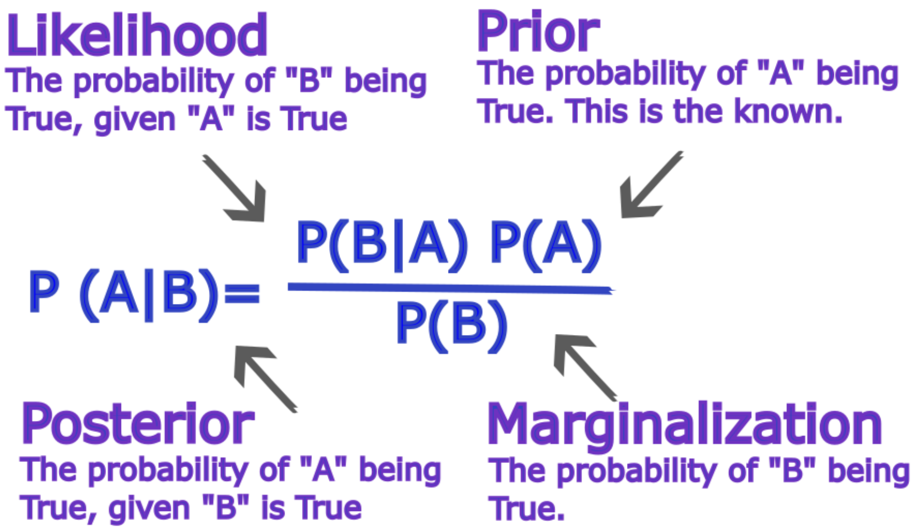
$$p(C \text{ AND } D) = p(C|D)p(D)$$

(The probability the person is at the doctor multiplied by the probability they have a cough if they are at the doctor.)

or

$$p(C \text{ AND } D) = p(C|D)p(D)$$

(The probabilitiy the person has a cough multiplied by the probabilitiy they are at the doc-



tor if they have a cough.)

Thus:

$$p(D|C) = \frac{p(C|D)p(D)}{P(C)}$$

Now, you can calculate $p(D|C)$ (in this case, the probability that you are waiting to see a doctor given that you have a cough.) if you know:

- $p(C|D)$ (The probability that you have a cough given that you are waiting to see a doctor)
- $p(D)$ (The probability that you are waiting for a doctor for any reason.)
- $p(C)$ (The probability that you have a cough anywhere)

Pretty much all modern statistical methods (including most artificial intelligence) are based on this formula, which is known as Bayes' Theorem. It was written down by Thomas Bayes before he died in 1761. It was then found and published after his death.

2.2 Using Bayes' Theorem

Back to the example at the beginning. To review:

- There are two bags that look exactly the same.
- Bag W has 60 white marbles and 40 red marbles.

- Bag R has 10 white marbles and 90 red marbles.
- You pull one marble from the selected bag: it is red.

What is the probability that the selected bag is Bag R? Intuitively, you know that the probability is now more than 0.5. What is the exact number?

In terms of conditional probability, we say we are looking for "the probability that the selected bag is Bag R, given that you drew a red marble", or $p(B_R|D_R)$, where B_R is "the selected bag is Bag R" and D_R is "you drew a red marble from the selected bag".

From Bayes' Theorem, we can write:

$$p(B_R|D_R) = \frac{P(D_R|B_R)P(B_R)}{P(D_R)}$$

$P(D_R|B_R)$ is just the probability of drawing a red marble given that the selected bag is Bag R. That is easy to calculate: There are 100 marbles in the bag, and 90 are red. Thus, $P(D_R|B_R) = 0.9$.

$P(B_R)$ is just the probability that you chose Bag R before you drew out a marble. Both bags look the same, so $P(B_R) = 0.5$. This is called *the prior*, because it represents what you thought the probability was before you got more information.

$P(D_R)$ is the probability of drawing a red marble. There was 0.5 probability that you put your hand into Bag W (in which 40 of the 100 marbles are red) and a 0.5 probability that you put your hand into Bag R (in which 90 of the 100 marbles are red). So

$$P(D_R) = 0.5\frac{40}{100} + 0.5\frac{90}{100} = 0.65$$

Putting it together:

$$p(B_R|D_R) = \frac{P(D_R|B_R)P(B_R)}{P(D_R)} = \frac{(0.9)(0.5)}{0.65} = \frac{9}{13} \approx 0.69$$

Thus, given that you have pulled a red marble, there is about a 69% chance that you have selected the bag with 90 red marbles.

2.3 Confidence

Bayes' Theorem, then, is about updating your beliefs based on evidence. Before you drew out the red marble, you selected one bag thinking it might contain 90 red marbles. How certain were you? 0.0 being complete disbelief and 1.0 entirely confidence, you were 0.5. After pulling out the red marble, you were about 0.69 confident that you had chosen the bag with 90 red marbles.

The question "How confident are you in your guess?" is very important in some situations. For example in medicine, diagnoses often lead to risky interventions. Few diagnoses come with 100% confidence. All doctors should know how to use Bayes' Theorem.

In a trial, a jury is asked to determine if the accused person is guilty of a crime. Few jurors are ever 100% certain. In some trials, Bayes' Theorem is an exceptionally important tool.

Chapter 3

Antiderivatives

In your study of calculus, you have learned about derivatives, which allow us to find the rate of change of a function at any given point. Derivatives are powerful tools that help us analyze the behavior of functions. Now, we will explore another concept called antiderivatives, which are closely related to derivatives.

An antiderivative, also known as an integral or primitive, is the reverse process of differentiation. It involves finding a function whose derivative is equal to a given function. In simple terms, if you have a function and you want to find another function that, when differentiated, gives you the original function back, you are looking for its antiderivative. Consider the graph of $f(x)$ below. We can sketch a possible antiderivative of f by noting the slope of the antiderivative is equal to the value of f. We will refer to the antiderivative of f as $F(x)$ (that is, $F'(x) = f(x)$).

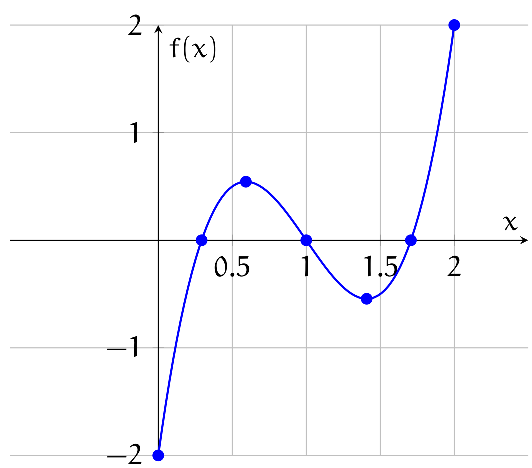


Figure 3.1: Plot of f with select points

If we are given a coordinate for $F(x)$, then we can use the graph of $f(x)$ to sketch $F(x)$. Suppose we know that $F(0) = 1$. From the graph of f, we also know that

x	$f(x) =$ slope of $F(x)$
0	-2
≈ 0.3	0
≈ 0.6	≈ 0.5
1	0
≈ 1.4	≈ -0.4
≈ 1.7	0
2	2

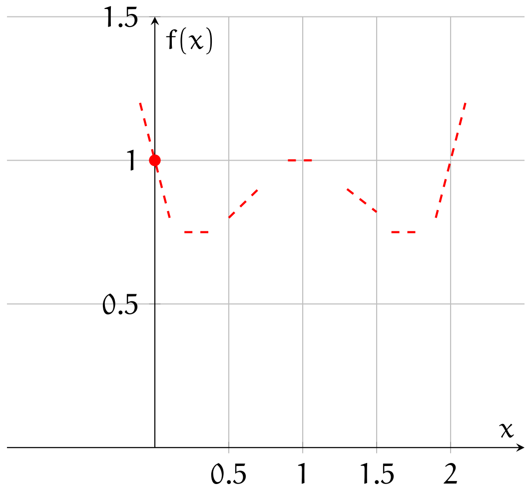


Figure 3.2: Beginning sketch of $F(x)$

We can then connect these slopes to have an approximate sketch of $F(x)$:

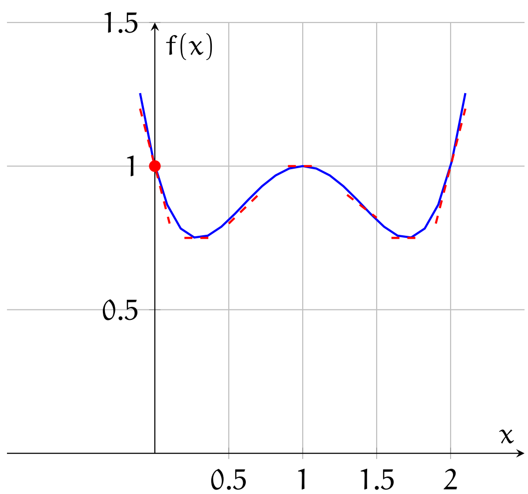


Figure 3.3: Sketch of $F(x)$

The symbol used to represent an antiderivative is $\int$. It is called the integral sign. For example, if $f(x)$ is a function, then the antiderivative of $f(x)$ with respect to x is denoted as $\int f(x)\,dx$. The dx at the end indicates that we are integrating with respect to x.

Another way to state this is that F is the antiderivative of f on an interval, I, if $F'(x) = f(x)$ over the interval. The relationship between f and F is discussed more in the chapter on the Fundamental Theorem of Calculus.

Finding antiderivatives requires using specific techniques and rules. Some common antiderivative rules include:

- The power rule: If $f(x) = x^n$, where n is any real number except -1, then the antiderivative of $f(x)$ is given by $\int f(x)\,dx = \frac{1}{n+1}x^{n+1} + C$, where C is the constant of integration.
- The constant rule: The antiderivative of a constant function is equal to the constant times x. For example, if $f(x) = 5$, then $\int f(x)\,dx = 5x + C$.
- The sum and difference rule: If $f(x)$ and $g(x)$ are functions, then $\int (f(x) + g(x))\,dx = \int f(x)\,dx + \int g(x)\,dx$. Similarly, $\int (f(x) - g(x))\,dx = \int f(x)\,dx - \int g(x)\,dx$.

Antiderivatives have various applications in mathematics and science. They allow us to calculate the total accumulation of a quantity over a given interval, compute areas under curves, and solve differential equations, among other things.

3.1 General Antiderivatives

It is important to note that an antiderivative is not a unique function. Since the derivative of a constant is zero, any constant added to an antiderivative will still be an antiderivative of the original function. This is why we include the constant of integration, denoted by C, in the antiderivative expression.

Stated formally, if F is an antiderivative of f on interval I, then the most general antiderivative of f on I is $F(x) + C$, where C is an arbitrary constant.

A concrete example of this is $f(x) = x^2$. Let us define $F(x)$ such that $F'(x) = f(x)$. That is, there is some function F such that the derivative of F is x^2. One possible solution for F is $F(x) = \frac{1}{3}x^3$. You can check using the power rule that $\frac{d}{dx}F(x) = f(x)$. What if we added or subtracted a constant from F? Let's define $G(x) = \frac{1}{3}x^3 + 2$. Well, $G'(x) = f(x)$ also! The same applies for $H(x) = \frac{1}{3}x^3 - 7$. Several possible antiderivatives of $f(x) = x^2$ are shown in figure 3.4.

Since taking a derivative "erases" any constant, you must always add back in the unknown constant, C, when finding the general antiderivative.

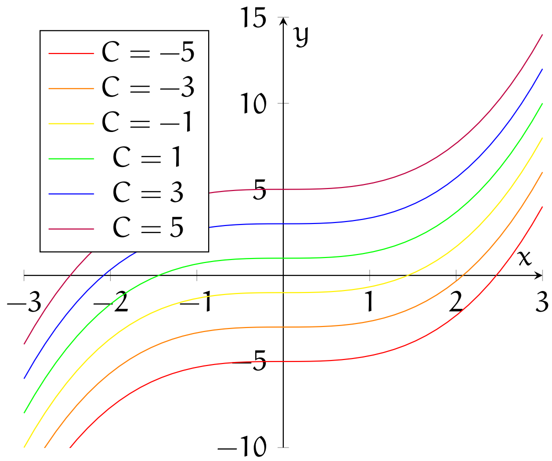


Figure 3.4: If $F'(x) = x^2$, then the general solution is $F(x) = \frac{1}{3}x^3 + C$

3.2 Specific Antiderivatives

If you are given a condition, you can often solve for C and find a specific antiderivative. For example, suppose that in addition to knowing that $F'(x) = x^2$, we also know that $F(3) = 2$. We can use the fact that F passes through $(3, 2)$ to find the value of C:

$$F(x) = \frac{1}{3}x^3 + C$$

$$F(3) = \frac{1}{3}(3)^2 + C = 2$$

$$39 + C = 2$$

$$C = -7$$

Therefore, the specific solution to $F'(x) = x^2$ with the condition that $F(3) = 2$ is $F(x) = \frac{1}{3}x^3 - 7$.

3.3 Antiderivatives of Trig Functions

We already know that $\frac{d}{dx} \sin x = \cos x$. Taking $\sin x$ to be $F(x)$ and $\cos x$ to be $f(x)$, we see that $F'(x) = f(x)$ and therefore, $\sin x$ is the antiderivative of $\cos x$.

Exercise 1

What is the antiderivative of $\sin x$? Explain your answer.

Working Space

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You should have found that the antiderivative of $\sin x$ is $-\cos x$. Other general antiderivatives of trigonometric functions are presented in the table below.

Function	Antiderivative
$\cos x$	$\sin x + C$
$\sin x$	$-\cos x + C$
$\sec^2 x$	$\tan x + C$
$\sec x \tan x$	$\sec x + C$
$-\csc^2 x$	$\cot x + C$
$-\csc x \cot x$	$\csc x + C$

Notice this is the flipped version of the derivatives of trigonometric functions presented in the Trigonometric Functions chapter. This hints at the relationship between derivatives and integrals: they are opposite processes.

3.4 Other Important Antidervatives

The power rule only applies when $n \neq -1$. So, what is the antiderivative of $f(x) = \frac{1}{x}$? Recall from the chapters on derivatives that $\frac{d}{dx} \ln x = \frac{1}{x}$ (see figure 3.5). Therefore, the general antiderivative of $\frac{1}{x}$ is $\ln |x| + C$. We have to take the absolute value because of the domain restrictions of $\ln x$. Notice that for $x < 0$, the slope of $\ln |x|$ is negative and decreasing (becoming more negative), and the value of $\frac{1}{x}$ is also negative and decreasing. Similarly, for $x > 0$, the slope of $\ln |x|$ is positive and decreasing (becoming less positive) and the value of $\frac{1}{x}$ is also positive and decreasing.

Since the derivative of e^x is e^x, it follows that the general antiderivative of e^x is $e^x + C$. What if there is a multiplying factor in the exponent, such as e^{kx}? Recall that $\frac{d}{dx} e^{kx} = ke^{kx}$. It follows that $\frac{d}{dx} \frac{1}{k} e^{kx} = e^{kx}$. Therefore, the general antiderivative of e^{kx} is $\frac{1}{k} e^{kx} + C$. (See figure 3.6 for an example where $k = 2$.)

Often, the base of an exponential function is not e. We can also find the general antiderivative of b^x, where $b \neq e$. Recall that $\frac{d}{dx} b^x = \ln b b^x$. Therefore, $\frac{d}{dx} \frac{1}{\ln b} b^x = b^x$, and

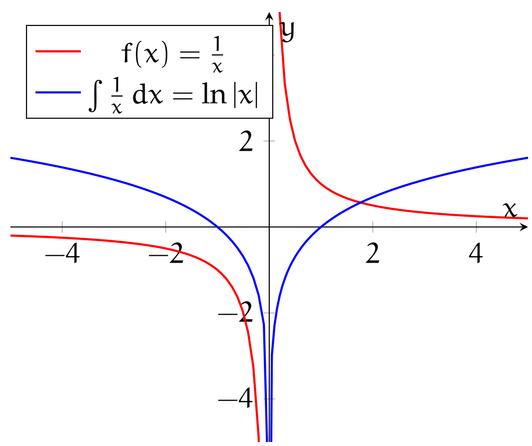

Figure 3.5: $\frac{1}{x}$ and its antiderivative, $\ln|x|$

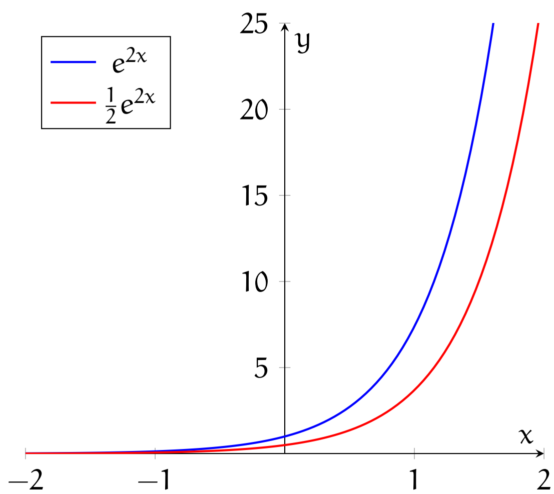

Figure 3.6: e^{2x} and its antiderivative $\frac{1}{2}e^{2x}$

the general antiderivative of b^x is $\frac{b^x}{\ln b}$.

3.5 Higher order antiderivatives

What if we are given the second order derivative, or a higher order? Take this example: $f''(x) = 2x + 3e^x$. The antiderivative of f'' is f'. Applying the power rule and knowing the antiderivative of e^x is e^x, we find that $f'(x) = x^2 + 3e^x + C_1$. We designate the constant as C_1, because we will have to determine the antiderivative a second time and we don't want to confuse our constants with each other. To find f, we apply the power rule again, and we find that $f(x) = \frac{1}{3}x^3 + 3e^x + C_1x + C_2$. You can check if this is correct by taking the derivative of $f(x)$ twice, which should yield the $f''(x)$ originally given.

In summary, antiderivatives are the reverse process of differentiation. They help us find functions whose derivatives match a given function. Understanding antiderivatives is crucial for various advanced calculus concepts and real-world applications.

Now, let's explore different techniques and methods for finding antiderivatives and discover how they can be applied in solving problems.

3.6 Additional Practice

Exercise 2

A particle moving in a straight line has an acceleration given by $a(t) = 6t+4$ (in units of $\frac{cm}{s^s}$). If its initial velocity is $-6\frac{cm}{s}$ and its initial position is $9cm$, what is the function $s(t)$ that describes the particle's position in cm?

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Exercise 3

Let $f'(x) = 2\sin x$. If $f(\pi) = 1$. Write an expression for $f(x)$.

Working Space

Answer on Page 39

Exercise 4

Find the general antiderivatives of the following functions:

1. $f(x) = x^2 + 2x - 4$
2. $g(x) = \sqrt[3]{x^2} + x\sqrt{x}$
3. $h(x) = \frac{1}{5} - \frac{2}{x}$
4. $r(\theta) = 2\sin\theta - \sec^2\theta$

Working Space

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Exercise 5

Find the f that satisfies the given conditions:

1. $f'(\theta) = \sin\theta + \cos\theta$, $f(\pi) = 2$
2. $f''(x) = 12x^2 + 6x - 4$, $f(0) = 4$ and $f(1) = 1$

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Chapter 4

Riemann Sums

4.1 The Meaning of the Area Under a Function

Let's look at the example of a hammer tossed in the air from a previous chapter. As you may recall, if a hammer is tossed up from the ground at 5 m/s, its velocity can be described as $v(t) = 5 - 9.8t$ (on Earth, where the acceleration due to gravity is approximately -9.8 $\frac{m}{s^2}$). The velocity function of our hammer from when it is tossed ($t = 0$) to when it hits the ground $t \approx 1.02$) is shown in figure 4.1.

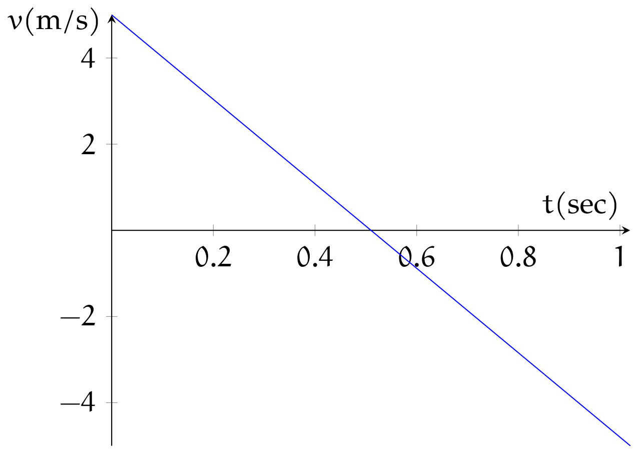


Figure 4.1: Velocity of a hammer thrown upwards at 5 m/s

Now, suppose we only have this velocity function, and we want to know how high above its initial position the hammer is tossed. Examine the graph: At approximately what time does the hammer reach its peak height? (Hint: what should the hammer's velocity be when it reaches its peak?). At the highest point of its flight, the hammer's velocity will be $0\ \frac{m}{s}$, which occurs at approximately $t = 0.5s$ (it's actually $t = 0.5102s$ but we don't need to be that precise for this example).

Now that we know *when* the hammer reaches its peak, how can we determine *how high* that peak is? Recall that velocity is the slope of the position-time graph. Since slope is change in position divided by change in time (in this case, as time is on the x-axis and position on the y-axis), then the slope must have units of [position]/[time] which could be $\frac{m}{s}$, $\frac{\text{miles}}{\text{hr}}$, and so on. These are units of velocity!

In figure 4.1, you can see that the units on the x-axis are seconds, and on the y-axis, the units are $\frac{m}{s}$. If we are looking for a *displacement* (that is, how far from its initial position the hammer has traveled), we are looking for a solution with units of meters. To yield an answer with those units, we wouldn't use the slope of the graph; this would yield an answer with units $\frac{m}{s^s}$, the units for acceleration. Instead, we need to *multiply*! The area between the velocity function and the x-axis (see figure 4.2) can be found this way:

$$Area = \frac{1}{2}bh$$

where b is the base of the triangle and h is the height.

$$Area = \frac{1}{2}(0.5s)(5\frac{m}{s})$$

$$Area = 1.25m$$

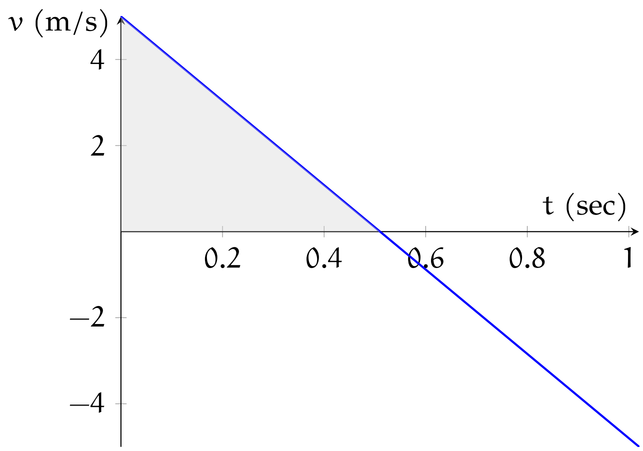


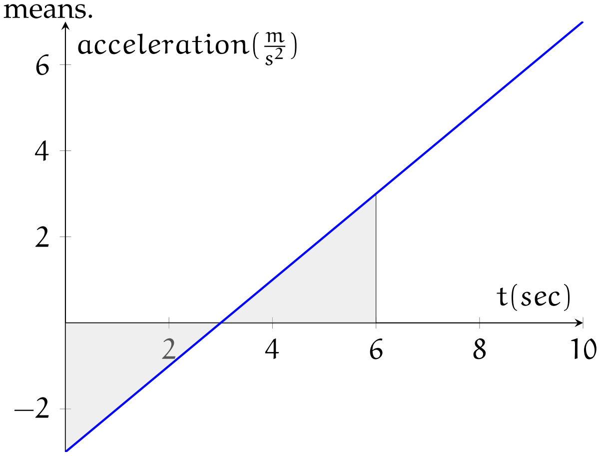
Figure 4.2: The area under $v(t)$ from $x = 0$ to $x = 0.5$ is equal to the displacement of the hammer

Notice that when multiplying the change in time (0.5 s) by the change in velocity (1.25 $\frac{m}{s}$), the seconds units cancel, yielding a result with units of meters. Therefore, the hammer reaches a peak height of ≈ 1.25 m, which you can confirm by examining the graph originally presented for the hammer toss in the chapter on graph shape.

4.1.1 Determining the Meaning of the Area with Units

Exercise 6

What units will the area shown in the graph have? Based on your answer, does the area represent a displacement, a net change in velocity, or a net change in acceleration? Calculate the shaded area [hint: areas below the x-axis are negative]. Write a sentence in plain English explaining what the are you calculated means.



Working Space

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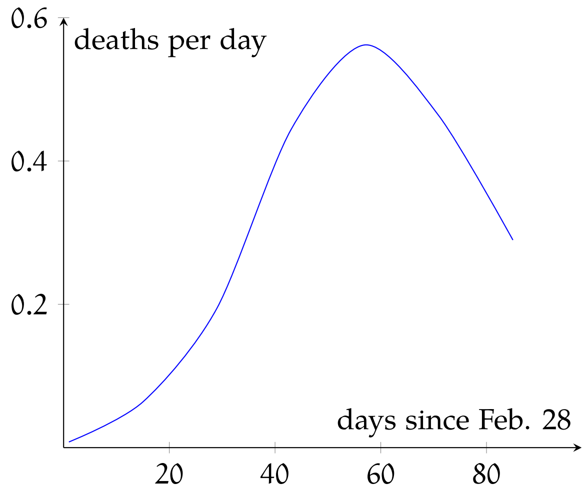
Exercise 7

The graph below shows historical data of the number of deaths due to SARS in Singapore over several months in 2003. What would the area under the curve represent?



Working Space

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Exercise 8

Oil leaked from a tank at a rate of $r(t)$ liters per hour. A site engineer recorded the leak rate over a period of 10 hours, shown in the table. Plot the data. How could you estimate the total volume of oil lost?

t(h)	0	2	4	6	8	10
$r(t)$(L/h)	8.7	7.6	6.8	6.2	5.7	5.3

Working Space

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4.2 Estimating the area under functions

In the hammer example above, it was easy to determine the area under the function, since the area took the shape of a triangle. However, what about finding the area under a more complex function, such as $f(x) = sin x + x$ (shown in figure 4.3)?

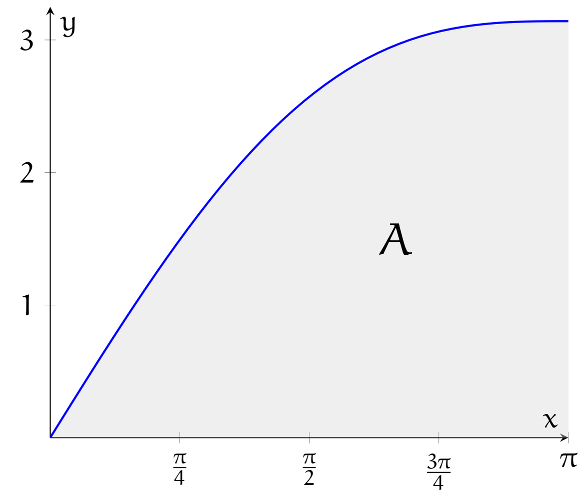

Figure 4.3: $f(x) = \sin x + x$

How can we determine the area under $f(x) = \sin x + x$ from $x = 0$ to $x = \pi$? We can *estimate* the area of that region by dividing the region into rectangles, finding the areas of the rectangles, and adding the areas. As an example, we will divide the region under $f(x) = \sin x + x$ into 4 intervals, shown in figure 4.4.

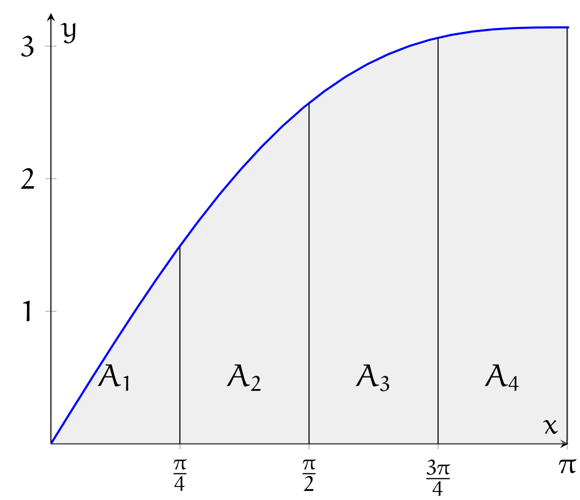

Figure 4.4: $f(x) = \sin x + x$ divided into 4 regions

As you can see in figure 4.4, each rectangle will have a width of $\frac{\pi}{4}$. But what about the height? One way is to use the value of the function at the rightmost value of each rectangle, as shown in figure 4.5.

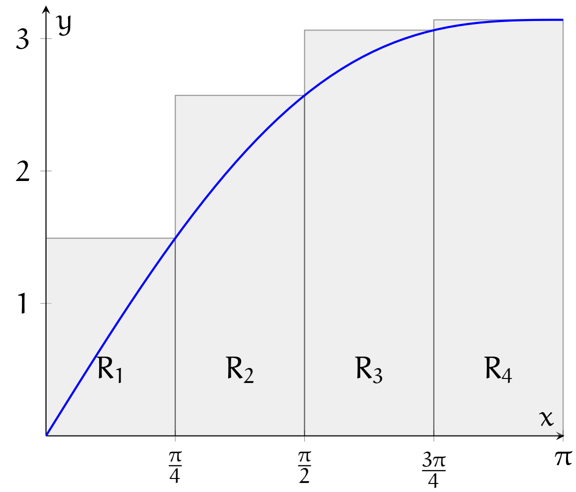

Figure 4.5: Four rectangle sections with heights determined by rightmost value of $f(x)$ on each interval

We can easily calculate the areas of each of these rectangles:

$$\frac{\pi}{4} \times f(\frac{\pi}{4}) + \frac{\pi}{4} \times f(\frac{\pi}{2}) + \frac{\pi}{4} \times f(\frac{3\pi}{4}) + \frac{\pi}{4} \times f(\pi)$$

$$\approx \frac{\pi}{4} \times (1.4925 + 2.5708 + 3.0633 + 3.1416) = 8.0646$$

Based on figure 4.5, will the calculated area be an overestimate or an underestimate? Each of the rectangles overshoots the function, so this will be an overestimate. What about using the leftmost value of $f(x)$ of each interval to determine the height of the rectangles? This is shown in figure 4.6.

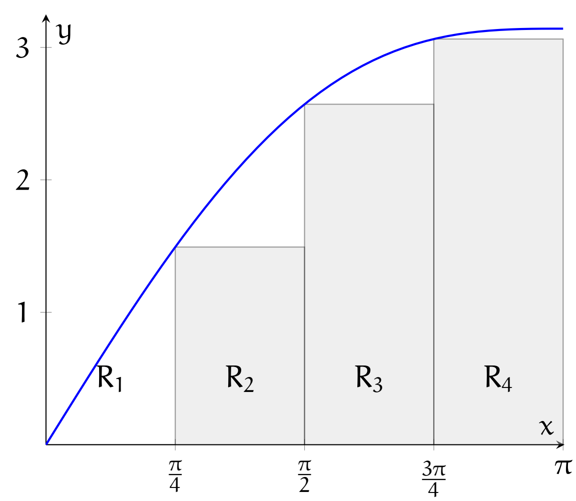

Figure 4.6: Four rectangle sections with heights determined by leftmost value of $f(x)$ on each interval

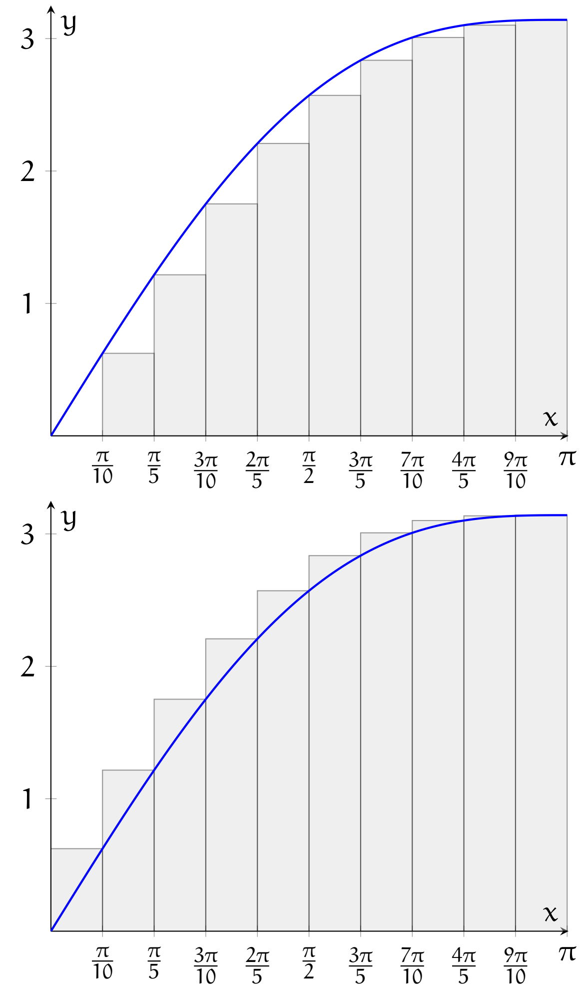

Figure 4.7: $\sin x + x$ broken into 10 intervals using either the left or right value to determine the height.

Notice that because $f(0) = 0$, the height of the first rectangle is zero, so we don't see it on the graph. To find the area of these rectangles:

$$\frac{\pi}{4} \times f(0) + \frac{\pi}{4} \times f(\frac{\pi}{4}) + \frac{\pi}{4} \times f(\frac{\pi}{2}) + \frac{\pi}{4} \times f(\frac{3\pi}{4})$$

$$\approx \frac{\pi}{4} \times (0 + 1.4925 + 2.5708 + 3.0633) = 5.5972$$

This is an underestimate. Therefore, the true value of the area under $f(x) = \sin x + x$ is between 5.5972 and 8.0646. This is an awfully wide window! We can narrow our estimate by increasing the number of intervals. Graphs of $f(x)$ with 10 intervals are shown in figure 4.7.

The total area for the left-determined rectangles is ≈ 6.4248, and for the right-determined, it is ≈ 7.4118. Therefore, we have narrowed the range for the true area under the curve to $6.4248 < A < 7.4118$. In general, as you increase the number of intervals, you get closer to the true area.

For a strictly increasing function, the right sum will be an overestimate and the left sum will be an underestimate of the true area under the curve. In the exercise below, you will examine a strictly decreasing function:

Exercise 9

Estimate the area under the graph of $f(x) = \frac{1}{x}$ from $x = 1$ to $x = 2$ using four rectangles and right endpoints. Sketch the graph and the rectangles. Is your estimate an overestimate or an underestimate? Repeat using left endpoints.

Working Space

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You should have found that for the strictly decreasing function $f(x) = \frac{1}{x}$, the right-determined sum is an *underestimate*, while the left-determined sum is an *overestimate*.

4.3 The Riemann Sum

In the previous section, we estimated the area under functions by dividing the area into approximating rectangles. This method is called a *Riemann Sum*. We will use a general example to formally define the Riemann sum. Consider a generic function divided into strips of equal width (shown in figure 4.8). The width of each strip is

$$\Delta x = \frac{b-a}{n}$$

, where a is the left endpoint of the interval, b is the right endpoint of the interval, and n is the number of strips. So, the right endpoints of the sections are

$$x_1 = a + \Delta x$$

.

$$x_2 = a + 2\Delta x$$

$$\ldots\ x_n = a + n\Delta x$$

As above, we can use the value of the function to determine the height of a rectangle whose area approximates the area of the section. (E.g. for the i^{th} strip, the width is Δx and the height is $f(x_i)$, see figure 4.9). So, the total area approximated by the rectangles is

$$R_n = f(x_1)\Delta x + f(x_2)\Delta x + \ldots + f(x_n)\Delta x$$

. This is the formal definition of the right Riemann sum. You can also take a left Riemann sum or a midpoint Riemann sum, as discussed below.

4.3.1 Right Riemann Sums

As seen above, a right Riemann sum uses the right-most value of $f(x)$ to determine the height of the rectangle (an example is shown in figure 4.10). We will refer to the right Riemann sum as R_n, where n is the number of intervals.

4.3.2 Left Riemann Sums

When taking a left Riemann sum, the height of the rectangle is determined by the value of the function at the lower (left-most) x-value. See figure 4.11. We will refer to left Riemann sums as L_n, where n is the number of intervals. So, the total area approximated by a Left

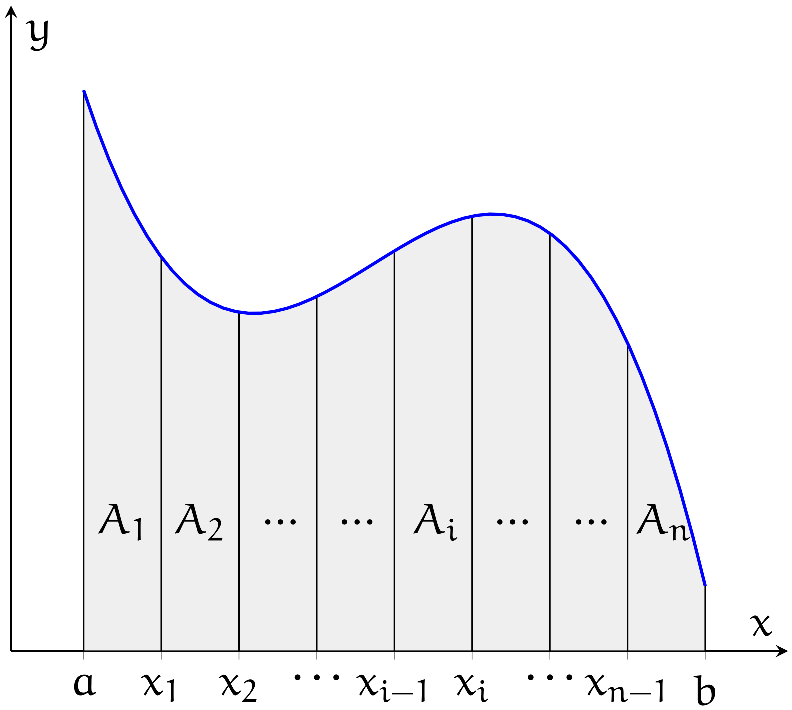


Figure 4.8: A representative function divided into n strips of equal width

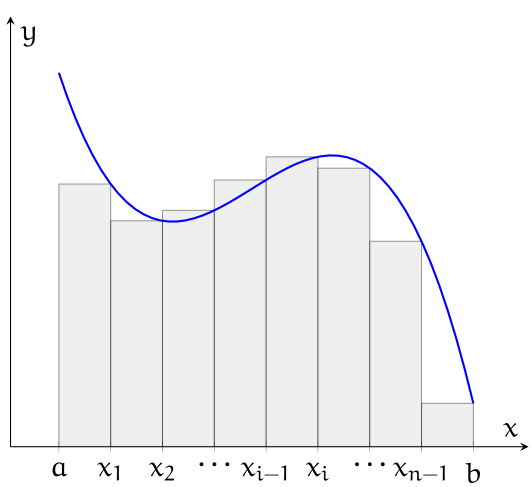


Figure 4.9: A representative function divided into n rectangles of equal width, with rectangle height determined by the right endpoint of the subinterval

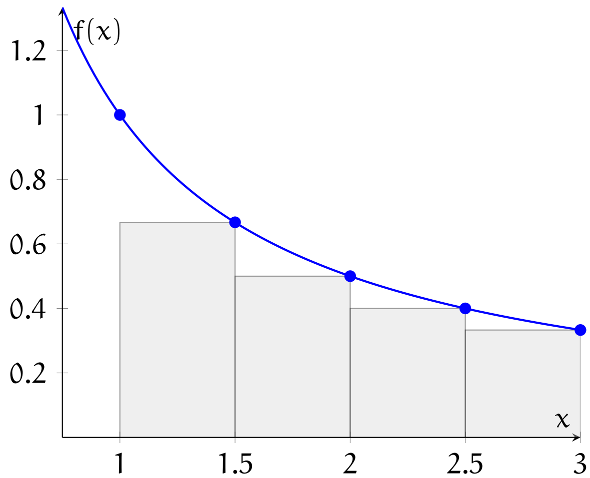


Figure 4.10: R_4 for $f(x) = \frac{1}{x}$

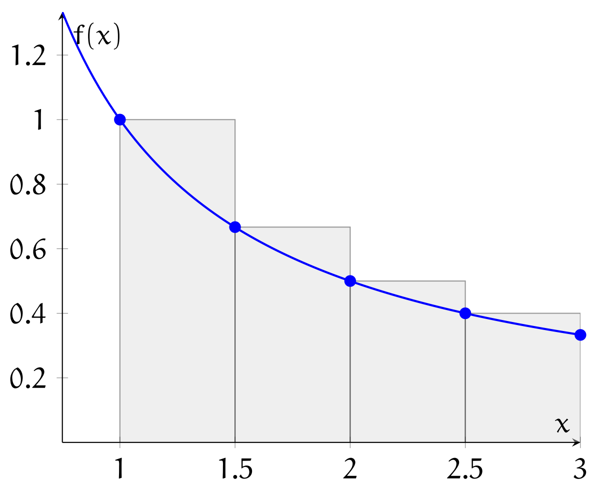


Figure 4.11: L_4 for $f(x) = \frac{1}{x}$

Riemann sum is

$$L_n = f(x_0)\Delta x + f(x_1)\Delta x + \ldots + f(x_{n-1})\Delta x$$

.

4.3.3 Midpoint Riemann Sums

A midpoint Riemann sum uses the value of $f(x)$ at the midpoint of the division to determine the height of the rectangle, as shown in figure 4.12. We will refer to the midpoint Riemann sum as M_n, where n is the number of intervals. So, the total area approximated

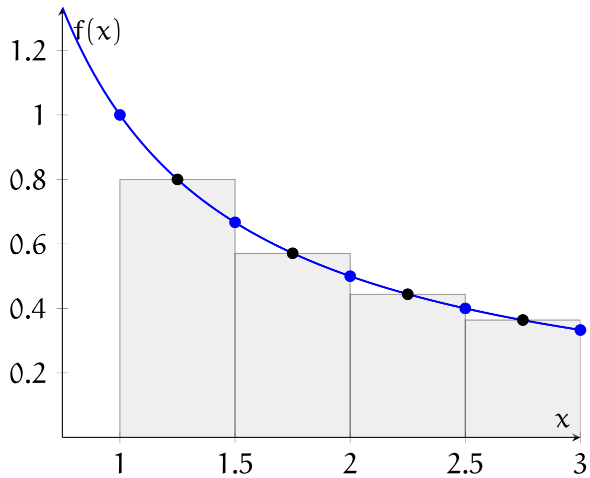


Figure 4.12: M_4 of $f(x) = \frac{1}{x}$

by the rectangles is

$$M_n = f(\frac{x_0 + x_1}{2})\Delta x + f(\frac{x_1 + x_2}{x})\Delta x + \ldots + f(\frac{x_{n-1} + x_n}{2})\Delta x$$

.

4.3.4 Riemann sum sigma notation

As you may recall, mathematicians use sigma notation to concisely express sums, such as Riemann sums. We can rewrite the definition of a right Riemann sum in sigma notation:

$$\sum_{i=1}^{n} f(x_i)\Delta x$$

where n is the number of subintervals. So, the actual area under the curve is the limit as n approaches ∞ of the above sum. Let's apply this by writing a sum that represents the area, A, of the region that lies between the x-axis and the function $f(x) = e^{-x}$ from $x = 0$ to $x = 2$.

First, we find an expression for Δx:

$$\Delta x = \frac{2-0}{n} = \frac{2}{n}$$

Recall that $x_i = a + i\Delta x$. Since a (the beginning of the interval) $= 0$, then the general expression for x_i in this case is $0 + i \times \frac{2}{n} = \frac{2i}{n}$. Substituting our expressions for Δx and x_i into the sum formula, we see that:

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} e^{\frac{-2i}{n}} \frac{2}{n}$$

We can also interpret a sum as the area under a specific function. Take the expression:

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{\pi}{n} \sin \frac{i\pi}{n}$$

There are two expressions in the sum: $\frac{\pi}{n}$ and $\sin \frac{i\pi}{n}$. It makes sense that $\Delta x = \frac{\pi}{n}$ and $f(x_i) = \sin \frac{i\pi}{n}$. Because $\Delta x = \frac{b-a}{n} = \frac{\pi}{n}$, it follows that the interval of the area has a width of π. We will need to examine the other expression, $\sin \frac{i\pi}{n}$, to determine an exact window.

Since $f(x_i) = \sin \frac{i\pi}{n}$, it follows that the function we are looking for is a sine function. Further, the expression for $x_i = \frac{i\pi}{n}$. Recall that $x_i = a + i\Delta x$, where a is the left-most boundary of the interval. Substituting what we have found already, we see that:

$$x_i = a + i\frac{\pi}{n} = \frac{i\pi}{n}$$

which implies that $a = 0$. Since we have established the interval is π wide, we can infer that $b = \pi$. Therefore, the limit $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{\pi}{n} \sin \frac{i\pi}{n}$ is equal to the area under $f(x) = \sin x$ from $x = 0$ to $x = \pi$.

Exercise 10

Use the formal definition of a Right Riemann sum to write a limit of a sum that is equal to the total area under the graph of f on the specified interval. Do not evaluate the limit.

1. $f(x) = \frac{2x}{x^2+1}$, $1 \le x \le 3$
2. $f(x) = x^2 + \sqrt{1 + 2x}$, $4 \le x \le 7$
3. $f(x) = \sqrt{\sin x}$, $0 \le x \le \pi$

Working Space

Answer on Page 42

Event	Time (s)	Velocity (ft/s)
Launch	0	0
Begin roll maneuver	10	185
End roll maneuver	15	319
Throttle to 89%	20	447
Throttle to 67%	32	742
Throttle to 104%	59	1325
Maximum dynamic pressure	62	1445
Solid rocket booster separation	125	4151

Figure 4.13: Speed of *Endeavour* from launch to booster separation

Exercise 11

Use the formal definition of a Right Riemann sum to find a region on a graph whose are is equal to the given limit. Do not evaluate the limit.

1. $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{3}{n}\sqrt{1+\frac{3i}{n}}$
2. $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{\pi}{4n} \tan \frac{i\pi}{4n}$

Working Space

Answer on Page 42

4.3.5 Real-world Riemann Sums

Sometimes we are working from real data, and the intervals aren't evenly spaced. That's ok! We can still use Riemann sums to make an estimate. Consider the velocity data from the 1992 launch of the space shuttle *Endeavour*, shown in tabular form in figure 4.13:

We can use a Riemann sum to estimate how far the space shuttle traveled in the first 62 seconds of flight. First, let's visualize our data (see figure 4.14). There are 7 time intervals from the data, but we only need the first 6. We can find a reasonable range for the distance the space shuttle travels by finding the left and right Riemann sums. Remember: Becayse this data is strictly increasing, the left sum will be our lower bound and the right sum will be our upper bound.

First, we'll find L_6. The width of the first interval is 10 seconds $(10-0=10)$ and the height

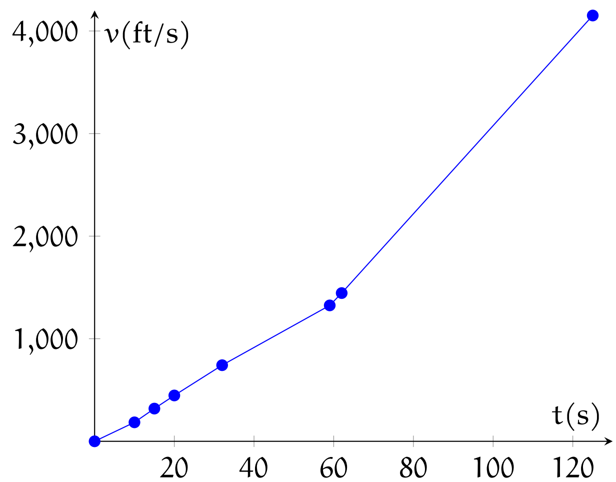


Figure 4.14: Plot of time, velocity data for the *Endeavour*

of the rectangle will be $v(0) = 0$. Calculations for the additional intervals are shown in the table:

Interval	Width(s)	Height(ft/s)	Area(ft)
1	10	0	0
2	5	185	925
3	5	319	1595
4	12	447	5364
5	27	742	20034
6	3	1325	3975

Adding the areas, we find the lower limit for the distance traveled is 31,893 feet. We can determine the upper bound, R_6, in a similar manner:

Interval	Width(s)	Height(ft/s)	Area(ft)
1	10	185	1850
2	5	319	1595
3	5	447	2235
4	12	742	8904
5	27	1325	35775
6	3	1445	4335

Adding the areas, we find the upper limit for the distance traveled is 54,694 feet. Therefore, the *Endeavour* traveled between 31,893 and 54,694 feet during the first 62 seconds of this flight.

4.4 Code for a Riemann Sum

You can create a program that automatically calculates a Riemann sum. Create a file called `riemann.py` and type the following into it:

```
import matplotlib.pyplot as plt
import sys
import math

from matplotlib.table import Rectangle

# Did the user supply two arguments?
if len(sys.argv) != 3:
    print(f"Usage: {sys.argv[0]} <stop> <divisions>")
    print(f"Numerically integrates 1/x from 1 to <stop>.")
    print(f"Calculates the value of 1/x at <divisions> spots in the range.")
    exit(1)

# Check to make sure the number of divisions is greater than zero?
divisions = int(sys.argv[2])
if divisions <= 0:
    print("ERROR: Divisions must be at least 1.")
    exit(1)

# Is the stopping point after 1.0?
stop = float(sys.argv[1])
if stop <= 1.0:
    print("ERROR: Stopping point must be greater than 1.0")
    exit(1)

start = 1.0
step_size = (stop - start)/divisions

print(f"Step size is {step_size:.5f}.")
x_values = []
y_values = []
sum = 0.0
for i in range(divisions):
    current_x = start + i * step_size
    current_y = 1.0/current_x
    area = current_y * step_size
    print(f"{i}: 1 / {current_x:.3f} = {current_y:4f}, area of rect = {area:8f} ")

    x_values.append(current_x)
    y_values.append(current_y)
```

```
    sum += area
    print(f"\tCumulative={sum:.3f}, ln({current_x:.3f})={math.log(current_x):.3f}")

print(f"Numerical integration of 1/x from 1.0 to {stop:.4f} is {sum:.4f}")
print(f"The natural log of {stop:.4f} is {math.log(stop):.4f}")

# Create data for the smooth 1/x line
SMOOTH_DIVISIONS = 200
smooth_start = start - 0.15
smooth_stop = stop + 1.0
smooth_step = (smooth_stop - smooth_start)/SMOOTH_DIVISIONS
smooth_x_values = []
smooth_y_values = []
for i in range(SMOOTH_DIVISIONS):
    current_x = smooth_start + i * smooth_step
    current_y = 1.0/current_x
    smooth_x_values.append(current_x)
    smooth_y_values.append(current_y)

# Put it on a plot
fig, ax = plt.subplots()
ax.set_xlim((smooth_x_values[0], smooth_x_values[-1]))
ax.set_ylim((0, smooth_y_values[0]))
ax.set_title("Riemann Sums for 1/x")

# Make the Riemann rects
for i in range(divisions):
    current_x = x_values[i]
    next_x = current_x + step_size
    current_y = y_values[i]
    rect = Rectangle((current_x, 0), step_size, current_y, edgecolor="green", facecolor=
    ax.add_patch(rect)

# Make the true 1/x curve
ax.plot(smooth_x_values, smooth_y_values, c="k", label="1/x")

# Show the user
plt.show()
```

This program will calculate and display a graph of the left Riemann sum of $\frac{1}{x}$ from 1 to the provided stop value with the indicated number of subintervals. When you run it, you will see a graph in a new window and something like this in the terminal:

```
Step size is 0.40000.
```

```
0: 1 / 1.000 = 1.000000, area of rect = 0.400000
        Cumulative=0.400, ln(1.000)=0.000
1: 1 / 1.400 = 0.714286, area of rect = 0.285714
        Cumulative=0.686, ln(1.400)=0.336
2: 1 / 1.800 = 0.555556, area of rect = 0.222222
        Cumulative=0.908, ln(1.800)=0.588
3: 1 / 2.200 = 0.454545, area of rect = 0.181818
        Cumulative=1.090, ln(2.200)=0.788
4: 1 / 2.600 = 0.384615, area of rect = 0.153846
        Cumulative=1.244, ln(2.600)=0.956
5: 1 / 3.000 = 0.333333, area of rect = 0.133333
        Cumulative=1.377, ln(3.000)=1.099
6: 1 / 3.400 = 0.294118, area of rect = 0.117647
        Cumulative=1.495, ln(3.400)=1.224
7: 1 / 3.800 = 0.263158, area of rect = 0.105263
        Cumulative=1.600, ln(3.800)=1.335
8: 1 / 4.200 = 0.238095, area of rect = 0.095238
        Cumulative=1.695, ln(4.200)=1.435
9: 1 / 4.600 = 0.217391, area of rect = 0.086957
        Cumulative=1.782, ln(4.600)=1.526
Numerical integration of 1/x from 1.0 to 5.0000 is 1.7820
The natural log of 5.0000 is 1.6094
```

Exercise 12

Use the Python program you created to find L_{10}, L_{50}, L_{100}, L_{500}, L_{1000}, and L_{5000} for the function $\frac{1}{x}$ from $x = 1$ to $x = 5$. What do you notice about the results?

Working Space

Answer on Page 43

4.5 Riemann Sum Practice

Exercise 13

t (hours)	4	7	12	15
$R(t)$ (L/hr)	6.5	6.2	5.9	5.6

A tank contains 50 liters of water after 4 hours of filling. Water is being added to the tank at rate $R(t)$. The value of $R(t)$ at select times is shown in the table. Using a right Riemann sum, estimate the amount of water in the tank after 15 hours of filling.

Working Space

Answer on Page 43

Exercise 14

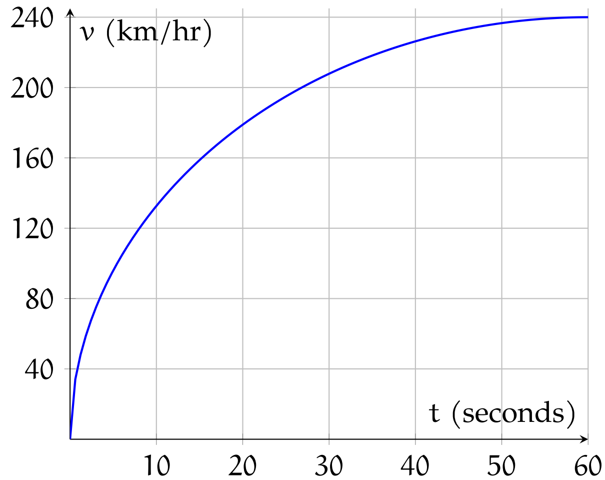
Let $f(x) = x - 2\ln x$. Estimate the area under f from $x = 1$ to $x = 5$ using four rectangles and the value of f at the midpoint of each interval. Sketch the curve and your approximating rectangles.

Working Space

Answer on Page 43

Exercise 15

A graph of a car's velocity over a period of 60 seconds is shown. Estimate the distance traveled during this period.



Working Space

Answer on Page 44

Appendix A

Answers to Exercises

Answer to Exercise 1 (on page 15)

Since we are finding the antiderivative of $\sin x$, we will define $f(x) = \sin x$. We are looking for a F, such that $F'(x) = \sin x$. The derivative of $\cos x$ is $-\sin x \neq f(x)$. However, the derivative of $-\cos x = \sin x = f(x)$. Since $\frac{d}{dx}[-\cos x] = \sin x$, the antiderivative of $\sin x$ is $-\cos x$.

Answer to Exercise 2 (on page 17)

First, we will find $v(t)$ by taking the antiderivative of $a(t)$ and using the initial condition $v(0) = -6$:

$$\int 6t + 4 \, , dt = 3t^2 + 4t + C = v(t)$$

$$v(0) = 3(0)^2 + 4(0) + C = -6$$

$$C = -6$$

Therefore, the velocity function is $v(t) = 3t^2 + 4t - 6$. Next, we repeat the process to find $s(t)$:

$$\int 3t^2 + 4t - 6 \, , dt = t^3 + 2t^2 - 6t + C = s(t)$$

$$s(0) = (0)^3 + 2(0)^2 - 6(0) + C = 9$$

$$C = 9$$

Therefore, the position function is $s(t) = t^3 + 2t^2 - 6t + 9$.

Answer to Exercise 3 (on page 18)

The antiderivative of $\sin x$ is $-\cos x$; therefore, the general solution is $f(x) = -2\cos x + C$. We use the given condition, $f(\pi) = 1$ to find C:

$$f(\pi) = -2\cos\pi + C = 1$$

$$C = 1 + 2\cos\pi = 1 + 2(-1) = -1$$

Therefore, the specific solution is $f(x) = -2\cos x - 1$

Answer to Exercise 4 (on page 18)

1. By the power rule, the antiderivative of x^2 is $\frac{1}{3}x^3$, the antiderivative of $2x$ is x_2, and the antiderivative of 4 is $4x$. So the general antiderivative of $f(x)$ is $\frac{1}{3}x^3 + x^2 - 4x + C$
2. We can rewrite $g(x)$ to more clearly see the powers of x. $g(x) = x^{\frac{2}{3}} + x^{\frac{3}{2}}$. Applying the power rule, we find the general antiderivative of $g(x)$ is $\frac{3}{5}x^{\frac{5}{3}} + \frac{2}{5}x^{\frac{5}{2}} + C$.
3. Recalling that the antiderivative of $\frac{1}{x}$ is $\ln|x|$, the general antiderivative of $h(x)$ is $\frac{1}{5}x - 2\ln|x| + C$
4. The antiderivative of $\sin\theta$ is $-\cos\theta$ and the antiderivative of $\sec^2\theta$ is $\tan x$. Therefore, the general antiderivative of $r(\theta)$ is $-2\cos\theta - \tan\theta + C$

Answer to Exercise 5 (on page 18)

1. The antiderivative of $\sin\theta$ is $-\cos\theta$, and the antiderivative of $\cos\theta$ is $\sin\theta$. The general form of f is $f(\theta) = -\cos\theta + \sin\theta + C$. Substituting $\theta = \pi$, we find that $f(\pi) = -\cos\pi + \sin\pi + C = 1 + 0 + C = 2$, which implies $C = 1$. Therefore, $f(\theta) = -\cos\theta + \sin\theta + 1$.
2. The general antiderivative of f'' is $f'(x) = 4x^3 + 3x^2 - 4x + C_1$. We don't have a condition for f', so we continue to find f. The antiderivative of f' is $f(x) = x^4 + x^3 - 2x^2 + C_1x + C_2$. We can find C_2 with the condition $f(0) = 4$. $f(0) = C_2 = 4$, so we know $f(x) = x^4 + x^3 - 2x^2 + C_1x + 4$. Using the condition $f(1) = 1$, we find that $C_1 = 3$. Therefore, the specific solution is $f(x) = x^4 + x^3 - 2x^2 + 3x - 4$.

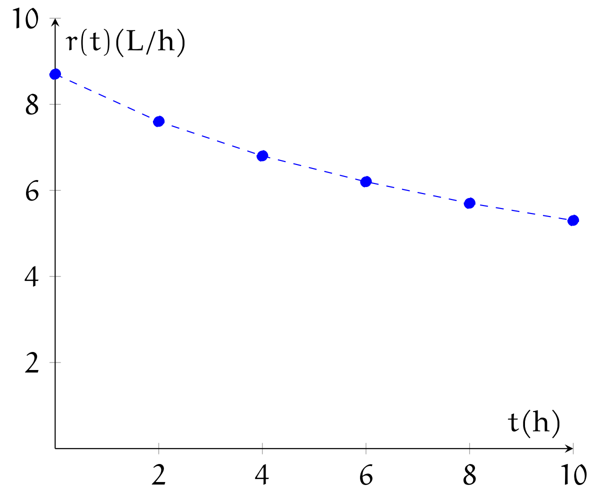
Answer to Exercise 6 (on page 21)

The units on the x-axis are s, and the units on the y-axis are $\frac{m}{s^2}$. The area would then have units of $s \times \frac{m}{s^2} = \frac{m}{s}$. Based on the units, the area represents a net change in velocity. The area above and below the axis are equal $(4.5\frac{m}{s})$; therefore, the total area is 0. This means the object's starting and ending velocity are the same.

Answer to Exercise 7 (on page 22)

The units of the area will be $days \times \frac{deaths}{day} = deaths$. The area under the curve represents the total number of people who died of SARS in Singapore during the time period represented [from March 1 to May 24 (if you took the time to do the math for the dates)].

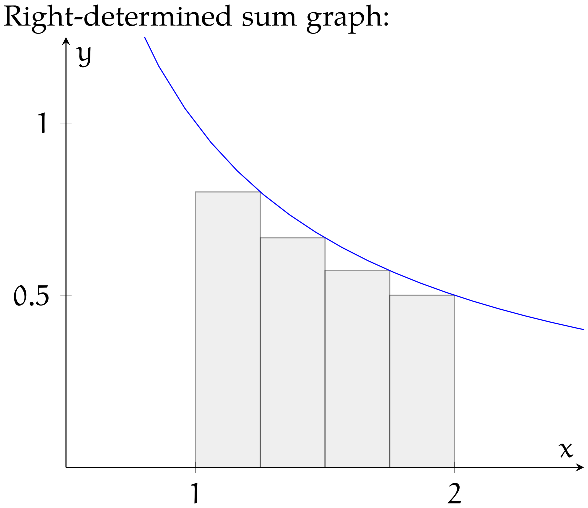
Answer to Exercise 8 (on page 22)



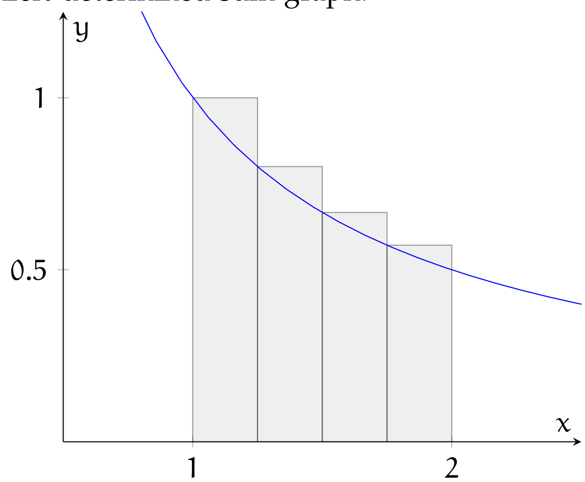
Based on the units, the area under the data would represent the total oil lost. One way to estimate this area would be to create rectangles, but there are other valid methods.

Answer to Exercise 9 (on page 26)

Right-determined sum graph:



The area of the right-determined sum is $0.25 \times (0.8+0.6667+0.5714+0.5) = 0.4202$. This is an underestimate of the actual area.
Left-determined sum graph:



The area of the left-determined sum is $0.25 \times (1+0.8+0.6667+0.5714) = 0.7595$. This is an overestimate of the actual area.

Answer to Exercise 10 (on page 31)

1. $\Delta x = \frac{3-1}{n} = \frac{2}{n}$ and $x_i = 1+i\frac{2}{n} = 1+\frac{2i}{n}$. Substituting, we get $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{2(1+\frac{2i}{n})}{(1+\frac{2i}{n})^2+1} \cdot \frac{2}{n}$
2. $\Delta x = \frac{7-4}{n} = \frac{3}{n}$ and $x_i = 4+\frac{3i}{n}$. Substituting, we get $\lim_{n\to\infty} \sum_{i=1}^{n} [(4+\frac{3i}{n})^2 + \sqrt{1+2(4+\frac{3i}{n})}]\frac{3i}{n}$
3. $\Delta x = \frac{\pi-0}{n} = \frac{\pi}{n}$ and $x_i = \frac{i\pi}{n}$. Substituting, we get $\lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{\sin\frac{i\pi}{n}}\frac{\pi}{n}$

Answer to Exercise 11 (on page 32)

1. $\Delta x = \frac{3}{n}$, which implies $b-a=3$. We could interpret $\sqrt{1+\frac{3i}{n}}$ two ways: either $f(x) = \sqrt{1+x}$ and $x_i = \frac{3i}{n}$ or $f(x) = \sqrt{x}$ and $x_i = 1+\frac{3i}{n}$. In the first case, we can find that $a=0$ and $b=3$, so the limit of the sum represents the area under $f(x) = \sqrt{1+x}$ from $x=0$ to $x=3$. For the second case, we can find that $a=1$ and $b=4$, so the limit of the sum represents the area under $f(x) = \sqrt{x}$ from $x=1$ to $x=4$.
2. $\Delta x = \frac{\pi}{4n}$, which implies $b-a=\frac{\pi}{4}$. We can see that $x_i = \frac{i\pi}{4n}$, which implies $a=0$ and therefore also that $b=\frac{\pi}{4}$. Therefore, the limit of the sum represents the total area under $f(x) = \tan x$ from $x=0$ to $x=\frac{\pi}{4}$.

Answer to Exercise 12 (on page 36)

Number of Intervals	Calculated Area
10	1.7820
50	1.6419
100	1.6256
500	1.6126
1000	1.6110
5000	1.6098

The area approaches the natural log of the endpoint, $\ln 5 \approx 1.6094$.

Answer to Exercise 13 (on page 37)

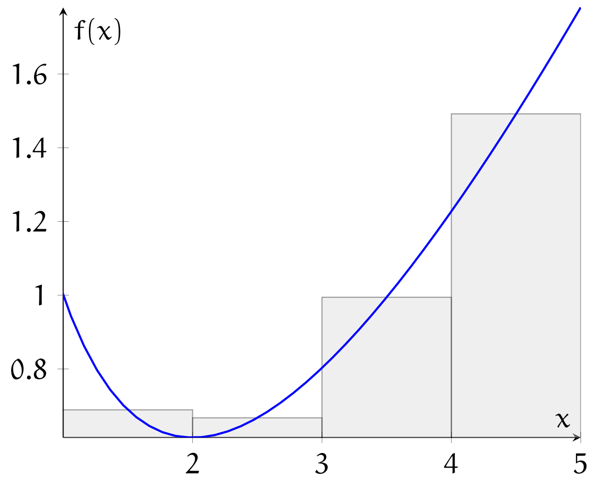
The volume of water will be the amount of water at 4 hours (50 liters) plus the area under the graph of $R(t)$ from $t = 4$ to $t = 15$. We will estimate this area with a right Riemann sum. The approximate volume added from $t = 4$ to $t = 7$ is $(7 - 4) * (6.2) = 18.6$ liters. The approximate volume added from $t = 7$ to $t = 12$ is $(12 - 7) * (5.9) = 29.5$ liters. The approximate volume added from $t = 12$ to $t = 15$ is $(15-12)*(5.6) = 16.8$ liters. Therefore, the approximate total volume of water in the tank at $t = 15$ is $50+18.6+29.5+16.8 = 114.9$ liters.

Answer to Exercise 14 (on page 37)

We will divide the area from $x = 1$ to $x = 5$ into four intervals at $x = 2$, $x = 3$, and $x = 4$. Next, we will find the value of $f(x)$ at the midpoint of each interval:

Interval	Midpoint	Value of $f(x)$ at midpoint
1	1.5	≈ 0.68907
2	2.5	≈ 0.66742
3	3.5	≈ 0.99447
4	4.5	≈ 1.49185

Using the values in the table, we can make a possible sketch of $f(x)$:

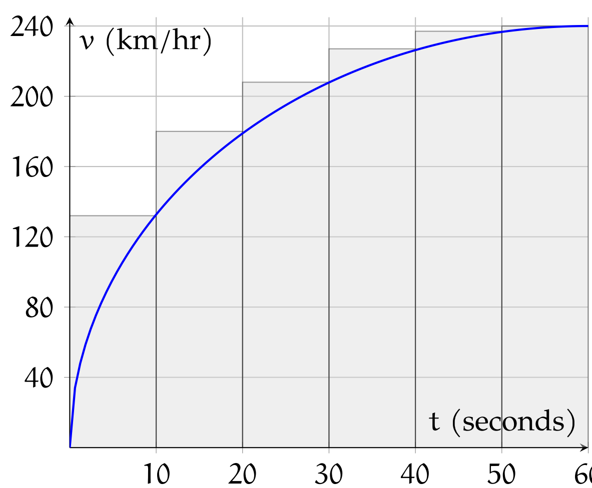


And we calculate the total area in the rectangles:

$$1 \times (0.68907 + 0.66742 + 0.99447 + 1.49185 = 3.84281$$

Answer to Exercise 15 (on page 38)

The question allows the student to choose the type of sum (left, right, or midpoint) and the number of intervals. A possible solution is given, but there are many ways to answer the question.
The tricky part here is noticing the units! In order to have a solution in kilometers, we will need to convert km/hr to m/s when we calculate the areas. A possible solution is to divide the graph into 6 intervals (one every 10 seconds) and use a right Riemann sum.



We can use the graph to *estimate* the height of each rectangle. Some reasonable estimates are $f(10) = 130\frac{\text{km}}{\text{hr}} \approx 36.1\frac{\text{ms}}{\text{s}}$, $f(20) = 180\frac{\text{km}}{\text{hr}} = 50\frac{\text{m}}{\text{s}}$, $f(30) = 210\frac{\text{km}}{\text{hr}} \approx 58.3\frac{\text{m}}{\text{s}}$, $f(40) = 230\frac{\text{km}}{\text{hr}} \approx 63.9\frac{\text{m}}{\text{s}}$, $f(50) = 235\frac{\text{km}}{\text{hr}} \approx 65.3\frac{\text{m}}{\text{s}}$, and $f(60) = 240\frac{\text{km}}{\text{hr}} \approx 66.7\frac{\text{m}}{\text{s}}$. [Any values within

± 5 of the listed values are reasonable.] Noting that each interval is 10 sec wide and usingß the estimates of $f(x)$ listed, we can estimate that the distance traveled is $10\ sec \times (36.1\frac{m}{s} + 50\frac{m}{s} + 58.3\frac{m}{s} + 63.9\frac{m}{s} + 65.3\frac{m}{s} + 66.7\frac{m}{s}) = 3403$ meters.

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