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Temperature and the Gas Laws

1.1 The Physics of Gases

Why does a bike tire feel warm after you pump it up? Why does an aerosol can get cold when you spray it? How does an air conditioner pump heat out of a hot house? The answers all come from thinking about a gas as the motion of trillions of tiny molecules.

Imagine a balloon full of helium. Each helium atom zips around, bouncing off the others and off the rubber. Now, let us say you start to heat the helium inside the balloon. As the temperature goes up, the molecules inside will start to move faster.

Remember that the kinetic energy of an object with mass m and velocity v is given by

$$k = \frac{1}{2}mv^2 \quad (1.1)$$

So, you could say “As the temperature of the gas increases, the kinetic energy of the molecules increases.” But a physicist would say “The temperature of the gas is how we measure its kinetic energy.”

A Statistical Look At Temperature

If you say “This jar of argon gas is 25 degrees Celsius,” you have told me about the *average* kinetic energy of the molecules in the jar. However, some molecules are moving very slowly. Others are moving really, really fast.

We could plot the probability distribution of the speeds of the molecules. For argon at 25 degrees Celsius, it would look like this:

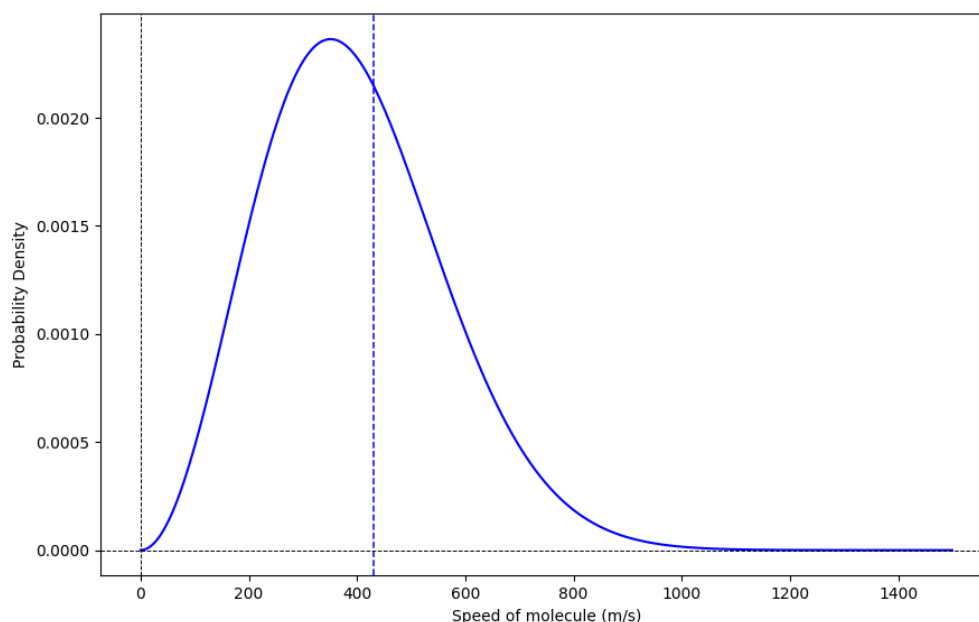


Figure 1.1: Probability distribution of molecular speeds for argon gas at 25 degrees Celsius.

The temperature, remember, is determined by the average kinetic energy of the molecules. Some molecules are moving slowly, and have less kinetic energy than the average. Some molecules are moving very quickly, and have more kinetic energy. The dotted line is the divider between the two groups: molecules moving at speeds to the left of the line have less kinetic energy than average; those on the right have more kinetic energy than average.

Where is that line? That is the RMS of the speeds of the molecules. That is, if we measured all the speeds of all the molecules $s_1, s_2, s_3, \dots, s_n$, that line would be given by the root of the mean of the squares:

$$v_{\text{rms}} = \sqrt{\frac{1}{n} (s_1^2 + s_2^2 + s_3^2 + \dots s_n^2)}$$

If you have the same gas at a lower temperature, the distribution shifts toward zero: the peak becomes taller and narrower, and the tail of very fast molecules becomes much shorter. In other words, fewer molecules have high speeds, and more molecules cluster near lower speeds. Let us look at an example.

Here is probability distribution of molecular speeds for argon gas at 25 degrees and -100 degrees Celsius.

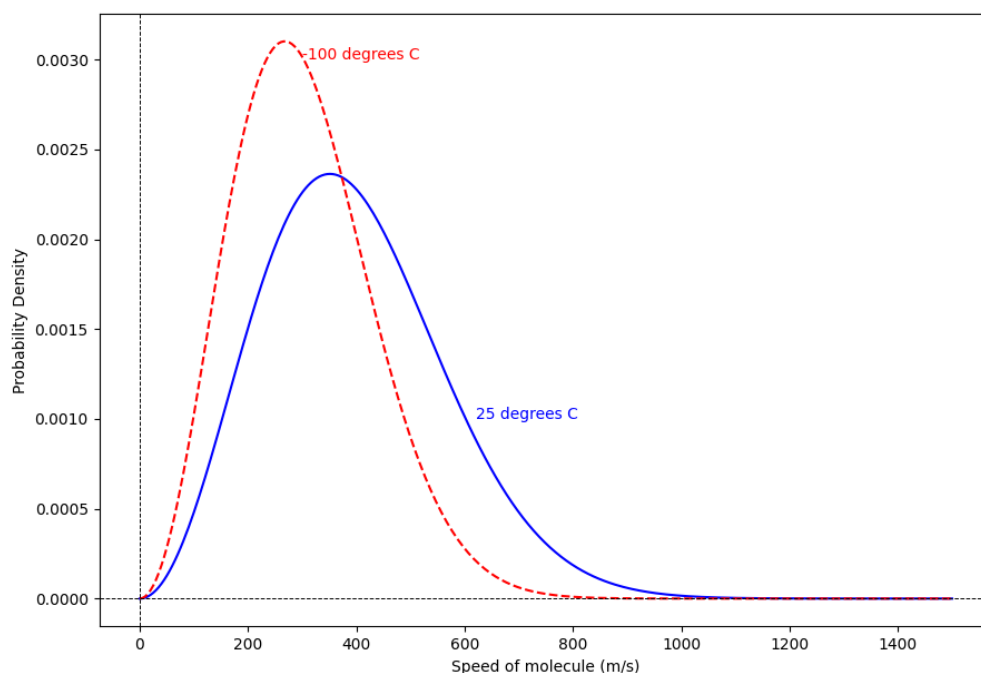


Figure 1.2: Probability distribution of molecular speeds for argon gas at 25 degrees and -100 degrees Celsius.

Absolute Zero and Degrees Kelvin

If you keep lowering the temperature, eventually, all the molecules stop moving. This is known as *absolute zero* — you cannot make anything colder than absolute zero. Absolute zero is -273.15° Celsius or -459.67° Fahrenheit.

In addition to Celsius and Fahrenheit, there is a third temperature system: Kelvin. Kelvin has the same scale as Celsius, but it starts at absolute zero (such that every value differs by 273.15). So, 0 degrees Celsius is 273.15 kelvins, and 100 degrees Celsius is 373.15 kelvins.

Any time you are working with the physics of temperature, you will use Kelvin.

Sometimes, when reading about gases, you will see “STP” which stands for “Standard Temperature and Pressure.” STP is defined to be 0° Celsius and 100 kPa.

1.1.1 Temperature and Volume

Let us say it is dawn, and a person is lying on a half-full air mattress in a field. The weight of the person will keep the pressure of the air inside constant (or pretty close).

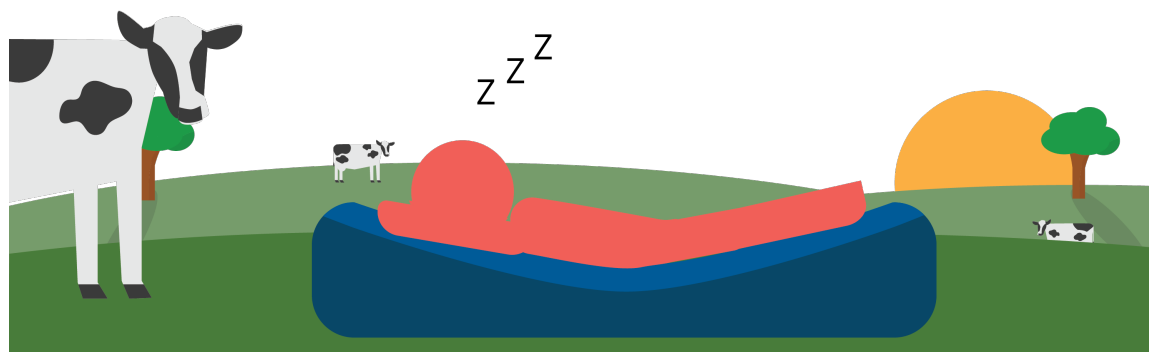


Figure 1.3: A person sleeping on an air mattress as the sun rises.

The molecules in the mattress are not entering or leaving that mattress. However, as the sun rises, the air inside will get warmer and expand. The person will be gently lifted by the expanding air. You might wonder: how much will the air expand?

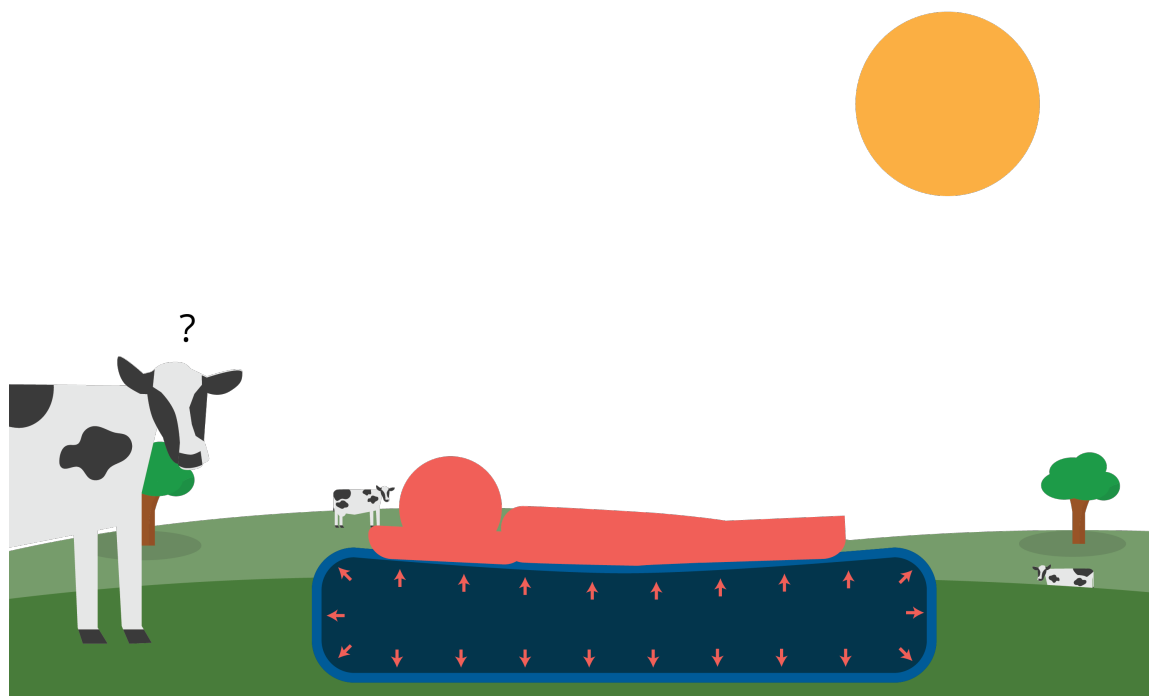


Figure 1.4: The air mattress inflates as the sun reaches its peak height in the sky.

If you have constant pressure and a constant number of molecules, the volume of the gas is proportional to the temperature in Kelvin:

$$V \propto T$$

This is known as Charles's Law.

Exercise 1 Temperature and Volume

Working Space

At dawn, the air inside the mattress has a volume of 1000 liters and a temperature of 12 degrees Celsius.

At noon, that same air has a temperature of 28 degrees Celsius. The pressure on the gas has not changed at all.

What is the volume of the gas at noon?

Answer on Page 79

Note: Volume and temperature are only proportional as long as the substance is a gas. We will talk about liquids and solids soon.

1.1.2 Pressure and Volume

As you increase the pressure on a gas, the molecules will get pushed closer together, and the volume will decrease.

For example, if you put the cap on an empty plastic bottle and squeeze it. As you put the gas inside the bottle under pressure, its volume will decrease.

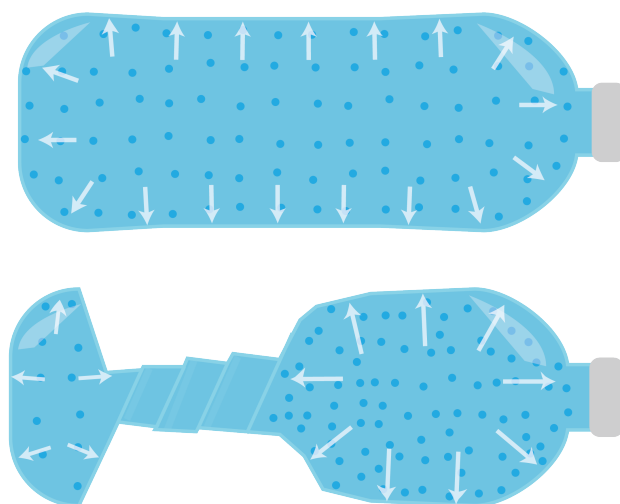


Figure 1.5: The water bottle shrinks as it is put under pressure.

If you keep the number of molecules and the temperature constant, the pressure of the gas and its volume are inversely proportional:

$$P \propto \frac{1}{V}$$

This is known as Boyle's Law. "But," you say with disbelief, "if I increase the pressure on my empty water bottle from 5 kPa to 10 kPa, the volume inside will not decrease by half!"

Do not forget that the air inside the bottle is under 101 kPa of atmospheric pressure before you even start to squeeze it.

Exercise 2 Pressure and Volume

Working Space

At an altitude where the atmospheric pressure is 100 kPa, you seal air in a 1 liter water bottle.

Squeezing the water bottle, you raise the internal pressure by 20 kPa. What is the

volume inside the bottle now?

Answer on Page 79

1.1.3 The Ideal Gas Law

You are gradually getting an intuition for the relationship between the number of molecules, the volume, the pressure, and the temperature of a gas. We can actually bring these together in one handy equation.

Ideal Gas Law

$$PV = nRT$$

where:

- P is the pressure in pascals
- V is the volume in cubic meters
- n is the number of molecules in moles
- R is the molar gas constant: 8.31446
- T is the temperature in Kelvin

(You can remember this as the "Pivnert.")

From the name, you might predict the following: The Ideal Gas Law is not 100% accurate. However, for most purposes, it works remarkably well.

Notice that the ideal gas law says nothing about what kind of gas it is; it works regardless.

Exercise 3 Ideal Gas Law

Working Space

You have a cylinder containing O_2 . The chamber inside has a radius of 12 cm and a length of 50 cm. The temperature in-

side the cylinder is 20 degrees Celsius.
The pressure inside the tank is 600 kPa.

How many moles of O_2 are inside?

Answer on Page 80

1.1.4 Molecules Like To Stay Close to Each Other

When two molecules get close to each other a few things can happen:

- They can undergo a chemical reaction: electrons are exchanged or shared and a different molecule or molecules come into existence. This is the realm of chemistry, and we will not go into it in this course.
- One or both of them have so much kinetic energy that they just pass each other or bounce off each other. This is what happens in a gas.
- The two molecules can "stick" together. This is what happens in a liquid or a solid.

Why do they stick together if they are not combined in a chemical reaction?

First, they do not get *too close*. If they get too close, their electron clouds repel each other with a strong force. This is what happens in a gas when two molecules bounce off of each other.

However, if the molecules are quite close to each other, there are forces that will attract them toward each other. These intermolecular forces are beyond the scope of this course, but they are called Van der Waals forces and hydrogen bonds. The strength of these forces vary based on the two molecules involved.

Which is why some of the matter around you is in gas form (molecules that do not stick together at the temperature and pressure you are living in because they have weak attractive forces) and some is non-gas (gangs of molecules with stronger attraction that makes them clump together as a liquid or a solid at that same temperature and pressure).

However, what if we change the temperature and pressure? We can change if and how the molecules clump together. This is known a *phase change*; We will cover it soon.

1.2 Kinetic Energy and Temperature of a Gas

As mentioned earlier in this chapter, for a particular gas, the temperature (in Kelvin) is proportional to the average kinetic energy of the individual molecules.

Perhaps you want to warm 3 moles of helium gas (trapped in a metal cylinder) from 10 degrees Celsius to 30 degrees Celsius. How would you compute exactly how many Joules of energy this would require?

The amount of energy necessary to raise one mole of a molecule by one degree is known as *molar heat capacity*. (The molar heat capacity of liquid water, for example, is 75.38 J per mole-degree.)

With gases, there are actually two different possible situations:

1. Constant volume: As you heat the gas, the pressure and the temperature increase. This molar heat capacity is usually denoted as $C_{V,m}$.
2. Constant pressure: As you heat the gas, the temperature and the volume increase. This molar heat capacity is usually denoted as $C_{P,m}$.

All gases made up of one atom (Helium, for example, is a monoatomic gas) have the same values for $C_{V,m}$ and $C_{P,m}$:

$$C_{V,m} = \frac{3}{2}R \approx 12.47 \text{ Joules per mole-degree}$$

$$C_{P,m} = \frac{5}{2}R \approx 20.8 \text{ Joules per mole-degree}$$

(Remember that R is the ideal gas constant ≈ 8.31446 Joules per mole-degree.)

Exercise 4 Warming Helium

Working Space

You have 3 moles of helium.

1. How many Joules would be required to warm 3 moles of helium gas by 20 degrees Celsius at constant volume?
2. How many Joules would be required

to warm 3 moles of helium gas by 20 degrees Celsius at constant pressure?

Answer on Page 80

1.2.1 Molecule Shape and Molar Heat Capacity

We told you that gases made up of one atom have the same values for $C_{V,m}$ and $C_{P,m}$:

$$C_{V,m} = \frac{3}{2}R \approx 12.47 \text{ Joules per mole-degree}$$

$$C_{P,m} = \frac{5}{2}R \approx 20.8 \text{ Joules per mole-degree}$$

For any molecule, it is generally true that

$$C_{P,m} \approx C_{V,m} + R$$

It is also true that for any molecule, there is some integer d such that

$$C_{V,m} \approx \frac{d}{2}R$$

For example, for all monoatomic gases, $d = 3$. For diatomic gases (like N_2 and O_2 , d is 5.

d is known as the *degree of freedom* of the molecule. When you study chemistry, they will teach you to predict d based on the shape of the molecule.

Here are the relevant numbers for some gases you are likely to work with. (For molecules with three or more atoms, vibrations also store energy, so $C_{V,m}$ is only roughly a multiple of $\frac{R}{2}$.)

Gas	type	$C_{V,m}$	$C_{P,m}$	d
He	monoatomic	12.4717	20.7862	3
Ar	monoatomic	12.4717	20.7862	3
O_2	diatomic	21.0	29.38	5
N_2	diatomic	20.8	29.12	5
H_2O (water vapor)	3 atoms	28.03	37.47	7
CO_2	3 atoms	28.46	36.94	7

1.2.2 Kinetic Energy and Temperature

For a sample of a gas, we can calculate its kinetic energy based on its molar heat capacity, the number of molecules, and the temperature:

$$E_K = C_{V,m}nT$$

where

- E_K is the kinetic energy in Joules
- $C_{V,m}$ is the molar heat capacity of the gas at constant volume
- n is the number of molecules in moles
- T is the temperature in Kelvin

Exercise 5 Warming Helium Revisited

Working Space

How much kinetic energy does 3 moles of helium have at 10 degrees Celsius?

How much kinetic energy does 3 moles of helium have at 30 degrees Celsius?

What is the difference?

Answer on Page 80

1.2.3 Why is $C_{V,m}$ different from $C_{P,m}$?

What if, instead of keeping the volume constant while we heat the molecules in the helium tank, we keep the pressure constant and let the gas expand? The change in kinetic energy is the same: 748 Joules.

However, we know that the molar heat capacity if we keep pressure constant is $\frac{5}{2}R$, so heating will require $\frac{5}{2}R(3)(20) = 1247$ Joules.

What happened to the 499 missing Joules!? Thermodynamics tells us energy is neither created nor destroyed, so it must have gone somewhere.

That energy was used pushing against the pressure as the gas expanded. For example, maybe the sample was in a balloon in space — the extra energy stretched the surface of the balloon.

The 499 joules went into the work of pushing the surroundings out of the way.

1.2.4 Work of Creating Volume Against Constant Pressure

Let us imagine that you had a total vacuum (zero pressure) with a piston. As you pulled the piston out, you would be pulling against the atmospheric pressure. How much energy would that require?

If you increased the volume of the vacuum by V against a pressure of P , you would do VP work.

Let us check to make sure the 499 Joules mentioned above makes sense with this in mind.

No initial pressure was given in the problem, so let us just make one up and see how things work out: 100 kPa. Using the ideal gas law, the initial volume would be:

$$V_1 = \frac{nRT}{P} = \frac{(3)(8.31446)(283.15)}{100,000} = 0.07063 \text{ cubic meters}$$

The volume after we heated the gas and let it expand against 100 kPa would be:

$$V_2 = \frac{nRT}{P} = \frac{(3)(8.31446)(303.15)}{100,000} = 0.07562 \text{ cubic meters}$$

So, the volume increased by $0.07562 - 0.07063 = 0.00499$ cubic meters. Multiplying that by 100,000 pa, we get 499 Joules, as we expected!

1.2.5 Why does a gas get hotter when you compress it?

Now, imagine that there is gas inside the piston and you push on the piston to compress that air. The work that you do is converted into kinetic energy, and that kinetic energy raises the temperature of the gas.

For example, if you had two moles of argon gas in the piston, and you pushed the piston

0.1 meters with an average force of 50 newtons, you will have done 5 Joules of work.

How much would 5 Joules raise the temperature of 2 mole of a monoatomic gas?

$$\Delta T = \frac{5}{(2)(C_{V,m})} = 0.2 \text{ K}$$

It works both ways: Compression makes a gas hotter; decompression makes a gas colder. You can sometimes experience the heat of compression when you pump up a bicycle tire — as you pump the tire will get warmer.

If you compress or decompress a gas without letting any heat enter or depart, we say the compression or decompression was *adiabatic*. In order to solve any interesting problems about heating/cooling due to compression/decompression, you will need to assume the process was adiabatic.

When a spacecraft enters the atmosphere, it has to deal with a lot heat. Some people assume that heat is due to friction of the air rubbing against the spacecraft at over 7,000 meters per second. Actually, most of the heat is due to the compression of the air as it gets pushed out of the way of the spacecraft.

1.2.6 How much hotter?

Let us say you have an accordion-like container filled with helium at 100 kPa (about 1 atmosphere) and 300 kelvins. It holds 2 cubic meters. You then you put it in a vice and quickly compress it down to 0.5 cubic meters. Assuming it was adiabatic, how hot would the gas inside be after the compression?

Here is the challenging part: As you crush the container, the temperature and the pressure in the container are both increasing. So, as you go, it gets harder and harder to crush. Each milliliter of volume that you eliminate requires a little more work than the milliliter before.

Let us simulate the process in Python, after which we will give you the formula.

In the simulation, you will start with an initial volume of 2 cubic meters and crush it down to 0.5 cubic meters in 40 steps. At each step you will recalculate the temperature and pressure.

Next, you will plot the results. Make a file called `gas_crunch.py`:

```
1 import numpy as np
2 import matplotlib.pyplot as plt
```

```

3
4 V_initial = 2.0 # cubic meters
5 V_final = 0.5 # cubic meters
6 step_count = 40 # steps
7
8 T_initial = 300.0 # kelvin
9 P_initial = 100000 # pascals
10
11 # Constants
12 R = 8.314462618 # ideal gas constant
13 C_v = 3.0 * R / 2.0 # molar heat capacity (constant volume)
14
15 # Compute the number of moles
16 n = P_initial * V_initial / (R * T_initial)
17 print(f"The container holds {n:.2f} moles of helium")
18
19 # How much volume do we need to eliminate in each step?
20 # (in cubic meters)
21 step_size = (V_initial - V_final) / step_count
22
23 # For recording the state for each step
24 data_log = np.zeros((step_count, 3))
25
26 # Variables to update in the loop
27 V_current = V_initial
28 T_current = T_initial
29 P_current = P_initial
30
31 for i in range(step_count):
32     # Record the current state
33     data_log[i,:] = [T_current, V_current, P_current/1000.0]
34
35     # Find how much energy to make the step at the current pressure
36     E_step = step_size * P_current
37
38     # Find how big the change in temperature will be from that energy
39     delta_T = E_step / (n * C_v)
40
41     # Update the current temperature, volume, and pressure
42     T_current += delta_T
43     V_current -= step_size
44     P_current = n * R * T_current / V_current
45
46     print(f"Iterative:{T_current:0.3f} K, {V_current:0.3f} m3, {P_current/1000.0:0.3f}
47     ↪ kPa")
48
49 fig, axs = plt.subplots(3,1,sharex=True, figsize=(8, 6))
50 axs[0].set_xlim((0,step_count))
51 axs[0].plot(data_log[:,0], 'k', lw=0.2)
52 axs[0].plot(data_log[:,0], 'r.')
53 axs[0].set_ylabel("Temperature (K)")
54
55 axs[1].plot(data_log[:,1], 'k', lw=0.2)

```

```
55  axs[1].plot(data_log[:,1], 'r.')
56  axs[1].set_ylabel("Volume (cubic m)")
57
58  axs[2].plot(data_log[:,2], 'k', lw=0.2)
59  axs[2].plot(data_log[:,2], 'r.')
60  axs[2].set_ylabel("Pressure (kPa)")
61
62  axs[2].set_xlabel("Step")
63
64  fig.savefig('tvpplot.png')
```

When you run this, you will see how many moles of gas there are and reasonable estimates of the temperature, volume, and pressure:

```
1  > python3 gas_crunch.py
2  The container holds 80.18 moles of helium
3  Iterative:733.499 K, 0.500 m3, 977.999 kPa
```

And a good plot of the intermediate values:

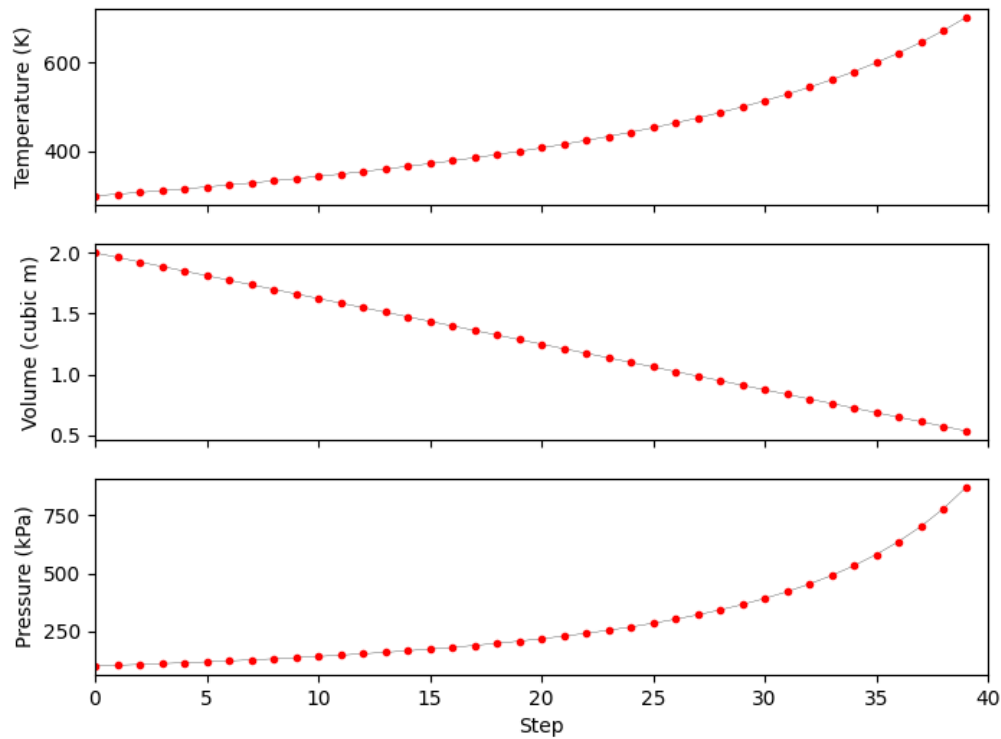


Figure 1.6: Temperature, volume, and pressure during the compression, simulated in 40 steps.

However, we will get better estimates if we break it up into 400 steps instead of 40. Change the line that defines the number of steps:

```
1 step_count = 400 # steps
```

Now the predicted temperature and pressure should be something like 753.603° K and 1004.803 kPa. (This is much closer to the correct result: 755.953 K and 1007.937 kPa.)

What if you break it into 400,000 steps? Now, the result should be really, really close to correct. And the plot is quite accurate:

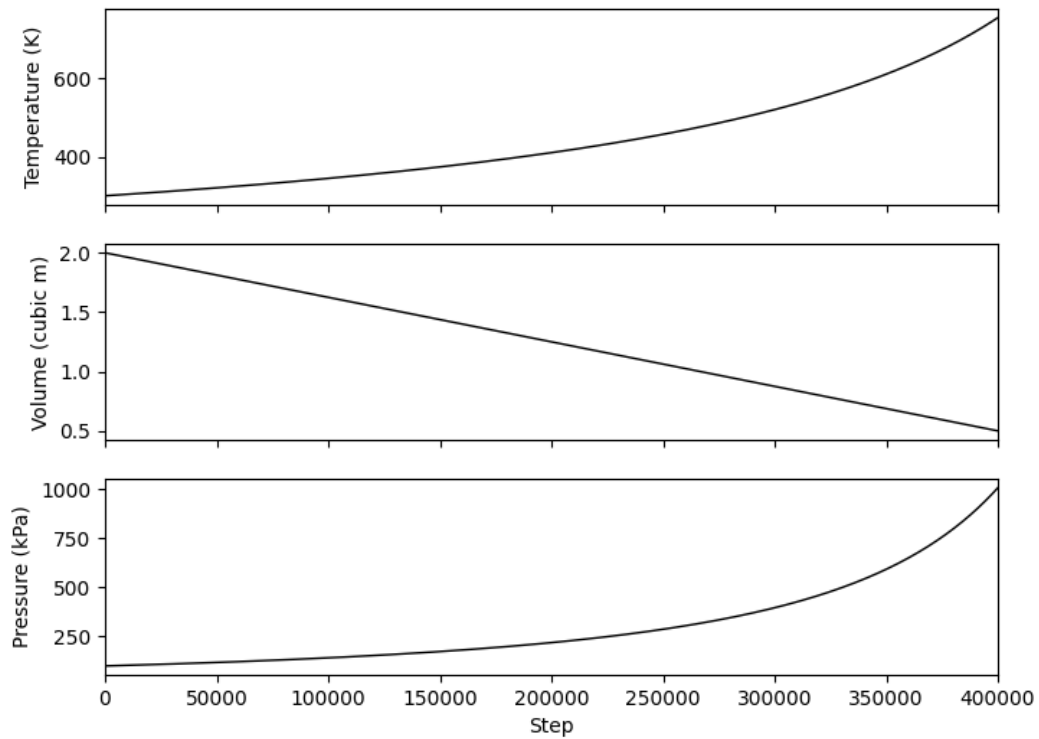


Figure 1.7: The same compression simulated with many more steps.

(You can comment out the lines that make the red dots on the graphs. No one wants to see 400,000 red dots.)

It is inefficient to have to do long simulations to guess the final temperature and pressure. Fortunately, there are two handy rules you can use to skip this:

Adiabatic Compression and Decompression

Let

$$\gamma = \frac{C_{P,m}}{C_{V,m}}$$

In an adiabatic compression or decompression, P and V change, but

$$P(V^\gamma)$$

stays constant.

Also

$$T \left(V^{(\gamma-1)} \right)$$

stays constant

For a monoatomic gas:

$$\gamma = \frac{C_{P,m}}{C_{V,m}} = \frac{5}{3}$$

$$\gamma - 1 = \frac{2}{3}$$

Before the compression:

$$T \left(V^{(\gamma-1)} \right) = 300 \left(2^{0.6667} \right) = 476.22$$

After the compression it has to be the same:

$$T \left(V^{(\gamma-1)} \right) = T \left(0.5^{0.6667} \right) = 476.22$$

Thus

$$T = 755.95 \text{ K}$$

We can then use the ideal gas law to solve for the final pressure:

$$P = \frac{nRT}{V} = \frac{(80.18)(8.31446)(755.95)}{0.5} = 1007937 \text{ pascals}$$

That is hot! As you let it cool back down to 300 kelvins, how much heat would be released?

$$E = C_{V,m}n\Delta T = (12.47)(80.2)(755.95 - 300) \approx 456 \text{ kJ}$$

1.2.7 How an Air Conditioner Works

Once again, imagine the accordion-like container filled with helium. Let us say you walked it outside and compressed it from 2 cubic meters to 0.5 cubic meters in a vise. The container would get to 755.95 kelvins. You keep it compressed in the vise, but let it cool down outside. When it gets back to 300 kelvins, you walk it back inside.

Now, without letting any molecules in or out of the container, you release the vise. The gas is decompressed and gets very cold. How cold? Cold enough to accept about 456 kJ of kinetic energy from your house. That is, it would absorb heat from your house until the gas inside was the same temperature as your house.

So, you walk outside with your accordion and your vise, and repeat:

1. Compress the gas outside.
2. Let the hot gas cool down outside.
3. Walk the room-temperature compressed gas inside.
4. Decompress the gas inside.
5. Let the cold gas warm up inside.

You could keep your house cool on a hot day this way. This is not unlike how an air conditioner works.

There is a hose filled with refrigerant that does a loop:

- Outside, the refrigerant is compressed and allowed to cool to the outside temperature. (Usually there is a big fan blowing on a coil of refrigerant to speed the process.) Inside, the refrigerant is decompressed and allowed to warm to the inside temperature. (Usually there is a big fan blowing the air of the home past a coil of refrigerant to speed the process.)

In each pass of the loop, the refrigerant absorbs some of the kinetic energy from inside the house, and releases it on the outside.

This same mechanism can be used to heat your house. (Units that both heat and cool are known as *heat pumps*.) The heat pump does the process backwards: The hot compressed refrigerant cools down inside. The cold decompressed refrigerant warms up outside.

The next chapter follows heat as it flows between objects and as it melts and boils matter.

Heat and Phases of Matter

Put a cold metal spoon in hot cocoa, and the spoon warms while the cocoa cools. Leave ice in a glass, and it melts without getting any warmer than 0°C . A rice cooker somehow knows exactly when the rice is done. In the Temperature and the Gas Laws chapter, you saw that temperature is the motion of molecules. This chapter follows that energy as it flows from hot things to cold things, and as it melts, boils, and freezes matter.

2.1 Heat

All mass in the universe has heat, whether you are looking at a block of dry ice (frozen CO_2 , -78.5°C) or the surface of the sun ($5,600^{\circ}\text{C}$). As long as the mass is above absolute zero — the coldest possible temperature in the universe — there is some amount of heat in it.

2.1.1 How Heat Works

As you heat up an object, you are imparting energy into it. Where does this energy go? The atoms take this energy and they begin to move, vibrating and bumping into each other, causing the heat to spread throughout. Every time the atoms collide and bounce off of each other, they emit a tiny amount of energy as light. In most cases, that light is in the infrared spectrum, but in extreme cases can be visible, such as with molten lava or hot metal.

As objects interact, they either put heat into colder objects or take heat from warmer objects. That is what allows you to heat up anything in the first place. The hot air from a stove or bunsen burner interacts with the pan or test tube you are heating, passing the air's heat on. How could you model this?

2.1.2 Specific Heat Capacity

If you are heating something, the amount of energy you need to transfer to it depends on three things: the mass of the thing you are heating, the amount of temperature change you want, and the *specific heat capacity* of that substance.

Energy in Heat Transfer

The energy moved in a heat transfer is given by

$$E = mc\Delta T$$

where m is the mass, ΔT is the change in temperature, and c is the specific heat capacity of the substance.

(Note that this assumes there is not a phase change. For example, this formula works nicely on warming liquid water, but it gets more complicated if you warm the water past its boiling point.)

Can we guess the specific heat capacity of a substance? It is very, very difficult to guess the specific heat of a substance, so we determine it by experimentation.

For example, it takes 0.9 joules to raise the temperature of solid aluminum one degree Celsius. So we say “The specific heat capacity of aluminum is 0.9 J/g °C.”

The specific heat capacity of liquid water is about 4.2 J/g °C.

Let us say you put a 1 kg aluminum pan that is 80° C into 3 liters of water that is 20° C. Energy, in the form of heat, will be transferred from the pan to the water until they are at the same temperature. We call this “thermal equilibrium”.

What will the temperature of the water be?

To answer this question, the amount of energy given off by the pan must equal the amount of energy absorbed by the water. They also need to be the same temperature at the end. Let T be the final temperature of both.

3 liters of water weighs 3,000 grams, so the change in energy in the water will be:

$$E_W = mc\Delta T = (3000)(4.2)(T - 20) = 12600T - 252000 \text{ joules}$$

The pan weighs 1000 grams, so the change in energy in the pan will be::

$$E_P = mc\Delta T = (1000)(0.9)(T - 80) = 900T - 72000 \text{ joules}$$

The total energy stays the same, so $E_W + E_P = 0$. This means you need to solve

$$(12600T - 252000) + (900T - 72000) = 0$$

And find that the temperature at equilibrium will be

$$T = 24^{\circ}\text{C}$$

Exercise 6 Thermal Equilibrium

Working Space

Just as you put the aluminium pan in the water as described above, someone also puts a 1.2 kg block of copper cooled to 10°C . The specific heat of solid copper is about $0.4\text{ J/g}^{\circ}\text{C}$.

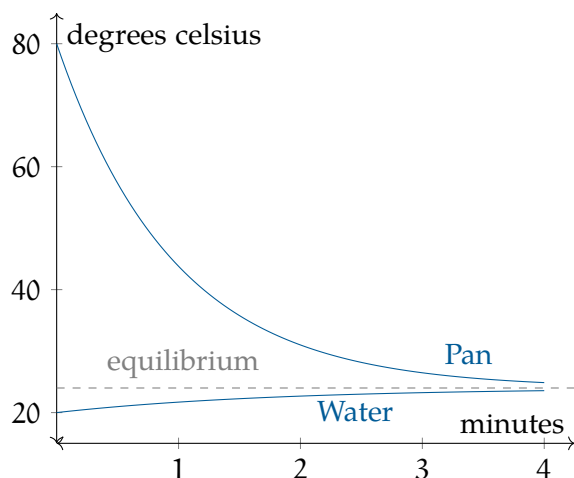
What is the new temperature at equilibrium?

Answer on Page 81

2.1.3 Getting to Equilibrium

When two objects with different temperatures are touching, the speed at which they exchange heat is proportional to the differences in their temperatures. As their temperatures get closer together, the heat exchange slows down.

In our example, the pan and the water will get close to equilibrium quickly, but they may never actually reach equilibrium.



Exercise 7 Cooling Your Coffee

Working Space

You have been given a ridiculously hot cup of coffee and a small pitcher of chilled milk.

You need to start chugging your coffee in three minutes, and you want it as cool as possible at that time. When should you add the milk to the coffee?

Answer on Page 81

2.1.4 Specific Heat Capacity Details

For any given substance, the specific heat capacity often changes a great deal when the substance changes state. For example, ice is $2.1 \text{ J/g } ^\circ\text{C}$, whereas liquid water is $4.2 \text{ J/g } ^\circ\text{C}$.

Even within a given state, the specific heat capacity varies a bit based on the temperature and pressure. If you are trying to do these sorts of calculations with great accuracy, you will want to find the specific heat capacity that matches your situation. For example, you might look for the specific heat capacity for water at 22°C at 1 atmosphere of pressure (atm).

2.2 Phases of Matter

You have experienced H_2O in three phases of matter:

- Ice is H_2O in the solid phase. At standard pressure, when the temperature of H_2O is below 0°C , it is a solid.
- Water is H_2O in the liquid phase. At standard pressure, when the temperature of H_2O is between 0°C and 100°C , it is a liquid.
- Water vapor (or steam) is the gas phase. At standard pressure, when the temperature of H_2O is above 100°C , it is a gas.

Let us look at some of the properties of the three phases:

Gas	Liquid	Solid
Assumes the volume and shape of its container	Assumes the shape, but not the volume, of its container	Retains its shape and volume
Compressible	Not compressible	Not compressible

2.2.1 Thinking Microscopically About Phase

As mentioned earlier in this chapter, there are intermolecular forces that attract molecules to each other. A pair of molecules will have very strong intermolecular forces or very weak intermolecular forces, depending on what atoms they are made of.

For example, two helium atoms are only very weakly attracted to each other. The particles in NaCl (table salt), on the other hand, attract each other very strongly.

In a gas, the molecules have lots of room to roam and lots of kinetic energy: The intermolecular attraction has very little effect.

In a liquid, the molecules are sticking close together, but are still moving around, sort of like bees in a hive.

In a solid, the molecules are not changing their configuration, and the kinetic energy they have is just expressed as vibrations within that configuration. You can imagine them like eggs vibrating in a carton.

As you would expect, molecules with strong intermolecular attraction require more kinetic energy to change phases. For example, helium is a liquid below -269°C . NaCl , on the other hand, is a liquid between 801° and $1,413^\circ\text{C}$.

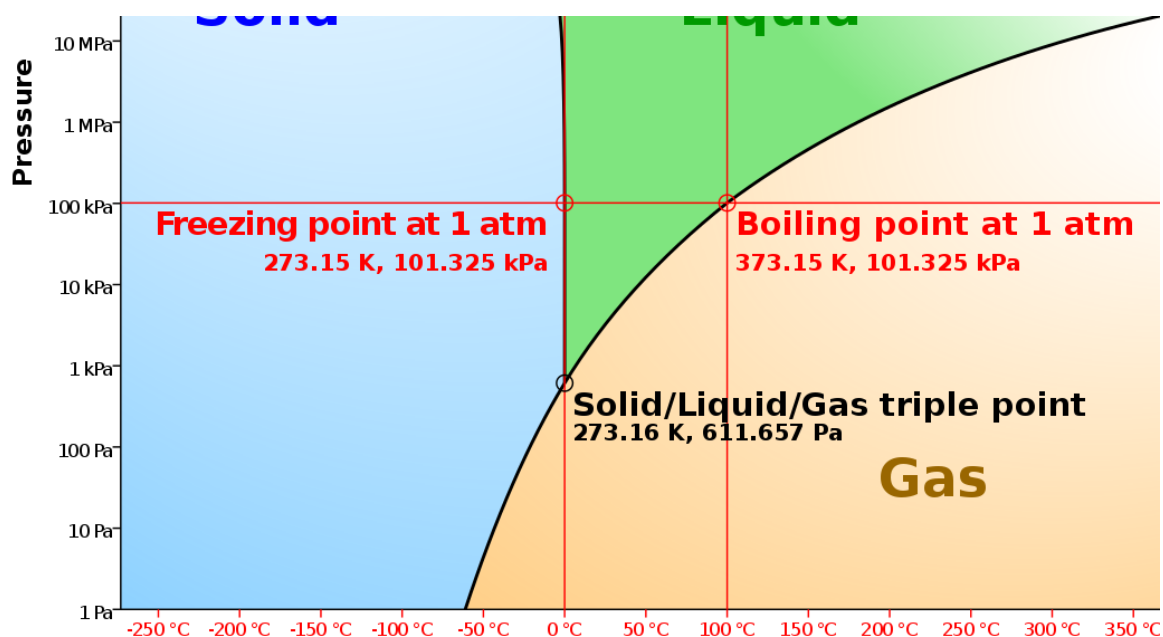
These temperatures are at standard pressure (100 kPa or 1 atm). Pressure also has a role

in phase change. In low pressure environments, it is much easier for the molecules to make the jump to being a gas.

For example, if you climb a mountain until the atmospheric pressure is 70 kPa, your water will boil at about 90° C.

If you rise in a balloon until the atmospheric pressure is 500 Pa, water cannot be liquid at any temperature. Ice turns directly into vapor, and liquid water never appears. There is no liquid water at 500 Pa!

For any molecule, we could observe its phase at a wide range of temperatures and pressures. This would let us create a phase diagram. Here is the phase diagram for H_2O :



2.2.2 Phase Changes and Energy

The molar heat capacity of ice is about 37.7 J/mol-K. That is, it takes about 37.7 Joules of energy to raise the temperature of one mole of ice by one degree kelvin.

The molar heat capacity of liquid water is about 75.4 J/mol-K. For water vapor, it is about 36.6 J/mol-K.

Imagine you have mole of ice at 173 K, and you are going to gradually add kinetic energy into it until you have steam at 473 K. You might guess (incorrectly) that the temperature vs. energy applied would look like this:

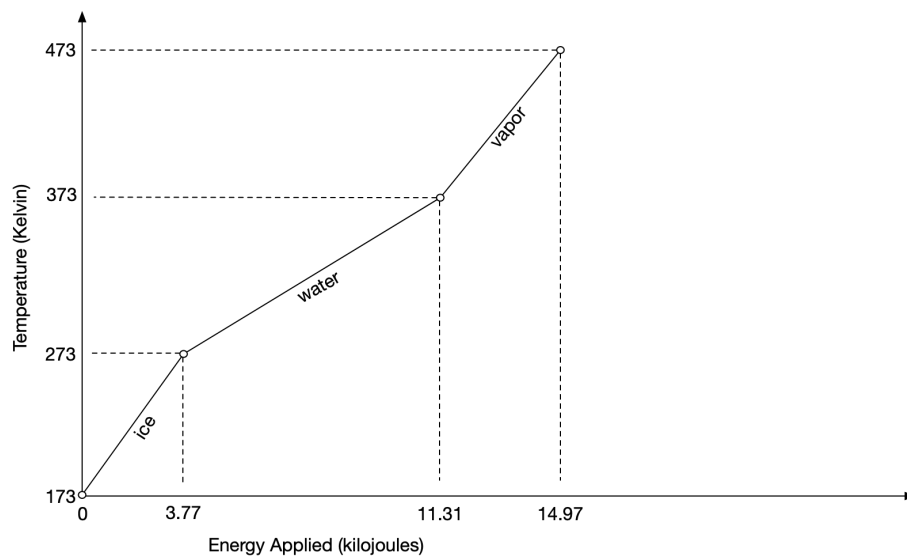


Figure 2.1: Assumption of temperature vs. energy applied.

However, once molecules are nestled into their solid state (like eggs in cartons), it takes extra energy to make them move like a liquid. How much more energy? For water, it is 6.01 kilojoules per mole. This is known as *the latent heat of melting* or *the heat of fusion*.

Similarly, the transition from liquid to gas takes energy. At standard pressure, converting a mole of liquid water to vapor requires 40.7 kilojoules per mole. This is known as *the latent heat of vaporization*. So, the graph would actually look like this:

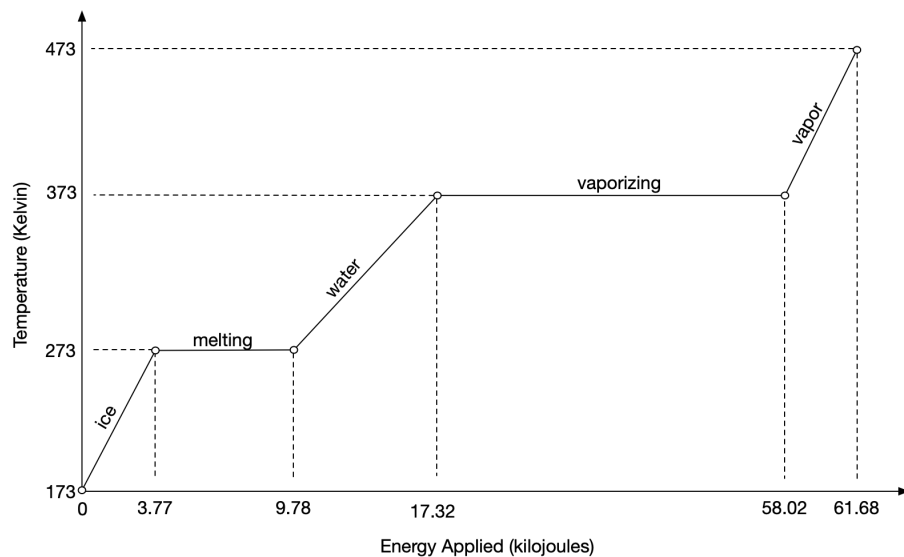


Figure 2.2: Actual temperature vs. energy applied.

Note that just as melting and vaporizing require energy, going the other way (freezing and condensing, respectively) give off energy. Thus, we can store energy using the phase change.

2.2.3 How a Rice Cooker Works

A rice cooker is a bowl with a lid and an electric heating element. You put rice and water into the bowl and turn on the heating element. The heating element pushes kinetic energy into the water, which gets warmer and eventually starts to boil.

How does the rice cooker know when to turn off (or at least down to a low-heat “keep warm” mode) before the rice starts to burn?

As long as there is a little liquid water in the bottom of the bowl, the rice will not burn, so the question is really “How does it know when there is no more liquid water in the bottom of the bowl?”

There is a mechanism (and there have been a few different versions of this mechanism) that monitors the temperature of the surface of the bowl. As long as there is liquid water in the bowl, it *cannot* go above 100°C ! When all the water has been absorbed by the rice or turned to steam, the temperature rises above 100°C , and the mechanism cuts off the heat.

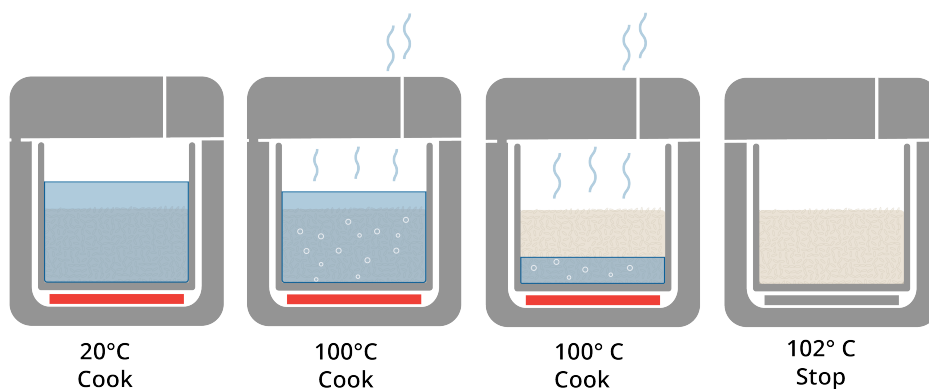


Figure 2.3: The various phases of a rice cooker.

Exercise 8 Using Water For Thermal Energy Storage

Tom is building a passive solar house; the front of the house is a greenhouse, and Tom likes to eat dinner there. Tom

Working Space

plans to use barrels (painted black) that hold 159 liters of water. The plan is to let the sun heat the barrels to 33°C by the time the sun goes down. (Any warmer and it would be unpleasant to eat dinner near them.)

At night, air will circulate past the barrels and through the house. Tom is OK with the house and the barrels dropping to 17°C .

Looking at the house's insulation and the expected nighttime temperatures, Tom estimates that the barrels need to store 300,000 kJ of energy in the barrels.

A mole of water is about 0.018 liters.

The molar heat capacity of water is about 75.38 J/mole-K .

How many barrels of water does Tom need to install in the greenhouse?

Answer on Page 81

Exercise 9 Using Mirabilite For Thermal Energy Storage

Working Space

Water barrels are going to take up too much of Tom's greenhouse!

There is a substance known as mirabilite, or Glauber's salt, or sodium sulfate decahydrate. It is relatively cheap to produce, and it has a melting point of 32.4°C .

The molar heat capacity of mirabilite is 550 J/mole-K .

The latent heat of melting mirabilite is 82 kJ per mole.

Mirabilite comes in a powder form. Assume that a mole of mirabilite occupies about 0.22 liters.

If Tom fills the barrels with mirabilite instead, how many barrels are needed?

Answer on Page 82

2.2.4 Thinking Statistically About Phase Change

We like to say simple things like "At 100° C, water changes to vapor." However, remember what temperature is: Temperature tells you how much average kinetic energy the water molecules have. The key word here is *average*; Some molecules are going faster than average and some are going slower.

A puddle in the street on a warm night will evaporate. It is not 100° C. Why would the puddle turn to vapor?

While the *average* molecule in the puddle does not have enough energy to escape the intermolecular forces, some of the molecules do. When a molecule on the surface has enough velocity (toward the sky!) to escape the intermolecular forces, it becomes vapor and leaves the puddle.

What happens to the temperature of the puddle during this sort of evaporation? Temperature is proportional to the average kinetic energy of the molecules. If a bunch of molecules with a lot of kinetic energy leave, the average kinetic energy (of the molecules that remain) will decrease.

The most obvious example of this process is sweating: When your body is in danger of getting too hot, sweat comes out of your pores and covers your body. The fastest moving molecules escape your body, taking the excess kinetic energy with them.

Evaporative Cooling Systems

An evaporative cooling system (also called a "Swamp Cooler") uses this idea to cool air. You can imagine a fan drawing warm air through a duct from the outside. Before the air is released inside, it passes very close to a cloth that is soaked with water. The warm air molecules (which have a lot of kinetic energy) slam into the water molecules, some of which get enough kinetic energy to become vapor. The cool air and the water vapor then enter the room.

"Wait, wait, wait," you say, "The heat has not gone away. It is just transferred into the water molecules."

Remember that it takes 40.7 kilojoules to change a mole of liquid water at 100° C to water vapor at 100°. Escaping those intermolecular bonds takes a great deal of energy!

Thus, if a mole of water evaporates, it has absorbed 40.7 kJ of heat that does not raise its temperature. The energy goes into breaking the molecules free of their intermolecular bonds. Physicists call this *latent heat*.

2.2.5 Humidity and Condensation

When a puddle is evaporating on a warm day, there might be some water vapor already in the air. Even as the water molecules in the puddle are evaporating, some water molecules in the air are crashing into the puddle and become liquid again. (We say they *condensed*, thus the word *condensation* to describe the water that accumulates on a cold glass on a warm day.)

When there is a lot of water vapor in the air, the puddle will evaporate more slowly. In fact, if there is enough water vapor in the air, the puddle will not evaporate at all. At this point, we say "The relative humidity is 100%." That is, relative to the amount of water the air will hold, it already has 100% that amount.

Neither sweating nor evaporative cooling systems work well when the relative humidity is high.

As the temperature goes up, the air can hold more water. We usually notice it when it goes the other way: the air cools and has more water vapor than it can hold. Some of the water vapor condenses into water droplets. If the droplets land on something, we call it "dew". If it is high in the sky, we call the droplets "a cloud". If it is near the ground, we call it "fog".

The Piston Engine

A car engine at highway speed sets off about 100 small explosions every second, and each one shoves a piston down with the force of a small car sitting on it. Turning those explosions into smooth motion is a triumph of engineering.

Most cars, airplanes, and chainsaws get their power from burning hydrocarbons in a combustion chamber. We say they have *internal combustion engines*. There are many types of internal combustion engines: jet engines, rotary engines, diesel engines, and more. In this chapter, we are going to explain how one type, piston engines, work. Most cars have piston engines.

Most piston engines burn gasoline, which is a blend of liquid hydrocarbons. Hydrocarbons are molecules made of hydrogen and carbon (and maybe a little oxygen). In the presence of oxygen and heat, hydrocarbons burn — the carbon combines with oxygen to become CO_2 , and the hydrogen combines with oxygen to become H_2O . In the process, heat is released, which causes the gases in the cylinder to create a lot of pressure on the piston.

3.1 Parts of the Engine

The engine block is a big hunk of metal. There are cylindrical holes bored into the engine block. A piston can slide up and down the cylinder. There are two valves in the wall of the cylinder:

- Before the burn, one valve opens to let fuel and air into the cylinder.
- After the burn, the other valve opens to let the exhaust out.

There is also a spark plug, which creates the spark that triggers the burn.

As you give the engine more gas, the cylinder does more frequent burns. When the engine is just idling, the cylinder fires about 9 times per second. When you depress the gas pedal all the way down, it is more like 40 times per second.

Each piston has a rod that connects it to the crank shaft. As the pistons move back and forth, the crank shaft turns around and around. Sometimes a piston is pushing the crankshaft, and sometimes the crank shaft is pushing or pulling the piston. All the cylinders share one crank shaft.

How many cylinders does a car have? Nearly all car models have between 3 and 8 cylinders. The opening of the valve and the firing of the spark plugs are timed so that cylinders all do their burns at different times. This makes the total power delivered to the crank shaft smoother.

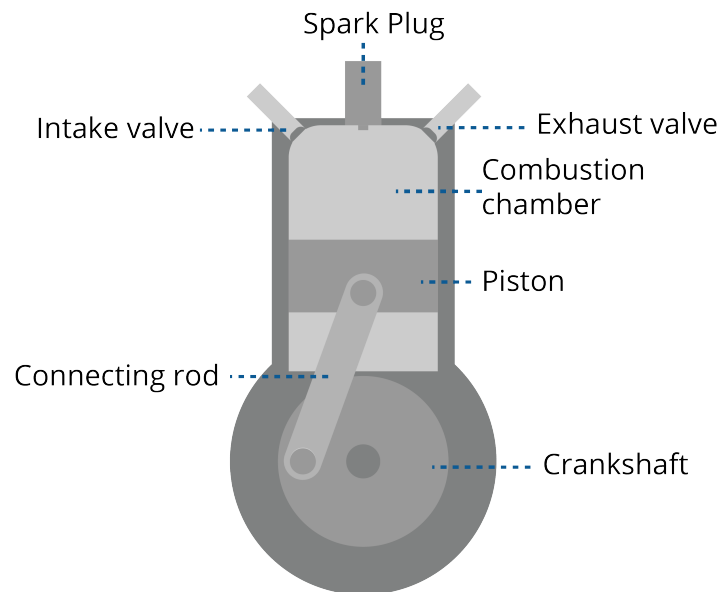


Figure 3.1: A cross-section of an 8 cylinder engine.

3.2 The Four-Stroke Process

Cars have four-stroke engines — this means for every two rotations of the crank shaft, each cylinder fires once. Smaller engines, like those in chainsaws, are often two-stroke engines — every cylinder fires every time the crank shaft rotates. For now, let us focus on four-stroke engines.

Here is the cycle of a single cylinder:

- As the crank shaft turns, it pulls the piston down. The intake valve opens and lets the gas/air mixture into the combustion chamber. See Figure 3.2.

Intake

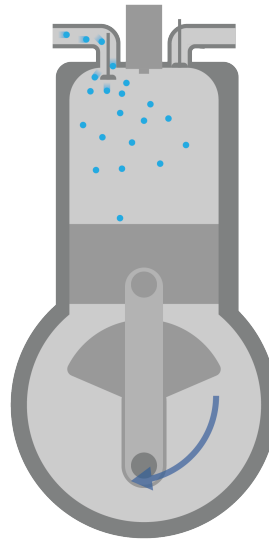


Figure 3.2: The intake stroke draws the fuel-air mixture into the combustion chamber.

- As the piston reaches the bottom of the stroke, the intake valve closes.
- Now, the crank shaft starts to push the piston up, compressing the fuel and air. See [Figure 3.3](#).

Compression

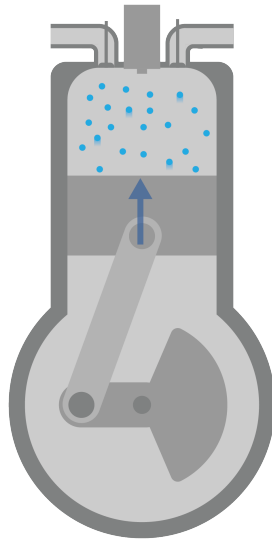


Figure 3.3: The compression stroke squeezes the fuel-air mixture.

- As the piston reaches the top of its stroke, the spark plug creates a spark. The fuel and oxygen burn quickly. The cool liquid fuel becomes hot carbon dioxide and water vapor.

Combustion

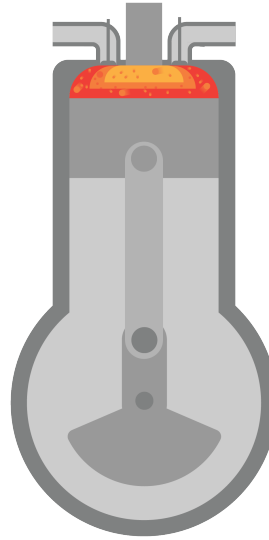


Figure 3.4: On the power stroke, the spark ignites the mixture, and the hot gases drive the piston down.

- There is now very high pressure inside the cylinder. It pushes hard on the piston, which pushes the crank shaft.
- When the piston reaches the bottom of this stroke, the exhaust valve opens.
- As the crank shaft pushes the piston up, the carbon dioxide and water vapor is pushed out.

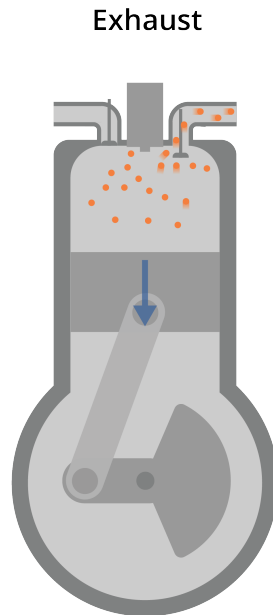


Figure 3.5: On the exhaust stroke, the burnt gases are expelled from the cylinder.

- When the piston reaches the top of this stroke, the exhaust valve is closed.

3.3 Dealing with Heat

Burning fuel inside a block of metal generates a large amount of heat. If there is too much heat, parts of the engine will start to melt. So, modern car engines are liquid cooled — there are arteries in the engine block carrying a liquid (called “coolant”). The hot coolant is pumped through the radiator (where the air passing through takes away the heat), then back into the engine.

Note that the heat that is carried away by the coolant is wasted energy. In fact, of the total energy created in burning the fuel, most car engines only transfer about 30% to turning the crank shaft. About 35% of the heat goes out with the exhaust. About 30% is carried away by the coolant. The remaining waste (usually about 5%) is lost to friction.

3.4 Dealing with Friction

From the description, it is clear that there is a great deal of metal sliding against metal, which would grind the engine up quickly if there were no lubrication. In a modern car, the moving parts in the engine are constantly bathed in oil. There is an oil pump that causes it to get sprayed on the crankshaft, the connecting rod, and in the cylinder under

the piston (that is, not on the combustion side).

The oil eventually falls back into a pan at the bottom of the engine. The oil pump sucks the oil up, pushes it through a filter (so bits of metal are not pumped back into the engine), and is then sprayed on the moving parts again.

3.5 Challenges

With a piston engine, there are a lot of things that can go wrong. Let us enumerate a few:

- *The seal around the piston leaks.* Mechanics say "We are not getting any compression." The cylinder does not get much power to the drive train.
- *The valves open or the spark plug fires at the wrong time.* This is known as a timing problem.
- *The spark plug does not make a spark.* The spark plug has two prongs of metal and electrons jump from one to the other. For a good spark, the prongs need to be a very precise distance apart. Sometimes you need to bend one of the prongs to get the right gap. This is known as *gapping*.
- *The mix of fuel and oxygen is wrong.* If there is too much fuel and not enough oxygen (so not all the fuel burns), we say the mix is too rich. If there is not enough fuel (so the burn produces less pressure than it could), we say the mix is too lean.

3.6 How We Measure Engines

If you look up the specs on an engine, you will see the following:

- The number of cylinders
- The cylinder bore, which is the diameter of the cylinder
- The piston stroke, which is the distance the piston travels in the cylinder
- The compression ratio, which is the ratio between the maximum volume of the combustion chamber and the minimum volume of the combustion chamber
- What fuel it runs on

The difference between the minimum and maximum volume of the cylinder is known as its *displacement*. The displacement represents the volume of air/fuel sucked into the intake valve before the compression begins.

We often talk about the displacement of the entire engine, which is the cylinder's displacement times the number of cylinders. The displacement of an engine can give you a

good idea of how much power it can produce.

For motorcycles, the displacement is often part of the name. For example, the Kawasaki Ninja 650 has about 650 cubic centimeters of displacement.

3.7 The Ford Model T and Ethanol

The Ford Model T was the first mass-produced car. It came out in 1908 and remained in production until 1927. It had a four-cylinder engine that would run on ethanol, benzene, or kerosene. For the purposes of this exercise, let us assume you are running yours on ethanol.

A molecule of ethanol has 2 carbon atoms, 6 hydrogen atoms, and 1 oxygen atom. The oxygen in the atmosphere is O_2 . When one molecule of ethanol combines with 3 molecules of O_2 , 2 molecules of CO_2 and 3 molecules of H_2O are created. A large amount of heat is also created — 1330 kilojoules for every mole of ethanol burned.

The engine block is usually very hot once the engine has been running. That is important because the ethanol will be completely vaporized at that temperature.

In any sample of air, 21

Exercise 10 Fuel Mix for the Model T

Working Space

On the Model T, a carburetor mixed the fuel and air before it went into the cylinder. The question to answer in this exercise is: How rich should the mix be at sea level (100 kPa)?

On the Wikipedia page for the Model T, we see the following facts:

- Cylinder bore: 9.525 cm
- Piston stroke: 10.16 cm

You can assume that the air/fuel mixture is 80°C before the pre-burn compression starts. (Thus, the ethanol, which boils at 78°C , is in its vapor phase.)

The questions, then, are:

- What is the displacement of a single cylinder?
- How many moles of gas (80°C and 100 kPa) will get sucked through the intake valve?
- How many moles of vaporized ethanol should be part of that?
- How many moles of CO_2 and H_2O are created in each burn?
- How much heat is created in each burn?

(This exercise has a lot of steps, but nothing you do not already know. You will need the Ideal Gas Law to figure out how many moles of air get dragged into the cylinder.)

Answer on Page 82

3.8 Compression Ratio

A large part of the inefficiency of an engine is heat that escapes through the exhaust valve. If your piston stroke were long enough, you could keep increasing the volume (which would cool the gases inside) until the gases inside were the same temperature as the outside world. Then there would be no wasted heat in the exhaust.

For this reason, generally, engines with a higher compression ratios tend to waste less energy through the heat of the exhaust. The Model T had a compression ratio close to 4:1. Modern car engines typically have compression ratios between 8:1 and 12:1. (The next chapter, on entropy and the second law of thermodynamics, shows why no engine can turn all of its heat into work, however long its stroke.)

Cars with especially high compression ratios often need fuel with a high octane rating (*high octane*). Octane measures how well a fuel resists igniting too early. When the fuel-air mix is compressed very hard, a low-octane fuel can ignite on its own before the spark,

which causes loud knocking.

It turned out that an easy way to boost the octane of the gasoline was to add a chemical called tetraethyl lead. Gasoline containing tetraethyl lead was known as “leaded gasoline”, and was intended to prevent the knocking. It is difficult to overstate the damages caused by putting large amounts of lead in the air. Japan banned leaded gasoline in 1986, and the last countries stopped using it around 2021.

3.9 The Choke and Fuel Injection

Most cars built before 1990 will have a carburetor, which ensures that the ratio between fuel and oxygen is constant, regardless of the amount of fuel released by the throttle.

If you go to start an old car on a cold morning, the cold engine will not vaporize the fuel properly, and the engine may not have enough power to start. For this reason, most carburetors have a *choke valve* that makes the mix richer. (If you pull the choke valve, be sure to push it back after the engine warms up.)

The carburetor was a common source of engine problems and inefficiencies. During the 1980s, carburetors were replaced by *fuel injection*: nozzles that spray a precise amount of fuel into the incoming air just before the intake valve. Many newer engines go a step further with *direct injection*, spraying the fuel straight into the cylinder.

In modern cars, the fuel injection system is controlled by a computer (an *Engine Control Module* or ECM), which delivers the fuel at the perfect time with the perfect amount based on variables like the temperature of the engine and the barometric pressure (which depends on the altitude at which the engine is operating).

Exercise 11 Hot from Compression

A modern engine compresses air with a compression ratio of 10:1. If the air starts at 300 K and the compression is fast enough to be adiabatic, how hot does it get? (For air, $\gamma \approx 1.4$.) Compare the Model T, with a ratio of about 4:1.

Working Space

Answer on Page 83

Entropy and the Second Law of Thermodynamics

Your bedroom gets messy all by itself, but it never cleans itself. A mug of hot cocoa left on your desk cools to room temperature; it never heats itself back up by pulling warmth out of the room. Drop an egg and it splatters, but you will never see a splattered egg leap back into its shell.

None of these would break the law of conservation of energy. The cocoa could, in principle, take heat from the room without creating or destroying any energy. Something else decides which way things go. That something is *entropy*, and the rule it follows is the *second law of thermodynamics*. The same rule tells you why no car engine can ever turn all of its fuel into motion, and why a heat pump can heat a house so cheaply.

4.1 The First Law, Briefly

You already know the first law: energy is never created or destroyed. For a gas in a cylinder, the change in its internal energy U (the kinetic energy of its molecules) is the heat Q that flows in minus the work W the gas does on the outside world:

$$\Delta U = Q - W$$

The first law is an accounting rule. It tells you that the books must balance, but it says nothing about which transactions actually happen. For that, we need the second law.

4.2 Which Way Does Heat Flow?

Put a hot pan in cool water, and heat flows from the pan to the water until they reach the same temperature, as you saw in the Heat and Phases of Matter chapter. It never goes the other way on its own.

The Second Law (Heat Version)

Heat flows on its own from a hotter object to a colder one, never from colder to hotter.

An air conditioner seems to break this rule: it moves heat from your cool house to the hot

outdoors. But it does not do it *on its own*. It needs a compressor, plugged into the wall, doing work. Unplug it, and the heat flows back in.

Why should heat care about direction? The answer comes from counting.

4.3 Counting Microstates

Flip 4 coins. If you only care about how many came up heads, there are 5 possible results: 0, 1, 2, 3, or 4 heads. Each of these is a *macrostate*: a description of the big picture. But each macrostate can happen in several ways. Each exact arrangement, like HTHT, is a *microstate*. You learned to count them in the Counting chapter: the number of ways to get k heads from n coins is $\binom{n}{k}$.

Heads	Microstates	Count
0	TTTT	1
1	HTTT, THTT, TTHT, TTTH	4
2	HHTT, HTHT, HTTH, THHT, THTH, TTHH	6
3	THHH, HTHH, HHTH, HHHT	4
4	HHHH	1

Every microstate is equally likely, so “2 heads” wins simply because it has the most ways to happen.

Now flip 100 coins. There is exactly 1 way to get all heads. The number of ways to get 50 heads is

$$\binom{100}{50} \approx 1.01 \times 10^{29}$$

The half-and-half macrostate is not just more likely; it overwhelms everything else.

Gas molecules work the same way. Imagine a box with a wall down the middle and 100 molecules crowded into the left half. Pull out the wall. Each molecule is equally likely to be on either side, so the chance that all 100 are on the left at any moment is

$$\left(\frac{1}{2}\right)^{100} \approx 7.9 \times 10^{-31}$$

A real room holds about 10^{27} molecules, so the chance that all the air in your classroom crowds into one corner is too small to write down. Nothing forbids it; it just never happens.

4.3.1 Watching It Happen

Here is a simple model of the box. At each step, pick a random molecule and move it to the other side. Create a file called `two_rooms.py`:

```

1 import random
2 import matplotlib.pyplot as plt
3
4 plt.rcParams["font.size"] = 14
5 random.seed(1)
6 n = 100                                # number of molecules
7 left = [True] * n                      # every molecule starts on the left
8 counts = [n]
9
10 for step in range(1000):
11     i = random.randrange(n)            # pick a random molecule...
12     left[i] = not left[i]              # ...and move it to the other side
13     counts.append(sum(left))
14
15 print("after 1000 steps:", counts[-1], "on the left")
16 print("average of the last 500 steps:", sum(counts[500:]) / len(counts[500:]))
17 print("all on the left again?", n in counts[1:])
18
19 plt.plot(counts)
20 plt.axhline(n / 2, color="gray", linestyle="--")
21 plt.xlabel("step")
22 plt.ylabel("molecules on the left")
23 plt.ylim(0, n)
24 plt.tight_layout()
25 plt.savefig("two_rooms.png", dpi=150)

```

When you run it, you should see:

```

after 1000 steps: 56 on the left
average of the last 500 steps: 50.45109780439122
all on the left again? False

```

No rule in the program pushes molecules to the right. Each move is completely random. The gas spreads out only because there are vastly more half-and-half arrangements than lopsided ones. That is the second law: systems drift toward the macrostates with the most microstates.

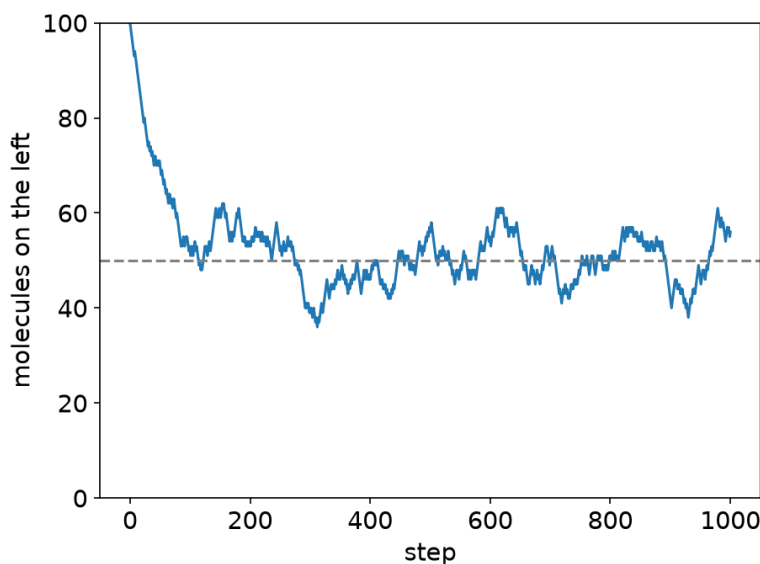


Figure 4.1: The molecules spread out until about half are on each side, then jiggle around 50 forever.

4.3.2 Entropy Is Counting

Ludwig Boltzmann turned this idea into a formula. If a macrostate can happen in W ways, its entropy is

Boltzmann's Entropy

$$S = k \ln W$$

where $k = 1.38 \times 10^{-23}$ J/K is Boltzmann's constant.

Boltzmann thought this formula was so important that it is carved on his tombstone.

Why the logarithm? Put two systems side by side. If one can be arranged in W_1 ways and the other in W_2 ways, together they can be arranged in $W_1 W_2$ ways. The logarithm turns that product into a sum, $\ln(W_1 W_2) = \ln W_1 + \ln W_2$, so entropies add, just as masses and energies do.

The Second Law (Entropy Version)

The total entropy of an isolated system never decreases.

4.4 Entropy and Heat

Counting microstates of 10^{27} molecules is hopeless, but there is a shortcut. When a small amount of heat Q flows into an object at temperature T (in kelvins), its entropy changes by

$$\Delta S = \frac{Q}{T}$$

Adding heat gives molecules more ways to share energy, so entropy rises. The same heat matters more to a cold object than to a hot one, which is why T is in the denominator.

Example: Your cocoa is at 350 K and your room is at 300 K. Suppose 1,000 J of heat flows from the cocoa to the room. (Both temperatures barely change during this small flow.) What happens to the total entropy?

Solution: The cocoa loses heat, so its entropy falls:

$$\Delta S_{\text{cocoa}} = \frac{-1000}{350} \approx -2.86 \text{ J/K}$$

The room gains the same heat at a lower temperature, so its entropy rises by more:

$$\Delta S_{\text{room}} = \frac{1000}{300} \approx 3.33 \text{ J/K}$$

The total rises by about 0.48 J/K. If the heat flowed the other way, from the room into the cocoa, every sign would flip and the total would *fall*. That is forbidden, so cocoa never heats itself up. The heat version and the entropy version of the second law are the same law.

Example: In the Heat and Phases of Matter chapter, you learned that melting ice takes 6.01 kJ per mole, which is about 334 J per gram. What is the entropy change when 10 g of ice melts at 273 K?

Solution: $Q = (10)(334) = 3340 \text{ J}$, all absorbed at 273 K, so

$$\Delta S = \frac{3340}{273} \approx 12.2 \text{ J/K}$$

Liquid water has more entropy than ice: the molecules are no longer locked in a crystal, so there are far more ways to arrange them.

4.5 Heat Engines and the Carnot Limit

In the Piston Engine chapter, you learned that a car engine turns only about 30% of its fuel's energy into turning the crankshaft. Better engineering helps, but no engine, however perfect, can reach 100%. Entropy explains why.

Every *heat engine* works the same way. It takes in heat Q_H from something hot (burning fuel, steam), turns part of it into work W , and dumps the rest, Q_C , into something cold (the air, a river).

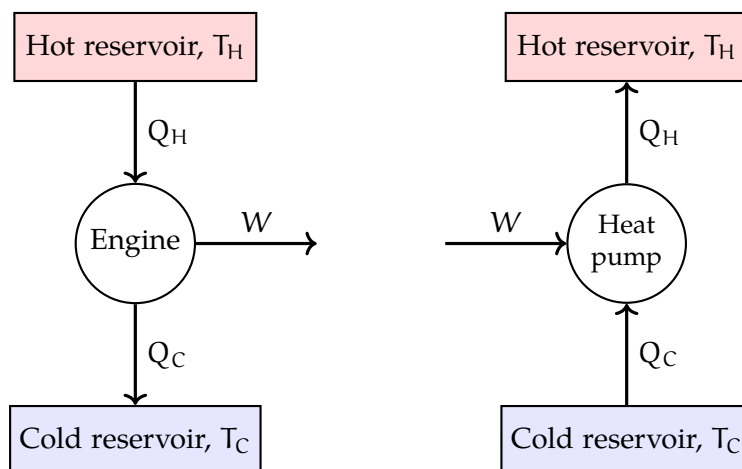


Figure 4.2: Left: a heat engine turns part of the heat flowing from hot to cold into work. Right: a refrigerator or heat pump uses work to push heat from cold to hot.

The *efficiency* is the fraction of the heat that becomes work:

$$e = \frac{W}{Q_H} = \frac{Q_H - Q_C}{Q_H}$$

Now track the entropy. The hot reservoir loses Q_H/T_H of entropy. The cold reservoir gains Q_C/T_C . The engine itself returns to the same state after every cycle, so its entropy does not change. Work carries no entropy. The second law says the total cannot fall:

$$\frac{Q_C}{T_C} \geq \frac{Q_H}{T_H} \quad \text{so} \quad \frac{Q_C}{Q_H} \geq \frac{T_C}{T_H}$$

The engine *must* dump some heat into the cold reservoir, or the entropy would go down. Substitute into the efficiency:

The Carnot Limit

$$e \leq 1 - \frac{T_C}{T_H}$$

Temperatures must be in kelvins.

Sadi Carnot, a French engineer, found this limit in 1824 by thinking about steam engines, decades before anyone understood entropy. An engine that reaches the limit exactly is called a *Carnot engine*. It would have to run infinitely slowly, so real engines always fall short.

Example: A power plant heats steam to 800 K and cools it with river water at 300 K. What is the best possible efficiency?

Solution:

$$e \leq 1 - \frac{300}{800} = 0.625$$

At most 62.5%. Real plants like this one reach about 40%. If this plant sends 1,000 MW of electricity to the grid at 40% efficiency, it burns fuel at $1000/0.4 = 2,500$ MW and dumps 1,500 MW of heat into the river and air. That is what those giant cooling towers are for.

The Carnot limit explains a lesson from the Piston Engine chapter: a higher compression ratio makes the burning gas hotter, which raises T_H , which raises the limit. To make any engine more efficient, make the hot side hotter or the cold side colder.

4.6 Refrigerators and Heat Pumps

Run an engine backwards and you get a refrigerator: work W pulls heat Q_C out of the cold inside and dumps $Q_H = Q_C + W$ into your warm kitchen. The coils on the back of your fridge are warm because they are getting rid of that heat.

For a heat pump warming a house in winter, what you want is the heat delivered inside, Q_H . Its *coefficient of performance* is

$$\text{COP} = \frac{Q_H}{W}$$

The same entropy argument (try it in an exercise) gives the limit

$$\text{COP} \leq \frac{T_H}{T_H - T_C}$$

Example: It is 0°C (273 K) outside, and you want to keep your house at 20°C (293 K). What is the best possible COP?

Solution:

$$\text{COP} \leq \frac{293}{293 - 273} = \frac{293}{20} \approx 14.7$$

A perfect heat pump would deliver almost 15 joules of heat for each joule of electricity. Real ones deliver about 3. Even so, that beats an electric space heater, which turns each joule of electricity into exactly 1 joule of heat. A heat pump does not make heat; it moves heat that is already outside, and moving heat is cheaper than making it.

Notice that the COP gets worse as the outside gets colder: the bigger $T_H - T_C$ is, the harder the pump must work.

4.7 Order Without Breaking the Law

If entropy always increases, how can a snowflake form, a seed grow into a tree, or you clean your room? Each of these makes something more orderly, which lowers its entropy.

The answer is the word *isolated*. A snowflake freezes by releasing heat into the cold air, and that heat raises the air's entropy by more than the snowflake loses. A tree uses sunlight. You burn breakfast to clean your room, and give off heat as you do it. Order can grow in one place, as long as more disorder is pushed somewhere else.

The second law also tells time which way to run. Almost every other law of physics works just as well backwards: a video of two billiard balls colliding looks fine played in reverse. A video of an egg unbreaking does not. The direction we call "the future" is the direction in which entropy increases. Followed to the end, it leads to the heat death of the universe, which the chapter on matter and energy mentioned: everything at one temperature, with no flow of heat left to run anything.

Exercise 12 Six Coins

You flip 6 coins. How many microstates have exactly 3 heads? What is the probability of getting exactly 3 heads? What is the probability of getting all heads?

Working Space

Answer on Page 83

Exercise 13 Ice in Your Drink

You drop 20 g of ice at 273 K into a drink at 293 K. Assume all of the ice melts, absorbing 334 J/g, and that the drink stays near 293 K. Find the entropy change of the ice, of the drink, and of the two together.

Working Space

Answer on Page 84

Exercise 14 An Inventor's Claim

An inventor says their new engine runs between a hot reservoir at 400 K and a cold one at 300 K, with an efficiency of 30%. Should you invest?

*Working Space**Answer on Page 84***Exercise 15 Geothermal Power**

A geothermal plant uses underground water at 450 K and cools it with air at 300 K. What is its maximum efficiency? Why do geothermal plants have lower efficiencies than plants that burn fuel?

*Working Space**Answer on Page 84***Exercise 16 A Cold Snap**

The outside temperature drops to -10°C (263 K), and the house is still kept at 293 K. Find the best possible COP of the heat pump, and compare it with the 0°C example.

*Working Space**Answer on Page 84*

Exercise 17 Open the Fridge

On a hot day, could you cool your kitchen by leaving the refrigerator door open?

Working Space

Answer on Page 84

Exercise 18 Proving the Heat Pump Limit

Use the entropy argument from the Carnot section to prove that a heat pump's COP is at most $T_H/(T_H - T_C)$. (Hint: now the cold reservoir loses Q_C/T_C and the hot reservoir gains Q_H/T_H , with $Q_C = Q_H - W$.)

Working Space

Answer on Page 84

Exercise 19 More Molecules

Make your simulation in `two_rooms.py` use 1,000 molecules and 10,000 steps. After the gas has spread out, the count on the left wanders around 500. Is the wandering bigger or smaller, as a fraction of all the molecules, than it was with 100?

Working Space

Answer on Page 85

u-Substitution

The chain rule lets you differentiate almost anything: the derivative of $\sin(x^3)$ is $\cos(x^3) \cdot 3x^2$. But what about going backward? If someone hands you $\int 3x^2 \cos(x^3) dx$, can you recognize it as the result of a chain rule? *U-substitution*, also known as the method of substitution, is the chain rule run in reverse.

Suppose we have an integral of the form:

$$\int f(g(x)) \cdot g'(x) dx \quad (5.1)$$

The u-substitution method suggests letting a new variable u be equal to the inside function $g(x)$, i.e.,

$$u = g(x) \quad (5.2)$$

Next, the differential of u , du , is given by:

$$du = g'(x) dx \quad (5.3)$$

Substituting u and du back into the integral gives us a simpler integral:

$$\int f(u) du \quad (5.4)$$

This new integral can often be simpler to evaluate. Once the antiderivative of $f(u)$ is found, we can substitute $u = g(x)$ back into the antiderivative to get the antiderivative of the original function in terms of x .

The method of u-substitution is a powerful tool for evaluating integrals, especially when combined with other techniques like integration by parts, partial fractions, and trigonometric substitutions.

Example: Find $\int 2x^2 \cos(x^3 - 3) dx$.

Solution: Integrating $\cos(x^3 - 3)$ is not so straightforward, so let us try the substitution $u = x^3 - 3$. Then:

$$du = 3x^2 \, dx$$

We do not have $3x^2$ in the integral, but we do have $2x^2$:

$$\frac{2}{3} du = 2x^2 \, dx$$

Substituting:

$$\begin{aligned} \int 2x^2 \cos(x^3 - 3) \, dx &= \int \frac{2}{3} \cos(u) \, du \\ &= \frac{2}{3} \sin(u) + C \end{aligned}$$

Now that we have the antiderivative of $f(u)$, we can back-substitute in for u :

$$\frac{2}{3} \sin(u) + C = \frac{2}{3} \sin(x^3 - 3) + C$$

We can check our answer by taking its derivative: we should get the original integrand back:

$$\begin{aligned} \frac{d}{dx} \left[\frac{2}{3} \sin(x^3 - 3) + C \right] &= \frac{2}{3} \cos(x^3 - 3) \cdot \left[\frac{d}{dx} (x^3 - 3) \right] \\ &= \frac{2}{3} \cos(x^3 - 3) \cdot (3x^2) = 2x^2 \cos(x^3 - 3) \end{aligned}$$

Sometimes, the right substitution takes a little thinking. Consider the following example:

Example: Find $\int \sqrt{x^2 - 1} \, x^5 \, dx$.

Solution: We can guess that $u = x^2 - 1$ could be an appropriate substitution, as that is what is under the square root. What to do with x^5 ? First, let us look at how the u -substitution for $x^2 - 1$ works out:

$$u = x^2 - 1$$

$$du = 2x \, dx$$

$$\frac{du}{2} = x \, dx$$

Then we will need to use one of the x 's in x^5 for the square root u -substitution. What can we do with the remaining x^4 ? Well, we see that if $u = x^2 - 1$, then $u + 1 = x^2$ and therefore

$(u + 1)^2 = x^4$. Substituting this all in:

$$\begin{aligned}\int \sqrt{x^2 - 1} x^5 dx &= \int x^4 \sqrt{x^2 - 1} x dx \\ &= \frac{1}{2} \int (u + 1)^2 \sqrt{u} du\end{aligned}$$

We can expand this to find the antiderivative:

$$\begin{aligned}&= \frac{1}{2} \int (u^2 + 2u + 1) u^{1/2} du = \frac{1}{2} \int u^{5/2} + 2u^{3/2} + u^{1/2} du \\ &= \frac{1}{2} \left[\frac{2}{7} u^{7/2} + \frac{4}{5} u^{5/2} + \frac{2}{3} u^{3/2} \right] + C \\ &= \frac{1}{7} u^{7/2} + \frac{2}{5} u^{5/2} + \frac{1}{3} u^{3/2} + C \\ &= \frac{1}{7} (x^2 - 1)^{7/2} + \frac{2}{5} (x^2 - 1)^{5/2} + \frac{1}{3} (x^2 - 1)^{3/2} + C\end{aligned}$$

Exercise 20 Indefinite Integrals and u-substitution

Use u-substitution to evaluate the following indefinite integrals. Confirm your answer by taking the derivative of the result.

Working Space

1. $\int \sin x \sqrt{1 + \cos x} dx$
2. $\int \frac{\cos(\pi/x)}{x^2} dx$
3. $\int 2x^2 (9 - x^3)^{2/3} dx$
4. $\int 3x^2 \sqrt{1 + x} dx$
5. $\int \frac{3x^2}{x^3 - 1} dx$

Answer on Page 85

5.1 The Substitution Rule for Definite Integrals

How do we use u -substitution for definite integrals? We will apply the fundamental theorem of calculus to answer this question. We define f and F such that F is the antiderivative of f . Then:

$$\int_a^b f(g(x)) \cdot g'(x) \, dx = F(g(x)) \Big|_a^b = F(g(b)) - F(g(a))$$

This represents the method of finding the indefinite antiderivative and evaluating from the original limits of integration.

We can also see that:

$$F(g(b)) - F(g(a)) = F(u) \Big|_{g(a)}^{g(b)} = \int_{g(a)}^{g(b)} f(u) \, du$$

Therefore, if g' is continuous on $[a, b]$ and f is continuous on the range of $u = g(x)$, then:

$$\int_a^b f(g(x)) \cdot g'(x) \, dx = \int_{g(a)}^{g(b)} f(u) \, du$$

This represents *changing the limits of integration* into the new variable, u , then evaluating the integral. While the second method is preferable, both methods yield the same answer.

Example: Evaluate $\int_0^5 \sqrt{3x+1} \, dx$ using both methods outlined above.

Solution: We start with the first method. We will use the substitution $u = 3x + 1$, and therefore $du/3 = dx$:

$$\int_0^5 \sqrt{3x+1} \, dx = \frac{1}{3} \int_{x=0}^{x=5} \sqrt{u} \, du$$

(We write the limits as $x = \dots$ to remind us the limits are for x , not u .)

$$\frac{1}{3} \int_{x=0}^{x=5} \sqrt{u} \, du = \frac{1}{3} \left[\frac{2}{3} u^{3/2} \right]_{x=0}^{x=5}$$

Now we substitute back in for u and evaluate:

$$\begin{aligned} \frac{2}{9} (3x+1)^{3/2} \Big|_0^5 &= \frac{2}{9} (16)^{3/2} - \frac{2}{9} (1)^{3/2} \\ &= \frac{2}{9} (64 - 1) = \frac{2 \cdot 63}{9} = 14 \end{aligned}$$

Let us compare this to the second, preferred method. We already know the u -substitution we will make, so next we need to find $g(0)$ and $g(5)$ (recall that we choose u such that $u = g(x)$):

$$g(x) = 3x + 1$$

$$g(0) = 1$$

$$g(5) = 16$$

Now we can make our substitution *and* change the limits of integration:

$$\begin{aligned} \int_0^5 \sqrt{3x+1} \, dx &= \frac{1}{3} \int_1^{16} \sqrt{u} \, du \\ &= \frac{1}{3} \left[\frac{2}{3} u^{3/2} \right]_1^{16} = \frac{2}{9} [16^{3/2} - 1^{3/2}] \\ &= \frac{2}{9} (64 - 1) = 14 \end{aligned}$$

With the second method, we get the same answer in fewer steps.

Exercise 21 Definite Integrals and u -substitution

Use u -substitution to evaluate the following definite integrals.

Working Space

1. $\int_0^{\pi/2} \cos x \sin(\sin x) \, dx$

2. $\int_0^{13} \frac{1}{\sqrt[3]{(1+2x)^2}} \, dx$

3. $\int_1^2 \frac{e^{1/x}}{x^2} \, dx$

4. $\int_0^{\pi/6} \frac{\sin x}{\cos^2 x} \, dx$

5. $\int_0^4 \frac{x}{\sqrt{1+2x}} \, dx$

Answer on Page 87

Methods of Integration

6.1 u-substitution

Every differentiation rule has an integration technique that runs it backward. The chain rule gives u-substitution, which you met in the last chapter: if $u = g(x)$, then $\int f(g(x))g'(x) dx = \int f(u) du$. The product rule will give integration by parts. Along the way, algebra (partial fractions) and trigonometric identities turn hard integrals into easy ones. Here is one more substitution example to warm up.

Let us apply u-substitution to a definite integral:

Example: Evaluate $\int_e^{e^4} \frac{1}{x\sqrt{\ln x}} dx$.

Solution: Recall that $\frac{d}{dx} \ln x = \frac{1}{x}$. Letting $\ln x = u$, it follows that $\frac{dx}{x} = du$. Rearranging the integral and substituting:

$$\begin{aligned} \int_e^{e^4} \frac{1}{x\sqrt{\ln x}} dx &= \int_e^{e^4} \frac{1}{\sqrt{\ln x}} \frac{dx}{x} \\ &= \int_{x=e}^{x=e^4} \frac{1}{\sqrt{u}} du \end{aligned}$$

Proceeding from here, there are two options: you can find the value of u at $x = e$ and $x = e^4$ and change the limits of the integral OR you can evaluate the integral, resubstitute back for x and then evaluate the result with the original limits. We will show both to demonstrate each method and show they have the same result.

Method 1: change the limits of integration When $x = e$, $u = \ln e = 1$. And when $x = e^4$, $u = \ln e^4 = 4$. Therefore, we can change the limits of the integral to:

$$\int_1^4 \frac{1}{\sqrt{u}} du = 2\sqrt{u} \Big|_1^4 = 2 \left[\sqrt{4} - \sqrt{1} \right] = 2(2 - 1) = 2$$

Method 2: keep the limits of integration and resubstitute for u :

$$\int_{x=e}^{x=e^4} \frac{1}{\sqrt{u}} du = 2\sqrt{u} \Big|_{x=e}^{x=e^4} = 2\sqrt{\ln x} \Big|_e^{e^4}$$

$$= 2 \left[\sqrt{\ln e^4} - \sqrt{\ln e} \right] = 2(\sqrt{4} - \sqrt{1}) = 2(2 - 1) = 2$$

Which is the same result as method 1. When done correctly, either method will yield the correct result. Choose the method you prefer.

Exercise 22

Using the substitution $u = x^2 - 3$, re-write $\int_{-1}^4 x(x^2 - 3)^5 dx$ in terms of u .

Working Space

Answer on Page 88

Exercise 23

Evaluate $\int_1^\infty x e^{-x^2} dx$.

Working Space

Answer on Page 88

6.2 Partial Fractions

We can integrate rational functions by using partial fractions to decompose a complex rational function into simpler ones. Suppose we wanted to integrate $f(x) = \frac{4x+5}{x^2+x-2}$:

$$\begin{aligned} \int \frac{4x+5}{x^2+x-2} dx &= \int \left(\frac{3}{x-1} + \frac{1}{x+2} \right) dx \\ &= 3 \ln|x-1| + \ln|x+2| + C \end{aligned}$$

Example: Find $\int \frac{x^2+x+1}{(x+1)^2(x+2)} dx$

Solution: We start by defining:

$$\frac{x^2+x+1}{(x+1)^2(x+2)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x+2}$$

Multiplying both sides by $(x + 1)^2(x + 2)$:

$$x^2 + x + 1 = A(x + 1)(x + 2) + B(x + 2) + C(x + 1)^2$$

Since there are only 2 roots to $(x + 1)^2(x + 2)$, we will equate the coefficients to find A, B, and C.

$$x^2 + x + 1 = A(x^2 + 3x + 2) + B(x + 2) + C(x^2 + 2x + 1)$$

$$x^2 + x + 1 = Ax^2 + 3Ax + 2A + Bx + 2B + Cx^2 + 2Cx + C$$

$$x^2 + x + 1 = (A + C)x^2 + (3A + B + 2C)x + (2A + 2B + C)$$

For this equation to be true, we know that:

$$A + C = 1$$

$$3A + B + 2C = 1$$

$$2A + 2B + C = 1$$

Solving for each, you should find that:

$$A = -2$$

$$B = 1$$

$$C = 3$$

And therefore,

$$\frac{x^2 + x + 1}{(x + 1)^2(x + 2)} = \frac{-2}{x + 1} + \frac{1}{(x + 1)^2} + \frac{3}{x + 2}$$

Substituting this into our integral,

$$\begin{aligned} \int \frac{x^2 + x + 1}{(x + 1)^2(x + 2)} dx &= \int \left[\frac{-2}{x + 1} + \frac{1}{(x + 1)^2} + \frac{3}{x + 2} \right] dx \\ &= -2 \ln|x + 1| + \frac{-1}{x + 1} + 3 \ln|x + 2| + C = \ln \left| \frac{(x + 2)^3}{(x + 1)^2} \right| - \frac{1}{x + 1} + C \end{aligned}$$

Example: Evaluate $\int \frac{2x^2 - x + 4}{x^3 + 4x} dx$

Solution: We begin by factoring the denominator:

$$x^3 + 4x = x(x^2 + 4)$$

Which cannot be factored further. Therefore, we define:

$$\frac{2x^2 - x + 4}{x(x^2 + 4)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 4}$$

$$2x^2 - x + 4 = A(x^2 + 4) + (Bx + C)x$$

$$2x^2 - x + 4 = Ax^2 + 4A + Bx^2 + Cx$$

Which implies that:

$$2 = A + B$$

$$C = -1$$

$$4A = 4$$

Therefore, $A = 1$, $B = 1$, and $C = -1$ and we can say that:

$$\begin{aligned} \int \frac{2x^2 - x + 4}{x^3 + 4x} dx &= \int \left[\frac{1}{x} + \frac{x-1}{x^2+4} \right] dx \\ &= \int \left[\frac{1}{x} + \frac{x}{x^2+4} - \frac{1}{x^2+4} \right] dx \\ &= \ln|x| + \frac{1}{2} \ln(x^2+4) - \frac{1}{2} \arctan\left(\frac{x}{2}\right) + C \end{aligned}$$

A useful identity that we used here is

$$\int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \arctan\left(\frac{x}{a}\right) + C$$

Exercise 24

Evaluate $\int_0^1 \frac{5x+8}{x^2+3x+2} dx$ without a calculator.

Working Space

Answer on Page 88

Exercise 25

Use the method of partial fractions to evaluate the following integrals:

Working Space

1. $\int \frac{4x}{x^3+x^2+x+1} dx$

2. $\int_{-1}^0 \frac{x^3-4x+1}{x^2-3x+2} dx$

3. $\int \frac{x^3+2x}{x^4+4x^2+3} dx$

Answer on Page 88

6.3 Integrating Powers of Trigonometric Functions

6.3.1 Odd powers of sine and cosine

Consider the integral:

$$\int \cos^5 x \, dx$$

To evaluate this integral, we may first try u -substitution, with $u = \cos x$. But then $du = -\sin x \, dx$, and there is no sine function in the integral. Instead, let us take advantage of the identity:

$$\cos^2 x = 1 - \sin^2 x$$

We begin by separating $\cos^5 x$ into three terms:

$$\int \cos x \cdot \cos^2 x \cdot \cos^2 x \, dx$$

Now we substitute in $\cos^2 x = 1 - \sin^2 x$:

$$\begin{aligned} \int \cos x (1 - \sin^2 x) (1 - \sin^2 x) \, dx \\ = \int (1 - \sin^2 x)^2 \cos x \, dx \end{aligned}$$

Now we have sine and cosine functions, which makes u -substitution possible. Let $u = \sin x$, then $du = \cos x \, dx$:

$$\begin{aligned} \int (1 - u^2)^2 \, du &= \int (1 - 2u^2 + u^4) \, du \\ &= u - \frac{2}{3}u^3 + \frac{1}{5}u^5 + C \Big|_{u=\sin x} = \sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x + C \end{aligned}$$

Exercise 26 Integrating odd powers of sine and cosine

Use the identity $\sin^2 \theta + \cos^2 \theta = 1$ and u-substitution to evaluate the following integrals:

Working Space

1. $\int \sin^2 x \cos^3 x \, dx$
2. $\int_0^{\pi/2} \sin^7 \theta \cos^5 \theta \, d\theta$
3. $\int \sin^3 \theta \cos^4 \theta \, d\theta$
4. $\int_0^{\pi/2} \sin^5 x \, dx$
5. $\int t \cos^5(t^2) \, dt$

Answer on Page 89

6.3.2 Even powers of sine and cosine

Consider $\int \cos^4 x \, dx$. Let us try applying the identity $\cos^2 x = 1 - \sin^2 x$:

$$\int \cos^4 x \, dx = \int \cos^2 x (1 - \sin^2 x) \, dx$$

We cannot use u-substitution for this. If we were to set $u = \sin x$, we see $du = \cos x \, dx$, and we would have an extra $\cos x$ left. Instead, let us use the double-angle identity $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$:

$$\begin{aligned} \int \cos^4 x \, dx &= \int (\cos^2 x)^2 \, dx \\ &= \frac{1}{4} \int (1 + \cos 2x)^2 \, dx = \frac{1}{4} \int (1 + 2\cos 2x + \cos^2 2x) \, dx \end{aligned}$$

We can apply the double-angle identity again, noting that $\cos^2 2x = \frac{1}{2}(1 + \cos 4x)$:

$$\begin{aligned} &= \frac{1}{4} \int \left[1 + 2\cos 2x + \frac{1}{2}(1 + \cos 4x) \right] \, dx \\ &= \frac{1}{4} \left[x + \frac{1}{2} \sin 2x + \frac{1}{2} \left(x + \frac{1}{4} \sin 4x \right) \right] + C \end{aligned}$$

$$= \frac{1}{4} \left[\frac{3}{2}x + \sin 2x + \frac{1}{8} \sin 4x \right] + C$$

In general, when the powers of sine and cosine are both even, apply the double-angle identities

$$\sin^2 x = \frac{1}{2} (1 - \cos 2x)$$

$$\cos^2 x = \frac{1}{2} (1 + \cos 2x)$$

Until the powers are reduced sufficiently to be able to evaluate the integral.

Exercise 27 Integrating even powers of sine and cosine

Use the double-angle identities to evaluate the following integrals:

Working Space

1. $\int_0^{\pi/2} \cos^2 \theta \, d\theta$
2. $\int_0^{\pi} \cos^4 2t \, dt$
3. $\int \sin^2 x \cos^2 x \, dx$
4. $\int_0^{2\pi} \sin^2 \frac{1}{3} \theta \, d\theta$
5. $\int_0^{\pi} \sin^2 t \cos^4 t \, dt$

Answer on Page 90

6.3.3 Integrating powers of secant and tangent

There are a few identities that we will not prove here, but will be useful:

1. $\int \tan x \, dx = \ln |\sec x| + C$
2. $\int \sec x \, dx = \ln |\sec x + \tan x| + C$

Consider the integral $\int \tan^6 x \sec^4 x \, dx$. We know that $\frac{d}{dx} \tan x = \sec^2 x$, so we want to arrange things such that we can use the u-substitution $u = \tan x$ and $du = \sec^2 x \, dx$. Let

us separate terms so we can easily see our intended du :

$$\int \tan^6 x \sec^2 x \sec^2 x \, dx$$

What to do with the extra $\sec^2 x$? We already know that $\sin^2 \theta + \cos^2 \theta = 1$. If we divide both sides by $\cos^2 \theta$, we get the identity $\tan^2 \theta + 1 = \sec^2 \theta$. So we can replace the extra $\sec^2 x$ with $\tan^2 x + 1$:

$$\begin{aligned} \int \tan^6 x (\tan^2 x + 1) \sec^2 x \, dx &= \\ \int \tan^8 x \sec^2 x \, dx + \int \tan^6 x \sec^2 x \, dx \end{aligned}$$

We can now apply our u -substitution, as outlined above:

$$\begin{aligned} &= \int u^8 \, du + \int u^6 \, du = \frac{1}{9}u^9 + \frac{1}{7}u^7 + C \\ &= \frac{1}{9}\tan^9 x + \frac{1}{7}\tan^7 x + C \end{aligned}$$

This method works best when the secant is raised to an even power. Reserve a $\sec^2$ to use for the u -substitution and express the remaining secants in terms of the tangent.

We can use a similar method when the power of tangent is odd. We will evaluate $\int \tan^5 x \sec^3 x \, dx$. First, remember that $\frac{d}{dx} \sec x = \sec x \tan x$. We will use this for a u -substitution, so let us pull out a secant factor and a tangent factor:

$$\int \tan^5 x \sec^3 x \, dx = \int \tan^4 x \sec^2 x \sec x \tan x \, dx$$

We re-write $\tan^4 x$ in terms of $\sec x$, since we will use the u -substitution $u = \sec x$ and $du = \sec x \tan x \, dx$:

$$\int \tan^4 x \sec^2 x \sec x \tan x \, dx = \int (\sec^2 x - 1)^2 \sec^2 x \sec x \tan x \, dx$$

(using $\tan^2 x = \sec^2 x - 1$)

$$\begin{aligned} &= \int (u^2 - 1)^2 u^2 \, du = \int (u^4 - 2u^2 + 1) u^2 \, du \\ &= \int (u^6 - 2u^4 + u^2) \, du = \frac{1}{7}u^7 - \frac{2}{5}u^5 + \frac{1}{3}u^3 + C \\ &= \frac{1}{7}\sec^7 x - \frac{2}{5}\sec^5 x + \frac{1}{3}\sec^3 x + C \end{aligned}$$

When the power of tangent is even and the power of secant is odd, you will need to try additional methods, including using integration by parts (explained below) or the identities for $\int \tan x \, dx$ and $\int \sec x \, dx$ given above.

Exercise 28 Integrating Powers of Secant and Tangent

Working Space

1. $\int \tan x \sec^3 x \, dx$
2. $\int \tan^4 x \sec^6 x \, dx$
3. $\int_0^{\pi/4} \sec^6 \theta \tan^6 \theta \, d\theta$

Answer on Page 90

Some integrals, like $\int x e^x \, dx$, still resist every technique in this chapter. The next chapter, Integration by Parts, handles them.

Integration by Parts

The product rule tells you how to differentiate a product of two functions. Run it backwards, and you get one of the most useful tricks in all of integration. In the Methods of Integration chapter, you integrated rational functions and powers of trigonometric functions. This chapter handles products like $x e^x$ and $x \sin x$, and uses them to find the mean of a continuous probability distribution.

7.1 Integration by Parts

Recall the Product Rule for derivatives:

$$\frac{d}{dx} [f(x) \cdot g(x)] = f(x) \cdot g'(x) + f'(x) \cdot g(x)$$

If we integrate both sides, we find that:

$$\begin{aligned} f(x) \cdot g(x) &= \int [f(x) \cdot g'(x) + f'(x) \cdot g(x)] \, dx \\ f(x) \cdot g(x) &= \int f(x)g'(x) \, dx + \int f'(x)g(x) \, dx \end{aligned}$$

Rearranging,

$$\int f(x)g'(x) \, dx = f(x)g(x) - \int f'(x)g(x) \, dx$$

This identity allows us to perform **integration by parts**, a powerful method that allows us to evaluate integrals of complex functions.

Example: Evaluate $\int x \cos x \, dx$.

Solution: We may be tempted to try u -substitution, but that will not work because $\frac{d}{dx} \cos x$ is not proportional to x and $\frac{d}{dx} x$ is not proportional to $\cos x$. Let us define $f(x) = x$ and $g'(x) = \cos x$. This implies $f'(x) = 1$ and $g(x) = \sin x$. Then we can say that:

$$\int x \cos x \, dx = \int f(x)g'(x) \, dx$$

Using the identity $\int f(x)g'(x) \, dx = f(x)g(x) - \int f'(x)g(x) \, dx$ and substituting for $f(x)$, $f'(x)$, $g(x)$, and $g'(x)$, we see that:

$$\begin{aligned}\int x \cos x \, dx &= [x \sin x] - \int 1 \cdot \sin x \, dx \\ &= x \sin x - \int \sin x \, dx \\ &= x \sin x - (-\cos x + C) = x \sin x + \cos x + C\end{aligned}$$

(recall that C is the integration constant). You can check your results by taking the derivative: you should get the original integrand back. Let us check our result in this case:

$$\begin{aligned}\frac{d}{dx} [x \sin x + \cos x + C] &= \frac{d}{dx} [x \sin x] + \frac{d}{dx} \cos x \\ &= x \frac{d}{dx} (\sin x) + \sin x \frac{d}{dx} (x) - \sin x \\ &= x \cos x + \sin x - \sin x = x \cos x\end{aligned}$$

How did we choose that $f(x)$ should be x and $g(x)$ should be $\sin x$ in the example above? In general, you want to choose such that the resulting integral is simpler than the one we started with. This means you want to choose f such that f' is *less complex* or a *lower order* than f .

To illustrate this, let us re-evaluate the example above, but this time let $f(x) = \cos x$ and $g'(x) = x$. Then we can say that $f'(x) = -\sin x$ and $g(x) = \frac{1}{2}x^2$. Substituting this into the integration by parts identity, we find that:

$$\int x \cos x \, dx = \frac{1}{2}x^2 \cos x - \int -\frac{1}{2}x^2 \sin x \, dx$$

Now the integral on the right side is more complex than the one we started with (on the left)! A good general rule for integration by parts is that *if* the two functions in the original integral are a polynomial and a sine or cosine function, set the polynomial to be $f(x)$ and the trigonometric function to be $g'(x)$. The polynomial will be differentiated and become *less complex*, while integrating the trigonometric function will not make it *more complex*.

Integration by parts is valid for definite integrals as well. Mathematically, this means:

$$\int_a^b f(x)g'(x) \, dx = f(x)g(x)|_a^b - \int_a^b f'(x)g(x) \, dx$$

Which is the same as:

$$\int_a^b f(x)g'(x) \, dx = (f(b)g(b)) - (f(a)g(a)) - \int_a^b f'(x)g(x) \, dx$$

Let us see one more example that incorporates both u -substitution and integration by parts.

Example: Evaluate $\int \frac{\arcsin \ln x}{x} \, dx$

Solution: First, we notice that $\ln x$ and $\frac{1}{x}$ both appear in the integrand. Let us define $u = \ln x$. Then $du = \frac{dx}{x}$:

$$\int \arcsin \ln x \frac{dx}{x} = \int \arcsin u \, du$$

For integration by parts, if we let $\arcsin u = f(u)$ and $du = g'(u)$, it follows that $f'(u) = \frac{1}{\sqrt{1-u^2}}$ and $g(u) = u$. Then we can say that:

$$\int \arcsin u \, du = \arcsin u \cdot u - \int \frac{u}{\sqrt{1-u^2}} \, du$$

We can use u -substitution again to evaluate the second integral (we will use v , since we have already said that $u = \ln x$). Let $v = 1 - u^2$, which means that $\frac{dv}{2} = (-u)du$. Substituting:

$$= u \cdot \arcsin u + \int \frac{1}{2\sqrt{v}} \, dv = u \cdot \arcsin u + \sqrt{v}$$

Substituting back for v :

$$= u \cdot \arcsin u + \sqrt{1-u^2}$$

And substituting back for u :

$$= \ln x \cdot \arcsin \ln x + \sqrt{1 - \ln^2 x} + C$$

Exercise 29

Let f be a function such that $\int f(x) \sin x \, dx = -f(x) \cos x + \int 4x^3 \sin x \, dx$. Give a possible expression for $f(x)$.

Working Space

Answer on Page 91

Exercise 30

Evaluate the following integrals using integration by parts:

Working Space

1. $\int_0^1 x \sin \frac{\pi}{2} x \, dx$
2. $\int e^\theta \cos \theta \, d\theta$
3. $\int (1 - t)^2 \cos \beta t \, dt$

Answer on Page 91

7.2 Application: The Mean of a Probability Distribution

Integration by parts has an important use in probability. How can we find the average (mean) of a variable with a given probability distribution function? Consider the example of transaction processing times. If we knew the total time spent processing transactions and the total number of customers, then the mean processing time would be the total processing time divided by the total number of customers.

Suppose we have a PDF that describes processing time for electronic transactions, $f(t)$, where t is processing time and $f(t)$ is the probability that an individual transaction takes a particular amount of time, t . Electronic transactions are fast, and we can assume no

transaction takes longer than 250 milliseconds, so we restrict the domain of $f(t)$ to $0 \leq t \leq 250$. We can divide the PDF into intervals of width Δt (see figure ??).

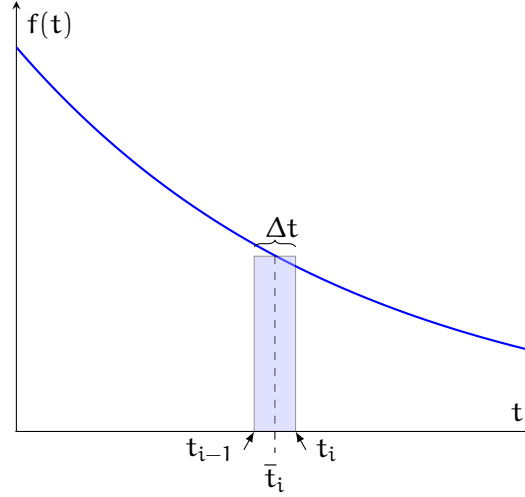


Figure 7.1: The probability that an individual transaction takes between t_{i-1} and t_i seconds is approximated by the shaded area

If a total of N transactions are processed, then the number of transactions that take between t_{i-1} and t_i is the probability of a transaction taking between t_{i-1} and t_i times N . That probability is approximately $f(\bar{t}_i)\Delta t$, and so the number of transactions taking between t_{i-1} and t_i is $Nf(\bar{t}_i)\Delta t$. Each of these transactions took approximately $\bar{t}_i$, and therefore the total processing time for these transactions is the number of transactions multiplied by the processing time:

$$\bar{t}_i [Nf(\bar{t}_i)\Delta t]$$

To find the total processing time for all transactions, we would add up the total processing time for each interval:

$$\sum_{i=1}^n N\bar{t}_i f(\bar{t}_i)\Delta t$$

And to find the average or mean processing time, we would divide this sum by the total number of customers, N :

$$\text{mean processing time} = \sum_{i=1}^n \bar{t}_i f(\bar{t}_i)\Delta t$$

Taking the limit as $n \rightarrow \infty$, the sum becomes an integral:

$$\text{mean processing time} = \int_0^{250} t \cdot f(t) dt$$

Recall that we use μ to represent the mean. Therefore, for a general PDF, $f(x)$, the mean is given by:

$$\mu = \int_{-\infty}^{\infty} x \cdot f(x) \, dx$$

Example: Find the mean value of the general exponential PDF,

$$f(x) = \begin{cases} 0, & \text{if } x < 0 \\ ce^{-cx}, & \text{if } x \geq 0 \end{cases}$$

Solution: Applying the definition of the mean:

$$\begin{aligned} \mu &= \int_{-\infty}^{\infty} xf(x) \, dx = \int_0^{\infty} xce^{-cx} \, dx \\ &= -xe^{-cx} \Big|_{x=0}^{x=\infty} - \int_0^{\infty} -e^{-cx} \, dx \\ &= 0 + \left[-\frac{1}{c}e^{-cx} \right]_{x=0}^{x=\infty} = \frac{1}{c} \end{aligned}$$

And since $\mu = 1/c$, we can re-write the general exponential PDF in terms of the mean:

$$f(x) = \begin{cases} 0, & \text{if } x < 0 \\ \mu^{-1}e^{-x/\mu}, & \text{if } x \geq 0 \end{cases}$$

Exercise 31 Onset of Symptoms

Researchers find that the time between infection by the COVID virus and the onset of symptoms can be described by the PDF:

$$f(t) = \begin{cases} \frac{1}{595}te^{-0.04t}, & \text{if } 0 \leq t \leq 120 \\ 0, & \text{otherwise} \end{cases}$$

What is the mean time between infection and the onset of symptoms?

Working Space

Answer on Page 91

Answers to Exercises

Answer to Exercise 1 (on page 9)

First, we convert the temperatures into Kelvin:

- Dawn: $12 + 273.15 = 285.15$
- Noon: $28 + 273.15 = 301.15$

So, the temperature T has increased by a factor of $\frac{301.15}{285.15} \approx 1.056$.

Thus, the volume of the air mattress has also increased by a factor of 1.056.

This means that the air mattress that had a volume of 1000 liters at dawn will have a volume 1056 liters at noon.

Answer to Exercise 2 (on page 10)

What is the pressure in kPa?

- Before squeezing: 100 kPa
- While squeezing: 120 kPa

So, the pressure P has increased by a factor of $\frac{120}{100} = 1.2$

$$1/1.2 \approx 0.833$$

The air in the bottle had a volume of 1 liter before squeezing, so it has a volume of 833 milliliters while being squeezed.

Answer to Exercise 3 (on page 11)

First, let us convert the known values to the right unit:

- Radius = 0.12 m
- Length = 0.5 m
- $T = 20 + 273.15 = 293.15$ kelvins
- $P = 600 \text{ kPa} = 600,000 \text{ Pa}$

The volume of the cylindrical chamber is $V = \pi r^2 h = \pi(0.12)^2(0.5) \approx 0.0226 \text{ m}^3$.

The Ideal Gas Law tells us that $PV = nRT$. We are solving for n .

$$n = \frac{PV}{RT} = \frac{(600,000)(0.0226)}{(8.31446)(293.15)} \approx 5.57 \text{ moles of O}_2$$

Answer to Exercise 4 (on page 13)

Constant volume: $E = C_{V,m} n \Delta T = (12.47)(3)(20) \approx 748$ joules.

Constant pressure: $E = C_{P,m} n \Delta T = (20.8)(3)(20) \approx 1247$ joules. (A change of 20 degrees Celsius is also a change of 20 kelvins.)

Answer to Exercise 5 (on page 15)

10 degrees Celsius is 283.15 kelvins. 30 degrees Celsius is 303.15.

For any gas:

$$E_K = C_{V,m} n T$$

And $C_{V,m} = 12.47$ for all monoatomic gases.

So, the energy at 10 degrees Celsius:

$$E_1 = (12.47)(3)(283.15) = 10,594 \text{ Joules}$$

The energy at 30 degrees Celsius:

$$E_2 = (12.47)(3)(303.15) = 11,342 \text{ Joules}$$

The difference?

$$E_2 - E_1 = 11,342 - 10,594 = 748 \text{ Joules}$$

Which is consistent with your earlier exercise.

Answer to Exercise 6 (on page 27)

$$E_C = (1200)(0.4)(T - 10) = 480T - 4800$$

Total energy stays constant:

$$0 = (12600T - 252000) + (900T - 72000) + (480T - 4800)$$

Solving for T gets you $T = 23.52^\circ \text{ C}$.

Answer to Exercise 7 (on page 28)

During the 3 minutes, you want the coffee to give off as much of its heat as possible, so you want to maximize the difference between the temperature of the coffee and the temperature of the room around it.

You wait until the last moment to put the milk in.

Answer to Exercise 8 (on page 32)

When one mole of water goes from 33° to 17° , it will give off $(75.38)(33 - 17) = 1,206$

Joules or 1.206 kJ.

Tom needs 300,000 kJ, which takes $300,000/1.206 = 248,739.72$ moles of water.

How many liters is that? $248,739.72 * 0.018 = 4,477.31$ liters.

How many barrels is that? $4,477.31/159 = 28.16$ barrels. Tom will need 29 barrels.

Answer to Exercise 9 (on page 33)

When one mole of mirabilite goes from 33° to 17° , it will give off $(550)(33 - 17) + 82,000 = 90,800$ Joules or 90.8 kJ.

Tom needs 300,000 kJ, which takes $300,000/90.8 = 3,304$ moles of mirabilite.

How many liters is that? $3,304 * 0.22 = 726.9$ liters.

How many barrels is that? $726.9/159 = 4.57$. Tom will need 5 barrels.

Answer to Exercise 10 (on page 44)

The pre-compression temperature is $80 + 273.15 = 353.15$ K.

The radius of the cylinder is $9.525/2 = 4.7625$ cm.

The area of a cross section of the cylinder is $\pi r^2 = \pi(4.7625)^2 \approx 71.26$ cm².

So, the change in volume between the minimum and maximum volume is $(71.26)(10.16) = 724$ ml, or 0.724 liters.

(With four cylinders, the total displacement of a Model T is thus $(4)(724) = 2,896$ cc.)

Now we use the ideal gas to figure out how many moles of gas will fit into 0.724 liters at 100 kPa and 353.15 K. (Kilopascals times liters gives joules, so the units work out.)

$$n = \frac{PV}{RT} = \frac{(100)(0.724)}{(8.314)(353.15)} = 0.02466 \text{ moles of air+fuel}$$

So, if we suck n_a moles of air and n_e moles of vaporized ethanol in to the cylinder:

$$n_a + n_e = 0.02466$$

So

$$n_a = 0.02466 - n_e$$

21% of n_a is O_2 :

$$n_{O_2} = 0.21n_a = 0.21(0.02466 - n_e) = 0.005178 - 0.21n_e$$

For a clean burn, we need 3 times as many O_2 molecules as ethanol molecules. Thus:

$$3n_e = n_{O_2} = 0.005178 - 0.21n_e$$

Solving for n_e :

$$n_e = \frac{0.005178}{3.21} = 0.001613 \text{ moles of ethanol}$$

For every molecule of ethanol that burns we get 2 molecules of CO_2 : 0.003226 moles.

For every molecule of ethanol that burns we get 3 molecules of H_2O : 0.004839 moles.

How much heat? $(0.001613)(1330) = 2.145$ kilojoules from each burn.

Answer to Exercise 11 (on page 46)

From the Temperature and the Gas Laws chapter, $TV^{\gamma-1}$ stays constant, so $T_2 = T_1 \left(\frac{V_1}{V_2}\right)^{\gamma-1} = 300 \times 10^{0.4} \approx 754$ K, about 480° C. For the Model T, $300 \times 4^{0.4} \approx 522$ K. Diesel engines use even higher ratios (around 20:1), and they need no spark plug at all: the compressed air is hot enough to ignite the fuel by itself.

Answer to Exercise 12 (on page 54)

$\binom{6}{3} = 20$ microstates have 3 heads. There are $2^6 = 64$ microstates in all, so the probability

is $20/64 \approx 0.31$. All heads is 1 microstate, so its probability is $1/64 \approx 0.016$.

Answer to Exercise 13 (on page 54)

$Q = (20)(334) = 6680$ J. The ice gains $6680/273 \approx 24.5$ J/K. The drink loses $6680/293 \approx 22.8$ J/K. The total increases by about 1.7 J/K, as the second law requires.

Answer to Exercise 14 (on page 55)

No. The Carnot limit is $1 - 300/400 = 0.25$, or 25%. No engine working between these temperatures can reach 30%.

Answer to Exercise 15 (on page 55)

$e \leq 1 - 300/450 \approx 0.33$, or 33%. Underground water is much cooler than burning fuel or steam from a boiler, so T_H is low and the Carnot limit is low.

Answer to Exercise 16 (on page 55)

$\text{COP} \leq 293/(293 - 263) = 293/30 \approx 9.8$, down from about 14.7. The colder it gets outside, the more work each joule of heat costs. This is why some homes keep a backup heater for the coldest days.

Answer to Exercise 17 (on page 56)

No. The fridge pulls heat Q_C from the air in front of it and dumps $Q_H = Q_C + W$ out the back, into the same kitchen. The kitchen gains a net W , the electrical energy the fridge uses, so it gets *warmer*. An air conditioner works only because it dumps Q_H outside.

Answer to Exercise 18 (on page 56)

The total entropy cannot fall, so $Q_H/T_H \geq Q_C/T_C = (Q_H - W)/T_C$. Multiply by $T_H T_C$: $Q_H T_C \geq (Q_H - W) T_H$, so $W T_H \geq Q_H (T_H - T_C)$. Dividing,

$$\text{COP} = \frac{Q_H}{W} \leq \frac{T_H}{T_H - T_C}$$

Answer to Exercise 19 (on page 56)

Smaller. With 100 molecules, the count typically strays about 5 from 50, which is 5% of the molecules. With 1,000, it strays about 16 from 500, only about 1.6%. The typical wandering grows like $\sqrt{n}$, so as a fraction it shrinks like $1/\sqrt{n}$. With 10^{27} molecules, the fluctuations are far too small to ever notice, which is why the second law looks perfectly strict for everyday objects.

Answer to Exercise 20 (on page 59)

1. Let $u = 1 + \cos x$. Then $du = -\sin x \, dx$ and $-du = \sin x \, dx$. Substituting:

$$\begin{aligned}\int \sin x \sqrt{1 + \cos x} \, dx &= \int -\sqrt{u} \, du = -\frac{2}{3} u^{3/2} + C \\ &= -\frac{2}{3} (1 + \cos x)^{3/2} + C\end{aligned}$$

Taking the derivative:

$$\begin{aligned}\frac{d}{dx} \left[-\frac{2}{3} (1 + \cos x)^{3/2} + C \right] &= -\frac{2}{3} \left[\frac{d}{dx} (1 + \cos x)^{3/2} + \frac{d}{dx} C \right] \\ &= -\frac{2}{3} \left[\frac{3}{2} (1 + \cos x)^{1/2} \cdot \frac{d}{dx} (1 + \cos x) \right] = -1 \sqrt{1 + \cos x} \cdot (-\sin x) = \sin x \sqrt{1 + \cos x}\end{aligned}$$

2. Let $u = \pi/x$. Then $du = (-\pi/x^2)dx$ and $-du/\pi = (1/x^2)dx$. Substituting:

$$\int \frac{\cos(\pi/x)}{x^2} \, dx = -\frac{1}{\pi} \int \cos u \, du = -\frac{1}{\pi} \sin u + C = -\frac{1}{\pi} \sin(\pi/x) + C$$

Taking the derivative:

$$\begin{aligned}\frac{d}{dx} \left[-\frac{1}{\pi} \sin(\pi/x) + C \right] &= -\frac{1}{\pi} \left[\frac{d}{dx} \sin(\pi/x) + \frac{d}{dx} C \right] \\ &= -\frac{1}{\pi} \left[\cos(\pi/x) \cdot \frac{d}{dx} \left(\frac{\pi}{x} \right) \right] = -\frac{1}{\pi} \left[\cos(\pi/x) \cdot \left(\frac{-\pi}{x^2} \right) \right] \\ &= \frac{\cos(\pi/x)}{x^2}\end{aligned}$$

3. Let $u = 9 - x^3$. Then $du = -3x^2 \, dx$ and $-\frac{2}{3}du = 2x^2 \, dx$. Substituting:

$$\int 2x^2 (9 - x^3)^{2/3} \, dx = -\frac{2}{3} \int (u)^{2/3} \, du$$

$$= -\frac{2}{3} \left[\frac{3}{5} u^{5/3} + C \right] = -\frac{2}{5} (9 - x^3)^{5/3} + C$$

Taking the derivative:

$$\begin{aligned} \frac{d}{dx} \left[-\frac{2}{5} (9 - x^3)^{5/3} + C \right] &= -\frac{2}{5} \left[\frac{d}{dx} (9 - x^3)^{5/3} \right] + \frac{d}{dx} C \\ &= -\frac{2}{5} \left[\frac{5}{3} (9 - x^3)^{2/3} \cdot \frac{d}{dx} (9 - x^3) \right] = -\frac{2}{3} (9 - x^3)^{2/3} (-3x^2) = 2x^2 (9 - x^3)^{2/3} \end{aligned}$$

4. Let $u = 1 + x$. Then $du = dx$. Additionally, $u - 1 = x$ and $x^2 = (u - 1)^2$. Substituting:

$$\begin{aligned} \int 3x^2 \sqrt{1+x} \, dx &= \int 3(u-1)^2 \sqrt{u} \, du = 3 \int (u^2 - 2u + 1) \sqrt{u} \, du \\ &= 3 \int u^{5/2} - 2u^{3/2} + u^{1/2} \, du = 3 \left[\frac{2}{7} u^{7/2} - \frac{2 \cdot 2}{5} u^{5/2} + \frac{2}{3} u^{3/2} + C \right] \\ &= \frac{6}{7} u^{7/2} - \frac{12}{5} u^{5/2} + 2u^{3/2} + C \\ &= \frac{6}{7} (1+x)^{7/2} - \frac{12}{5} (1+x)^{5/2} + 2(1+x)^{3/2} + C \end{aligned}$$

Taking the derivative:

$$\begin{aligned} \frac{d}{dx} \left[\frac{6}{7} (1+x)^{7/2} - \frac{12}{5} (1+x)^{5/2} + 2(1+x)^{3/2} + C \right] \\ &= \frac{6}{7} \left(\frac{7}{2} (1+x)^{5/2} \right) - \frac{12}{5} \left(\frac{5}{2} (1+x)^{3/2} \right) + 2 \left(\frac{3}{2} (1+x)^{1/2} \right) \\ &= 3(1+x)^{5/2} - 6(1+x)^{3/2} + 3(1+x)^{1/2} \\ &= \left[3(1+x)^2 - 6(1+x) + 3 \right] (1+x)^{1/2} \\ &= \left[3(1+2x+x^2) - 6 - 6x + 3 \right] \sqrt{1+x} \\ &= \left[3+6x+3x^2-6-6x+3 \right] \sqrt{1+x} = 3x^2 \sqrt{1+x} \end{aligned}$$

5. Let $u = x^3 - 1$. Then $du = 3x^2 \, dx$. Substituting:

$$\begin{aligned} \int \frac{3x^2}{x^3-1} \, dx &= \int \frac{1}{u} \, du = \ln u + C \\ &= \ln(x^3 - 1) + C \end{aligned}$$

Taking the derivative:

$$\begin{aligned}\frac{d}{dx} [\ln(x^3 - 1) + C] &= \frac{d}{dx} \ln(x^3 - 1) + \frac{d}{dx} C \\ &= \frac{1}{x^3 - 1} \cdot \left[\frac{d}{dx} (x^3 - 1) \right] = \frac{3x^2}{x^3 - 1}\end{aligned}$$

Answer to Exercise 21 (on page 61)

1. Let $g(x) = u = \sin x$. Then $du = \cos x \, dx$. Additionally, $g(0) = \sin 0 = 0$ and $g(\pi/2) = \sin(\pi/2) = 1$. Substituting and changing the limits of integration:

$$\int_0^{\pi/2} \cos x \sin(\sin x) \, dx = \int_0^1 \sin u \, du = -\cos u \Big|_0^1 = \cos 0 - \cos 1 = 1 - \cos 1$$

2. Let $g(x) = u = 1 + 2x$. Then $du = 2 \, dx$ and $\frac{du}{2} = dx$. Additionally, $g(0) = 1$ and $g(13) = 1 + 2(13) = 27$. Substituting and changing the limits of integration:

$$\begin{aligned}\int_0^{13} \frac{1}{\sqrt[3]{(1+2x)^2}} \, dx &= \frac{1}{2} \int_1^{27} \frac{1}{\sqrt[3]{u^2}} \, du = \frac{1}{2} \int_1^{27} u^{-2/3} \, du \\ &= \frac{1}{2} \left[3u^{1/3} \right]_1^{27} = \frac{3}{2} \left[\sqrt[3]{27} - \sqrt[3]{1} \right] = \frac{3}{2} (3 - 1) = 3\end{aligned}$$

3. Let $g(x) = u = 1/x$. Then $du = (-1/x^2) \, dx$ and $-du = dx/x^2$. Additionally, $g(1) = 1$ and $g(2) = 1/2$. Substituting and changing the limits of integration:

$$\begin{aligned}\int_1^2 \frac{e^{1/x}}{x^2} \, dx &= - \int_1^{1/2} e^u \, du = \int_{1/2}^1 e^u \, du \\ &= e^u \Big|_{1/2}^1 = e - \sqrt{e}\end{aligned}$$

4. Let $g(x) = u = \cos x$. Then $du = -\sin x \, dx$ and $-du = \sin x \, dx$. Additionally, $g(0) = \cos 0 = 1$ and $g(\pi/6) = \cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}$. Substituting and changing the limits of integration:

$$\begin{aligned}\int_0^{\pi/6} \frac{\sin x}{\cos^2 x} \, dx &= - \int_1^{\sqrt{3}/2} \frac{1}{u^2} \, du = \int_{\sqrt{3}/2}^1 \frac{1}{u^2} \, du \\ &= -\frac{1}{u} \Big|_{\sqrt{3}/2}^1 = \frac{2}{\sqrt{3}} - 1 = \frac{2\sqrt{3} - 3}{3}\end{aligned}$$

5. Let $g(x) = u = 1 + 2x$. Then $du = 2 \, dx$ and $\frac{du}{2} = dx$. And if $u = 1 + 2x$, then $x = \frac{u-1}{2}$. Additionally, $g(0) = 1$ and $g(4) = 9$. Substituting and changing the limits of

integration:

$$\begin{aligned}
 \int_0^4 \frac{x}{\sqrt{1+2x}} dx &= \frac{1}{2} \int_1^9 \frac{\frac{u-1}{2}}{\sqrt{u}} du = \frac{1}{4} \int_1^9 \frac{u-1}{\sqrt{u}} du \\
 &= \frac{1}{4} \int_1^9 \sqrt{u} - \frac{1}{\sqrt{u}} du = \frac{1}{4} \left[\frac{2}{3} u^{3/2} - 2u^{1/2} \right]_1^9 \\
 &= \frac{1}{4} \left[\frac{2}{3} (9^{3/2} - 1^{3/2}) - 2(\sqrt{9} - \sqrt{1}) \right] = \frac{1}{4} \left[\frac{2}{3} (26) - 2(2) \right] \\
 &= \frac{1}{4} \left[\frac{52}{3} - 4 \right] = \frac{1}{4} \left[\frac{52-12}{3} \right] = \frac{1}{4} \left[\frac{40}{3} \right] = \frac{10}{3}
 \end{aligned}$$

Answer to Exercise 22 (on page 64)

If $u = x^2 - 3$, then $du = 2x dx$ and $x(x^2 - 3)^5 dx = \frac{1}{2} u^5 du$. When $x = -1$, $u = -2$ and when $x = 4$, $u = 13$. Putting it all together, we find an equivalent integral is $\frac{1}{2} \int_{-2}^{13} u^5 du$.

Answer to Exercise 23 (on page 64)

Letting $u = -x^2$, then $du = -2x dx$ and $x dx = \frac{-1}{2} du$. Substituting u and du into the integral, we have $\int_{x=1}^{x=\infty} \frac{-1}{2} e^u du$, which equals $\frac{-1}{2} e^u = \frac{-1}{2} e^{-x^2} \Big|_1^\infty$. Evaluating the statement, we get $\frac{-1}{2} (e^{-\infty} - e^{-1}) = \frac{-1}{2} (0 - \frac{1}{e}) = \frac{1}{2e}$.

Answer to Exercise 24 (on page 66)

We cannot use u -substitution because $\frac{d}{dx}(x^2 + 3x + 2) \neq n(5x + 8)$. We will use partial fractions to simplify the integrand. Setting up: $\frac{5x+8}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2}$. Rearranging, we find $5x + 8 = A(x + 2) + B(x + 1)$. Letting $x = -2$, we find that $B = 2$. And taking $x = -1$, we find $A = 3$. Therefore, $\int_0^1 \frac{5x+8}{x^2+3x+2} dx = \int_0^1 \frac{3}{x+1} dx + \int_0^1 \frac{2}{x+2} dx$. Evaluating the integrals, we get $3 \ln(x+1) \Big|_0^1 + 2 \ln(x+2) \Big|_0^1 = 3(\ln 2 - \ln 1) + 2(\ln 3 - \ln 2) = 3 \ln 2 + 2 \ln \frac{3}{2} = \ln 8 + \ln \frac{9}{4} = \ln \frac{8 \cdot 9}{4} = \ln 18$.

Answer to Exercise 25 (on page 66)

- Let $\frac{4x}{x^3+x^2+x+1} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1}$. Rearranging, we see that $4x = A(x^2 + 1) + (Bx + C)(x + 1)$. Which means that $4x = Ax^2 + A + Bx^2 + Bx + Cx + C$, which implies that $A + B = 0$ and $B + C = 4$ and $A + C = 0$. Solving this system of equations, we see that $A = -2$,

$B = 2$, and $C = 2$. So we can say that $\int \frac{4x}{x^3+x^2+x+1} dx = \int \left[\frac{-2}{x+1} + \frac{2x}{x^2+1} + \frac{2}{x^2+1} \right] dx$. Which evaluates to $-2 \ln|x+1| + \ln|x^2+1| + 2 \arctan(x) + K$, where K is the constant of integration.

- Since the order of x is greater in the numerator, first we divide and see that $\frac{x^3-4x+1}{x^2-3x+2} = (x+3) + \frac{3x-5}{x^2-3x+2}$. Now let $\frac{3x-5}{x^2-3x+2} = \frac{A}{x-2} + \frac{B}{x-1}$, which means that $3x-5 = A(x-1) + B(x-2)$. Solving, we find that $A = 1$ and $B = 2$. Therefore, $\int_{-1}^0 \frac{x^3-4x+1}{x^2-3x+2} dx = \int_{-1}^0 \left[x+3 + \frac{1}{x-2} + \frac{2}{x-1} \right] dx$ which evaluates to $\frac{1}{2}x^2 + 3x + \ln|x-2| + 2 \ln|x-1| \Big|_{x=-1}^{x=0} = \frac{5}{2} - \ln 6$.
- Note that $\frac{x^3+2x}{x^4+4x^2+3} = \frac{x^3+2x}{(x^2+1)(x^2+3)}$. Then let $\frac{x^3+2x}{(x^2+1)(x^2+3)} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2+3}$. Then $x^3 + 2x = (A+C)x^3 + (B+D)x^2 + (3A+C)x + (3B+D)$ which implies that $A+C=1$, $B+D=0$, $3A+C=2$, and $3B+D=0$. Solving this system of equations, we see that $A=C=\frac{1}{2}$ and $B=D=0$, which means that $\frac{x^3+2x}{x^4+4x^2+3} = \frac{x}{2(x^2+1)} + \frac{x}{2(x^2+3)}$. And therefore, $\int \frac{x^3+2x}{x^4+4x^2+3} dx = \int \left[\frac{x}{2(x^2+1)} + \frac{x}{2(x^2+3)} \right] dx = \frac{1}{4} \ln|x^2+1| + \frac{1}{4} \ln|x^2+3| + K$, where K is the constant of integration.

Answer to Exercise 26 (on page 68)

- $\int \sin^2 x \cos^3 x dx = \int \sin^2 x \cos x (1 - \sin^2 x) dx = \int \sin^2 x \cos x dx - \int \sin^4 x \cos x dx$. Applying the u -substitution $u = \sin x$ and $du = \cos x dx$: $\int \sin^2 x \cos x dx - \int \sin^4 x \cos x dx = \int u^2 du - \int u^4 du = \frac{1}{3}u^3 - \frac{1}{5}u^5 + C \Big|_{u=\sin x} = \frac{1}{3} \sin^3 x - \frac{1}{5} \sin^5 x + C$
- $\int_0^{\pi/2} \sin^7 \theta \cos^5 \theta d\theta = \int_0^{\pi/2} \sin^7 \theta (\cos^2 \theta)^2 \cos \theta d\theta = \int_0^{\pi/2} \sin^7 \theta (1 - \sin^2 \theta)^2 \cos \theta d\theta = \int_0^{\pi/2} \sin^7 \theta (1 - 2\sin^2 \theta + \sin^4 \theta) \cos \theta d\theta = \int_0^{\pi/2} (\sin^7 \theta - 2\sin^9 \theta + \sin^{11} \theta) \cos \theta d\theta$. Substituting $u = \sin \theta$ and $du = \cos \theta d\theta$: $\int_0^{\pi/2} (\sin^7 \theta - 2\sin^9 \theta + \sin^{11} \theta) \cos \theta d\theta = \int_{u=0}^{u=\pi/2} (u^7 - 2u^9 + u^{11}) du = \left[\frac{1}{8}u^8 - \frac{2}{10}u^{10} + \frac{1}{12}u^{12} \right]_{u=0}^{u=\pi/2} = \left[\frac{1}{8} \sin^8 \theta - \frac{2}{10} \sin^{10} \theta + \frac{1}{12} \sin^{12} \theta \right]_{\theta=0}^{\theta=\pi/2} = \frac{1}{8} - \frac{2}{10} + \frac{1}{12} = \frac{1}{120}$
- $\int \sin^3 \theta \cos^4 \theta d\theta = \int \sin \theta (1 - \cos^2 \theta) \cos^4 \theta d\theta = \int (\cos^4 \theta - \cos^6 \theta) \sin \theta d\theta$. Substituting $u = \cos \theta$ and $du = -\sin \theta d\theta$: $\int (u^6 - u^4) du = \frac{1}{7}u^7 - \frac{1}{5}u^5 + C = \frac{1}{7} \cos^7 \theta - \frac{1}{5} \cos^5 \theta + C$
- $\int_0^{\pi/2} \sin^5 x dx = \int_0^{\pi/2} \sin^4 x \sin x dx = \int_0^{\pi/2} (1 - \cos^2 x)^2 \sin x dx = \int_0^{\pi/2} (1 - 2\cos^2 x + \cos^4 x) \sin x dx$. Substituting $u = \cos x$ and $du = -\sin x dx$: $\int_{x=0}^{x=\pi/2} (-1 + 2u^2 - u^4) du = \left[-u + \frac{2}{3}u^3 - \frac{1}{5}u^5 \right]_{x=0}^{x=\pi/2} = \left[-\cos x + \frac{2}{3} \cos^3 x - \frac{1}{5} \cos^5 x \right]_{x=0}^{x=\pi/2} = 1 - \frac{2}{3} + \frac{1}{5} = \frac{8}{15}$
- First, we substitute for t^2 : $u = t^2$ and $\frac{du}{2} = (t)dt$. $\int t \cos^5(t^2) dt = \int \frac{1}{2} \cos^5 u du = \frac{1}{2} \int \cos^4 u \cos u du = \frac{1}{2} \int (1 - \sin^2 u)^2 \cos u du$. Substituting again, we set $v = \sin u$ and $dv = \cos u du$: $\frac{1}{2} \int (1 - 2v^2 + v^4) dv = \frac{1}{2} \left[v - \frac{2}{3}v^3 + \frac{1}{5}v^5 \right] + C$. Back-substituting $v = \sin u$: $\frac{1}{2} \left(\sin u - \frac{2}{3} \sin^3 u + \frac{1}{5} \sin^5 u \right) + C$. And back-substituting $u = t^2$: $\frac{1}{2} \left(\sin t^2 - \frac{2}{3} \sin^3 t^2 + \frac{1}{5} \sin^5 t^2 \right) + C = \frac{1}{2} \sin t^2 - \frac{1}{3} \sin^3 t^2 + \frac{1}{10} \sin^5 t^2 + C$.

Answer to Exercise 27 (on page 69)

1. $\int_0^{\pi/2} \cos^2 \theta \, d\theta = \int_0^{\pi/2} \frac{1}{2} (1 + \cos 2\theta) \, d\theta = \frac{1}{2} [\theta + \frac{1}{2} \sin 2\theta]_{\theta=0}^{\theta=\pi/2} = \frac{1}{2} [(\frac{\pi}{2} - 0) + \frac{1}{2} (\sin 2 \cdot \frac{\pi}{2} - \sin 0)] = \frac{1}{2} [\frac{\pi}{2} + \frac{1}{2} \sin \pi] = \frac{1}{2} \cdot \frac{\pi}{2} = \frac{\pi}{4}$
2. $\int_0^{\pi} \cos^4 2t \, dt = \int_0^{\pi} (\cos^2 2t)^2 \, dt = \int_0^{\pi} [\frac{1}{2} (1 + \cos 4t)]^2 \, dt = \frac{1}{4} \int_0^{\pi} (1 + 2 \cos 4t + \cos^2 4t) \, dt = \frac{1}{4} \int_0^{\pi} [1 + 2 \cos 4t + \frac{1}{2} (1 + \cos 8t)] \, dt = \frac{1}{4} \int_0^{\pi} [\frac{3}{2} + 2 \cos 4t + \frac{1}{2} \cos 8t] \, dt = \frac{1}{4} [\frac{3}{2}t + \frac{1}{2} \sin 4t + \frac{1}{16} \sin 8t]_{t=0}^{t=\pi} = \frac{1}{4} [(\frac{3\pi}{2} - 0) + \frac{1}{2} (\sin 4\pi - \sin 0) + \frac{1}{16} (\sin 8\pi - \sin 0)] = \frac{3\pi}{8}$
3. $\int \sin^2 x \cos^2 x \, dx = \int \frac{1}{2} (1 - \cos 2x) \frac{1}{2} (1 + \cos 2x) \, dx = \frac{1}{4} \int (1 - \cos^2 2x) \, dx = \frac{1}{4} \int [1 - \frac{1}{2} (1 + \cos 4x)] \, dx = \frac{1}{4} \int (\frac{1}{2} - \frac{1}{2} \cos 4x) \, dx = \frac{1}{8} \int 1 - \cos 4x \, dx = \frac{1}{8} [x - \frac{1}{4} \sin 4x] + C = \frac{1}{32} [4x - \sin 4x] + C$
4. $\int_0^{2\pi} \sin^2 (\frac{1}{3}\theta) \, d\theta = \int_0^{2\pi} \frac{1}{2} (1 - \cos (\frac{2}{3}\theta)) \, d\theta = \frac{1}{2} [\theta - \frac{3}{2} \sin (\frac{2}{3}\theta)]_{\theta=0}^{\theta=2\pi} = \frac{1}{2} [(2\pi - 0) - \frac{3}{2} (\sin \frac{4\pi}{3} - \sin 0)] = \frac{1}{2} (2\pi - \frac{3}{2} (-\frac{\sqrt{3}}{2})) = \pi + \frac{3\sqrt{3}}{8}$
5. $\int_0^{\pi} \sin^2 t \cos^4 t \, dt = \int_0^{\pi} \frac{1}{2} (1 - \cos 2t) (\frac{1}{2} (1 + \cos 2t))^2 \, dt = \frac{1}{8} \int_0^{\pi} (1 - \cos 2t) (1 + 2 \cos 2t + \cos^2 2t) \, dt = \frac{1}{8} \int_0^{\pi} (1 + 2 \cos 2t + \cos^2 2t - \cos 2t - 2 \cos^2 2t - \cos^3 2t) \, dt = \frac{1}{8} \int_0^{\pi} (1 + \cos 2t - \cos^2 2t - \cos^3 2t) \, dt = \frac{1}{8} \int_0^{\pi} (1 + \cos 2t - \frac{1}{2} (1 + \cos 4t)) \, dt - \frac{1}{8} \int_0^{\pi} \cos 2t \cos^2 2t \, dt = \frac{1}{8} \int_0^{\pi} (\frac{1}{2} + \cos 2t - \frac{1}{2} \cos 4t) \, dt - \frac{1}{16} \int_0^{\pi} \cos 2t (1 - \sin^2 2t) \, dt$. Let $u = \sin 2t$, then $du = 2 \cos 2t \, dt$ and $\cos 2t \, dt = \frac{du}{2}$. Then the previous integrals are equal to $\frac{1}{8} [\frac{t}{2} + \frac{1}{2} \sin 2t - \frac{1}{8} \sin 4t]_{t=0}^{t=\pi} - \frac{1}{16} \int_{t=0}^{t=\pi} \frac{1}{2} (1 - u^2) \, du = \frac{1}{8} [(\frac{\pi}{2} - 0) + \frac{1}{2} (\sin 2\pi - \sin 0) - \frac{1}{8} (\sin 4\pi - \sin 0)] - \frac{1}{32} [u - \frac{1}{3} u^3]_{t=0}^{t=\pi} = \frac{1}{8} (\frac{\pi}{2}) - \frac{1}{32} [\sin 2t - \frac{1}{3} \sin^3 2t]_{t=0}^{t=\pi} = \frac{\pi}{16} - \frac{1}{32} [(\sin 2\pi - \sin 0) - \frac{1}{3} (\sin^3 2\pi - \sin^3 0)] = \frac{\pi}{16} - \frac{1}{32} (0) = \frac{\pi}{16}$

Answer to Exercise 28 (on page 71)

1. $\int \tan x \sec^3 x \, dx = \int \sec^2 x \tan x \sec x \, dx$. Let $u = \sec x$, then $du = \sec x \tan x \, dx$. Substituting, $\int \sec^2 x \tan x \sec x \, dx = \int u^2 \, du = \frac{1}{3} u^3 + C = \frac{1}{3} \sec^3 x + C$
2. $\int \tan^4 x \sec^6 x \, dx = \int \tan^4 x \sec^4 x \sec^2 x \, dx = \int \tan^4 x (1 + \tan^2 x)^2 \sec^2 x \, dx$. Let $u = \tan x$, then $du = \sec^2 x \, dx$. Substituting, $\int \tan^4 x (1 + \tan^2 x)^2 \sec^2 x \, dx = \int u^4 (1 + u^2)^2 \, du = \int u^4 (1 + 2u^2 + u^4) \, du = \int u^4 + 2u^6 + u^8 \, du = \frac{1}{5} u^5 + \frac{2}{7} u^7 + \frac{1}{9} u^9 + C = \frac{1}{5} \tan^5 x + \frac{2}{7} \tan^7 x + \frac{1}{9} \tan^9 x + C$
3. $\int_0^{\pi/4} \sec^6 \theta \tan^6 \theta \, d\theta = \int_0^{\pi/4} \sec^4 \theta \tan^6 \theta \sec^2 \theta \, d\theta = \int_0^{\pi/4} (1 + \tan^2 \theta)^2 \tan^6 \theta \sec^2 \theta \, d\theta$. Let $u = \tan \theta$ and $du = \sec^2 \theta \, d\theta$. Substituting, $\int_0^{\pi/4} (1 + \tan^2 \theta)^2 \tan^6 \theta \sec^2 \theta \, d\theta = \int_{\theta=0}^{\theta=\pi/4} (1 + u^2)^2 u^6 \, du = \int_{\theta=0}^{\theta=\pi/4} (1 + 2u^2 + u^4) u^6 \, du = \int_{\theta=0}^{\theta=\pi/4} u^6 + 2u^8 + u^{10} \, du = [\frac{1}{7} u^7 + \frac{2}{9} u^9 + \frac{1}{11} u^{11}]_{\theta=0}^{\theta=\pi/4} = [\frac{1}{7} \tan^7 \theta + \frac{2}{9} \tan^9 \theta + \frac{1}{11} \tan^{11} \theta]_{\theta=0}^{\theta=\pi/4} = \frac{1}{7} (1^7 - 0^7) + \frac{2}{9} (1^9 - 0^9) + \frac{1}{11} (1^{11} - 0^{11}) = \frac{1}{7} + \frac{2}{9} + \frac{1}{11} = \frac{316}{693}$

Answer to Exercise 29 (on page 76)

This question takes the form of integration by parts. That is, $\int f(x)g'(x) dx = f(x)g(x) - \int g(x)f'(x) dx$. If we let $g(x) = -\cos x$, then $g'(x) = \sin x$. The structure of the equation implies that $f'(x) = 4x^3$ and therefore that f could be $f(x) = x^4$.

Answer to Exercise 30 (on page 76)

1. $\frac{4}{\pi^2}$. Let $f = x$ and $g' = \sin \frac{\pi}{2}x dx$. Then $f' = dx$ and $g = -\frac{2}{\pi} \cos \frac{\pi}{2}x$. Which implies that $\int_0^1 x \sin \frac{\pi}{2}x dx = \left[\frac{-2x}{\pi} \cos \frac{\pi}{2}x \right]_{x=0}^{x=1} - \int_0^1 \frac{-2}{\pi} \cos \frac{\pi}{2}x dx$. Evaluating $\left[\frac{-2x}{\pi} \cos \frac{\pi}{2}x \right]_{x=0}^{x=1} = \left(\frac{-2}{\pi} \cos \frac{\pi}{2} \right) - (0 \cos 0) = 0 - 0 = 0$. Therefore, $\int_0^1 x \sin \frac{\pi}{2}x dx = \int_0^1 \frac{2}{\pi} \cos \frac{\pi}{2}x dx = \frac{2}{\pi} \left[\frac{2}{\pi} \sin \frac{\pi}{2}x \right]_0^1 = \frac{4}{\pi^2} \left[\sin \frac{\pi}{2} - \sin 0 \right] = \frac{4}{\pi^2}$.
2. $\frac{e^\theta}{2} (\sin \theta + \cos \theta)$. Let $f = e^\theta$ and $g' = \cos \theta d\theta$. Then $f' = e^\theta d\theta$ and $g = \sin \theta$ and according to integration by parts $\int e^\theta \cos \theta d\theta = e^\theta \sin \theta - \int e^\theta \sin \theta d\theta$. We can also evaluate $\int e^\theta \sin \theta d\theta$ using integration by parts. Let $f = e^\theta$ and $g' = \sin \theta d\theta$. Then $f' = e^\theta d\theta$ and $g = -\cos \theta$ and according to integration by parts $\int e^\theta \cos \theta d\theta = e^\theta \sin \theta - [-e^\theta \cos \theta - \int -e^\theta \cos \theta d\theta] = e^\theta \sin \theta + e^\theta \cos \theta - \int e^\theta \cos \theta d\theta$. We can rearrange this to solve for $\int e^\theta \cos \theta d\theta$: $2 \int e^\theta \cos \theta d\theta = e^\theta \sin \theta + e^\theta \cos \theta \rightarrow \int e^\theta \cos \theta d\theta = \frac{e^\theta}{2} (\sin \theta + \cos \theta)$.
3. $\frac{(1-t)^2}{\beta} \sin \beta t + \frac{2(t-1)}{\beta^2} \cos \beta t - \frac{2}{\beta^3} \sin \beta t$. Let $f = (1-t)^2$ and $g' = \cos \beta t dt$. Then $f' = -2(1-t)dt$ and $g = \frac{1}{\beta} \sin \beta t$. Then using integration by parts $\int (1-t)^2 \cos \beta t dt = \frac{(1-t)^2}{\beta} \sin \beta t - \int \frac{(-2)(1-t)}{\beta} \sin \beta t dt = \frac{(1-t)^2}{\beta} \sin \beta t + \frac{2}{\beta} \int (1-t) \sin \beta t dt$. We use integration by parts again to evaluate $\int (1-t) \sin \beta t dt$. Let $f = 1-t$ and $g' = \sin \beta t dt$. Then $f' = -dt$ and $g = -\frac{1}{\beta} \cos \beta t$. Then $\int (1-t) \sin \beta t dt = (1-t) \left(-\frac{1}{\beta} \right) \cos \beta t - \int \left(-\frac{1}{\beta} \right) \cos \beta t - dt = \frac{t-1}{\beta} \cos \beta t - \int \frac{\cos \beta t}{\beta} dt = \frac{t-1}{\beta} \cos \beta t - \frac{1}{\beta^2} \sin \beta t$. Substituting this back in for $\int (1-t) \sin \beta t dt$, we see that $\int (1-t)^2 \cos \beta t dt = \frac{(1-t)^2}{\beta} \sin \beta t + \frac{2}{\beta} \left[\frac{t-1}{\beta} \cos \beta t - \frac{1}{\beta^2} \sin \beta t \right] = \frac{(1-t)^2}{\beta} \sin \beta t + \frac{2(t-1)}{\beta^2} \cos \beta t - \frac{2}{\beta^3} \sin \beta t$.

Answer to Exercise 31 (on page 78)

$$\mu = \int_{-\infty}^{\infty} tf(t) dt = \frac{1}{595} \int_0^{120} t^2 e^{-0.04t} dt$$

Integrate by parts twice, or use the antiderivative

$$\int t^2 e^{-at} dt = -e^{-at} \left(\frac{t^2}{a} + \frac{2t}{a^2} + \frac{2}{a^3} \right)$$

with $a = 0.04$. Evaluating from 0 to 120 gives

$$\mu = \frac{1}{595} \left(31250 - 541250 e^{-4.8} \right) = \frac{6250 - 108250 e^{-4.8}}{119} \approx 45.04$$

Therefore, the mean time until the onset of symptoms is approximately 45 hours.



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