Contents

Chapter 1

Introduction to Linear Algebra

Welcome to the world of linear algebra, a branch of mathematics that relies on vectors, matrices, and linear transformations. You are familiar with most of these concepts, so in this workbook you will see how you can use them together to solve problems.

Let's review what you know.

- **Vectors**. In workbook 7, you saw how vectors can represent forces, as well as how to add and multiply them to figure out such things as rocket engine force and direction.
- **Matrices**. In workbook 3, you learned to use spreadsheets to solve problems numerically, such as how to figure out the number of barrels a cooper has to produce to achieve a certain take-home pay. Spreadsheets are essentially matrices — a row by column structure that contains values.
- **Linear transformations**. When you studied congruence in workbook 5, you were introduced to a few linear transformations, such as translation and reflection.

1.1 What's With the Linear?

You might be thinking, "Hey, haven't I been doing algebra already?"

You have! You have come a long way in your problem solving journey. You have used algebra to solve simple equations like $7x + 10 = 24$ and quadratic equations like $4x^2 + 9x + 31 = 0$. What distinguishes linear algebra is the focus on linear combinations. Any equation with a power greater than 1, such as a quadratic, is nonlinear. Want to see what we will be covering? Take a look at 3blue1brown's website with papers and videos on Linear Algebra: `https://www.3blue1brown.com/topics/linear-algebra`

We will first take a look at linear combinations

1.2 Linear Combinations

You won't see any **sin**, **cos**, or **tan** operations in this section. Linear operations do not use trigonometric functions; those operations are all nonlinear. A linear combination consists solely of addition and scalar multiplication. You will see that linear combinations allow you to solve many types of problems in science and engineering. Before we get deep into the numbers, let's take a look at a few linear operations you can perform on images. This will give you an intuition for the underlying math. After that, we'll take a look at some numbers.

1.3 Image Operations

The simplest image, a bitmap, can be represented by a two-dimensional matrix of values — either 0 for black, or 1 for white. Grayscale images are also represented by a two-dimensional matrix of values, but the values typically range from 0 to 255. 0 is black, 255 is white, and the values in between represent shades of gray.

Color images are more complex. The simplest color image is a three-dimensional matrix of values. You can think of it as three 2D matrices, one to represent red values (R), another for green values (G), and the third for blue values (B). The combination of R, G, and B determines the color you see.

Working with images means working with millions of pixels. Fortunately, modern techniques make this a snap. Let's look at some common operations on an image of a rocket. Flipping is a linear operation.

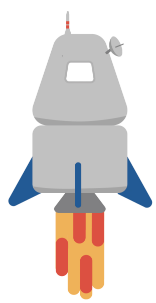


Figure 1.1: Rocket image.

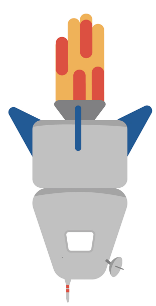


Figure 1.2: The rocket being flipped involves a binary linear equation.

Next, the image is rotated 90 degrees. This rotation is linear, but if you want to rotate it at an angle that isn't a multiple of 90, you would need trigonometry. This would be treading into nonlinear territory, but that happens in the field of linear algebra. You will learn about nonlinear extensions later, which use trigonometric functions and imaginary numbers.

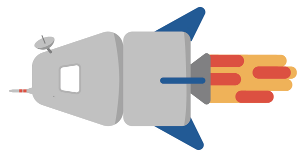


Figure 1.3: Rotating the rocket by 90°

Inversion is an interesting linear operation that involves redefining the red, green, and blue values, such that the new value is 1.0 minus the old value. The resulting black background gives the impression the rocket is in deep space, don't you think?

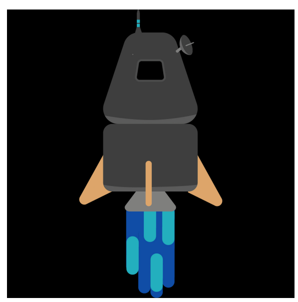


Figure 1.4: Inverting the rocket replaces each pixel with a "opposite" color.

It is possible to redefine the red, green, and blue values in many ways. Visit the NASA website and search for false color images. NASA and other scientists redefine colors to communicate such things as the amount of vegetation or water in an area, the temperatures of the sun's surface, and so on. Photographers often do this for artistic effect. For example, the image on the left was taken with an infrared camera. (This infrared is not the same as thermal kind you have likely seen before. This is the infrared that is emitted by living plants.) The image is further processed to swap channels. For example, the matrix representing red might be swapped with the matrix representing blue. The image on the right shows the image after swapping color values. All these swapping operations are linear.

Figure 1.5: Left: image with infrared camera. Right: image shown post swapping color values

1.4 The Numbers Behind Some Image Operations

You will see a few *matrices* in this section. Let's first look at how a spreadsheet can be represented as a matrix. Recall the barrel-making shop example. This is part of that spreadsheet.

	A	B	C	D
1	**Barrels Produced** (per month)	115	120	125
2	**Materials cost** (per barrel)	$45.00	$45.00	$45.00
3	**Sale price** (per barrel)	$100.00	$100.00	$100.00
4	**Rent** (per month)	$2,000.00	$2,000.00	$2,000.00
5	**Pretax Earnings** (per month)	$4,325.00	$4,600.00	$4,875.00
6	**Taxes** (per month)	$865.00	$920.00	$975.00
7	**Take home pay** (per month)	$3,460.00	$3,680.00	$3,900.00

Figure 1.6: Spreadsheet of the barrel example.

Represented as a matrix, it looks like the following. Note the differences: A matrix contains only values, no labels. This matrix uses floating point values, hence the inclusion of decimal points.

$$\begin{bmatrix} 115.0 & 120.0 & 125.0 \\ 45.0 & 45.0 & 45.0 \\ 100.0 & 100.0 & 100.0 \\ 2000.0 & 2000.0 & 2000.0 \\ 4325.0 & 4600.0 & 4875.0 \\ 865.0 & 920.0 & 975.0 \\ 3460.0 & 3680.0 & 3900.0 \end{bmatrix}$$

A matrix that represents an image contains only pixel values, whereas the barrel-making shop matrix represents seven kinds of variables: barrels produced, materials cost, sales price, rent, pretax earnings, taxes, and take home pay.

The simplest image to create is a bitmap, because that requires a matrix of zeros and ones. This is a matrix for a 10 pixel by 10 pixel image. Why use decimal points when this is obviously a matrix of integers? It turns out that when you use tools like Python to process matrices, you must be conscious of data types. Most of the Python methods we use for image operations expect floating points. A few expect integer types, but you'll see how to handle type conversion later, in the section on Python.

$$\begin{bmatrix} 0. & 0. & 0. & 0. & 1. & 0. & 0. & 0. & 1. & 0 \\ 0. & 0. & 0. & 0. & 1. & 1. & 1. & 1. & 1. & 1 \\ 0. & 0. & 0. & 0. & 1. & 0. & 0. & 0. & 1. & 0 \\ 0. & 0. & 0. & 0. & 1. & 0. & 0. & 0. & 1. & 0 \\ 0. & 0. & 0. & 0. & 1. & 0. & 0. & 0. & 1. & 0 \\ 0. & 0. & 0. & 0. & 1. & 0. & 0. & 0. & 1. & 0 \\ 0. & 0. & 0. & 0. & 1. & 0. & 0. & 0. & 1. & 0 \\ 0. & 0. & 0. & 0. & 1. & 1. & 1. & 1. & 1. & 1 \\ 0. & 0. & 0. & 0. & 1. & 0. & 0. & 0. & 1. & 0 \\ 0. & 0. & 0. & 0. & 1. & 0. & 0. & 0. & 1. & 0 \end{bmatrix}$$

When converted to an image, it is very tiny. This is an enlarged version, so you can see the pattern.

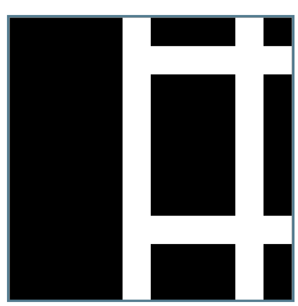


Figure 1.7: Resulting bitmap vectors as a black and white image.

We can create an inverse of this image by changing all the values in the matrix so that 0 becomes 1 and 1 becomes 0. (Technically this is not the way you would invert a matrix, as you will see in the Python section. For this example, you will get the same visual result, but when you start inverting matrices programmatically, you will learn the formal definition.)

$$\begin{bmatrix} 1. & 1. & 1. & 1. & 0. & 1. & 1. & 1. & 0. & 1 \\ 1. & 1. & 1. & 1. & 0. & 0. & 0. & 0. & 0. & 0 \\ 1. & 1. & 1. & 1. & 0. & 1. & 1. & 1. & 0. & 1 \\ 1. & 1. & 1. & 1. & 0. & 1. & 1. & 1. & 0. & 1 \\ 1. & 1. & 1. & 1. & 0. & 1. & 1. & 1. & 0. & 1 \\ 1. & 1. & 1. & 1. & 0. & 1. & 1. & 1. & 0. & 1 \\ 1. & 1. & 1. & 1. & 0. & 1. & 1. & 1. & 0. & 1 \\ 1. & 1. & 1. & 1. & 0. & 0. & 0. & 0. & 0. & 0 \\ 1. & 1. & 1. & 1. & 0. & 1. & 1. & 1. & 0. & 1 \\ 1. & 1. & 1. & 1. & 0. & 1. & 1. & 1. & 0. & 1 \end{bmatrix}$$

When converted to an image and enlarged, it looks like this:

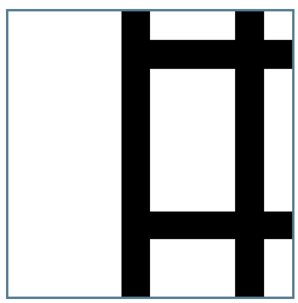


Figure 1.8: Inversion of the linearr matrix.

Rotating the original matrix by 90 degrees gives this:

$$\begin{bmatrix} 0. & 1. & 0. & 0. & 0. & 0. & 0. & 1. & 0. & 0. \\ 1. & 1. & 1. & 1. & 1. & 1. & 1. & 1. & 1. & 1. \\ 0. & 1. & 0. & 0. & 0. & 0. & 0. & 1. & 0. & 0. \\ 0. & 1. & 0. & 0. & 0. & 0. & 0. & 1. & 0. & 0. \\ 0. & 1. & 0. & 0. & 0. & 0. & 0. & 1. & 0. & 0. \\ 1. & 1. & 1. & 1. & 1. & 1. & 1. & 1. & 1. & 1. \\ 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \end{bmatrix}$$

This is the resulting enlarged image:

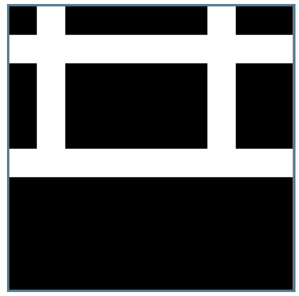


Figure 1.9: Rotating the image by 90deg.

You'll transpose many matrices in the upcoming pages. It requires swapping rows for columns.

$$
\begin{bmatrix}
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\
1. & 1. & 1. & 1. & 1. & 1. & 1. & 1. & 1. & 1. \\
0. & 1. & 0. & 0. & 0. & 0. & 0. & 1. & 0. & 0. \\
0. & 1. & 0. & 0. & 0. & 0. & 0. & 1. & 0. & 0. \\
0. & 1. & 0. & 0. & 0. & 0. & 0. & 1. & 0. & 0. \\
1. & 1. & 1. & 1. & 1. & 1. & 1. & 1. & 1. & 1. \\
0. & 1. & 0. & 0. & 0. & 0. & 0. & 1. & 0. & 0.
\end{bmatrix}
$$

The resulting image looks like this:

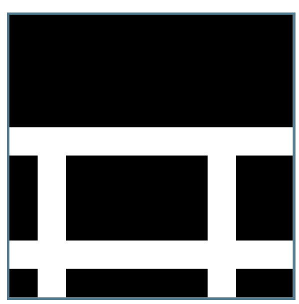


Figure 1.10: Transposed image reversing rows and columns.

What about adding images? That fits the definition of a linear combination. Recall that grayscale images have values from 0 to 255. To make things simple, let's define two matrices with values ranging from 0.0 to 1.0. When we want a grayscale image, it is easy to multiply the matrix by 255.

Let's call this matrix f.

$$\begin{bmatrix} 1.0 & 0.5 & 0.0 \\ 1.0 & 1.0 & 1.0 \\ 0.5 & 0.0 & 0.0 \end{bmatrix}$$

When multiplied by 255 and converted to a grayscale image:

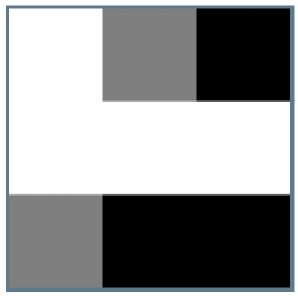


Figure 1.11: Bitmap of matrix f.

Let's call this matrix g:

$$\begin{bmatrix} 0.5 & 0.0 & 0.0 \\ 1.0 & 0.5 & 1.0 \\ 1.0 & 1.0 & 1.0 \end{bmatrix}$$

When multiplied by 255 and converted to a grayscale image:

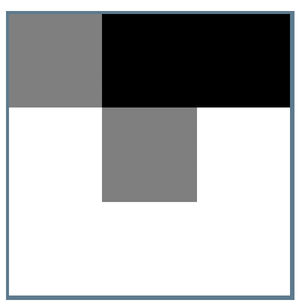


Figure 1.12: Bitmap of matrix g.

When we add f and g we get k:

$$\begin{bmatrix} 1.5 & 0.5 & 0.0 \\ 2.0 & 1.5 & 2.0 \\ 1.5 & 1.0 & 1.0 \end{bmatrix}$$

However, the values in k exceed the range of 0.0 to 1.0, so we normalize by dividing the matrix by 2.0:

$$\begin{bmatrix} 0.75 & 0.25 & 0.00 \\ 1.00 & 0.75 & 1.00 \\ 0.75 & 0.50 & 0.50 \end{bmatrix}$$

When multiplied by 255 and converted to grayscale, we get:

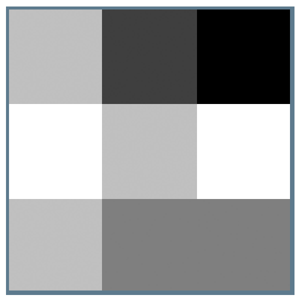


Figure 1.13: Adding $f + g$ represented as a bitmap.

Let's go back to the first small grayscale image (from Figure 1.11):

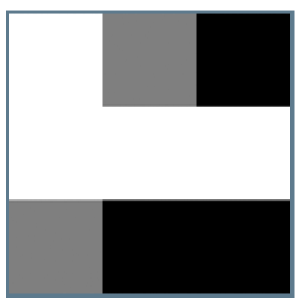


Figure 1.14: Figure 11, repeated.

If you want to keep the pattern in the first column, you could multiply the matrix by a vector, [1.0 0.0 0.0]. The 1.0 will keep the values in the first column, but the 0.0 will knock out the other values because 0.0 times anything equals 0.0.

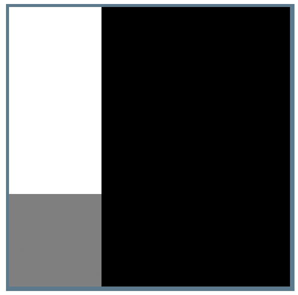


Figure 1.15: Multiplying channels by 0 results in them being represented as black bitmaps, while 1 keeps them the same.

What do you think will happen if you use the vector [0.0 1.0 0.0] or [0.0 0.0 1.0]? You'll get a chance later to use Python to perform image operations.

All the operations we performed on these images satisfy the requirement for linear combinations: preserving addition and scalar multiplication.

1.5 Applications of Linear Algebra

So far you've seen how linear operations on matrices can process images by:

- multipling a matrix using a scalar (e.g., normalize, change the range)
- adding one matrix to another to get a composite image
- multipling two matrices to perform a transform (e.g., flipping)
- mulitpling a matrix with a vector (isolating a channel)

Many areas in engineering and science rely on the matrix operations defines by linear algebra. Besides image processing, linear algebra is used for:

- **Computer Graphics**. When you play a video game or watch the latest CG animation, matrix operations transform objects in the scene to make them appear as if moving, getting closer, and so on. You can represent the vertices of objects as vectors, and then apply a transformation matrix.
- **Data Analysis**. We live in an era in which it's easy to collect so much data that it's difficult to make sense of the data by just looking at it. You can represent the data in matrix form and then find a solution vector. For example, scientists use this technique to figure out the effectiveness of drug treatments on disease.
- **Economics**. Take a look at financial section of any news organization and you'll see headlines such as "Economic Data Points to Faster Growth" or "Is the Inflation

Battle Won?" Economists can use systems of linear equations to represent economic indicators, such as consumer consumption, government spending, investment rate, and gross national product. By using various methods that you'll learn about later, they can get a good idea of the state of the economy.

- **Engineering**. Engineers couldn't do without linear algebra. For example, the orbital dynamics of space travel relies on it. Engineers must predict and calculate the the motion of planetary bodies, satellites, and spacecraft. By solving systems of linear equations engineers can make sure that a spacecraft travels to its destination without running into a satellite or space rock.

1.6 Let's Observe the Sun!

India recently sent the Aditya spacecraft on a mission to study the Sun. Without a thorough understanding of linear algebra (among other things), the engineers would not have accomplished the amazing feat of getting Aditya in a stable orbit around a Lagrange point.

In previous chapters you learned about gravity and its effects. A Lagrange point is a point in space between two bodies (e.g. Earth and Sun) where there is gravitational equilibrium. With the right trajectory, a spacecraft will orbit around a Lagrange point in a stable position that doesn't require much energy to maintain. That's called a Halo orbit. Because that there are no fueling stations in space, a Halo orbit will allow Aditya to maintain position for about 5 years. Pretty good mileage!

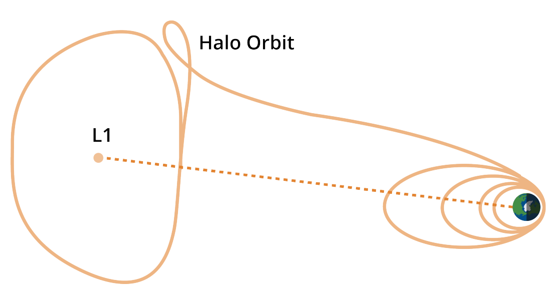


Figure 1.16: The Aditya Orbit.

Aditya's engineers had to calculate a looping maneuver that would precisely inject the Aditya spacecraft into the Halo orbit. They determined the angles and burn times for the thrust engine. If they were wrong in one direction, the spacecraft would fly off to the sun. The other direction would send the spacecraft back in the direction of Earth. Their success is due to a solid understanding of vectors and linear algebra.

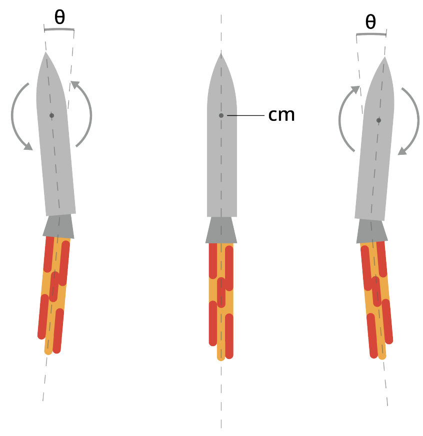


Figure 1.17: The gimbal of the rocket engine.

1.7 Images in Python

One of the wonderful things about python is the availability of libraries for specialized computation. The Python Imaging Library, PIL, is what you'll use to create images, read existing images from disk, and perform operation on images. To create and manipulate arrays, you will use NumPy.

Create a file called `image_creation.py` and enter this code:

```
# Import necessary modules
import numpy as np
import PIL
from PIL import Image
from PIL import ImageOps

# Create a 10 by 10 pixel bitmap Image.
# Using a decimal point ensure python see the values as floating point numbers
# Some image operations assume floats

bitmapArray = np.array([
[0., 0., 0., 0., 1., 0., 0., 0., 1., 0.],
[0., 0., 0., 0., 1., 1., 1., 1., 1., 1.],
[0., 0., 0., 0., 1., 0., 0., 0., 1., 0.],
[0., 0., 0., 0., 1., 0., 0., 0., 1., 0.],
[0., 0., 0., 0., 1., 0., 0., 0., 1., 0.],
[0., 0., 0., 0., 1., 0., 0., 0., 1., 0.],
[0., 0., 0., 0., 1., 0., 0., 0., 1., 0.],
[0., 0., 0., 0., 1., 1., 1., 1., 1., 1.],
[0., 0., 0., 0., 1., 0., 0., 0., 1., 0.],
```

```
[0., 0., 0., 0., 1., 0., 0., 0., 1., 0.]])

# Image.fromarray assumes a range of 0 to 255, so scale by 255

myImage = Image.fromarray(bitmapArray*255)
myImage.show()

# A window opens with an image so tiny you might think nothing is there
# Zoom in to see the pattern

# Transpose the array, create an image, and then show it.
# Note that you operate on the original array (not the image)
# Remember to zoom in to see the pattern

myImageTransposed = bitmapArray.transpose()
myImageTransposed.show()

# Invert the array. You'll use the NumPy invert method.
# The invert method assumes integer values. You need to convert the data type
# Numpy has a method for that

intBitmapArray = np.asarray(bitmapArray,dtype="int")
invertedArray = np.invert(intBitmapArray)

# Take a look at the array

invertedArray

# The values range from -2 to -1. Image values are positive.
# You need to change the range so the values are from 0 to 255
# Further you need to change back to floating point values because
# the PIL method requires them

invertedArray = (invertedArray + 2)*1.0
invertedImage = Image.fromarray(255*invertedArray)
invertedImage.show()

# Zoom in on the image and compare the pattern with the original
```

FIXME this needs improvement

1.8 Exercise

Create a python program that creates matrix f and matrix g from the previous section, and then performs all the operations shown in that section. If you are not sure how to accomplish something, consult the online documentation for the PIL and NumPy python libraries.

Chapter 2

Vectors and Matrices

The last chapter provided an overview of linear algebra, using several image examples. In this chapter, we will focus primarily on vector-matrix multiplications. First, we will show how matrices can be used to represent a set of linear equations. Then, we will provide you with a general definition of vector-matrix multiplication, followed by a few examples. You will have an opportunity to solve a problem manually, then by using Python. In this chapter, we will use two-dimensional matrices for simplicity, but a matrix can have any number of dimensions.

2.1 Matrices

We've been looking at vectors. We've seen them in physics as a straight line comprised of x and y components, or represented as a column of numbers. For example, while we may write $\mathbf{v} = [1, 2, 3]$ in line, the vector is really:

$$\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

A matrix can be made of many columns, like the 3×3 matrix shown below:

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 4 & 8 \\ 3 & 9 & 27 \end{bmatrix}$$

We describe the size and shape of matrices by saying *an* $m \times n$ *matrix*, where m is the number of *rows* and n is the number of *columns*. A *vector* is simply a one-column matrix. For example, the vectors $\mathbf{v}$ above is of size3×1. Matrices aren't restricted to 2 dimensions: a matrix can be 3, 4, or any number of dimensions. For example, a $3 \times 2 \times 4$ matrix would be made of 4 stacked 3×2 matrices. Visually, however, that is hard to represent, and even worse to compute.

A matrix is considered *square* when it has the same number of rows and columns. Most matrix operations have the restriction that the matrix must be square in order to perform operations, such as determinant calculation, finding an inverse, or eigenvector/eigenvalues, all of which you will learn in future chapters.3

Exercise 1 Matrix Dimensions 1

Write the dimensions of the following matrices:

1. $\begin{bmatrix} -3 & 0 & 4 & -2 & -4 \\ -1 & 5 & 3 & 4 & -2 \\ -3 & 2 & 3 & -5 & 1 \end{bmatrix}$
2. $\begin{bmatrix} -3 & 1 \end{bmatrix}$
3. $\begin{bmatrix} -3 & 2 & -3 \\ 4 & 0 & -3 \\ -5 & -4 & 1 \\ 0 & -2 & 2 \end{bmatrix}$

Working Space

Answer on Page 83

Exercise 2 Matrix Dimensions 2

Create a matrix with the indicated dimensions.

1. 1×3
2. 2×4
3. 4×3

Working Space

Answer on Page 83

2.1.1 Zero Matrices

Recall that we can represent a generic zero vector as $\mathbf{0}$ or $\vec{0}$ (you may see both), which indicates a vector of any number of dimensions filled with zeros. Just like vectors, there are *zero matrices*, which can be any number of dimensions, all filled with zeros. In two dimensions, zero matrices are denoted as $0_{m \times n}$, where the subscript is the dimension of the matrix. The subscript can be expanded to denote any number of dimensions.

2.2 Operations of Matrices

2.2.1 Adding and Subtracting Matrices

Matrices that are the same dimension can be added and subtracted. Just like vectors, to add matrices you add the elements in the same position:

$$\begin{bmatrix} -2 & -1 \\ 2 & 4 \end{bmatrix} + \begin{bmatrix} 5 & 2 \\ -1 & -4 \end{bmatrix} = \begin{bmatrix} -2+5 & -1+2 \\ 2+-1 & 4+-4 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & 0 \end{bmatrix}$$

And to subtract matrices, you subtract the elements in the same position:

$$\begin{bmatrix} -2 & -1 \\ 2 & 4 \end{bmatrix} - \begin{bmatrix} 5 & 2 \\ -1 & -4 \end{bmatrix} = \begin{bmatrix} -2-5 & -1-2 \\ 2-(-1) & 4-(-4) \end{bmatrix} = \begin{bmatrix} -7 & -3 \\ 3 & 8 \end{bmatrix}$$

Formally, for 2-dimensional matrices, we can say that:

Adding and Subtracting Matrics

For two $m \times n$ matrices, the sum of the matrices is the matrix of the sums of the elements in analogous positions:

$$\begin{bmatrix} x_{11} & x_{12} & x_{13} & \cdots & x_{1n} \\ x_{21} & x_{22} & x_{23} & \cdots & \vdots \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ x_{m1} & x_{m2} & x_{m3} & \cdots & x_{mn} \end{bmatrix} + \begin{bmatrix} y_{11} & y_{12} & y_{13} & \cdots & y_{1n} \\ y_{21} & y_{22} & y_{23} & \cdots & \vdots \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ y_{m1} & y_{m2} & y_{m3} & \cdots & y_{mn} \end{bmatrix} =$$

$$\begin{bmatrix} x_{11}+y_{11} & x_{12}+y_{12} & x_{13}+y_{13} & \cdots & x_{1n}+y_{1n} \\ x_{21}+y_{21} & x_{22}+y_{22} & x_{23}+y_{23} & \cdots & \vdots \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ x_{m1}+y_{m1} & x_{m2}+y_{m2} & x_{m3}+y_{m3} & \cdots & x_{mn}+y_{mn} \end{bmatrix}$$

To subtract matrices, simply add the negative of the second matrix (that is, $\boldsymbol{A} - \boldsymbol{B} = \boldsymbol{A} + -\boldsymbol{B}$). Additionally, matrix addition is commutative ($\boldsymbol{A} + \boldsymbol{B} = \boldsymbol{B} + \boldsymbol{A}$).

Matrices of different dimensions cannot be added or subtracted.

Exercise 3 Adding and Subtracting Matrices

Find $A + B$, A - B, and B - A.

Working Space

1. $A = \begin{bmatrix} 0 & 4 & 0 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 3 & -2 & 5 \end{bmatrix}$
2. $A = \begin{bmatrix} 4 & -4 & -2 \\ 1 & -3 & 5 \\ -5 & 3 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 0 & -1 \\ -5 & -3 & -2 \\ -5 & 3 & -4 \end{bmatrix}$.
3. $A = \begin{bmatrix} -2 & -1 & -5 & -1 \\ 5 & -4 & 4 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} -5 & -2 & 3 & -5 \\ 0 & 5 & -4 & -3 \end{bmatrix}$.

Answer on Page 83

2.2.2 Multiplying Matrices

Matrix multiplication has dimensional limits. We cannot multiply any two matrices; the first matrix must have the same number of columns as the second has number of rows. Let's examine the origin of the dimension limits on matrix multiplication. We begin with a review of the vector dot product.

Recall that in order to find the dot product of two vectors, they must be the same length (that is, the same number of dimensions). The result is always a scalar: one number. You can review finding the dot product of vectors and practice the dimension limits on the vector dot product in the next exercise.

Exercise 4 Vector Dot Product Review

Working Space

Find all possible pairs of vectors that can be used to find a dot product, then find the dot products.

1. $\mathbf{a} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$
2. $\mathbf{b} = \begin{bmatrix} -3 \\ 3 \\ 5 \\ -5 \end{bmatrix}$
3. $\mathbf{c} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$
4. $\mathbf{d} = \begin{bmatrix} -5 \\ -1 \end{bmatrix}$
5. $\mathbf{e} = \begin{bmatrix} 1 \\ -5 \\ 3 \\ 1 \end{bmatrix}$
6. $\mathbf{f} = \begin{bmatrix} 4 \\ 1 \\ -3 \end{bmatrix}$

Answer on Page 84

To multiply two matrices, it is helpful to think of the rows of the first matrix and the columns of the second matrix as vectors. Let's see how this shakes out for two 2×2 matrices in Figure 2.1:

$$\begin{array}{c} \\ \mathbf{a}_1 \rightarrow \\ \mathbf{a}_2 \rightarrow \end{array}\begin{bmatrix} 5 & 4 \\ -5 & 1 \end{bmatrix} \times \overset{\begin{array}{cc}\mathbf{b}_1 & \mathbf{b}_2 \\ \downarrow & \downarrow\end{array}}{\begin{bmatrix} -1 & -2 \\ -5 & -4 \end{bmatrix}} = \begin{bmatrix} \mathbf{a}_1 \cdot \mathbf{b}_1 & \mathbf{a}_1 \cdot \mathbf{b}_2 \\ \mathbf{a}_2 \cdot \mathbf{b}_1 & \mathbf{a}_2 \cdot \mathbf{b}_2 \end{bmatrix}$$

$$\boldsymbol{A} \quad \times \quad \boldsymbol{B}$$

Figure 2.1: Each entry in $\boldsymbol{C}$, c_{ij}, is the dot product of the i^{th} row of $\boldsymbol{A}$, a_i, and the j^{th} column of $\boldsymbol{B}$, b_j.

Let's look at this more concretely. For two-dimensional matrices, it can be helpful to move your left index finger across the row and right index finger down the column, as shown in Figure 2.2.

$$\begin{bmatrix} 5 & 4 \\ -5 & 1 \end{bmatrix} \times \begin{bmatrix} -1 & -2 \\ -5 & -4 \end{bmatrix} = \begin{bmatrix} 5\times\text{-}1+4\times\text{-}4 & \\ & \end{bmatrix}$$

$$\begin{bmatrix} 5 & 4 \\ -5 & 1 \end{bmatrix} \times \begin{bmatrix} -1 & -2 \\ -5 & -4 \end{bmatrix} = \begin{bmatrix} \text{-}21 & 5\times\text{-}2+4\times\text{-}4 \\ & \end{bmatrix}$$

$$\begin{bmatrix} 5 & 4 \\ -5 & 1 \end{bmatrix} \times \begin{bmatrix} -1 & -2 \\ -5 & -4 \end{bmatrix} = \begin{bmatrix} \text{-}21 & \text{-}26 \\ \text{-}5\times\text{-}1+1\times\text{-}5 & \end{bmatrix}$$

$$\begin{bmatrix} 5 & 4 \\ -5 & 1 \end{bmatrix} \times \begin{bmatrix} -1 & -2 \\ -5 & -4 \end{bmatrix} = \begin{bmatrix} \text{-}21 & \text{-}26 \\ 0 & \text{-}5\times\text{-}2+1\times\text{-}4 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 4 \\ -5 & 1 \end{bmatrix} \times \begin{bmatrix} -1 & -2 \\ -5 & -4 \end{bmatrix} = \begin{bmatrix} \text{-}21 & \text{-}26 \\ 0 & 6 \end{bmatrix}$$

Figure 2.2: You can use your fingers to trace across matrix $\boldsymbol{A}$ and down matrix $\boldsymbol{B}$ to find $\boldsymbol{A} \cdot \boldsymbol{B}$.

Since each entry in the product matrix is the dot product between a row of the first matrix and a column of the second matrix, the first matrix must have the same number of elements in each row as the second has in each column. Another way to say this is that the number of columns of the first matrix must match the number of rows in the second matrix.

Matrix Multiplication

For two-dimensional matrices, the inner dimensions must match in order to carry out matrix multiplication. That is, if we want to find $\boldsymbol{A}\times\boldsymbol{B}$, and $\boldsymbol{A}$ has dimensions $m \times n$, then $\boldsymbol{B}$ must have dimensions $n \times p$, where m, n, and p are integers. The resulting matrix will have dimensions $m \times p$ (m and p may be equal or unequal). Note that we use $\times$ to differentiate from the dot product, represented by $\cdot$.

Exercise 5 Multiplying Matrices 1

Multiply the matrices.

Working Space

1. $\begin{bmatrix} -5 & -2 & 2 & 1 \end{bmatrix} \times \begin{bmatrix} -3 & -1 & -5 \\ 3 & 0 & 3 \\ 4 & -1 & -4 \\ -1 & -4 & 2 \end{bmatrix}$

2. $\begin{bmatrix} 1 \\ 5 \\ -5 \\ 4 \\ 1 \end{bmatrix} \times \begin{bmatrix} 0 & 5 & 1 \end{bmatrix}$

3. $\begin{bmatrix} -1 & 4 & -4 \\ 5 & -3 & 5 \\ -1 & -4 & 4 \\ -4 & 1 & 4 \end{bmatrix} \times \begin{bmatrix} -3 & 5 & 1 \\ -3 & 0 & -3 \\ 0 & 3 & 0 \end{bmatrix}$

Answer on Page 84

Exercise 6 Multiplying Matrices 2

Find $A \times B$ and $B \times A$.

1. $A = \begin{bmatrix} -2 \\ 2 \\ 1 \\ -2 \end{bmatrix}$ and $B = \begin{bmatrix} -4 & 3 & -5 & -2 \end{bmatrix}$
2. $A = \begin{bmatrix} -4 & -2 \\ 2 & 5 \\ -3 & -4 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & -2 & -4 \\ 1 & -4 & 0 \end{bmatrix}$
3. $A = \begin{bmatrix} 2 & 0 & 1 & 4 \\ -4 & 0 & -5 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & -3 \\ -4 & -1 \\ -2 & 3 \\ -5 & 1 \end{bmatrix}$

Working Space

Answer on Page 84

What have you noticed about the results of $A \times B$ as compared to $B \times A$? You should have noticed that the product matrices are *different dimensions*. This leads us to the next unusual property of matrix multiplication: it is *non-commutative*. That is, the *order* in which you multiply matrices affects the result. This is very different from scalar values!

As you saw in the second matrix multiplication exercise, $\boldsymbol{A}$ is a 2×4 matrix and $\boldsymbol{B}$ is a 4×2 matrix, then $\boldsymbol{AB}$ is a 2×2 matrix, while $\boldsymbol{BA}$ is a 4×4 matrix. It is obvious, then, that $A \cdot B \neq B \cdot A$. What if $\boldsymbol{A}$ and $\boldsymbol{B}$ are square matrices?

Exercise 7 Are Matrices Commutative?

Find $A \cdot B$ and $B \cdot A$ if $A = \begin{bmatrix} -3 & 5 \\ -1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 1 \\ 4 & -3 \end{bmatrix}$.

Working Space

Answer on Page 85

As you can see, even if $\boldsymbol{A}$ and $\boldsymbol{B}$ are square, matrix multiplication is still not commutative.

Non-Commutation of Matrix Multiplication

For two matrices $\boldsymbol{A}$ and $\boldsymbol{B}$, where neither is an identity matrix or a zero matrix:

$$A \times B \neq B \times A$$

Properties of the Zero Matrix

Just like the number 0, the zero matrix, $\boldsymbol{0}$ has unique mathematical properties:

Properties of the Zero Matrix

For a matrix, $\boldsymbol{A}$, and a zero matrix, $\boldsymbol{0}$

1. $A + 0e = A$
2. $A + -A = 0e$
3. $0 \cdot A = 0e$

The Identity Matrix

There is another special matrix, called the *identity matrix*, usually denoted with $\boldsymbol{I}$. An identity matrix is all zeroes except for a diagonal line of ones. A 3×3 identity matrix is shown below:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

All identity matrices are square (that is, they have the same number of rows as they do columns). The identity matrix has the special property that whenever a vector or matrix is multiplied by $\boldsymbol{I}$, it doesn't change. Let's look at some examples:

Example: If $\mathbf{x} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$, what is $I\mathbf{x}$? (Take $\boldsymbol{I}$ to be a 2×2 identity matrix.)

Solution:

$$I\mathbf{x} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 \cdot (2) + 0 \cdot (-3) \\ 0 \cdot (2) + 1 \cdot (-3) \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

Example: If $B = \begin{bmatrix} -2 & 5 \\ 3 & -4 \end{bmatrix}$, what is $I \times B$?

Solution:

$$I \times B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} -2 & 5 \\ 3 & -4 \end{bmatrix} = \begin{bmatrix} 1 \cdot (-2) + 0 \cdot (5) & 1 \cdot (5) + 0 \cdot (-4) \\ 0 \cdot (-2) + 1 \cdot (5) & 0 \cdot (5) + 1 \cdot (-4) \end{bmatrix} = \begin{bmatrix} -2 & 5 \\ 3 & -4 \end{bmatrix}$$

Properties of the Identity Matrix

An $n \times n$ identity matrix, $\boldsymbol{I}$, does not change any vectors or matrices it multiplies. That is:

1. $I \times \mathbf{x} = \mathbf{x}$
2. $I \times B = B$

where $\mathbf{x}$ is an $n \times 1$ vector and $\boldsymbol{B}$ is an $n \times p$ matrix (p may be, but is not necessarily, equal to n).

2.2.3 Transposing a Matrix

The **transpose** of a matrix is an operation that flips a matrix over its diagonal. In other words, rows become columns and columns become rows. If a matrix A has entries a_{ij}, then its transpose, written $A^\top$, sometimes A^T, has entries a_{ji}.

Transpose of a Matrix

If

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix},$$

then the **Transpose of A** is

$$A^T = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix}.$$

That is, entry a_{ij} becomes a_{ji}.

Example: Let

$$A = \begin{bmatrix} 2 & -1 & 4 \\ 0 & 3 & 5 \end{bmatrix}.$$

Then the transpose of A is

$$A^\top = \begin{bmatrix} 2 & 0 \\ -1 & 3 \\ 4 & 5 \end{bmatrix}.$$

Notice how the first *row* of A becomes the first *column* of $A^\top$, and so on.

2.2.4 Row-Major Order, Column-Major Order, and Computer Memory

Note that the longer the diagonal of $m \times n$, the longer this computation could take, especially by hand! For very large matrices, we love using computers to do the heavy lifting. Just as matrices are large sets of numbers, computer memory is essentially a large matrix, just stored an a very long string of characters. Remember how data is stored as a long, contiguous array? We can use this fact to simplify the expense of this operation. Instead of creating a whole new matrix, we can just swap the indices of the existing entries. This way, we don't have to move any data around in memory, just change how we access it. Many libraries and programming languages (Python, C++, Assembly, etc) default to **row-major order**, storing the matrix consecutively in rows in memory. Other languages (Fortran, MATLAB, R, etc) use **column-major order**, storing the matrix consecutively in columns in memory. When transposing a matrix stored in row-major order, we can just access the data as if it were stored in column-major order, and vice versa. This is as simple as swapping the bounds of the row/column operations. This trick saves us from having to move data around in memory, making the transpose operation much faster!

(a) Row-major ordering (b) Column-major ordering

Figure 2.3: Row-major and column-major ordering of matrix elements.

2.2.5 Properties of Transposes

The transpose operation behaves nicely with matrix addition and multiplication:

(1)

(2) $(A^T)^T = A$

(3) $(A + B)^T = A^T + B^T$

(4) $(cA)^T = cA^T$ for any scalar c

(5) $(ABC)^T = C^T B^T A^T$ (order reverses)

Property 4 is especially important and often surprises students—the transpose of a product reverses the order.

Transposing a matrix is used throughout linear algebra, especially when:

- converting rows into column vectors (and vice versa),
- defining dot products using matrix multiplication,
- forming symmetric matrices ($A = A^T$),
- working with orthogonal matrices ($Q^T Q = I$),

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

More generally, a matrix with m rows and n columns is referred to as an $m \times n$ matrix, or simply an m-by-n matrix; m and n are its dimensions.

The general form of a 2×3 matrix A is:

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \end{bmatrix}$$

2.3 Types of Matrices

Matrices can be described by their shape:

Row Matrix A matrix of size $1 \times n$ with 1 row and n columns.

Column Matrix A matrix of size $n \times 1$ with n rows and 1 column, typically used to represent vectors.

Square Matrix A matrix of size $n \times n$, containing the same number of rows and columns

Rectangular Matrix A matrix of size $m \times n$, that has an unequal number of rows and columns.

They can also be described by their unique numerical properties. Special matrices that come in handy for certains types of computations. These are a few of the most common special matrices:

Zero Matrix A matrix that only contains entries that are zero.

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

The matrix above could also be

$$0_{2\times2}$$

Diagonal Matrix A square matrix with nonzero entries along the diagonal, and zeroes everywhere else.

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

Identity Matrix A diagonal matrix with entries of 1 along the diagonal.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Symmetric Matrix A Symmetric matrix is a matrix, when transposed, becomes itself.

$$A = \begin{bmatrix} 20 & 40 & 60 \\ 40 & 50 & 80 \\ 60 & 80 & 100 \end{bmatrix}$$

Triangular Matrix This is a special square matrix that can be upper triangular or lower triangular. If upper, the main diagonal and all entries above it are nonzero while the lower entries are all zero. If lower, the main diagonal and all the entries below it are nonzero, while the upper entries are all zero. A matrix that is both upper and lower triangular becomes a diagonal matrix.

Upper Triangular Matrix

$$\begin{bmatrix} 2 & 3 & 1 \\ 0 & 5 & 4 \\ 0 & 0 & 6 \end{bmatrix}$$

Lower Triangular matrix

$$\begin{bmatrix} 7 & 0 & 0 \\ 2 & 5 & 0 \\ 4 & 3 & 9 \end{bmatrix}$$

2.3.1 Symmetric Matrices

If you want to find out if a square matrix is symmetric, you need to transpose it. If the transpose is equal to the original matrix, then the matrix is symmetric.

To transpose a matrix, flip it over its diagonal so that the rows and columns are switched, like this:

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{bmatrix}$$

After transposing:

$$A^T = \begin{bmatrix} a_{1,1} & a_{2,1} & a_{3,1} \\ a_{1,2} & a_{2,2} & a_{3,2} \\ a_{1,3} & a_{2,3} & a_{3,3} \end{bmatrix}$$

Note that A^T means the transpose of A.

Let's see how this works for the following square matrix, A.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$$

Switch the rows and columns to get the transpose:

$$A^T = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$$

Notice that $A = A^T$, so the matrix is symmetric.

What about matrix B?

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 7 & 8 & 9 \end{bmatrix}$$

Switch the rows and columns to get the transpose:

$$B = \begin{bmatrix} 1 & 2 & 7 \\ 2 & 4 & 8 \\ 3 & 5 & 9 \end{bmatrix}$$

Note that $B \neq B^T$, so B is not symmetrical.

Exercise 8 Matrix Transposition

Find the transpose of this matrix. Is it symmetric?

$$A = \begin{bmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{bmatrix}$$

Working Space

Answer on Page 85

2.3.2 Can We Divide Matrices?

Matrices cannot be divided. Suppose we have a matrix, $\boldsymbol{A}$, a vector $\mathbf{x}$, and another vector $\mathbf{b}$ such that:

$$\boldsymbol{A} \cdot \mathbf{x} = \mathbf{b}$$

Now, if we know $\boldsymbol{A}$ and $\mathbf{x}$, it is easy to find $\mathbf{b}$. What if, on the other hand, we know $\boldsymbol{A}$ and $\mathbf{b}$ and want to find $\mathbf{x}$? We might be tempted to do something like this:

$$\mathbf{x} = \frac{\mathbf{b}}{\boldsymbol{A}}$$

While this would be correct if $\mathbf{x}$, $\mathbf{b}$, and $\boldsymbol{A}$ were scalars, but it is not for matrices. However, there is an analogy we can make. Instead of trying to divide by $\boldsymbol{A}$, we can multiply by its *inverse*:

Inverse Matrices

Given a matrix $\boldsymbol{A}$, and vectors $\mathbf{b}$ and $\mathbf{x}$, if

$$\boldsymbol{A} \times \mathbf{x} = \mathbf{b}$$

Then,

$$\mathbf{x} = A^{-1} \times \mathbf{b}$$

A^{-1} is called the *inverse matrix*. *But*, the inverse *does not always exist*! We will explore inverse matrices and how to find them in the next chapter.

Chapter 3

Linear Combinations of Vectors

In the introductory linear algebra chapter, you learned that vectors and matrices can be rotated, inverted, and added. In this chapter, we will explore linear combinations of vectors and the span of group of vectors. The **span** of a group of vectors is the set of vectors that can be made with linear combinations of the original group of vectors. We will offer mathematical and visual explanations later in the chapter. First, let's examine linear combinations.

A *linear combination* is simply the addition of vectors with leading scalar multipliers. In algebra, we can express polynomials similar to the form $5x + -4y$. In linear algebra, we do this with a vectors: $3\begin{bmatrix}2\\-1\end{bmatrix} + 2\begin{bmatrix}3\\5\end{bmatrix}$ is a linear combination of the vectors $\begin{bmatrix}2\\-1\end{bmatrix}$ and $\begin{bmatrix}3\\5\end{bmatrix}$. Another way to say this is:

Linear Combination of Vectors

A linear combination of a list of n vectors, $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}$ takes the form:

$$a_1\mathbf{v_1} + a_2\mathbf{v_2} + \cdots + a_n\mathbf{v_n}$$

where $a_1, a_2, \cdots, a_n \in \mathbb{R}$. You may see the list of vectors as $\mathbf{x_1}, \mathbf{x_2}, \ldots \mathbf{x_n}$, both are acceptable variables.

Example: Find a linear combination of $\begin{bmatrix}2\\1\\-3\end{bmatrix}$ and $\begin{bmatrix}1\\-2\\4\end{bmatrix}$ that gives the vector $\begin{bmatrix}17\\-4\\2\end{bmatrix}$.

Solution: We are looking for a_1 and a_2 such that:

$$a_1\begin{bmatrix}2\\1\\-3\end{bmatrix} + a_2\begin{bmatrix}1\\-2\\4\end{bmatrix} = \begin{bmatrix}17\\-4\\2\end{bmatrix}$$

Looking at each dimension separately, we get the system of equations:

$$2a_1 + 1a_2 = 17$$

$$1a_1 - 2a_2 = -4$$

$$-3a_1 + 4a_2 = 2$$

If we can solve this system of equations, we will find a_1 and a_2. Let's multiply the first equation by 2 and add it to the second equation:

$$2\,(2a_1 + a_2) + (a_1 - 2a_2) = 2(17) + (-4)$$

$$4a_1 + 2a_2 + a_1 - 2a_2 = 34 - 4$$
$$5a_1 = 30$$
$$a_1 = 6$$

Now we can take a_1 and substitute it back into any equation in our system to find a_2. Let's use the third equation:

$$-3(6) + 4a_2 = 2$$
$$-18 + 4a_2 = 2$$
$$4a_2 = 20$$
$$a_2 = 5$$

Since we used all 3 equations, we know $a_1 = 6$ and $a_2 = 5$ are solutions to all 3 equations. If we had only used the first two equations to find a_1 and a_2, we would want to substitute our values back into the third equation to make sure our solution holds for that equation also.

Therefore,

$$6\begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix} + 5\begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix} = \begin{bmatrix} 17 \\ -4 \\ 2 \end{bmatrix}.$$

Exercise 9 Linear Combinations

Find a linear combination of the first two vectors that yields the third vector.

1. $\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}-3\\1\end{bmatrix}, \begin{bmatrix}4\\5\end{bmatrix}$
2. $\begin{bmatrix}9\\4\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}-5\\3\end{bmatrix}$
3. $\begin{bmatrix}7\\-2\end{bmatrix}, \begin{bmatrix}-8\\4\end{bmatrix}, \begin{bmatrix}6\\-2\end{bmatrix}$

Working Space

Answer on Page 85

Sometimes, a set of vectors cannot be combined to make a specific vector. Take the pair of vectors we have looked at before: $\begin{bmatrix}2\\1\\-3\end{bmatrix}$ and $\begin{bmatrix}1\\-2\\4\end{bmatrix}$. Can we find a combination to make vector $\begin{bmatrix}17\\-4\\5\end{bmatrix}$? Let's try. We define a_1 and a_2 such that:

$$a_1 \begin{bmatrix}2\\1\\-3\end{bmatrix} + a_2 \begin{bmatrix}1\\-2\\4\end{bmatrix} = \begin{bmatrix}17\\-4\\5\end{bmatrix}$$

Which creates the system of equations:

$$2a_1 + a_2 = 17$$

$$a_1 - 2a_2 = -4$$
$$-3a_1 + 4a_2 = 5$$

We have two variables (a_1 and a_2) and three equations. Let's use the first two to find a_1 and a_2, then check our answers by substituting our solutions into the third equation. First, we'll multiply the second equation by -2 and add that to the first equation:

$$2a_1 + a_2 + (-2)(a_1 - 2a_2) = 17 + (-2)(-4)$$
$$2a_1 + a_2 - 2a_1 + 4a_2 = 17 + 8$$
$$5a_2 = 25$$
$$a_2 = 5$$

Substituting for a_2 back into the first equation and solving for a_1:

$$2a_1 + 5 = 17$$
$$2a_1 = 12$$
$$a_1 = 6$$

Now, let's check if $a_1 = 6$, $a_2 = 5$ is a solution to the third equation:

$$-3(6) + 4(5) = 5$$
$$-18 + 20 = 2 \neq 5$$

Therefore, there is no linear combination of the vectors $\begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}$ that yields $\begin{bmatrix} 17 \\ -4 \\ 5 \end{bmatrix}$.

Linear Combinations as systems.

We are aiming to know if $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 = \mathbf{w}$. We can rewrite this as a matrix multiplication:

$$[v_1 \; v_2] \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \mathbf{w}$$

Then, to check whether w is a linear combination of v_1 and v_2, solve the system of equations. If a solution exists, then yes; if not, then no. Note that the a_n values are only scalar numbers, while the v_n's remain vectors.

We will talk more about systems of equations in Chapter 4.

3.1 Visualizing Linear Combinations

First, let's look at what vectors can be made from linear combinations of the 2-dimensional unit vectors $\mathbf{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Suppose we are looking for a linear combination of $\mathbf{i}$ and $\mathbf{j}$ to create the vector $\begin{bmatrix} 3 \\ -4 \end{bmatrix}$. We can find such a linear combination:

$$3\mathbf{i} + (-4)\,\mathbf{j} = 3\begin{bmatrix} 1 \\ 0 \end{bmatrix} - 4\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \end{bmatrix}$$

In fact, with $\mathbf{i}$ and $\mathbf{j}$, we can create any 2-dimensional vector. To prove this, consider a generic vector, $\mathbf{z} = \begin{bmatrix} a \\ b \end{bmatrix}$, where $a, b \in \mathbb{R}$. We are looking for a linear combination of $\mathbf{i}$ and $\mathbf{j}$ such that:

$$c_1\mathbf{i} + c_2\mathbf{j} = \begin{bmatrix} a \\ b \end{bmatrix}$$

The above equation yields the system of equations:

$$c_1\,(1) + c_2\,(0) = a$$

$$c_1\,(0) + c_2\,(1) = b$$

And the solution to this system of equations is:

$$c_1 = a$$

$$c_2 = b$$

Therefore, using $\mathbf{i}$ and $\mathbf{j}$, we can construct any vector in $\mathbb{R}^2$ (that is, any vector in the xy-plane). What about combinations of other vectors?

Let's consider linear combinations of two vectors: $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$. The vectors are shown in figure 3.1.

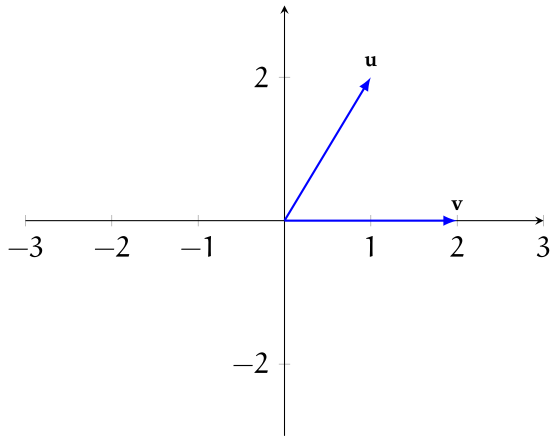
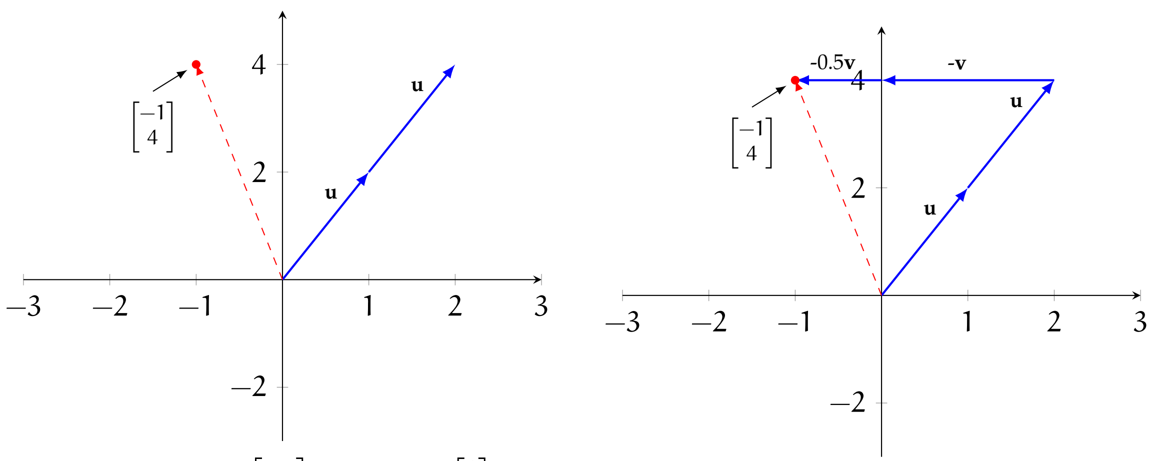


Figure 3.1: The vectors $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$.

Suppose we want to construct the vector $\begin{bmatrix} -1 \\ 4 \end{bmatrix}$. Since only $\mathbf{u}$ has value in the y-dimension, we can start by adding $\mathbf{u}$ vectors to reach $y = 4$ (see figure 3.2a). Next, we can use $\mathbf{v}$ vectors to reach $\begin{bmatrix} -1 \\ 4 \end{bmatrix}$ (see figure 3.2b).



(a) To create vector $\begin{bmatrix} 4 \\ -2 \end{bmatrix}$ with $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$, we begin by adding two $\mathbf{u}$ vectors to reach a y-value of 4.

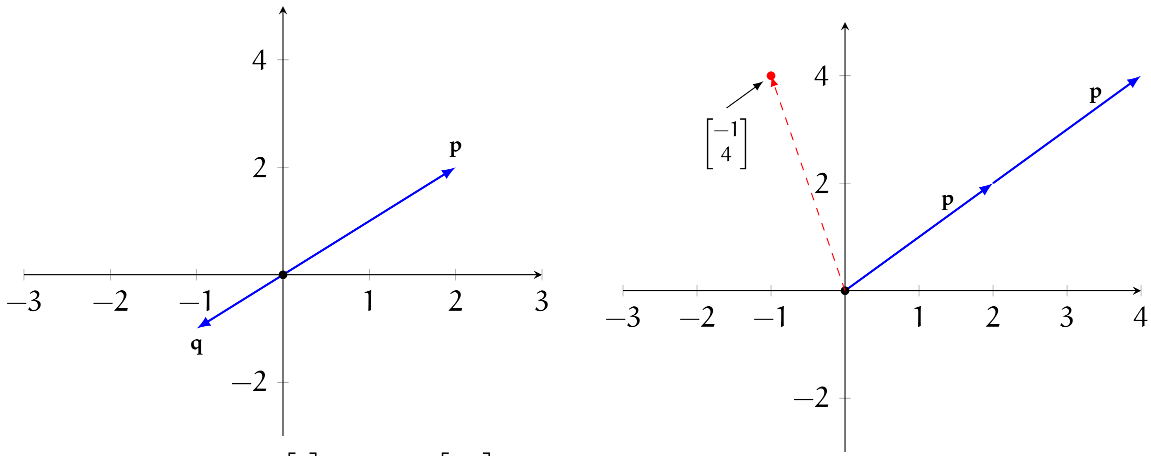
(b) If $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$, then $2\mathbf{u} - 1.5\mathbf{v} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$.

Figure 3.2: Using linear combinations of $\mathbf{u}$ and $\mathbf{v}$ to construct $\begin{bmatrix} -1 \\ 4 \end{bmatrix}$.

Using this method, we can imagine reaching any point in $\mathbb{R}^2$: we add or subtract as many $\mathbf{u}$ vectors as needed to reach the appropriate y-value, then add or subtract as many $\mathbf{v}$

vectors to reach the appropriate x-value. The vectors are *not multiples* of each other, so we can say that they span all of $\mathbb{R}^2$.

Let's look at another pair of vectors: $\mathbf{p} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ and $\mathbf{q} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$ (see figure 3.3a). Again, let's try to use $\mathbf{p}$ and $\mathbf{q}$ to construct the vector $\begin{bmatrix} -1 \\ 4 \end{bmatrix}$. We begin by using $\mathbf{p}$ to reach the y-value of 4 (see figure 3.3b).



(a) The vectors $\mathbf{p} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ and $\mathbf{q} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$.

(b) Adding two $\mathbf{p}$ vectors to reach $y = 4$.

Figure 3.3: Visualizing combinations of $\mathbf{p}$ and $\mathbf{q}$.

But now we run into a problem: no matter how many multiples of $\mathbf{q}$ vectors we add or subtract, we just move along the $\mathbf{p}$ vector and never reach our goal of $\begin{bmatrix} -1 \\ 4 \end{bmatrix}$ (see figure 3.4). Notice that $\mathbf{p}$ and $\mathbf{q}$ lie on the same line, (for a better visualization, refer back to 3.3a). In other words, the vectors are *scalar multiples* of each other: $\begin{bmatrix} 2 \\ 2 \end{bmatrix} = -2 \cdot \begin{bmatrix} -1 \\ -1 \end{bmatrix}$. When two vectors lie on the same line or a scalar multiples of each other, we call them *linearly dependent*.

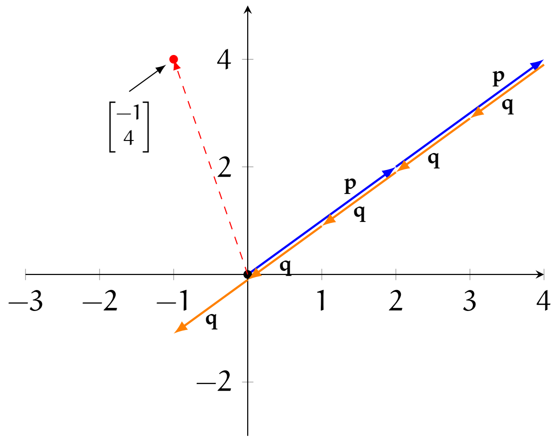


Figure 3.4: There is no linear combination of $\mathbf{p} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ and $\mathbf{q} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$ that yields the vector $\begin{bmatrix} -1 \\ 4 \end{bmatrix}$.

FIXME add summary

In the next chapter, we will be looking more at linear combinations, span, and independence in the form of systems of equations. Future chapters will look even further at these concepts, linear dependence and independence, and their applications to various subspaces.

Chapter 4

Systems of Equations as Matrices

4.1 Systems of Equations, REF, and RREF

We have talked a lot about matrices and vectors. Now we can use matrices and vectors to solve systems of equations. A system of linear equations can be written compactly using matrices and vectors. This matrix representation allows us to use algebraic tools such as row operations and matrix multiplication to analyze and solve the system more efficiently. Let's look at a very simple example from a previous section.

$$\begin{cases} 1x_1 + 2x_2 = -1 \\ 2x_1 + 0x_2 = 4 \end{cases}$$

We can write this in the form of a matrix! This may come as a surprise to some of you. How can we do this? Let's rewrite the the coefficients as a matrix and the x_n's as column vector, and the constant vector, or right-hand-side vector, called $\vec{b}$. Here is the coefficient matrix which we will call A:

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix}$$

Then we can write the column vector of x, which we will terminalize as $\vec{x}$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

And we can write the $\vec{b}$ as the following column vector:

$$\vec{b} = \begin{bmatrix} -1 \\ -4 \end{bmatrix}$$

Now, if we multiply A and $\vec{x}$,

$$A\vec{x} = \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1x_1 + 2x_2 \\ 2x_1 + 0x_2 \end{bmatrix}$$

We get exactly the left-hand sides of our original system of equations! Therefore, when we set this equal to the right-hand-side vector $\vec{b}$:

$$\vec{b} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$$

we get the fundamental system of equation formula represented as a matrix Equation

$$A\vec{x} = \vec{b} \tag{4.1}$$

This is a very important equation for linear algebra as a whole. Notice that the size of matrix has to respect the matrix multiplication rules:

$$\underset{m\times n}{A} \cdot \underset{n\times 1}{\vec{b}} = \underset{m\times 1}{\vec{p}}$$

Exercise 10 Matrix Equation

Rewrite the systems of equations in matrix form and identify A, $\vec{x}$, and $\vec{b}$.

$$\begin{cases} -4x_1 + 9x_2 - 8x_3 = 5 \\ -1x_1 + 0x_2 + 6x_3 = 7 \end{cases}$$

Working Space

Answer on Page 87

4.2 Row-Echelon Form

This brings up the question, how can we solve for the $\vec{x}$? There are multiple ways, but most commonly we form an augmented matrix and solve for Row-Echelon Form.

4.2.1 The Augmented Matrix

What is the augmented matrix? It is a way of represented the A matrix and $\vec{b}$ vector. We write the A matrix, separated by a vertical line, and then writing the $\vec{b}$ vector:

$$[A \mid \vec{b}] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & \mid & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & \mid & b_2 \\ \vdots & \vdots & & \vdots & \mid & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & \mid & b_m \end{bmatrix}$$

The left block contains all the coefficients of the system of equations, and the right block contains only the solutions. Then, to solve for $\vec{x}$, we perform *Elementary Row Operations* to find the *Row-Echelon Form*, mathematically stated as $\text{rref}(A|\vec{b})$. Let's continue with the matrix vector set we were experimenting with:

$$A\vec{x} = \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1x_1 + 2x_2 \\ 2x_1 + 0x_2 \end{bmatrix}$$

This can be written as an augmented matrix:

$$\left[A|\vec{b}\right] = \begin{bmatrix} 1 & 2 & \mid & -1 \\ 2 & 0 & \mid & 4 \end{bmatrix}$$

4.2.2 Elementary Row Operations

The Row-Echelon Form of a matrix is an *equivalent* form of a matrix obtained through Gaussian and Gauss-Jordan elimination. There are lots of videos on this process online, so multiple videos included in your digital resources to walk you through the process. The rules of Gaussian Elimination are simple, only 3 different elementary row operations, or ERO's can be performed:

- Swapping two rows
- multiply a row by a non-zero scalar
- dd a multiple of one row to another row

Why do these operations preserve the given linear system? Swapping two rows only changes the order in which the equations are written and therefore has no effect on the solution set. Multiplying a row by a nonzero scalar corresponds to multiplying both sides of an equation by the same scalar, which does not alter its solutions. Finally, if two equations are satisfied by the same solution, then any linear combination of those equations is also satisfied by that solution; in particular, adding a multiple of one equation to another preserves the solution set. Before and after each operation, the augmented matrix contains exactly the same information as the system of equations and therefore represents the same solution set.

With these ERO's, we are trying to achieve a matrix with the following properties:

- Leading entries (not necessarily 1) move to the right as you go down
- Rows of zeros, if exist, are at the bottom
- Each pivot has zeros in all positions below it in the same column.

Lets walk through the steps to simplify $\text{ref}(A|\vec{b})$:

$$\begin{bmatrix} 1 & 2 & | & -1 \\ 2 & 0 & | & 4 \end{bmatrix}$$

$$R_2 \to R_2 - 2R_1 \implies \begin{bmatrix} 1 & 2 & | & -1 \\ 0 & -4 & | & 6 \end{bmatrix}$$

As you can see, we set Row 2 to be Row 2 minus 2 times Row 1. This gives us a zero in the first column of Row 2. Technically, this matrix is now in Row-Echelon Form (REF). However, we can continue applying elementary row operations to obtain an even more structured form called the *Reduced Row-Echelon Form*.

4.2.3 Reduced Row Echelon Form

A matrix is in *Reduced Row-Echelon Form* (RREF) if it satisfies all of the conditions for Row-Echelon Form, and in addition:

- Each leading entry is equal to 1
- Each leading entry is the only nonzero entry in its column

In other words, not only do pivots move to the right as we move down the rows, but each pivot column contains zeros both above and below the pivot, and each pivot is normalized to 1. These additional conditions make RREF especially useful, since the solutions to the system can often be read directly from the matrix.

Continuing our example, we start with the matrix in REF:

$$\begin{bmatrix} 1 & 2 & | & -1 \\ 0 & -4 & | & 6 \end{bmatrix}$$

First, make the leading entry in Row 2 equal to 1 by dividing the row by -4:

$$R_2 \to -\frac{1}{4}R_2 \implies \begin{bmatrix} 1 & 2 & | & -1 \\ 0 & 1 & | & -\frac{3}{2} \end{bmatrix}$$

Next, eliminate the entry above the pivot in column 2:

$$R_1 \to R_1 - 2R_2 \implies \begin{bmatrix} 1 & 0 & | & 2 \\ 0 & 1 & | & -\frac{3}{2} \end{bmatrix}$$

The matrix is now in Reduced Row-Echelon Form.

4.2.4 Writing the Solution Vector

Recall that the augmented matrix represents a system of linear equations. After reducing the matrix to Reduced Row-Echelon Form, we obtain

$$\begin{bmatrix} 1 & 0 & | & 2 \\ 0 & 1 & | & -\frac{3}{2} \end{bmatrix}$$

Each row now corresponds directly to an equation in the variables x_1 and x_2:

$$\begin{aligned} x_1 &= 2 \\ x_2 &= -\frac{3}{2} \end{aligned}$$

Thus, the solution to the system can be written as the vector

$$\vec{x} = \begin{bmatrix} 2 \\ -\frac{3}{2} \end{bmatrix}$$

Since every variable corresponds to a pivot column, the system has a *unique solution*.

We have sucessfully solved the system of equations using matrix methods:

$$\begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ -\frac{3}{2} \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$$

This is a fundamental technique in linear algebra, and you will use it often in this module and beyond!

4.3 Larger RREF Example

Let's solve a larger system by finding $\vec{x}$ in the matrix equation

$$A\vec{x} = \vec{b}.$$

4.3.1 Write the System and Identify A, $\vec{x}$, and $\vec{b}$

Consider the following system of equations:

$$\begin{cases} x_1 + x_2 + x_3 + x_4 = 4 \\ 2x_1 + x_2 + 3x_3 + x_4 = 9 \\ x_1 + 3x_2 + 2x_3 + 2x_4 = 10 \\ 3x_1 + x_2 + 2x_3 + 4x_4 = 13 \end{cases}$$

The coefficient matrix, variable vector, and constant vector are:

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 1 & 3 & 1 \\ 1 & 3 & 2 & 2 \\ 3 & 1 & 2 & 4 \end{bmatrix}, \qquad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \qquad \vec{b} = \begin{bmatrix} 4 \\ 9 \\ 10 \\ 13 \end{bmatrix}.$$

So the matrix equation is:

$$A\vec{x} = \vec{b}.$$

4.3.2 Form the Augmented Matrix

$$\left[A|\vec{b}\right] = \begin{bmatrix} 1 & 1 & 1 & 1 & | & 4 \\ 2 & 1 & 3 & 1 & | & 9 \\ 1 & 3 & 2 & 2 & | & 10 \\ 3 & 1 & 2 & 4 & | & 13 \end{bmatrix}.$$

4.3.3 Row-Reduce to REF

Use the pivot in Row 1 to clear entries below it in column 1:

$$R_2 \to R_2 - 2R_1$$
$$R_3 \to R_3 - R_1$$
$$R_4 \to R_4 - 3R_1$$

$$\Longrightarrow \begin{bmatrix} 1 & 1 & 1 & 1 & | & 4 \\ 0 & -1 & 1 & -1 & | & 1 \\ 0 & 2 & 1 & 1 & | & 6 \\ 0 & -2 & -1 & 1 & | & 1 \end{bmatrix}.$$

Now use the pivot in Row 2 (column 2) to clear entries below it in column 2:

$$R_3 \to R_3 + 2R_2$$
$$R_4 \to R_4 - 2R_2$$

$$\Longrightarrow \begin{bmatrix} 1 & 1 & 1 & 1 & | & 4 \\ 0 & -1 & 1 & -1 & | & 1 \\ 0 & 0 & 3 & -1 & | & 8 \\ 0 & 0 & -3 & 3 & | & -1 \end{bmatrix}.$$

Next use the pivot in Row 3 (column 3) to clear the entry below it in column 3:

$$R_4 \to R_4 + R_3$$

$$\Longrightarrow \begin{bmatrix} 1 & 1 & 1 & 1 & | & 4 \\ 0 & -1 & 1 & -1 & | & 1 \\ 0 & 0 & 3 & -1 & | & 8 \\ 0 & 0 & 0 & 2 & | & 7 \end{bmatrix}.$$

At this point, the matrix is in Row-Echelon Form (REF).

4.3.4 Continue to RREF

Make the pivot in Row 4 equal to 1:

$$R_4 \to \frac{1}{2}R_4 \implies \begin{bmatrix} 1 & 1 & 1 & 1 & | & 4 \\ 0 & -1 & 1 & -1 & | & 1 \\ 0 & 0 & 3 & -1 & | & 8 \\ 0 & 0 & 0 & 1 & | & \frac{7}{2} \end{bmatrix}.$$

Clear the entries above the pivot in column 4:

$$R_1 \to R_1 - R_4$$
$$R_2 \to R_2 + R_4$$
$$R_3 \to R_3 + R_4$$

$$\implies \begin{bmatrix} 1 & 1 & 1 & 0 & | & \frac{1}{2} \\ 0 & -1 & 1 & 0 & | & \frac{9}{2} \\ 0 & 0 & 3 & 0 & | & \frac{23}{2} \\ 0 & 0 & 0 & 1 & | & \frac{7}{2} \end{bmatrix}.$$

Make the pivot in Row 3 equal to 1:

$$R_3 \to \frac{1}{3}R_3 \implies \begin{bmatrix} 1 & 1 & 1 & 0 & | & \frac{1}{2} \\ 0 & -1 & 1 & 0 & | & \frac{9}{2} \\ 0 & 0 & 1 & 0 & | & \frac{23}{6} \\ 0 & 0 & 0 & 1 & | & \frac{7}{2} \end{bmatrix}.$$

Clear the entries above the pivot in column 3:

$$R_1 \to R_1 - R_3$$
$$R_2 \to R_2 - R_3$$

$$\implies \begin{bmatrix} 1 & 1 & 0 & 0 & | & -\frac{10}{3} \\ 0 & -1 & 0 & 0 & | & \frac{2}{3} \\ 0 & 0 & 1 & 0 & | & \frac{23}{6} \\ 0 & 0 & 0 & 1 & | & \frac{7}{2} \end{bmatrix}.$$

Make the pivot in Row 2 equal to 1:

$$R_2 \to -R_2 \implies \begin{bmatrix} 1 & 1 & 0 & 0 & | & -\frac{10}{3} \\ 0 & 1 & 0 & 0 & | & -\frac{2}{3} \\ 0 & 0 & 1 & 0 & | & \frac{23}{6} \\ 0 & 0 & 0 & 1 & | & \frac{7}{2} \end{bmatrix}.$$

Clear the entry above the pivot in column 2:

$$R_1 \to R_1 - R_2$$

$$\implies \begin{bmatrix} 1 & 0 & 0 & 0 & | & -\frac{8}{3} \\ 0 & 1 & 0 & 0 & | & -\frac{2}{3} \\ 0 & 0 & 1 & 0 & | & \frac{23}{6} \\ 0 & 0 & 0 & 1 & | & \frac{7}{2} \end{bmatrix}.$$

This matrix is now in Reduced Row-Echelon Form (RREF).

4.3.5 Write the Solution Vector

Reading each row as an equation gives:

$$x_1 = -\frac{8}{3}$$
$$x_2 = -\frac{2}{3}$$
$$x_3 = \frac{23}{6}$$
$$x_4 = \frac{7}{2}$$

Thus the solution vector is:

$$\vec{x} = \begin{bmatrix} -\frac{8}{3} \\ -\frac{2}{3} \\ \frac{23}{6} \\ \frac{7}{2} \end{bmatrix}.$$

You can verify the solution by confirming that $A\vec{x} = \vec{b}$ using matrix multiplication.

Note that solution sets must be one of the following:

- No Solution (ie. parallel lines),
- Exactly One Unique Solution (ie. lines or planes intersecting at one point),
- or Infinitely Many Solutions (ie. same line or plane is redundantly in the set)

4.4 Free Variables

In the reduced row-echelon form (RREF) of a matrix, pivots determine which variables are constrained by the system.

4.4.1 Zero rows and redundant equations

A row of all zeros in RREF represents a redundant equation. That row corresponds to the equation

$$0x_1 + 0x_2 + \cdots + 0x_n = 0,$$

which is always true and therefore imposes no restriction on the variables. Such a row arises when one of the original equations is a linear combination of the others.

Every row of zeros is a row without a pivot. Consequently, if a matrix has at least one zero row in RREF:

- The number of pivots is less than the number of rows,
- The rank of the matrix is less than the number of rows,
- The rows of the matrix are linearly dependent.

4.4.2 Free variables

Pivots are associated with *columns*, not rows. A variable is called a *free variable* if its column does not contain a pivot.

Thus:

- A *row* without a pivot (a zero row) indicates a redundant equation,
- A *column* without a pivot corresponds to a free variable.

Free variables can take arbitrary values and lead to infinitely many solutions when solving a homogeneous system.

4.4.3 Infinitely Many Solutions

Example: Consider the augmented matrix in RREF:

$$\begin{bmatrix} 1 & 0 & 0 & 8 \\ 0 & 0 & 1 & -6 \end{bmatrix}$$

Labeling the variable columns (and ignoring the constants c column), we write:

$$\begin{bmatrix} x_1 & x_2 & x_3 & c \\ 1 & 0 & 0 & 8 \\ 0 & 0 & 1 & -6 \end{bmatrix}$$

The pivot positions are in the columns corresponding to x_1 and x_3. Since the column corresponding to x_2 contains no pivot, x_2 is a free variable.

The system of equations represented by this matrix is:

$$x_1 = 8,$$
$$x_3 = -6.$$

The variable x_2 does not appear in any equation and may take any real value. Since x_1 and x_3 are bound, we can say the solution is :

$$x_1 = 8, \qquad x_3 = -6, \qquad x_2 \in \mathbb{R}$$

writen as a vector this is: $\vec{x} = \begin{bmatrix} 8 \\ x_2 \\ -6 \end{bmatrix}$ or in set builder notation: $\left\{ \begin{pmatrix} 8 \\ x_2 \\ -6 \end{pmatrix} : x_2 \in \mathbb{R} \right\}$

Thus, the presence of a column without a pivot leads to a free variable and infinitely many solutions.

4.4.4 Inconsistent sets

Let's look at another example: **Example** The augmented matrix

$$\begin{bmatrix} 1 & 2 & 4 & 12 \\ 12 & 24 & 48 & 9 \\ 3 & 6 & 12 & -4 \end{bmatrix}$$

simplifies to

$$\begin{bmatrix} 1 & 2 & 4 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

we can interperite this As

$$x_1 + 2x_2 + 4x_3 = 0,$$
$$0x_1 + 0x_2 + 0x_3 = 1.$$

Note that the equation $0 = 1$ is never true, so the system is *inconsistent*, and there is *no solution*.

The presence of an all zero row signifies an inconsistent system.

FIXME exercises

4.5 Basis, Dimension, Rank

Let's define a few terms. These definitions are very basic, and will come back in the future:

Rank

Rank is the number of pivots in the RREF state of a matrix of size $m \times n$. In other words, the number of nonzero rows in any row-echelon form of a matrix A is denoted $\text{rank}(A)$. This has the constraints of:

- $\text{rank}(A) \leq m$
- $\text{rank}(A) \leq n$

Dimension

The **dimension** of the solution set of a system is the number of free variables. We calculate this with

$$\texttt{dimension} = \texttt{number of variables} - \texttt{rank}$$

or equivalently, for matrix A of size $m \times n$

$$\texttt{dimension} = n - \text{rank}(A)$$

Basis

A **basis** is a minimal set of direction vectors that describes all solutions. Specifically, this consists of $n - \text{rank}(A)$ vectors, one for each free variable

In summary,

$$\begin{aligned} \text{rank} &= \text{number of pivots} \\ \text{number of free variables} &= n - \text{rank} \\ \text{dimension of solution set} &= n - \text{rank} \\ \text{number of basis vectors} &= n - \text{rank} \end{aligned}$$

Exercises

Exercise 11 Augmented Matrix 1

Simplify the matrix:

$$\begin{bmatrix} 1 & 2 & -1 & 3 \\ 2 & 4 & -2 & 6 \\ 1 & 1 & 0 & 2 \end{bmatrix}$$

to RREF state and determine the type of solution set.

Working Space

Answer on Page 88

Exercise 12 Augmented Matrices 2

Put the augmented matrix

$$\begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & -1 & 3 \\ 2 & 1 & 0 & 1 \end{bmatrix}$$

into RREF and determine its solution.

Working Space

Answer on Page 88

Exercise 13 Augmented Matrices 3

Determine the solution for the following set:

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 9 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Working Space

Answer on Page 89

Exercise 14 Rank 1

Find the rank of

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$$

Working Space

Answer on Page 89

Exercise 15 Rank 2

Let A be a 4×6 matrix with exactly 3 pivots in RREF.

- What is $\text{rank}(A)$?
- How many free variables are there?

Working Space

Answer on Page 89

Exercise 16 Free Variables

The RREF of a system has the form:

$$\begin{bmatrix} 1 & 0 & -2 & 0 & 3 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

1. Identify the pivot variables
2. Identify the free variables
3. State the dimension of the solution set

Working Space

Answer on Page 90

4.6 Creating Matrices in Python

Recall that matrices can be represented by 2D arrays in Python. Let's explore this idea further with the module `numpy`. Create a file called `matrices_creation.py` and enter this code:

```
# import the python module that supports matrices
```

```
import numpy as np
# Use the function np.array to define a matrix that
# contains specific values that you supply.
A = np.array([[ 5, 1, 3],
              [ 1, -1, 8],
              [ 6, 2, 1]])
# The transpose function returns
A.transpose()
```

When you run it, you should see:

```
array([[ 5, 1, 6],
       [ 1, -1, 2],
       [ 3, 8, 1]])
```

As you can see, $A \neq A^T$, so A is not symmetric. Try another:

```
# create a matrix, B
B = np.array([[ 5, 1, 6],
              [ 1, -1, 2],
              [ 6, 2, 1]])
B.transpose()
```

When you run it, you should see:

```
array([[ 5, 1, 6],
       [ 1, -1, 2],
       [ 6, 2, 1]])
```

B is symmetric. You can actually transpose any matrix using this function, but a matrix cannot be symmetric unless it is square.

Try this code to see what happens when you transpose a rectangular matrix.

```
# create a matrix, J
J = np.array([[ 5, 1, 3, 0],
              [ 1, -1, 8, 11],
              [ 6, 2, 1,-7]])
J.transpose()
```

Note that transposing a rectangular matrix changes its dimension from 3 by 4 to 4 by 3. You should see a transposed matrix, but it's not symmetric.

```
array([[ 5,  1,  6],
       [ 1, -1,  2],
       [ 3,  8,  1],
       [ 0, 11, -7]])
```

4.6.1 Creating Special Matrices in Python

Use the same file to add this code for creating a zero matrix.

```
# create an 8 by 10 Zero matrix.
 C = np.zeros((8, 10))
 C
```

When you run it, you should see:

```
array([[0., 0., 0., 0., 0., 0., 0., 0., 0., 0.],
       [0., 0., 0., 0., 0., 0., 0., 0., 0., 0.],
       [0., 0., 0., 0., 0., 0., 0., 0., 0., 0.],
       [0., 0., 0., 0., 0., 0., 0., 0., 0., 0.],
       [0., 0., 0., 0., 0., 0., 0., 0., 0., 0.],
       [0., 0., 0., 0., 0., 0., 0., 0., 0., 0.],
       [0., 0., 0., 0., 0., 0., 0., 0., 0., 0.],
       [0., 0., 0., 0., 0., 0., 0., 0., 0., 0.]])
```

Add the following code to create an 8 by 8 Identity matrix.

```
# create an 8 by 8 Identity matrix
 D = np.eye(8)
 D
```

When you run it, you should see:

```
array([[1., 0., 0., 0., 0., 0., 0., 0.],
       [0., 1., 0., 0., 0., 0., 0., 0.],
       [0., 0., 1., 0., 0., 0., 0., 0.],
       [0., 0., 0., 1., 0., 0., 0., 0.],
       [0., 0., 0., 0., 1., 0., 0., 0.],
       [0., 0., 0., 0., 0., 1., 0., 0.],
       [0., 0., 0., 0., 0., 0., 1., 0.],
       [0., 0., 0., 0., 0., 0., 0., 1.]])
```

As you progress in your studies, you will learn the importance of diagonal matrices and of extracting the diagonal of a matrix. Let's see how to extract a diagonal, then create a diagonal matrix.

```
# create a  matrix
W = np.array([[1, 2, 3, 4],
              [5, 6, 7, 8],
              [-8, -7, -6, -5],
              [-4, -3, -2, -1]])
```

Extract the main diagonal using np.diag(<array>,<diagonal to extract>). Passing 0 as the second parameter specifies the main diagonal. A positive value extracts a diagonal from the upper part. A negative value extracts a diagonal from the lower part. Run this code then experiment passing other values to see what you get.

```
print(np.diag(W,0))
```

When you run it, you should see:

```
array([ 1,  6, -6, -1])
```

You can also use np.diag() to create a diagonal matrix from a 1D array. In this case, do not pass a second paramenter.

```
Q = np.array([1, 2, 3])
DiagArray = np.diag(Q))
print(DiagArray)
```

When you run it you should see;

```
[[1 0 0]
 [0 2 0]
 [0 0 3]]
```

Python has functions for extracting upper and lower triangular matrices. Try these:

```
print(np.triu(W))
print(np.tril(W))
```

You should see:

```
[[ 1  2  3  4]
 [ 0  6  7  8]
 [ 0  0 -6 -5]
 [ 0  0  0 -1]]

[[ 1  0  0  0]
 [ 5  6  0  0]
 [-8 -7 -6  0]
 [-4 -3 -2 -1]]
```

4.7 Conclusion

In this module, we have explored the fundamental concepts of matrices, including their types, properties, and how they can be used to represent and solve systems of linear equations. We have also learned how to perform row operations to convert matrices into Row-Echelon Form (REF) and Reduced Row-Echelon Form (RREF), which are essential techniques in linear algebra. In the next chapter 7, we will talk deeper about the concepts of vector spaces and subspaces and how to find RREF, building on our understanding of matrices and their applications.

Chapter 5

Systems of Linear Equations

In the chapter on linear combinations, we saw that we can linearly combine vectors to create other vectors. Consider 3 vectors:

$$\mathbf{x} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} \ \mathbf{y} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \ \mathbf{z} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

We can write a linear combination of these vectors:

$$c\mathbf{x} + d\mathbf{y} + e\mathbf{z}$$

Which we can expand to show the vectors:

$$c \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + d \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} + e \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -c - d - 2e \\ 2c + 2d + e \\ d \end{bmatrix}$$

We can also represent this combination with a matrix where each column is one of the vectors:

$$\begin{bmatrix} -1 & -1 & -2 \\ 2 & 2 & 1 \\ 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} c \\ d \\ e \end{bmatrix} = \begin{bmatrix} -c - d - 2e \\ 2c + 2d + e \\ d \end{bmatrix}$$

5.1 Trail Mix for Mars

Let's look at an applied problem. Three astronauts, Pat, Kai, and River, are getting ready for a trip to Mars. NASA food service is preparing trail mix for the voyage, tailored to each astronaut's taste. The chef needs to submit a budget based on the cost of the trail mix for each astronaut. The mix is made up of raisins, almonds, and chocolate.

Pat prefers a raisins:almonds:chocolate ratio of 6:10:4, Kai likes 2:3:15, and River wants 14:1:5. The chef can buy a kg of raisins for $7.50, a kg of almonds for $14.75, and a kg of chocolate for $22.25. Assuming each astronaut will get 20 kg of trail mix, which astronaut will cost more to feed?

First, set up a matrix to represent the raisins:almonds:chocolate ratios. (Conveniently, these ratios already add to 20.)

$$\text{MixRatios} = \begin{bmatrix} 6 & 10 & 4 \\ 2 & 3 & 15 \\ 14 & 1 & 5 \end{bmatrix}$$

Use a vector to represent the cost of each item:

$$\text{IngredientCost} = \begin{bmatrix} 7.50 \\ 14.75 \\ 22.25 \end{bmatrix}$$

To find the cost of trail mix for each astronaut, we simply find the dot product between the mix ratios and the ingredient costs to get:

Pat = \$281.50
Kai = \$615.50
River = \$231.00

Exercise 17 Vector Matrix Multiplication

Multiply the array A with the vector v. Compute this by hand, and make sure to show your computations.

$$A = \begin{bmatrix} 1 & -2 & 3 & 5 \\ -4 & 2 & 7 & 1 \\ 3 & 3 & -9 & 1 \end{bmatrix}$$

$$v = \begin{bmatrix} 2 \\ 2 \\ 6 \\ -1 \end{bmatrix}$$

Working Space

Answer on Page 90

Exercise 18 Using Vector Matrix Multiplication

A college professor offers three different methods of determining a student's final grade. In method A, the student's grade is 20% based on attendance, 50% homework, 15% midterm, and 15% final. This professor knows many students can learn the material without attending every class, so with method B the student's grade is 50% homework, 20% midterm, and 30% final. Last, the professor knows some students don't do the homework but still show they understand the material by doing well on the tests. With method C, a student's grade is 40% midterm and 60% final. The professor uses whatever method results in the highest grade to determine each student's final grade.

Suppose Suzy has attended 65% of classes, has an average homework grade of 30%, earned a 80% on the midterm, and earned a 75% on the final. What final grade will her professor post?

Working Space

Answer on Page 90

5.1.1 Vector-Matrix Multiplication in Python

Most real-world problems use very large matrices, where it becomes impractical to do calculations by hand. As long as you understand how matrix-vector multiplication is performed, you will be equipped to use a computing language, like Python, to do the calculations for you.

Create a file called `vectors_matrices.py` and enter this code:

```
# import the python module that supports matrices
import numpy as np
```

```
# create an array
a = np.array([[5, 1, 3, -2],
              [1, -1, 8, 4],
              [6, 2, 1, 3]])

# create a vector
b = np.array([1, 2, 3, -8])

# calculate the dot product
print(a.dot(b))
```

When you run it, you should see:

```
[16, 6, 8]
```

5.2 Where to Learn More

Watch this video from Khan Academy about matrix-vector products: https://rb.gy/frga5

Chapter 6

Linear Independence and Dependence

We have briefly mentioned the idea of linear independence and dependence in previous chapters. Let's explicitly define them using the concepts we have covered already.

6.1 Linear Independence

Linear Independence

Linear independent vectors are vectors that cannot be written as linear combinations (scalar multiples) of each other.

Mathematically, this means that

$$c_1 v_1 + c_2 v_2 \dots c_k v_k = 0$$

is only satified by

$$c_1 = c_2 = \cdots = c_k = 0$$

Linear Dependence

A set of vectors $\{v_1, \dots, v_k\}$ is **linearly dependent** if there exist scalars $c_1, \dots, c_k$, not all zero, such that

$$c_1 v_1 + \cdots + c_k v_k = \mathbf{0}.$$

For two vectors, this reduces to one being a scalar multiple of the other.

When vectors are arranged as columns of a matrix, linear independence means that no column can be written as a linear combination of the others.

We say that each vector contributes a new *direction* not captured by any previous ones, meaning that no vector can be written as a linear combination of the others. Each vector adds something genuinely new to the set, so there is no redundancy.

Exercise 19 Testing Linear Independence

Determine whether each pair of vectors is linearly independent or linearly dependent.

1. $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 4 \\ -2 \end{bmatrix}$
2. $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -6 \\ -2 \end{bmatrix}$
3. $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 5 \end{bmatrix}$
4. $\begin{bmatrix} -2 \\ 4 \\ 6 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -2 \\ -3 \end{bmatrix}$

Working Space

Answer on Page 91

Any set of vectors that includes the zero vector is automatically linearly dependent, since

$$1 \cdot \vec{0} = \vec{0}$$

gives a nontrivial solution.

Exercise 20 Zero Vector and Dependence

Determine whether each set of vectors is linearly independent.

1. $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$
2. $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$
3. $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Working Space

Answer on Page 91

Geometrically, in $\mathbb{R}^2$: independent vectors are not collinear. In $\mathbb{R}^2$, the vectors are generally perpendicular.

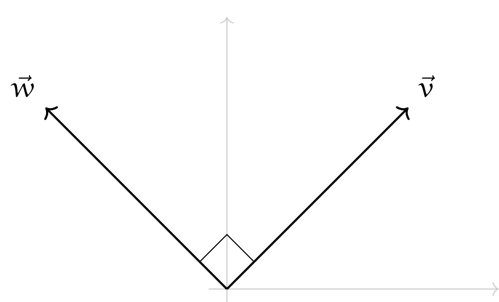


Figure 6.1: An example of independent vectors in $\mathbb{R}^2$.

In $\mathbb{R}^3$, two independent vectors lie in a plane, while three independent vectors do not lie in the same plane.

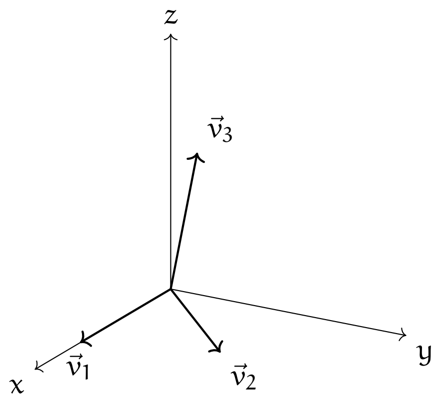


Figure 6.2: An example of linearly independent vectors in $\mathbb{R}^3$.

Exercise 21 Values Causing Dependence

Find all values of k for which the vectors are linearly dependent.

1. $\begin{bmatrix} 1 \\ k \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$

2. $\begin{bmatrix} k \\ 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 2k \\ 2 \\ 4 \end{bmatrix}$

Working Space

Answer on Page 92

6.1.1 Connection to Span

The *span* of a set of vectors is the collection of all vectors that can be written as linear combinations of those vectors. In other words, the span consists of all vectors that are reachable using the given set.

We now use this idea to better understand linear independence.

Consider the vector space $\mathbb{R}^3$. Let

$$V = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Any vector in $\mathbb{R}^3$ can be written as a linear combination of these three vectors. Therefore,

$$\text{span}(V) = \mathbb{R}^3.$$

Moreover, no vector in V can be written as a linear combination of the other two. Each vector contributes a new direction to the span. For this reason, the vectors in V are *linearly independent*.

Equivalently, a set of vectors is linearly independent if every vector in their span has a *unique* representation as a linear combination of those vectors.

This example illustrates an important idea: a linearly independent set contains no redundant vectors, and when such a set spans a space, the set is called a basis for the space.

A set B is a basis if its elements are linearly independent and every element of V is a linear combination of elements of B. In other words, a basis is a linearly independent spanning set. A vector space can have several bases; however all the bases have the same number of elements, called the dimension of the vector space. We will dive into this concept more in the subspaces chapter, so don't worry if this is a bit of a jump!

6.2 Linearly Dependent Vectors: Geometrically

Two vectors are linearly dependent if one is a multiple of the other. Mathematically,

Linearly dependent vectors in $\mathbb{R}^n$

Two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ are linearly dependent if there exists a scalar $a \in \mathbb{R}$ such that

$$\mathbf{v} = a\,\mathbf{u}.$$

Graphically, two linearly dependent vectors in $\mathbb{R}^2$ lie on the same line through the origin (or one of them is the zero vector).

If two vectors are linearly dependent, then linear combinations of them can only produce vectors lying on that same line. If they are *not* linearly dependent, they are called linearly *independent*, and their linear combinations can produce every vector in $\mathbb{R}^2$.

Example: Which of the following 3 vectors are linearly dependent, if any? $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} -3 \\ 4 \\ -1 \end{bmatrix}$, $\mathbf{w} = \begin{bmatrix} 6 \\ -8 \\ 2 \end{bmatrix}$.

Solution: Two vectors are linearly dependent if one is a scalar multiple of the other. Let's compare **u** and **v**. Since the first component of **u** is 1 and the first component of **v** is -3, let's multiply **u** by -3 to see if we get **v**:

$$-3\mathbf{u} = -3 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -3 \\ -6 \\ -9 \end{bmatrix} \neq \mathbf{v}$$

Therefore, **u** and **v** are *not* linearly dependent. Now let's examine **v** and **w**. Again, we will use the first components: the first component of **w** is 6, so let's see if multiplying **v** by -2 yields **w**:

$$-2\mathbf{v} = -2 \begin{bmatrix} -3 \\ 4 \\ -1 \end{bmatrix} = \begin{bmatrix} 6 \\ -8 \\ 2 \end{bmatrix} = \mathbf{w}$$

Therefore, **v** and **w** are linearly dependent. Since we already know that **u** and **v** are not linearly dependent, we also know that **u** and **w** are also not linearly dependent.

Exercise 22 Linear Dependence

Identify which, if any, of the following vectors are linearly dependent:

1. $\mathbf{a} = \begin{bmatrix} -4 \\ 1 \\ 4 \end{bmatrix}$
2. $\mathbf{b} = \begin{bmatrix} -4 \\ 5 \\ -3 \end{bmatrix}$
3. $\mathbf{c} = \begin{bmatrix} 2 \\ -4 \\ 6 \end{bmatrix}$
4. $\mathbf{d} = \begin{bmatrix} 1 \\ -\frac{1}{4} \\ -1 \end{bmatrix}$
5. $\mathbf{e} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$
6. $\mathbf{f} = \begin{bmatrix} -6 \\ \frac{3}{2} \\ 6 \end{bmatrix}$

Working Space

Answer on Page 92

6.3 What's next?

In this chapter, we defined linear independence and linear dependence, which describe whether a set of vectors contains redundancy. Linearly independent vectors introduce new directions, while dependent vectors can be written as linear combinations of others.

These ideas lead naturally to the study of subspaces, which describe collections of vectors that are closed under addition and scalar multiplication. In the next section, we formalize how sets of vectors generate larger structures and how linear independence helps describe them efficiently.

Chapter 7

Subspaces

Recall that, in Chapter 4, we established that all linear systems can be represented in matrix form as $A\vec{x} = \vec{b}$. In this chapter, we will explore the concept of subspaces, which are fundamental to understanding the structure of solutions to linear systems.

Quickly, let's review some vocabulary. The zero vector, denoted as $\vec{0}$, is the vector where all components are zero. It will be a column vector of appropriate size $n \times 1$.

If $A\vec{x} = \vec{0}$, where $\vec{b} = \vec{0}$, then the system is called **homogeneous**. If $A\vec{x} = \vec{b}$ where $\vec{b} \neq \vec{0}$, then the system is called **non-homogeneous**.

7.1 What is a Subspace?

A subspace V of $\mathbb{R}^n$ is a subset V of $\mathbb{R}^n$ satisfying the following properties:

- The zero vector $\vec{0}$ is in the subspace.
- If $\vec{u}$ and $\vec{v}$ are in the subspace, then their sum $\vec{u} + \vec{v}$ is also in the subspace.
- If $\vec{u}$ is in the subspace and c is a scalar, then the scalar multiple $c\vec{u}$ is also in the subspace.

In short, a subspace is exactly the set of all linear combinations of some collection of vectors. Additionally, every subpace is a span.

Subspace Span

If V is a subspace of $\mathbb{R}^n$, then there exists a set of vectors

$$\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\} \subseteq \mathbb{R}^n$$

such that

$$V = \text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$$

That is, every vector in V can be written as a linear combination of vectors in V.

Because V is closed under:

- vector addition, and
- scalar multiplication,

any linear combination of vectors in V must also lie in V.

So if $\vec{u}, \vec{v} \in V$ and $a, b \in \mathbb{R}$, then

$$a\vec{u} + b\vec{v} \in V$$

The converse is also true: any span is a subspace.

$$\text{span}(V) \text{ is a subspace of } \mathbb{R}^n$$

For example, vectors $\vec{v_1} = [1, 0]$ and $\vec{v_2} = [0, 1]$ span all of $\mathbb{R}^2$, since they can be scaled to fit all of $\mathbb{R}^2$.

- the span v_1 is a line through $\vec{0}$
- the span v_1, v_2 is a line or plane through $\vec{0}$
- the span of v_1, v_2, v_3 is a line or plane through $\vec{0}$, or all of $\mathbb{R}^3$

We need to review and reinforce some vocabulary that will come up often during this section. Let's look at Basis and Dimension.

7.2 Basis and Dimension

A basis of a subspace V is a set of linearly independent vectors that span V.

Basis

A set of vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\} \in \mathbb{R}^n$ is called a **basis** for $\mathbb{R}^n$ if

- the vectors span $\mathbb{R}^n$, and
- the vectors are linearly independent.

For example, a basis for $\mathbb{R}^3$ is given by the standard unit vectors:

$$V = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Note that the span of V is all of $\mathbb{R}^3$, and the vectors in V are linearly independent.

For a subspace V, there are often many possible bases. For example, another basis for $\mathbb{R}^3$ is given by the vectors:

$$W = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Put simply, a basis is a smallest possible set of vectors that can be used to build every vector in the space, with no redundancy. No vector in the basis can be written as a linear combination of the others.

A standard basis is a special type of basis that is often used for $\mathbb{R}^n$.

Standard Basis

The **standard basis** for $\mathbb{R}^n$ is the set of vectors

$$\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\},$$

where $\vec{e}_i$ is the vector in $\mathbb{R}^n$ with a 1 in the i-th position and 0 in all other positions.

An example of this is the standard basis for $\mathbb{R}^2$:

$$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

Recall we may often simplify this to i hat and j hat notation:

$$\left\{ \hat{i}, \hat{j} \right\} \qquad c_1\hat{i} + c_2\hat{j}$$

The standard basis consists of the vectors that point along the coordinate axes. Each standard basis vector measures exactly one coordinate.

The number of vectors in the basis is called the dimension of the subspace.

Dimension

The **dimension** of a vector space V is the number of vectors in any basis for V. If V has a basis consisting of k vectors, then we say that V has dimension k, and write

$$\dim(V) = k.$$

In particular, the dimension of $\mathbb{R}^n$ is n, since the standard basis contains exactly n vectors.

Properties of Dimension:

- A basis of a subspace $\mathbb{R}^n$ contains exactly n vectors.
- If W is a linear subspace of V, then $\dim(W) \leq \dim(V)$.
- If V is a finite-dimensional vector space and W is a linear subspace of V with $\dim(W) = \dim(V)$, then $W = V$.
- The space $\mathbb{R}^n$ has the standard basis $\{e_1, \dots, e_n\}$, where e_i is the i-th column of the corresponding identity matrix. Therefore, $\mathbb{R}^n$ has dimension n.

7.3 Nullspace

We now examine a fundamental example of a subspace: the **nullspace** of a matrix. Recall that a subspace is a subset of $\mathbb{R}^n$ that is closed under vector addition and scalar multiplication and contains the zero vector.

The nullspace of a matrix A, denoted $\text{Null}(A)$, is the set of all vectors $\vec{x}$ such that

$$A\vec{x} = \vec{0}.$$

That is, the nullspace is precisely the solution set of the homogeneous system $A\vec{x} = \vec{0}$.

Nullspace

The nullspace of a matrix A is defined as

$$\text{Null}(A) = \{\vec{x} \in \mathbb{R}^n : A\vec{x} = \vec{0}\}.$$

The $\vec{x}$, then represents all vectors that get flattened to origin $(\vec{0})$.

Because the equation $A\vec{x} = \vec{0}$ is homogeneous, its solution set always contains the zero vector. Moreover, if $\vec{x}_1$ and $\vec{x}_2$ are solutions, then any linear combination of the form

$$a\vec{x}_1 + b\vec{x}_2$$

is also a solution. For this reason, the nullspace of a matrix is always a subspace of $\mathbb{R}^n$.

7.3.1 Linear Combinations and Span

Recall that a **linear combination** of vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ is any vector of the form

$$a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_n\vec{v}_n,$$

where $a_1, a_2, \ldots, a_n \in \mathbb{R}$ are scalars.

The set of all possible linear combinations of a collection of vectors is called their **span**. If a subspace can be written as the span of one or more vectors, those vectors describe all possible directions within the subspace.

7.3.2 Example

Consider the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}.$$

To find the nullspace of A, we solve the homogeneous system $A\vec{x} = \vec{0}$:

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

This corresponds to the system of equations

$$x_1 + 2x_2 = 0, \qquad 2x_1 + 4x_2 = 0.$$

Solving for x_1 in terms of x_2 gives

$$x_1 = -2x_2.$$

Notice that x_1 can be written in terms of x_2. This implies *linear dependence*, the fact that the vectors can be written as combinations of each other. Thus, every vector in the nullspace can be written in the form

$$\vec{x} = x_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$

Therefore, the nullspace of A is

$$\text{Null}(A) = \text{span}\left\{ \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}.$$

Geometrically, this nullspace is a line through the origin in $\mathbb{R}^2$, which is a one-dimensional subspace of $\mathbb{R}^2$. The nullspace contains all scalar multiples of the vector $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$. This tells us that the solutions to the homogeneous system $A\vec{x} = \vec{0}$ form a line in $\mathbb{R}^2$. All vectors along this line produce the zero vector when substituted into the equation $A\vec{x} = \vec{0}$.

7.3.3 Example

Find a basis for the nullspace of the matrix

$$A = \begin{bmatrix} 1 & 2 & -1 & 0 \\ 2 & 4 & -2 & 0 \\ 0 & 1 & 1 & 3 \\ 0 & 2 & 2 & 6 \end{bmatrix}.$$

First, find the RREF form of A. Notice immediately that $R_2 = 2R_1$ and $R_4 = 2R_3$ in A, so we will definitely have free rows. We get that

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & -3 & -6 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Our $\vec{x}$ vector has to be the same size as our column, so it must live in $\mathbb{R}^4$. This gives us $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$.

So finding $A\vec{x} = \vec{0}$ provides the systems of equations:

$$\begin{cases} x_1 - 3x_3 - 6x_4 = 0 \\ x_2 + x_3 + 3x_4 = 0 \end{cases}$$

We can rewrite this as:

$$\begin{cases} x_1 = 3x_3 + 6x_4 \\ x_2 = -x_3 - 3x_4 \end{cases} \implies x_3 \begin{bmatrix} 3 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 6 \\ -3 \\ 0 \\ -1 \end{bmatrix}$$

So a basis for the nullspace is formed by

$$\text{Null}(A) = \left\{ \begin{bmatrix} 3 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 6 \\ -3 \\ 0 \\ -1 \end{bmatrix} \right\}$$

Exercise 23 Finding a Basis for a Nullspace

Working Space

Consider the matrix

$$A = \begin{bmatrix} 1 & -1 & 1 & 3 & 2 \\ 2 & -1 & 1 & 5 & 1 \\ 3 & -1 & 1 & 7 & 0 \\ 0 & 1 & -1 & -1 & -3 \end{bmatrix}$$

When put in reduced row echelon form,

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 & 2 & -1 \\ 0 & 1 & -1 & -1 & -3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

a Find a basis for the nullspace of A.

b Find a vector $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$ in the nullspace of A such that $x_1 = -3$, $x_3 = 1$, and $x_4 = 2$.

Answer on Page 92

Key Idea: Nullspace and Linear Independence

Solving for the *nullspace* of a matrix tells you whether a set of vectors is **linearly independent or linearly dependent**.

- If the nullspace contains *only* the zero vector, then the vectors are **linearly independent**.
- If the nullspace contains a *nonzero* vector, then the vectors are **linearly dependent**.

Practical test (via RREF):

- If every column has a *pivot*, the columns are linearly independent.
- If one or more columns lack a pivot, the columns are linearly dependent.

Equivalently, the equation $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution if and only if the columns of A are linearly dependent.

7.4 Row Space

The row space of a matrix A, denoted $\text{Row}(A)$, is the subspace of $\mathbb{R}^n$ spanned by the row vectors of A. Each row vector can be viewed as a vector in $\mathbb{R}^n$, and the row space consists of all linear combinations of these row vectors.

Row Space

The row space of a matrix A is defined as

$$\text{Row}(A) = \text{span}\{\text{row}_1, \text{row}_2, \ldots, \text{row}_m\},$$

where row_i represents the i-th row of the matrix A. The row space consists of all directions in $\mathbb{R}^n$ that can be built from the rows of the matrix. Note that we are observing the matrix before it is transformed into RREF or altered in any way.

Another way to think about it is that the row space represents all possible linear combinations of the equations represented by the rows of the matrix A. Each row is "tested" on by $\vec{x}$ to produce a component of the output vector $A\vec{x}$. Equivalently, the row space is the set of all vectors that can be formed by adding and scaling the rows of A.

When we compute:

$$A\vec{x}$$

each row of A gets dotted with $\vec{x}$ to produce a component of the output vector.

If each of the rows of A are denoted as $\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_m$, then $A\vec{x}$,

$$\begin{bmatrix} \vec{r}_1 \cdot \vec{x} \\ \vec{r}_2 \cdot \vec{x} \\ \vdots \\ \vec{r}_m \cdot \vec{x} \end{bmatrix}$$

The row space, then, is a subspace of $\mathbb{R}^n$ that captures all possible linear combinations of the rows of A. The row space consists of all possible tests on $\vec{x}$ that can be performed by the rows of A.

$$A = \begin{bmatrix} —\vec{r}_1— \\ —\vec{r}_2— \\ \vdots \\ —\vec{r}_m— \end{bmatrix} \quad \text{with } \vec{r}_i \in \mathbb{R}^n$$

Why does the row space live in $\mathbb{R}^n$?

- $\vec{x}$ has n components and lives in $\mathbb{R}^n$, and
- each row $\vec{r}_i$ has n components and lives in $\mathbb{R}^n$

7.4.1 Example

Consider the matrix

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 3 \end{bmatrix}$$

The row vectors of A are

$$\vec{r}_1 = \begin{bmatrix} 1 & 2 & -1 \end{bmatrix}, \quad \vec{r}_2 = \begin{bmatrix} 0 & 1 & 3 \end{bmatrix}$$

dotting them with an $\vec{x}$ of appropriate size gives

$$A\vec{x} = \begin{bmatrix} \vec{r}_1 \cdot \vec{x} \\ \vec{r}_2 \cdot \vec{x} \end{bmatrix}$$

The row space of A is the span of its row vectors:

$$\text{Row}(A) = \text{span}\left\{ \begin{bmatrix} 1 & 2 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 3 \end{bmatrix} \right.$$

7.5 Column Space

The column space of a matrix A, denoted $\text{Col}(A)$, is the subspace of $\mathbb{R}^m$ spanned by the column vectors of A. Each column of A can be viewed as a vector in $\mathbb{R}^m$, and the column space consists of all linear combinations of these column vectors.

Column Space

The **column space** of a matrix A is defined as

$$\text{Col}(A) = \text{span}\{\vec{c}_1, \vec{c}_2, \dots, \vec{c}_n\},$$

where $\vec{c}_i$ represents the i-th column of the matrix A. The column space consists of all vectors in $\mathbb{R}^m$ that can be formed as linear combinations of the columns of A.

Equivalently, the column space can be described as the set of all possible outputs of the matrix:

$$\text{Col}(A) = \{A\vec{x} : \vec{x} \in \mathbb{R}^n\}.$$

The columns of the *original* matrix A that correspond to pivot columns in $\text{RREF}(A)$ span the column space of A. We will discuss this idea further in a future chapter, but it corresponds to the fact that the pivot columns are linearly independent and form a basis for the column space.

7.5.1 Example

Consider the matrix

$$A = \begin{bmatrix} 1 & 3 & 1 & 4 \\ 2 & 7 & 3 & 9 \\ 1 & 5 & 3 & 1 \\ 1 & 2 & 0 & 8 \end{bmatrix}.$$

After applying elementary row operations, we find that the RREF of A is

$$\text{RREF}(A) = \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since there is a free row, there is a *dependent* row in the original matrix. The pivot columns are columns 1, 2, and 4. Thus, a basis for the column space of A is given by the original columns 1, 2, and 4:

$$\begin{bmatrix}1\\2\\1\\1\end{bmatrix}, \quad \begin{bmatrix}3\\7\\5\\2\end{bmatrix}, \quad \begin{bmatrix}4\\9\\1\\8\end{bmatrix}.$$

Thus the subspace spans $\mathbb{R}^4$ and consists of all linear combinations of these three vectors.

Exercise 24 Nullspace, Rowspace, and Columnspace

Working Space

Consider the matrix

$$A = \begin{bmatrix}1 & 2 & -1 & 3\\2 & 4 & -2 & 6\\1 & 1 & 0 & 2\\0 & 1 & 1 & 1\end{bmatrix}.$$

a Compute the reduced row echelon form of A.

b Identify the pivot columns and free columns of A. What does this tell you about the solutions?

c Find a basis for the **nullspace** $N(A)$. What is the dimension of $N(A)$?

d Find a basis for the **row space** $R(A)$. What is the dimension of $R(A)$?

e Find a basis for the **column space** $C(A)$. State clearly where the vectors for the basis come from.

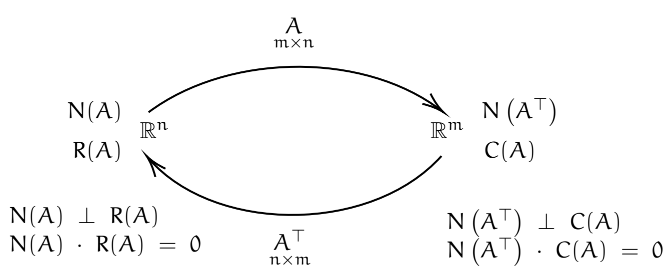
Answer on Page 93

7.6 Summary

In this chapter, we have established a few different common subspaces: the Nullspace, Rowspace, and Columnspace. Each of these subspaces captures different structural information about a matrix and the system of equations it represents. The nullspace describes all input vectors that are sent to the zero vector, the column space contains all possible outputs of the matrix, and the row space encodes the independent constraints imposed by the system.

We have now set the stage for a new way of viewing matrices. Rather than thinking of a matrix solely as a collection of numbers or equations, we can begin to interpret a matrix as a transformation – a rule that takes input vectors and maps them to output vectors. In the next few chapters, we will build on our understanding of subspaces to study how matrices act on vectors and reshape space, providing a unifying viewpoint for many of the ideas introduced so far. We will look at the geometric implications of matrices and how they *transform* space. This is the basis of many computer graphic programs, like calculating lengths of shadows in VR games or simulations!

Before ending this chapter, take a look at this graphic. It shows the relationship between the subspaces we talked about, plus a new one: $\text{Null}(A^T)$, the transpose of the nullspace. If matrix A has size $m \times n$, then A^T has size $n \times m$. Note which subspaces live in $\mathbb{R}^n$ and which live in $\mathbb{R}^m$.



Instead of asking what vectors solve $A\vec{x} = \vec{0}$, we will begin asking:

What does the matrix A do to an arbitrary vector?

The big take away: A matrix of size $m \times n$ can be viewed as a rule that takes vectors in $\mathbb{R}^n$ as inputs and produces vectors in $\mathbb{R}^m$as outputs.

Appendix A

Answers to Exercises

Answer to Exercise 1 (on page 18)

1. 3×5
2. 1×2
3. 4×3

Answer to Exercise 2 (on page 18)

1. The matrix should have 1 row and 3 columns. For example,

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$$

2. The matrix should have 2 rows and 4 columns. For example,

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix}$$

3. The matrix should have 4 rows and 3 columns. For example,

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix}$$

Answer to Exercise 3 (on page 20)

1. $\boldsymbol{A} + \boldsymbol{B} = \begin{bmatrix} -2 & 7 & -2 & 10 \end{bmatrix}$. $\boldsymbol{A} - \boldsymbol{B} = \begin{bmatrix} 2 & 1 & 2 & 0 \end{bmatrix}$. $\boldsymbol{B} - \boldsymbol{A} = \begin{bmatrix} -2 & -1 & -2 & 0 \end{bmatrix}$
2. $\boldsymbol{A} + \boldsymbol{B} = \begin{bmatrix} 9 & -4 & -3 \\ -4 & -6 & 3 \\ -10 & 6 & -4 \end{bmatrix}$. $\boldsymbol{A} - \boldsymbol{B} = \begin{bmatrix} -1 & -4 & -1 \\ 6 & 0 & 7 \\ 0 & 0 & 4 \end{bmatrix}$. $\boldsymbol{B} - \boldsymbol{A} = \begin{bmatrix} 1 & 4 & 1 \\ -6 & 0 & -7 \\ 0 & 0 & -4 \end{bmatrix}$.

3. $\boldsymbol{A}+\boldsymbol{B} = \begin{bmatrix} -7 & -3 & -2 & -6 \\ 5 & 1 & 0 & 0 \end{bmatrix}$. $\boldsymbol{A}-\boldsymbol{B} = \begin{bmatrix} 3 & 1 & -8 & 4 \\ 5 & -9 & 8 & 6 \end{bmatrix}$. $\boldsymbol{B}-\boldsymbol{A} = \begin{bmatrix} -3 & -1 & 8 & -4 \\ -5 & 9 & -8 & -6 \end{bmatrix}$.

Answer to Exercise 4 (on page 21)

It is possible to compute $\mathbf{a} \cdot \mathbf{d}$, $\mathbf{b} \cdot \mathbf{e}$, and $\mathbf{c} \cdot \mathbf{f}$:

1. $\mathbf{a} \cdot \mathbf{d} = 1(-5) + 2(-1) = -5 + (-2) = -7$
2. $\mathbf{b} \cdot \mathbf{e} = -3(1) + 3(-5) + 5(3) + -5(1) = -3 + (-15) + 15 - 5 = -8$
3. $\mathbf{c} \cdot \mathbf{f} = 1(4) + 2(1) + -1(-3) = 4 + 2 + 3 = 9$

Answer to Exercise 5 (on page 23)

1. $\begin{bmatrix} -2 & -1 & 5 \end{bmatrix}$
2. $\begin{bmatrix} 0 & 5 & 1 \\ 0 & 25 & 5 \\ 0 & -25 & -5 \\ 0 & 20 & 4 \\ 0 & 5 & 1 \end{bmatrix}$
3. $\begin{bmatrix} -9 & -17 & 11 \\ -6 & 40 & -4 \\ 15 & 7 & -13 \\ 9 & -8 & -1 \end{bmatrix}$

Answer to Exercise 6 (on page 24)

1. $A \times B = \begin{bmatrix} 8 & -6 & 10 & 4 \\ -8 & 6 & -10 & -4 \\ -4 & 3 & -5 & -2 \\ 8 & -6+1- & 4 & \end{bmatrix}$ and $B \times A = \begin{bmatrix} 13 \end{bmatrix}$
2. $A \times B = \begin{bmatrix} -2 & 16 & 16 \\ 5 & -24 & -8 \\ -4 & 22 & 12 \end{bmatrix}$ and $B \times A = \begin{bmatrix} 8 & 6 \\ -12 & -22 \end{bmatrix}$

3. $A \times B = \begin{bmatrix} -22 & 1 \\ 15 & -4 \end{bmatrix}$ and $B \times A = \begin{bmatrix} 12 & 0 & 15 & 3 \\ -4 & 0 & 1 & -15 \\ -16 & - & -17 & -11 \\ -14 & 0 & -10 & -21 \end{bmatrix}$

Answer to Exercise 7 (on page 24)

$$A \times B = \begin{bmatrix} -3 & 5 \\ -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} -1 & 1 \\ 4 & -3 \end{bmatrix} = \begin{bmatrix} -3(-1)+5(4) & -3(1)+5(-3) \\ -1(-1)+0(4) & -1(1)+0(-3) \end{bmatrix} = \begin{bmatrix} 23 & -18 \\ 1 & -1 \end{bmatrix}$$

$$B \times A = \begin{bmatrix} -1 & 1 \\ 4 & -3 \end{bmatrix} \cdot \begin{bmatrix} -3 & 5 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -1(-3)+1(-1) & -1(5)+1(0) \\ 4(-3)+-3(-1) & 4(5)+-3(0) \end{bmatrix} = \begin{bmatrix} 2 & -5 \\ -9 & 20 \end{bmatrix}$$

Answer to Exercise 8 (on page 31)

$$A = A^t = \begin{bmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{bmatrix}$$

Answer to Exercise 9 (on page 35)

1. We are looking for a_1 and a_2 such that:

$$a_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + a_2 \begin{bmatrix} -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

Which creates the system of equations:

$$a_1 - 3a_2 = 4$$

$$2a_1 + a_2 = 5$$

We can multiply the first equation by -2 and add it to the second to solve for a_2:

$$-2(a_1 - 3a_2) + 2a_1 + a_2 = -2(4) + 5$$

$$6a_2 + a_2 = -8 + 5$$

$$7a_2 = -3$$

$$a_2 = -\frac{3}{7}$$

Substituting a_2 back into an equation and solving for a_1:

$$a_1 - 3\left(-\frac{3}{7}\right) = 4$$

$$a_1 + \frac{9}{7} = 4$$

$$a_1 = \frac{19}{7}$$

Therefore, $\frac{19}{7}\begin{bmatrix}1\\2\end{bmatrix} - \frac{3}{7}\begin{bmatrix}-3\\1\end{bmatrix} = \begin{bmatrix}4\\5\end{bmatrix}$.

2. We are looking for a_1 and a_2 such that:

$$a_1\begin{bmatrix}9\\4\end{bmatrix} + a_2\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}-5\\3\end{bmatrix}$$

Which creates the system of equations:

$$9a_1 = -5$$

$$4a_1 + a_2 = 3$$

We can find a_1 from the first equation:

$$a_1 = -\frac{5}{9}$$

Substituting for a_1 back into the second equation and solving for a_2:

$$4\left(-\frac{5}{9}\right) + a_2 = 3$$

$$a_2 - \frac{20}{9} = 3$$

$$a_2 = \frac{47}{9}$$

Therefore, $-\frac{5}{9}\begin{bmatrix}9\\4\end{bmatrix} + \frac{47}{9}\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}-5\\3\end{bmatrix}$.

3. We are looking for a_1 and a_2 such that:

$$a_1\begin{bmatrix}7\\-2\end{bmatrix} + a_2\begin{bmatrix}-8\\4\end{bmatrix} = \begin{bmatrix}6\\-2\end{bmatrix}$$

Which yields the system of equations:

$$7a_1 - 8a_2 = 6$$

$$-2a_1 + 4a_2 = -2$$

Doubling the second equation and adding it to the first:

$$7a_1 - 8a_2 + 2(-2a_1 + 4a_2) = 6 + 2(-2)$$

$$7a_1 - 8a_2 - 4a_1 + 8a_2 = 6 - 4$$

$$3a_1 = 2$$

$$a_1 = \frac{2}{3}$$

Substituting for a_1 back into the second equation and solving for a_2:

$$-2\left(\frac{2}{3}\right) + 4a_2 = -2$$

$$-\frac{4}{3} + 4a_2 = -2$$

$$4a_2 = -\frac{2}{3}$$

$$a_2 = -\frac{1}{6}$$

Therefore, $\frac{2}{3}\begin{bmatrix} 7 \\ -2 \end{bmatrix} - \frac{1}{6}\begin{bmatrix} -8 \\ 4 \end{bmatrix} = \begin{bmatrix} 6 \\ -2 \end{bmatrix}$.

Answer to Exercise 10 (on page 42)

$$A = \begin{bmatrix} -4 & 9 & -8 \\ -1 & 0 & 6 \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$$

$$A\vec{x} = \vec{b} \implies \begin{bmatrix} -4 & 9 & -8 \\ -1 & 0 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$$

Answer to Exercise 11 (on page 53)

Notice immediately that $R_2 = 2R_1$, which automatically means redundancy.

$$\begin{bmatrix} 1 & 2 & -1 & 3 \\ 2 & 4 & -2 & 6 \\ 1 & 1 & 0 & 2 \end{bmatrix} \xrightarrow[R_3 \leftarrow R_3 - R_1]{R_2 \leftarrow R_2 - 2R_1} \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & -1 \end{bmatrix}$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_2 \leftarrow -R_2} \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_1 \leftarrow R_1 - 2R_2} \boxed{\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}}$$

The RREF is:

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This gives us the system:

$$x_1 + x_3 = 1,$$
$$x_2 - x_3 = 1.$$

So,

$$x_1 = 1 - x_3, \qquad x_2 = 1 + x_3$$

making x_3 free. We can write this as: $\vec{x} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \qquad x_3 \in \mathbb{R}$ Equivalently, as a set:

$$\left\{ \begin{pmatrix} 1 - x_3 \\ 1 + x_3 \\ x_3 \end{pmatrix} : x_3 \in \mathbb{R} \right\}$$

Since x_3 can be anything, there are infinitely many solutions. Equivalently, noticing that the row of all zeros implies infinitely many solution is a valid step to come to an answer.

Answer to Exercise 12 (on page 54)

$$\begin{bmatrix} 1 & 0 & 2 & -1 \\ 1 & 1 & 2 & -1 \\ 0 & 0 & 3 & 1 \end{bmatrix} \xrightarrow{R_2 \leftarrow R_2 - R_1} \begin{bmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 1 \end{bmatrix}$$

$$\xrightarrow{R_3 \leftarrow \frac{1}{3}R_3} \begin{bmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{1}{3} \end{bmatrix}$$

$$\xrightarrow{R_1 \leftarrow R_1 - 2R_3} \boxed{\begin{bmatrix} 1 & 0 & 0 & -\frac{5}{3} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{1}{3} \end{bmatrix}}$$

This gives us a unique solution:

$$x_1 = -\frac{5}{3}, \quad x_2 = 0, \quad x_3 = \frac{1}{3}$$

or as a vector $\vec{x} = \begin{bmatrix} -\frac{5}{3} \\ 0 \\ \frac{1}{3} \end{bmatrix}$

Answer to Exercise 13 (on page 54)

Notice that one row of the matrix states $\begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}$, which implies $0x_1 + 0x_2 + 0x_3 = 1$, which is not possible. Therefore the matrix represents an inconsistent system and there is no solution.

Answer to Exercise 14 (on page 54)

The RREF of this matrix is

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix} \xrightarrow[R_3 \leftarrow R_3 - 3R_1]{R_2 \leftarrow R_2 - 2R_1} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Note that there are two rows of zeroes, and the first column has a 1. This implies two dependent rows ($R_2 = 2R_1 \qquad R_3 = 3R_1$), and with only 1 pivot column, $\text{rank}(A) = 1$.

Answer to Exercise 15 (on page 55)

By definition in this chapter:

$$\text{rank}(A) = \text{number of pivots} = 3$$

A 4×6 matrix has $n = 6$ columns, so

$$\# \text{ free variables} = n - \text{rank}(A) = 6 - 3 = 3.$$

So, free variables $= 3$

Answer to Exercise 16 (on page 55)

There are $n = 4$ variables (x_1, x_2, x_3, x_4).

1. Pivot columns are columns $1, 2, 4$, so the pivot columns are $x_1,\ x_2,\ x_4$.
2. Free variables: the only non-pivot column is column 3, so the free variable is x_3
3. The dimension of the solution set is the number of free variables: 1. Equivalently, dimension $= n - \text{rank} = 4 - 3 = 1$

Answer to Exercise 17 (on page 60)

$$Av = (11\ 37\ \ -43)$$

Answer to Exercise 18 (on page 61)

The different methods can be represented in a matrix:

$$\begin{bmatrix} 0.20 & 0.50 & 0.15 & 0.15 \\ 0 & 0.50 & 0.20 & 0.30 \\ 0 & 0 & 0.4 & 0.6 \end{bmatrix}$$

And Suzy's individual grades can be represented by a vector:

$$\begin{bmatrix} 65 \\ 30 \\ 80 \\ 75 \end{bmatrix}$$

To see the results of the three different methods, we can multiply the matrix and the vector:

$$\begin{bmatrix} 0.20 & 0.50 & 0.15 & 0.15 \\ 0 & 0.50 & 0.20 & 0.30 \\ 0 & 0 & 0.4 & 0.6 \end{bmatrix} \cdot \begin{bmatrix} 65 \\ 30 \\ 80 \\ 75 \end{bmatrix} = \begin{bmatrix} 0.2(65) + 0.5(30) + 0.15(80) + 0.15(75) \\ 0(65) + 0.5(30) + 0.2(80) + 0.3(75) \\ 0(65) + 0(30) + 0.4(80) + 0.6(75) \end{bmatrix}$$

Which yields:

$$\begin{bmatrix} 51.25 \\ 53.5 \\ 77 \end{bmatrix}$$

Since method C yields the highest grade, the professor will post a final grade of 77.

Answer to Exercise 19 (on page 64)

Two vectors are linearly dependent if one is a scalar multiple of the other.

1. Dependent, since $\begin{bmatrix} 4 \\ -2 \end{bmatrix} = 2 \begin{bmatrix} 2 \\ -1 \end{bmatrix}$.
2. Dependent, since $\begin{bmatrix} -6 \\ -2 \end{bmatrix} = -2 \begin{bmatrix} 3 \\ 1 \end{bmatrix}$.
3. Independent, since neither vector is a scalar multiple of the other.
4. Dependent, since $\begin{bmatrix} 1 \\ -2 \\ -3 \end{bmatrix} = -\frac{1}{2} \begin{bmatrix} -2 \\ 4 \\ 6 \end{bmatrix}$.

Answer to Exercise 20 (on page 65)

Any set containing the zero vector is linearly dependent, since a nontrivial linear combination can equal the zero vector.

1. Dependent.
2. Dependent.
3. Dependent.

Answer to Exercise 21 (on page 66)

1. The vectors are dependent when $\begin{bmatrix}1\\k\end{bmatrix}$ is a scalar multiple of $\begin{bmatrix}2\\4\end{bmatrix}$. This occurs when $k = 2$.

2. The vectors are dependent for all values of k, since

$$\begin{bmatrix}2k\\2\\4\end{bmatrix} = 2\begin{bmatrix}k\\1\\2\end{bmatrix}.$$

Answer to Exercise 22 (on page 69)

We see that $\frac{\mathbf{a}}{-4} = -\frac{1}{4}\begin{bmatrix}-4\\1\\4\end{bmatrix} = \begin{bmatrix}1\\-\frac{1}{4}\\-1\end{bmatrix} = \mathbf{d}$. Additionally, $\frac{3}{2}\mathbf{a} = \frac{3}{2}\begin{bmatrix}-4\\1\\4\end{bmatrix} = \begin{bmatrix}-6\\\frac{3}{2}\\6\end{bmatrix} = \mathbf{f}$. Therefore, vectors **a**, **d**, and **f** are linearly dependent.

We also see that $\frac{1}{2}\mathbf{c} = \frac{1}{2}\begin{bmatrix}2\\-4\\6\end{bmatrix} = \begin{bmatrix}1\\-2\\3\end{bmatrix} = \mathbf{e}$. Therefore, vectors **c** and **e** are linearly dependent. Vector **b** is not linearly dependent to any of the other vectors.

Answer to Exercise 23 (on page 77)

a Solving $A\vec{x} = \vec{0}$ gives us two equations:

$$\begin{cases} x_1 + 2x_4 - x_5 = 0 \\ x_2 - x_3 - x_4 - 3x_5 = 0 \end{cases} \implies \begin{cases} x_1 = -2x_4 + x_5 \\ x_2 = x_3 + x_4 + 3x_5 \end{cases}$$

So we establish x_3, x_4, and x_5 as bound or fixed variables. So, every vector in the nullspace has the form

$$\vec{x} = \begin{bmatrix} -2x_4 + x_4 \\ x_3 + x_4 + 3x_4 \\ x_3 \\ x_4 \\ x_4 \end{bmatrix}$$

or, equivalently

$$\vec{x} = x_3 \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 3 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

So a basis can be formed by the set

$$\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

b We are given a vector $\vec{x} = \begin{bmatrix} -3 \\ x_2 \\ 1 \\ 2 \\ x_5 \end{bmatrix}$.

Recall that we have $x_1 = -2x_4 + x_5$, so inputting our knowns we can say $-3 = -2(2) + x_5 \implies = x_5 = 1$.

We can then solve $x_2 = x_3 + x_4 + 3x_5$ for x_2:

$$x_2 = 1 + 2 + 3(1) = 6$$

These two variables give us the $\vec{x}$

$$\vec{x} = \begin{bmatrix} -3 \\ 6 \\ 1 \\ 2 \\ 1 \end{bmatrix}$$

Answer to Exercise 24 (on page 81)

a

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

b Columns 1, 2, and 3 are pivot columns (containing a pivot). Column 4 is a non-pivot or free column. This tells you that there are 3 fixed variables, and 1 free one. The nullspace will be 1 dimensional.

c We have the system of equations:

$$\begin{aligned} x_1 + x_4 &= 0, \\ x_2 + x_4 &= 0, \\ x_3 &= 0. \end{aligned}$$

Letting x_4 be free, $\vec{x} = x_4 \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$. The $N(A) = \left\{ \begin{pmatrix} -1 \\ -1 \\ 0 \\ 1 \end{pmatrix} \right\}$ with a dimension of 1.

d The row space is the subspace spanned by the rows of A. A convenient basis for the row space is given by the nonzero rows of $\mathrm{rref}(A)$. Thus a basis is

$$\left\{ \begin{bmatrix} 1 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} \right\}.$$

These three rows are linearly independent, and they span the row space because row reduction did not create any new row space, just simplified the set. The rowspace has $\mathrm{rowspace}(A) = 3$.

e For the column space, we must take columns from the *original* matrix A. The usual rule is: pivot columns of the original A form a basis for $C(A)$. Since the pivot columns are $1, 2, 3$, a basis is the set of the first three columns of A

$$\mathbf{c}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{c}_2 = \begin{bmatrix} 2 \\ 4 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{c}_3 = \begin{bmatrix} -1 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

and the column space is the span of each original pivot column:

$$C(A) = \mathrm{span}\{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3\}$$

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