Contents

Chapter 1

Partial Derivatives and Gradients

This chapter will introduce you to partial derivatives and gradients, equipping you with the tools to study functions of multiple variables. We will explore how these concepts provide valuable insights into optimization, vector calculus, and various fields of science and engineering.

Partial derivatives come into play when dealing with functions that depend on multiple variables. Unlike ordinary derivatives that consider changes along a single variable, partial derivatives focus on how a function changes concerning each individual variable while holding the others constant. In essence, partial derivatives measure the rate of change of a function with respect to one variable, while keeping the other variables fixed.

The notation for a partial derivative of a function $f(x, y, \ldots)$ with respect to a specific variable, say x, is denoted as $\frac{\partial f}{\partial x}$. Similarly, $\frac{\partial f}{\partial y}$ represents the partial derivative with respect to y, and so on. It is essential to remember that when taking partial derivatives, we treat the other variables as constants during the differentiation process.

The gradient is a vector that combines the partial derivatives of a function. It provides a concise representation of the direction and magnitude of the steepest ascent or descent of the function. The gradient vector points in the direction of the greatest rate of increase of the function. By understanding the gradient, we gain insights into optimizing functions and finding critical points where the function reaches maximum or minimum values.

Throughout this chapter, we will explore the following key topics related to partial derivatives and gradients:

- Calculating partial derivatives: We will delve into the techniques and rules for computing partial derivatives of various functions, including polynomials, exponential functions, and trigonometric functions. We will also explore higher-order partial derivatives and mixed partial derivatives.
- Interpreting partial derivatives: Understanding the geometric and physical interpretations of partial derivatives is essential. We will discuss the notion of tangent planes, directional derivatives, and the relationship between partial derivatives and local linearity.
- Gradient vectors and their properties: We will introduce this concept, including it connection to the direction of steepest ascent, its relationship with partial derivatives,

and how it relates to level curves and level surfaces.

- Applications of partial derivatives and gradients: We will explore various applications of these concepts, including optimization problems, constrained optimization, tangent planes, linear approximations, and their relevance in fields like physics, economics, and engineering.

By grasping the concepts of partial derivatives and gradients, you will unlock a powerful mathematical framework for analyzing and optimizing functions of multiple variables. These tools will equip you to tackle advanced calculus problems and gain deeper insights into the behavior of functions in diverse fields.

1.1 Calculating Partial Derivatives

For a function of two variables, $f(x, y)$, we can take the derivative with respect to x or with respect to y. These are called the *partial derivatives* of f. Formally, the partial derivatives are defined as:

Limit Definition of Partial Derivatives

$$f_x(x, y) = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h}$$

$$f_y(x, y) = \lim_{h \to 0} \frac{f(x, y + h) - f(x, y)}{h}$$

Let's consider a polynomial function of two variables: $f(x, y) = 3x^2 + y^3 + 4xy$. We will use the limit definition to find the partial derivative with respect to x, then compare this to what we already know about derivatives of single-variable functions. Recall that if we can describe a function as a sum of two other functions, the derivative of the original function is the same as the sum of the derivatives of the other functions. That is,

$$\text{if } f(x) = g(x) + h(x)$$

$$\text{then } f'(x) = g'(x) + h'(x)$$

Let's then define $r(x, y) = 3x^2$, $s(x, y) = y^3$, and $t(x, y) = 4xy$. And so $f(x, y) = r(x, y) + s(x, y) + t(x, y)$, which means $f_x(x, y) = r_x(x, y) + s_x(x, y) + t_x(x, y)$. Then,

$$f_x(x, y) = \lim_{h \to 0} \frac{r(x + h, y) - r(x, y)}{h} + \lim_{h \to 0} \frac{s(x + h, y) - s(x, y)}{h} + \lim_{h \to 0} \frac{t(x + h, y) - t(x, y)}{h}$$

$$= \lim_{h \to 0} \frac{3(x + h)^2 - 3x^2}{h} + \lim_{h \to 0} \frac{y^3 - y^3}{h} + \lim_{h \to 0} \frac{4(x + h)y - 4xy}{h}$$

$$= \lim_{h\to 0} \frac{3x^2 + 6xh + h^2 - 3x^2}{h} + 0 + \lim_{h\to 0} \frac{4xy + 4hy - 4xy}{h}$$

Notice that $s_x(x, y) = 0$. This term only had y, and its derivative with respect to x is zero. Continuing,

$$f_x(x, y) = \lim_{h\to 0} \frac{6xh + h^2}{h} + \lim_{h\to 0} \frac{4hy}{h} = \lim_{h\to 0} 6x + h + \lim_{h\to 0} 4y$$
$$= 6x + 4y$$

As you can see, $r_x(x, y) = 6x$ and $t_x(x, y) = 4y$. Recall the polynomial rule for single derivatives. The derivative of $3x^2$ is $6x$, which is also what we see with the partial derivative in this case. What about the other term, $4xy$? Well, we know the derivative of bx, where b is a constant, is b. The partial derivative of $4xy$ with respect to x being $4y$ suggests the rule for determining partial derivatives:

Rule for Finding Partial Derivatives of $f(x, y)$

1. To find the partial derivative with respect to x, f_x, treat y as a constant and differentiate with respect to x.
2. To find the partial derivative with respect to y, f_y, treat x as a constant and differentiate with respect to y.

Let's check this by predicting f_y, then using the limit definition to confirm our prediction. Applying the polynomial rule, we predict that f_y is:

$$f_y(x, y) = 3y^2 + 4x$$

Which we found by treating x as a constant and taking the derivative of each term with respect to y. Let's see if we get the same result using the limit definition of the derivative with respect to y:

$$f_y(x, y) = \lim_{h\to 0} \frac{f(x, y + h) - f(x, y)}{h}$$
$$= \lim_{h\to 0} \frac{\left[3x^2 + (y + h)^3 + 4x(y + h)\right] - \left[3x^2 + y^3 + 4xy\right]}{h}$$
$$= \lim_{h\to 0} \frac{3x^2 + y^3 + 3y^2h + 3yh^2 + h^3 + 4xy + 4xh - 3x^2 - y^3 - 4xy}{h}$$
$$= \lim_{h\to 0} \frac{3y^2h + 3yh^2 + h^3 + 4xh}{h} = \lim_{h\to 0} 3y^2 + 3yh + h^2 + 4x = 3y^2 + 4x$$

Which is our expected result. In summary, you find the partial derivative with respect to a particular variable by treating all the other variables as constants and differentiating with respect to the particular variable, applying the rules of differentiation you've already learned.

1.1.1 Partial Derivative Notation

There are many ways to denote a partial derivative. We've already seen one way, f_x and f_y. Another common notation uses a lowercase Greek letter delta, and a further uses capital D. They are shown below:

Partial Derivative Notations

$$f_x(x, y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x, y) = D_x f$$
$$f_y(x, y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x, y) = D_y f$$

Exercise 1 First Partial Derivatives

Find f_x and f_y for the following functions.

1. $f(x, y) = 3x^4 + 4x^2y^3$
2. $f(x, y) = xe^{-y}$
3. $f(x, y) = \sqrt{3x + 4y^2}$
4. $f(x, y) = \sin x^2y$
5. $f(x, y) = \ln(x^y)$

Working Space

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1.1.2 Partial Derivatives of Functions of More than Two Variables

The above method of determining partial derivatives applies to functions with three, four, or any number of variables.

Example: Find all the first derivatives of the function $f(x, y, z) = y\cos\left(x^2 + 3z\right)$.

Solution:

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left[y\cos\left(x^2 + 3z\right)\right] = -y\sin\left(x^2 + 3z\right)\left(\frac{\partial}{\partial x}\left(x^2 + 3z\right)\right)$$

$$\frac{\partial f}{\partial x} = -2xy\sin\left(x^2 + 3z\right)$$

And

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left[y\cos\left(x^2+3z\right)\right]$$
$$\frac{\partial f}{\partial y} = \cos\left(x^2+3z\right)$$

And

$$\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}\left[y\cos\left(x^2+3z\right)\right] = -y\sin\left(x^2+3z\right)\left(\frac{\partial}{\partial z}\left(x^2+3z\right)\right)$$
$$\frac{\partial f}{\partial z} = -3y\sin\left(x^2+3z\right)$$

Exercise 2 Partial Derivatives with 3 or More Variables

Find all first partial derivatives of the following functions.

1. $f = \sin\left(x^2 - y^2\right)\cos\left(\sqrt{z}\right)$
2. $q = \sqrt[3]{t^3 + u^3 \sin(5v)}$
3. $w = x^z y^x$

Working Space

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1.1.3 Higher Order Partial Derivatives

Just like with single-variable equations, we can take the partial derivative more than once. There are also several notations for second partial derivatives.

Second Partial Derivative Notation

$$(f_x)_x = f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}$$

$$(f_x)_y = f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}$$

$$(f_y)_x = f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}$$

$$(f_y)_y = f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}$$

Notice that for $\left(\partial^2 f/\partial y \partial x\right)$, we first take the derivative with respect to x, then with respect to y.

Example: Find all the second order partial derivatives of $f(x, y) = 2x^2 - x^3y^2 + y^3$.

Solution: We begin by finding f_x and f_y:

$$f_x(x, y) = 4x - 3x^2y^2$$

$$f_y(x, y) = -2x^3y + 3y^2$$

We then take another partial derivative to find all the second order partial derivatives:

$$f_{xx}(x, y) = \frac{\partial}{\partial x} f_x(x, y) = \frac{\partial}{\partial x}\left(4x - 3x^2y^2\right) = 4 - 6xy^2$$

$$f_{xy}(x, y) = \frac{\partial}{\partial y} f_x(x, y) = \frac{\partial}{\partial y}\left(4x - 3x^2y^2\right) = -6x^2y$$

$$f_{yx}(x, y) = \frac{\partial}{\partial x} f_y(x, y) = \frac{\partial}{\partial x}\left(-2x^3y + 3y^2\right) = -6x^2y$$

$$f_{yy}(x, y) = \frac{\partial}{\partial y} f_y(x, y) = \frac{\partial}{\partial y}\left(-2x^3y + 3y^2\right) = -2x^3 + 6y$$

What do you notice about f_{xy} and f_{yx}? They are the same! This is not a coincidence of the particular function used in the example. For most functions, $f_{xy} = f_{yx}$, as stated by Clairaut's theorem.

Clairaut's Theorem

If f is defined on a disk D and f_{xy} and f_{yx} are both continuous on D, then $f_{xy} = f_{yx}$ on D.

This is also true for third, fourth, and higher-order derivatives.

Exercise 3 Clairaut's Theorem

Show that Clairaut's theorem holds for the following functions (show that $f_{xy} = f_{yx}$).

1. $f(x, y) = e^{2xy} \sin x$
2. $f(x, y) = \frac{x^2}{x+y}$
3. $f(x, y) = \ln(2x + 3y)$

Working Space

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Exercise 4 Second Order Partial Derivatives

Find all second order partial derivatives of the function.

1. $f(x, y) = x^5y^2 - 3x^3y^2$
2. $v = \sin\left(p^3 + q^2\right)$
3. $T = e^{-3r}\cos\theta^2$

Working Space

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1.1.4 The Chain Rule

For single-variable functions, where $y = f(x)$ and $x = g(t)$, we have seen that:

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$$

Which is the Chain Rule for single-variable functions. For multi-variable functions, there are several versions of the Chain Rule, depending on how the variables and functions are defined. First, we consider the case where $z = f(x, y)$ and $x = g(t)$ and $y = h(t)$ (i.e. f is a multi-variable function of x and y, while x and y are single-variable functions of t).

This means that z is an indirect function of t:

$$z = f(x, y) = f(g(t), h(t))$$

Then the derivative of z with respect to t is given by:

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

Example: If $z = xy^2 + 3x^4y$, where $x = 2\sin(t)$ and $y = \cos(3t)$, find dz/dt when $t = \pi/2$.

Solution: First, we apply the Chain Rule to z:

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial x}\frac{dy}{dt}$$

$$= \frac{\partial}{\partial x}\left[xy^2 + 3x^4y\right] \cdot \frac{d}{dt}[2\sin(t)] + \frac{\partial}{\partial y}\left[xy^2 + 3x^4y\right] \cdot \frac{d}{dt}[\cos(3t)]$$

$$= \left(y^2 + 12x^3y\right) \cdot (2\cos(t)) + \left(2xy + 3x^4\right) \cdot (-3\sin(3t))$$

When $t = \pi/2$, $\cos(t) = 0$, $\sin(3t) = -1$, $x = 2$, and $y = 0$. Substituting:

$$\frac{dz}{dt} = (0 + 0) \cdot (0) + \left(0 + 3(2)^4\right) \cdot (-3 \cdot -1)$$

$$= 3(2)^4 \cdot 3 = 144$$

Another case is where x and y are also multi-variable functions. Consider $z = f(x, y)$, $x = g(s, t)$, and $y = h(s, t)$. This means z is an indirect function of s and t:

$$z = f(x, y) = f(g(s, t), h(s, t))$$

In this case, there are two partial derivatives of z:

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

Example: Find $\partial z/\partial s$ and $\partial z/\partial t$ if $z = e^{2x}\cos y$, $x = s^2t$, and $y = st^2$.

Solution: First, let's find $\partial z/\partial s$:

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

$$= \frac{\partial}{\partial x}\left[e^{2x}\cos y\right] \cdot \frac{\partial}{\partial s}\left[s^2 t\right] + \frac{\partial}{\partial y}\left[e^{2x}\cos y\right] \cdot \frac{\partial}{\partial s}\left[st^2\right]$$

$$= \left(2e^{2x}\cos y\right) \cdot (2st) + \left(-e^{2x}\sin y\right) \cdot \left(t^2\right)$$

Substituting for x and y:

$$\frac{\partial z}{\partial s} = 4ste^{2s^2t}\cos\left(st^2\right) - t^2e^{2s^2t}\sin\left(st^2\right)$$

And finding $\partial z/\partial t$:

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

$$= \frac{\partial}{\partial x}\left[e^{2x}\cos y\right] \cdot \frac{\partial}{\partial t}\left[s^2 t\right] + \frac{\partial}{\partial y}\left[e^{2x}\cos y\right] \cdot \frac{\partial}{\partial t}\left[st^2\right]$$

$$= \left(2e^{2x}\cos y\right) \cdot \left(s^2\right) + \left(-e^{2x}\sin y\right) \cdot (2st)$$

Substituting for x and y:

$$\frac{\partial z}{\partial t} = 2s^2e^{2s^2t}\cos\left(st^2\right) - 2ste^{2s^2t}\sin\left(st^2\right)$$

Exercise 5 The Chain Rule for Multivariable Functions

Find dz/dt or $\partial z/\partial s$ and $\partial z/\partial t$.

Working Space

1. $z = \sin x \cos y$, $x = 3\sqrt{t}$, $y = 2/t$
2. $z = \sqrt{1 + xy}$, $x = \tan t$, $y = \arctan t$
3. $z = \arctan\left(x^2 + y^2\right)$, $x = t \ln s$, $y = se^t$
4. $z = \sqrt{x}e^{xy}$, $x = 1 + st$, $y = s^2 - t^2$

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1.2 Interpreting Partial Derivatives

What is the meaning of a partial derivative? Recall that $z = f(x, y)$ plots a surface, S. Consider the function $z = \cos y - x^2$, shown in figure 1.1.

We can see that $f(1, \pi/3) = -1/2$; therefore, the point $(1, \pi/3, -1/2)$ lies on the surface $z = \cos y - x^2$ (the black dot shown in figure **??**). If we fix y such that $y = \pi/3$, we are looking at the intersection between the surface and the plane $y = \pi/3$ (see figure **??**).

We can describe this intersection as $g(x) = f(x, \pi/3)$, so the slope of a tangent line to this

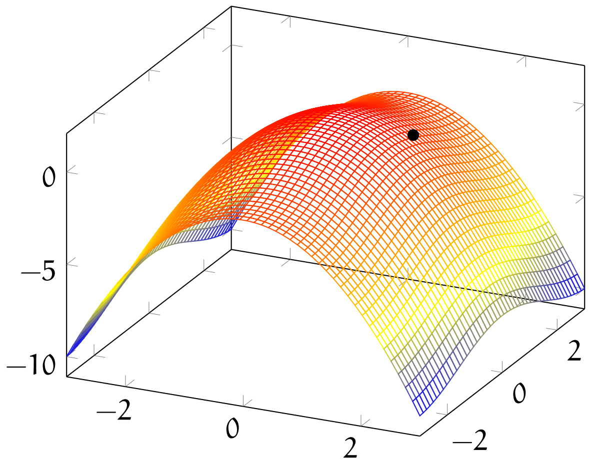

Figure 1.1: The surface $z = \cos y - x^2$

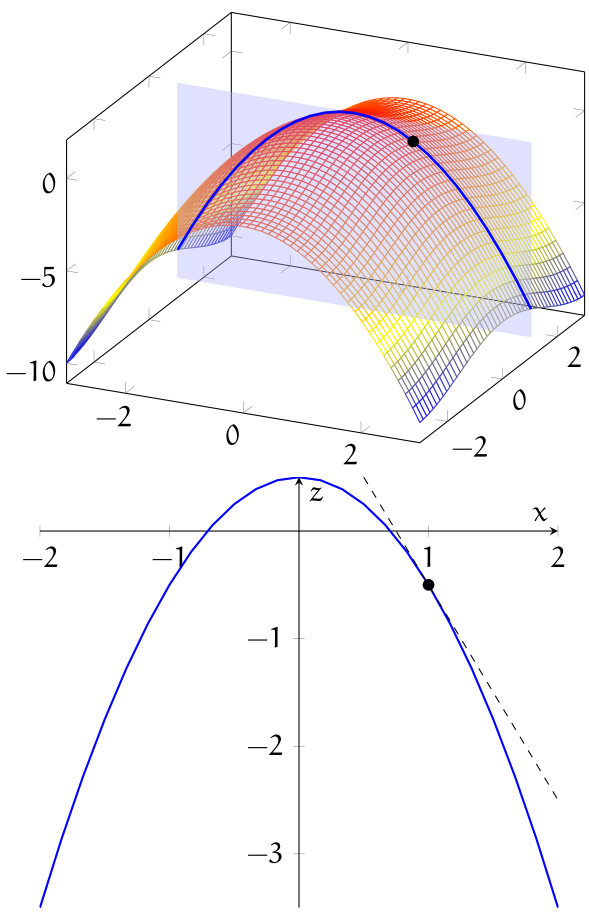

Figure 1.2: The intersection between the surface $z = \cos y - x^2$ and $y = \pi/3$ is the parabola $z(x) = 1/2 - x^2$

intersection is given by $g'(x) = f_x(x, \pi/3)$. This means, geometrically, $f_x(1, \pi/3)$ is the slope of the line that lies tangent to $z = f(x, y)$ at the point $(1, \pi/3, -1/2)$ and in the plane $y = \pi/3$ (see figure 1.2). Alternatively, you could think of f_x as the slope of the tangent line to the surface that is parallel to the x-axis.

Similarly, we can fix $x = 1$ and look at the intersection between the surface $z = \cos y - x^2$ and the plane $x = 1$ (see figure 1.3. Just like before, we can describe this intersection as $h(y) = f(1, y)$, which means the slope of a line tangent to the intersection is given by $h'(y) = f_y(1, y)$. Therefore, as with f_x, $f_y(a, b)$ gives the slope of a line tangent to the point $(a, b, f(a, b))$ and parallel to the y-axis.

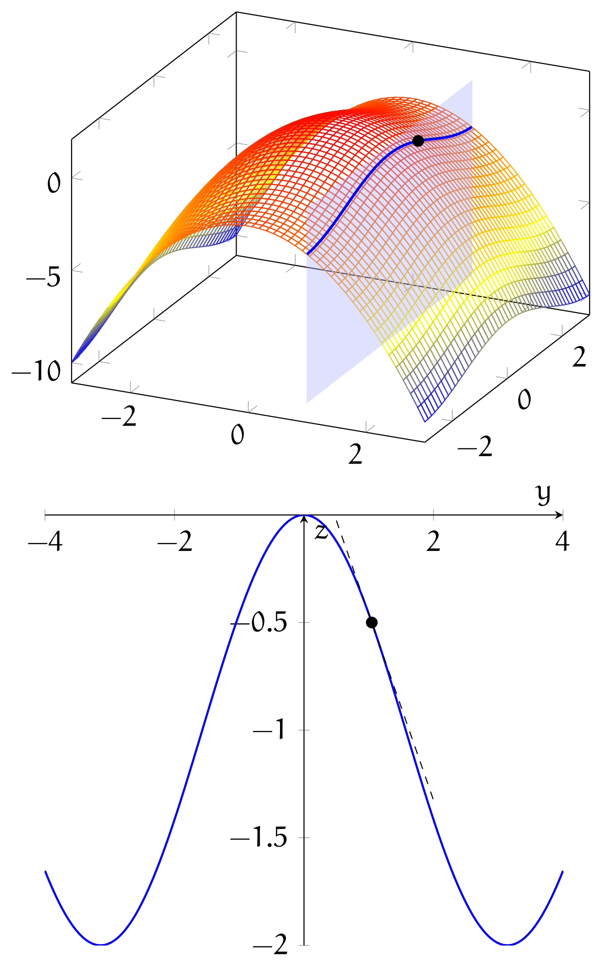

Figure 1.3: The intersection between the surface $z = \cos y - x^2$ and $x = 1$ is the trigonometric function $z = \cos y - 1$

Example: The density of bacterial growth at a point (x, y) on a flat agar plate is given by $D = 45/\left(2 + x^2 + y^2\right)$. Find the rate of change of bacterial density at the point $(1, 3)$ (a) in the x-direction and (b) in the y-direction. Interpret the meaning of your results.

Solution: The rate of change of a two-variable function in the x-direction is given by the partial derivative with respect to x:

$$D_x = \frac{\partial}{\partial x}\frac{45}{2+x^2+y^2} = \frac{-45\,(\partial/\partial x)\left(2+x^2+y^2\right)}{\left(2+x^2+y^2\right)^2}$$
$$= \frac{-90x}{\left(2+x^2+y^2\right)^2}$$

The rate of change in the x-direction at $(x, y) = (1, 3)$ is given by:

$$D_x(1,3) = \frac{-90(1)}{\left(2+1^2+3^2\right)^2} = \frac{-90}{(12)^2} = \frac{-90}{144} = -\frac{5}{8}$$

This means that at $(1, 3)$, the density of bacteria is decreasing as you move away $x = 0$ along the line $y = 3$.

Similarly, the rate of change in the y-direction is given by the partial derivative with respect to y:

$$D_y = \frac{\partial}{\partial y}\frac{45}{2+x^2+y^2} = \frac{-45\,(\partial/\partial y)\left(2+x^2+y^2\right)}{\left(2+x^2+y^2\right)^2}$$
$$= \frac{-90y}{\left(2+x^2+y^2\right)^2}$$

The rate of change in the y-direction at $(x, y) = (1, 3)$ is given by:

$$D_y(1,3) = \frac{-90(3)}{\left(2+1^2+3^2\right)^2} = \frac{-270}{144} = -\frac{15}{8}$$

This means that at $(1, 3)$ the density of bacteria is decreasing faster along the y-direction than along the x-direction.

Exercise 6 Using partial derivatives to find tangent lines

Find equations for tangent lines to the surface at the given xy-coordinate. In which direction is the function changing the fastest?

1. $z = x^2 e^{y/x}$, $(1, -1)$
2. $z = \cos x + y \sin y$, $(\pi, \pi/2)$
3. $z = x^2 y - 3xy^2$, $(3, 2)$

Working Space

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1.3 Gradient Vectors

The gradient vector is used to find the direction of the maximum rate of change of a surface (for example, the steepest part of a mountain). In order to understand the gradient, we must first discuss directional derivatives. Recall that the partial derivatives, f_x and f_y, can be used to define a plane tangent to the surface $z = f(x, y)$ (see figure 1.4). Directional derivatives allow us to find the slope of the tangent plane in directions other than the x- and y-directions.

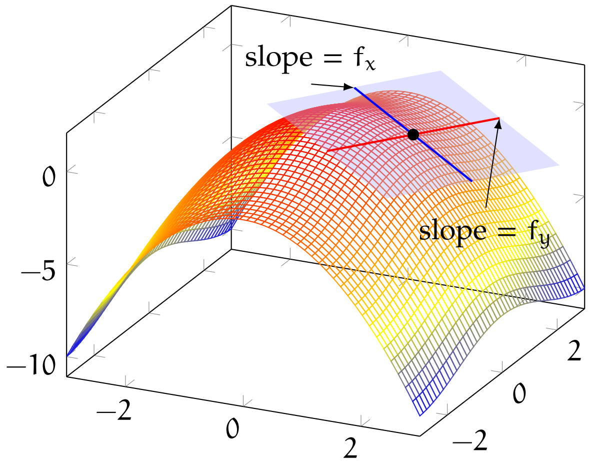

Figure 1.4: The directional derivatives, f_x and f_y define a tangent plane

1.3.1 Directional Derivatives

«««< HEAD The contour map in figure 1.5 shows the elevation, $f(x, y)$ ======= The contour map in figure 1.5 shows the elevation, $H(x, y)$, »»»> 75c26aca4245f06f19f767709f3697abe6d41eba for a mountain. You already know that you can use the partial derivatives, f_x and f_y to find the rate of change in elevation going east-west or north-south. But what about other directions? Suppose the hiking path you're on goes north-east. How can you predict the steepness (i.e. the rate of elevation change) along this path? The directional derivative allows us to find the rate of change in any direction.

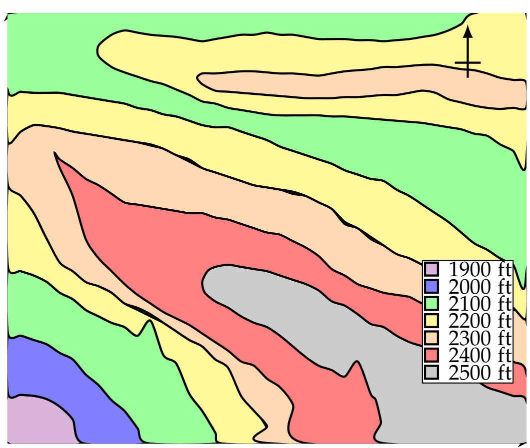

Figure 1.5: The contour plot shows the elevation of a mountain. f_x gives the slope going east, while f_y gives the slope going north

At some point, (x_o, y_o), the partial derivatives $f_x(x_o, y_o)$ and $f_y(x_o, y_o)$ give the rate of change of elevation in the east-west and north-south directions, respectively (see figure 1.6).

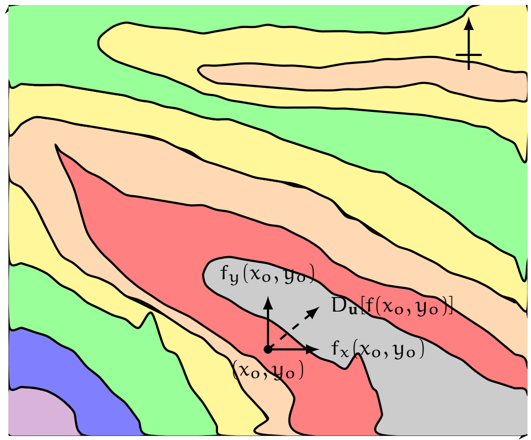

Figure 1.6: If **u** points north-east, then the directional derivative of $f(x, y)$ at (x_o, y_o), $D_{\mathbf{u}}[f(x_o, y_o)]$, tells the rate of change going north-east

To find the rate of change at (x_o, y_o), in the direction of some arbitrary unit vector, $\mathbf{u} = [a, b] = a\mathbf{i} + b\mathbf{j}$, we first note that the point $Q = (x_o, y_o, z_o)$, where $z_o = f(x_o, y_o)$, lies on the surface defined by $z = f(x, y)$. There is a vertical plane, P, that passes through Q and points in the direction of **u**. This intersection defines curve C, which lies on the surface, and the slope of this curve at $Q = (x_o, y_o, z_o)$ is the directional derivative of H in the direction of **u** (see figure 1.7).

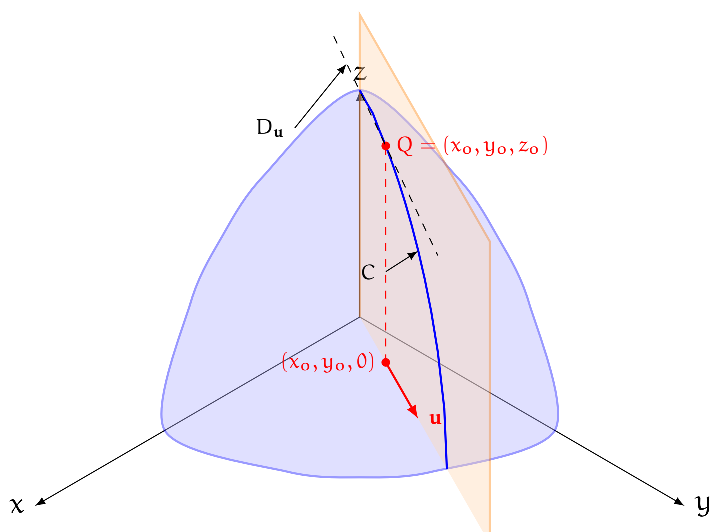

Figure 1.7: The slope of the curve formed between the plane parallel to **u** and the surface $z = f(x, y)$ is the directional derivative, $D_{\mathbf{u}}$

We can choose another point, $R = (x, y, z)$, the is h units away from Q along **u** (see 1.8). Then the change in x is $x - x_o = ha$ and the change is y is $y - y_o = hb$. And the slope from Q to R is given by:

$$\frac{\delta z}{h} = \frac{f(x, y) - f(x_o, y_o)}{h}$$

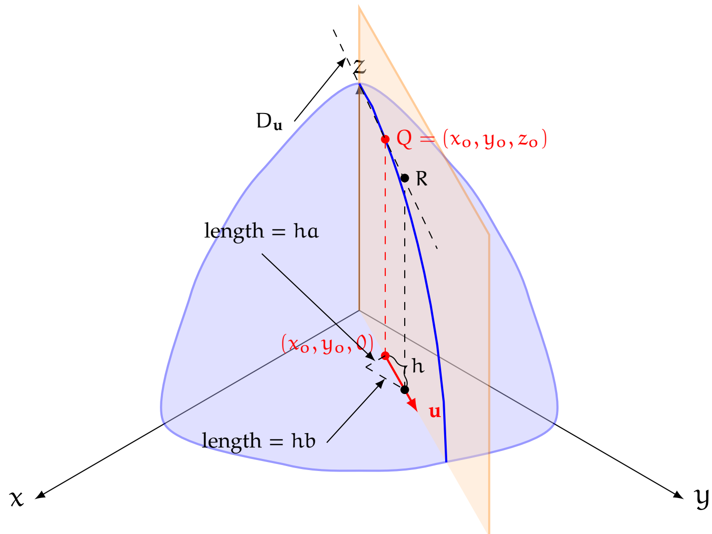

Figure 1.8: A second point, R, along **u** is h units away along **u**

We find the directional derivative by substituting for x and y and taking the limit as h goes to zero:

$$D_\mathbf{u} f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_o + ha, y_o + hb) - f(x_o, y_o)}{h}$$

How is this related to f_x and f_y? Let's define $g(h)$ such that $g(h) = f(x_o + ha, y_o + hb)$. Then

$$g'(0) = \lim_{h \to 0} \frac{g(h) - g(0)}{h}$$

$$= \lim_{h \to 0} = \frac{f(x_o + ha, y_o + hb) - f(x_o, y_o)}{h} = D_\mathbf{u} f(x_o, y_o)$$

We can also apply the Chain Rule to $g(h)$:

$$g'(h) = \frac{\partial f}{\partial x}\frac{dx}{dh} + \frac{\partial f}{\partial y}\frac{dy}{dh} = f_x(x, y)a + f_y(x, y)b$$

Substituting $h = 0$, $x = x_o$, and $y = y_o$, we see that:

$$g'(0) = f_x(x_o, y_o)a + f_y(x_o, y_o)b$$

Which means that:

$$D_\mathbf{u} f(x_o, y_o) = f_x(x_o, y_o)a + f_y(x_o, y_o)b$$

So a directional derivative is:

The Directional Derivative

Let f be a differentiable function and $\mathbf{u}$ be a unit vector, «««< HEAD $\mathbf{u} = [a, b]$. Then the directional derivative in the direction of $\mathbf{u}$ is:

$$D_{\mathbf{u}}f(x,y) = f_x(x,y)a + f_y(x,y)b = \mathbf{u}_x\left[\frac{\partial}{\partial x}f(x,y)\right] + \mathbf{u}_y\left[\frac{\partial}{\partial y}f(x,y)\right]$$

======= $\mathbf{u} = [a, b]$. This means the directional derivative in the direction of $\mathbf{u}$ is:

$$D_u f(x,y) = f_x(x,y)a + f_y(x,y)b = \mathbf{u}_x\left[\frac{\partial}{\partial x}f(x,y)\right] + \mathbf{u}_y\left[\frac{\partial}{\partial y}f(x,y)\right]$$

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Where $\mathbf{u}_x$ and $\mathbf{u}_y$ are the x- and y-components of $\mathbf{u}$, respectively.

Example: Find the directional derivative $D_{\mathbf{u}}f(x,y)$ if $f(x,y) = y^3 - 3xy + 4x^2$ and $\mathbf{u}$ is the unit vector given by the angle $\theta = \pi/3$. What is the rate of change in the direction of $\mathbf{u}$ at $(1,2)$?

Solution: We can describe $\mathbf{u}$ thusly:

$$\mathbf{u} = \left[\cos\frac{\pi}{3}, \sin\frac{\pi}{3}\right] = \left[\frac{1}{2}, \frac{\sqrt{3}}{2}\right]$$

And therefore:

$$D_{\mathbf{u}}f(x,y) = f_x(x,y)\left(\frac{1}{2}\right) + f_y(x,y)\left(\frac{\sqrt{3}}{2}\right)$$

$$= \frac{\partial}{\partial x}\left(y^3 - 3xy + 4x^2\right)\left(\frac{1}{2}\right) + \frac{\partial}{\partial y}\left(y^3 - 3xy + 4x^2\right)\left(\frac{\sqrt{3}}{2}\right)$$

$$= \frac{1}{2}(-3y + 8x) + \frac{\sqrt{3}}{2}\left(3y^2 - 3x\right)$$

$$= \frac{-3}{2}y + 4x + \frac{3\sqrt{3}}{2}y^2 - \frac{3\sqrt{3}}{2}x = \frac{3\sqrt{3}}{2}y^2 + \frac{8 - 3\sqrt{3}}{2}x - \frac{3}{2}y$$

And therefore $D_{\mathbf{u}}f(1,2)$ is:

$$= \frac{3\sqrt{3}}{2}(2)^2 + \frac{8 - 3\sqrt{3}}{2}(1) - \frac{3}{2}(2) = 6\sqrt{3} + 4 - \frac{3\sqrt{3}}{2} - 3$$

$$= 1 + \frac{9\sqrt{3}}{2}$$

1.3.2 Unit Vectors in Two Dimensions

What if the given vector is not a unit vector? We can scale the given vector to find a unit vector in the same direction:

Example: Find the directional derivative of $f(x, y) = 3x\sqrt{y}$ at $(1, 4)$ in the direction of $\mathbf{v} = [2, 1]$.

Solution: First, we need to find a unit vector in the same direction as $\mathbf{v}$. There are several ways to do this. In two dimensions, a unit vector in the same direction as $\mathbf{v}$ can be found using trigonometry (see figure 1.9 for an illustration).

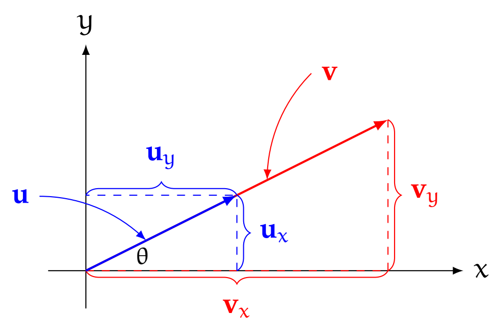

Figure 1.9: $\mathbf{u}$ is a unit vector in the same direction as $\mathbf{v}$

We know that $\theta = \arctan(\mathbf{v}_y/\mathbf{v}_x)$. Therefore, the x-component of the unit vector, $\mathbf{u}$, is given by:

$$\mathbf{u}_x = |\mathbf{u}| \cos\theta = \cos\left(\arctan\frac{\mathbf{v}_y}{\mathbf{v}_x}\right)$$

Similarly, we know that:

$$\mathbf{u}_y = |\mathbf{u}| \sin\theta = \sin\left(\arctan\frac{\mathbf{v}_y}{\mathbf{v}_x}\right)$$

(Recall that since $\mathbf{u}$ is a unit vector, $|\mathbf{u}| = 1$).

Let's use this method to find a unit vector, $\mathbf{u}$, in the same direction as $\mathbf{v} = [2, 1]$:

$$\mathbf{u}_x = \cos\left(\arctan\frac{1}{2}\right) \approx \cos(0.464) = \frac{2}{\sqrt{5}}$$

$$\mathbf{u}_y = \sin\left(\arctan\frac{1}{2}\right) \approx \sin(0.464) = \frac{1}{\sqrt{5}}$$

Therefore, a unit vector in the same direction as $\mathbf{v}$ is $\mathbf{u} = \left[2/\sqrt{5}, 1/\sqrt{5}\right]$.

And we can find the directional derivative:

$$D_{\mathbf{u}}(x, y) = \mathbf{u}_x \left[\frac{\partial}{\partial x} f(x, y)\right] + \mathbf{u}_y \left[\frac{\partial}{\partial y} f(x, y)\right]$$

$$D_{\mathbf{u}}(x, y) = \left(\frac{2}{\sqrt{5}}\right)\left[\frac{\partial}{\partial x}(3x\sqrt{y})\right] + \left(\frac{1}{\sqrt{5}}\right)\left[\frac{\partial}{\partial y}(3x\sqrt{y})\right]$$

$$D_{\mathbf{u}}(x, y) = \left(\frac{2}{\sqrt{5}}\right)(3\sqrt{y}) + \left(\frac{1}{\sqrt{5}}\right)\left(\frac{3x}{2\sqrt{y}}\right)$$

$$D_{\mathbf{u}}(x, y) = \frac{12y + 3x}{2\sqrt{5y}}$$

To find the magnitude of the directional derivative at $(1, 4)$, we substitute for x and y:

$$D_{\mathbf{u}}(1, 4) = \frac{12(4) + 3(1)}{2\sqrt{5(4)}} = \frac{51}{4\sqrt{5}} \approx 5.702$$

1.3.3 Unit Vectors in Higher Dimensions

The trigonometric explanation for finding unit vectors is more difficult to visualize in higher dimensions. However, there is another method that works well in 2, 3, and higher dimensions. Recall that the magnitude of a vector, $\mathbf{v} = [\mathbf{v}_x, \mathbf{v}_y]$ is given by $|\mathbf{v}| = \sqrt{(\mathbf{v}_x)^2 + (\mathbf{v}_y)^2}$. For a vector with n dimensions, $\mathbf{v} = [\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n]$, the magnitude is given by $|\mathbf{v}| = \sqrt{(\mathbf{v}_1)^2 + (\mathbf{v}_2)^2 + \cdots + (\mathbf{v}_n)^2}$.

To find a unit vector, $\mathbf{u}$, in the same direction as $\mathbf{v}$, we can scale $\mathbf{v}$ up or down so that its magnitude is 1. We can do this by dividing by $\mathbf{v}$'s magnitude. Consider the two-dimensional vector used in the last example, $\mathbf{v} = [2, 1]$. Its magnitude is:

$$|\mathbf{v}| = \sqrt{(2)^2 + (1)^2} = \sqrt{5}$$

Let's check if $\mathbf{v}/|\mathbf{v}|$ is a unit vector:

$$\frac{\mathbf{v}}{|\mathbf{v}|} = \left(\frac{1}{\sqrt{5}}\right)[2, 1] = \left[\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right]$$

And the magnitude of this scaled vector is:

$$\left|\frac{\mathbf{v}}{|\mathbf{v}|}\right| = \sqrt{\left(\frac{2}{\sqrt{5}}\right)^2 + \left(\frac{1}{\sqrt{5}}\right)^2} = \sqrt{\frac{4}{5} + \frac{1}{5}} = \sqrt{1} = 1$$

Notice our unit vector is the same as we found using the trigonometric method above.

«««< HEAD Another way to think of the question is: what factor, k, can we multiply $\mathbf{v}$ by to yield a vector with a magnitude of 1? Let's see this method for the 3-dimensional vector $\mathbf{v} = [3, 2, 1]$. We are looking for a k such that: ======= Another way to think of the question is: What factor, k, can we multiply $\mathbf{v}$ by to yield a vector with a magnitude of 1? Let's see this method for the 3-dimensional vector $\mathbf{v} = [3, 2, 1]$. We are looking for a k such that: »»»> 75c26aca4245f06f19f767709f3697abe6d41eba

$$|k\mathbf{v}| = 1$$

$$|k\mathbf{v}| = |[3k, 2k, 1k]| = \sqrt{(3k)^2 + (2k)^2 + (1k)^2}$$
$$= \sqrt{9k^2 + 4k^2 + k^2} = k\sqrt{14} = 1$$

«««< HEAD Which implies that $k = 1/\sqrt{14}$, which is $1/|\mathbf{v}|$. And therefore a unit vector in the same direction as $\mathbf{v} = [3, 2, 1]$ is:

$$\mathbf{u} = \frac{1}{\sqrt{14}}[3, 2, 1] = \left[\frac{3}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{1}{\sqrt{14}}\right]$$

======= Which implies that $k = 1/\sqrt{14}$, which is $1/|\mathbf{v}|$. Therefore, a unit vector in the same direction as $\mathbf{v} = [3, 2, 1]$ is:

$$\mathbf{u} = \frac{1}{\sqrt{14}}[3, 2, 1] = \left[\frac{3}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{1}{\sqrt{14}}\right]$$

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Exercise 7 Finding Directional Derivatives

Find the directional derivative of the function at the given point in the direction of the given vector.

1. $f(x, y) = e^{3x} \sin 2y$, $(0, \pi/6)$, $\mathbf{v} = [-3, 4]$
2. $f(x, y) = x^2y + xy^3$, $(2, 4)$, $\mathbf{v} = 2\mathbf{i} - \mathbf{j}$
3. $f(x, y, z) = \ln\left(x^2 + 3y - z\right)$, $(2, 2, 1)$, $\mathbf{v} = [1, 1, 1]$

Working Space

Answer on Page 79

1.3.4 Maximizing the Gradient

The directional derivative can be written as the dot product of two vectors:

$$D_u f(x, y) = af_x(x, y) + bf_y(x, y) = [f_x(x, y), f_y(x, y)] \cdot \mathbf{u}$$

«««< HEAD The first vector, $[f_x(x, y), f_y(x, y)]$, is called *the gradient of f*, and is noted as ∇f. ======= The first vector, $[f_x(x, y), f_y(x, y)]$ is called *the gradient of f* and is noted as ∇f. The gradient operator, ∇, is the derivative of a scalar function that results in a vector which shows the magnitude and direction of the greatest rate of change. »»»> eabbbcc8b878e818b0fd1aeab6d3a95ed058f052

The Gradient

For a two-variable function, $f(x, y)$, the gradient of f is the vector:

$$\nabla f = [f_x(x, y), f_y(x, y)] = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

Where **i** and **j** are the unit vectors in the x- and y-directions, respectively.

Think back to the elevation example we opened the chapter with. What if we wanted to complete our ascent as quickly as possible? We would want to know the direction in which the elevation is changing the fastest. This occurs when the direction we are going is the same direction as the gradient vector, ∇f.

Recall that the dot product is defined as:

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}|\,|\mathbf{v}| \cos\theta$$

Where θ is the angle between the vectors **u** and **v**. Applying this to the directional derivative, we see that:

$$D_u f = \nabla f \cdot \mathbf{u} = |\nabla f|\,|\mathbf{u}| \cos\theta = |\nabla f| \cos\theta$$

Which is at its maximum when ∇f and **u** point in the same direction (because $\cos(0) = 1$). Therefore, the gradient vector points in the direction of maximum change and the magnitude of that vector is the rate of maximum change.

Example: Find the maximum rate of change of $f(x, y) = 4y\sqrt{x}$ at $(4, 1)$. In what direction does the maximum change occur?

Solution: We begin by finding ∇f:

$$\nabla f = \left[\frac{\partial}{\partial x}\left(4y\sqrt{x}\right), \frac{\partial}{\partial y}\left(4y\sqrt{x}\right)\right]$$

$$\nabla f = \left[\frac{2y}{\sqrt{x}}, 4\sqrt{x}\right]$$

And thus,

$$\nabla f(4,1) = \left[\frac{2(1)}{\sqrt{4}}, 4\sqrt{4}\right] = [1,8]$$

Therefore, the maximum value of ∇f at $(4,1)$ is:

$$|\nabla f| = \sqrt{1^2 + 8^2} = \sqrt{65}$$

in the direction of the vector $[1,8]$.

Exercise 8 Using the Gradient to find Maximum Change

Suppose you are climbing a mountain whose elevation is described by $z = 3000 - 0.01x^2 - 0.02y^2$. Take the positive x-direction to be east and the positive y-direction to be north.

1. If you are at $(x, y) = (50, 50)$, what is your elevation?
2. If you walk south, will you ascend or descend?
3. If you walk northwest, will you ascend or descend? Will the rate of elevation change be greater or less than if you walked south?
4. In what direction should you walk for the steepest ascent? What will your ascension rate be?

Working Space

Answer on Page 81

1.4 Applications of Partial Derivatives and Gradients

1.4.1 Laplace's Equation

A partial differential equation that has applications in fluid dynamics and electronics is Laplace's equation. Solutions to Laplace's equation are called *harmonic functions*.

Laplace's Equation

Consider a twice-differentiable function, f. In two dimensions, Laplace's Equation is given by:

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

And in three dimensions,

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$$

Another way to represent Laplace's equation is:

$$\delta f = \nabla^2 f = \nabla \cdot \nabla f = 0$$

Where $\nabla^2 = \delta$ is called the *Laplace operator*.

Example: Determine whether or not $f = x^2 + y^2$ is a solution to Laplace's equation.

Solution: We are checking to see if $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$ for $f(x, y) = x^2 + y^2$. Finding $\partial^2 f / \partial x^2$:

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}\left(x^2 + y^2\right)\right]$$
$$= \frac{\partial}{\partial x}(2x) = 2$$

And finding $\partial^2 f / \partial y^2$:

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left[\frac{\partial}{\partial y}\left(x^2 + y^2\right)\right]$$
$$= \frac{\partial}{\partial y}(2x) = 2$$

Then $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 2 + 2 = 4 \neq 0$. Therefore, $f(x, y) = x^2 + y^2$ is not a solution to Laplace's equation.

Exercise 9 Solutions to Laplace's Equation

Determine whether the function is a solution to Laplace's equation.

1. $f(x, y) = x^2 - y^2$
2. $f(x, y) = \sin x \cosh y + \cos x \sinh y$
3. $f(x, y) = e^{-x} \cos y - e^{-y} \cos x$

Working Space

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1.4.2 The Wave Equation

Another useful equation with partial derivatives is the Wave Equation:

$$\frac{\partial^2 f}{\partial t^2} = a^2 \frac{\partial^2 f}{\partial x^2}$$

Where f is a function of x and t and a is a constant. This equation describes waves, such as a vibrating string, light waves, or sound waves.

Example: Show that $f(x, t) = \sin(x - at)$ satisfies the wave equation.

Solution: First, we find the second partial derivatives:

$$\frac{\partial^2 f}{\partial t^2} = \frac{\partial}{\partial t}\left[\frac{\partial}{\partial t}(\sin(x - at))\right] = \frac{\partial}{\partial t}[-a\cos(x - at)] = -a^2 \sin(x - at)$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}(\sin(x - at))\right] = \frac{\partial}{\partial x}[\cos(x - at)] = -\sin(x - at)$$

And we see that:

$$a^2\frac{\partial^2 f}{\partial x^2} = -a^2 \sin(x - at) = \frac{\partial^2 f}{\partial t^2}$$

Therefore, this function satisfies the wave equation.

Exercise 10 The Wave Equation

Show that the following functions satisfy the wave equation:

1. $f(x, t) = \cos(kx)\cos(akt)$
2. $f(x, t) = \sin(x - at) + \ln(x + at)$
3. $f(x, t) = \frac{t}{a^2t^2 - x^2}$

Working Space

Answer on Page 83

1.4.3 Cobb-Douglas Production Function

The Cobb-Douglas function describes the marginal utility of capital and labor as theorized by the economists Charles Cobb and Paul Douglas. Capital investments are things like new machinery, expanded factories, or raw materials. Labor investments involve hiring more workers or improving working conditions to improve work rates. We can describe total production, P, as a function of labor, L, and capital, K. Cobb and Douglas posit three conditions:

1. Without either labor or capital, production will cease.
2. The marginal utility of labor is proportional to the amount of production per unit of labor.
3. The marginal utility of capital is proportional to the amount of production per unit of capital.

The marginal utility of labor is given by the partial derivative, $\partial P/\partial L$ and the production per unit of labor is given by P/L. Therefore, statement 2 says that:

$$\frac{\partial P}{\partial L} = \alpha \frac{P}{L}$$

where α is some constant. Keeping K constant at $K = K_0$, we have the differential equation:

$$\frac{dP}{dL} = \alpha \frac{P}{L}$$

Solving, we find that:

$$P(L, K_0) = C_1(K_0)\, L^{\alpha}$$

We make C_1 a function of K_0 because it could depend on K_0. In a similar manner to above, we can write statement 3 as a mathematical statement:

$$\frac{\partial P}{\partial K} = \beta \frac{P}{K}$$

where β is also a constant. Keeping $L = L_0$ and solving, we see that:

$$P(L_0, K) = C_2(L_0) K^{\beta}$$

again, we assume C_2 is a function of the fixed labor, L_0. Combining these equations, we get:

$$P(L, K) = bL^{\alpha}K^{\beta}$$

where b is a constant independent of capital and labor. Additionally, from statement 1, we know that $\alpha > 0$ and $\beta > 0$. What happens if both labor and capital are increased by a factor of n? Let's examine the effect on P:

$$P(nL, nK) = b\,(nL)^{\alpha}\,(nK)^{\beta}$$

$$P(nL, nK) = n^{\alpha+\beta} bL^{\alpha}K^{beta} = n^{\alpha+\beta}P(L, K)$$

Cobb and Douglas noted that if $\alpha + \beta = 1$, then $P(nL, nK) = nP(L, K)$, and therefore increasing labor and capital by a factor of n increases production by a factor of n as well. Therefore, the Cobb-Douglas equation assumes $\alpha + \beta = 1$ and can be written as:

$$P(L, K) = bL^{\alpha}K^{1-\alpha}$$

Exercise 11 Cobb-Douglas Production Model

Cobb and Douglas modeled production in the US from 1900 to 1922 with the equation $P(L, K) = 1.01L^{0.75}K^{0.75}$.

1. Express the marginal utility of labor as a function of L and K.
2. Express the marginal utility of capital as a function of L and K.
3. In 1916, $L = 382$ and $K = 126$ (compared to initial values of 100 in 1900). What is the marginal utility of labor in 1916? Of capital?
4. Based on your answer to the previous question, would you invest in capital or labor if you owned a factory in 1916? Why?

Working Space

Answer on Page 84

Chapter 2

Introduction to Linear Algebra

Welcome to the world of linear algebra, a branch of mathematics that relies on vectors, matrices, and linear transformations. You are familiar with most of these concepts, so in this workbook you will see how you can use them together to solve problems.

Let's review what you know.

- **Vectors**. In workbook 5, you saw how vectors can represent forces, as well as how to add and multiply them to figure out such things as rocket engine force and direction.
- **Matrices**. In workbook 8, you learned to use spreadsheets to solve problems numerically, such as how to figure out the number of barrels a cooper has to produce to achieve a certain take-home pay. Spreadsheets are essentially matrices — a row by column structure that contains values.
- **Linear transformations**. When you studied congruence in workbook 4, you were introduced to a few linear transformations, such as translation and reflection.

2.1 What's With the Linear?

You might be thinking, "Hey, haven't I been doing algebra already?"

You have! You have come a long way in your problem solving journey. You have used algebra to solve simple equations like $7x + 10 = 24$ and quadratic equations like $4x^2 + 9x + 31 = 0$. What distinguishes linear algebra is the focus on linear combinations. Any equation with a power greater than 1, such as a quadratic, is nonlinear.

We will first take a look at linear combinations

2.2 Linear Combinations

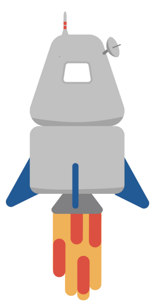
You won't see any **sin**, **cos**, or **tan** operations in this section. Linear operations do not use trigonometric functions; those operations are all nonlinear. A linear combination preserves addition and scalar multiplication. You will see that linear combinations allow you to solve many types of problems in science and engineering. Before we get deep into the numbers, let's take a look at a few linear operations you can perform on images. This will give you an intuition for the underlying math. After that, we'll take a look at some numbers.

2.3 Image Operations

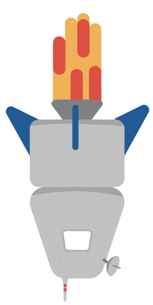
The simplest image, a bitmap, can be represented by a two-dimensional matrix of values — either 0 for black, or 1 for white. Grayscale images are also represented by a two-dimensional matrix of values, but the values typically range from 0 to 255. 0 is black, 255 is white, and the values in between represent shades of gray.

Color images are more complex. The simplest color image is a three-dimensional matrix of values. You can think of it as three 2D matrices, one to represent red values (R), another for green values (G), and the third for blue values (B). The combination of R, G, and B determines the color you see.

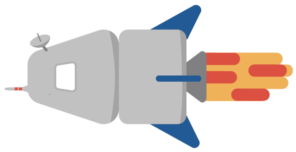
Working with images means working with millions of pixels. Fortunately, modern techniques make this a snap. Let's look at some common operations on an image of a rocket.



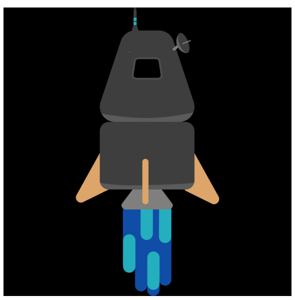
Flipping is a linear operation.



Next, the image is rotated 90 degrees. This rotation is linear, but if you want to rotate it at an angle that isn't a multiple of 90, you would need trigonometry. This would be treading into nonlinear territory, but that happens in the field of linear algebra. You will learn about nonlinear extensions later, which use trigonometric functions and imaginary numbers.



Inversion is an interesting linear operation that involves redefining the red, green, and blue values, such that the new value is 1.0 minus the old value. The resulting black background gives the impression the rocket is in deep space, don't you think?



It is possible to redefine the red, green, and blue values in many ways. Visit the NASA website and search for false color images. NASA and other scientists redefine colors to communicate such things as the amount of vegetation or water in an area, the temperatures of the sun's surface, and so on. Photographers often do this for artistic effect. For example, the image on the left was taken with an infrared camera. (This infrared is not the same as thermal kind you have likely seen before. This is the infrared that is emitted by living plants.) The image is further processed to swap channels. For example, the matrix representing red might be swapped with the matrix representing blue. The image on the right shows the image after swapping color values. All these swapping operations are linear.

2.4 The Numbers Behind Some Image Operations

You will see a few matrices in this section. Let's first look at how a spreadsheet can be represented as a matrix. Recall the barrel-making shop example. This is part of that spreadsheet.

	A	B	C	D
1	**Barrels Produced** (per month)	115	120	125
2	**Materials cost** (per barrel)	$45.00	$45.00	$45.00
3	**Sale price** (per barrel)	$100.00	$100.00	$100.00
4	**Rent** (per month)	$2,000.00	$2,000.00	$2,000.00
5	**Pretax Earnings** (per month)	$4,325.00	$4,600.00	$4,875.00
6	**Taxes** (per month)	$865.00	$920.00	$975.00
7	**Take home pay** (per month)	$3,460.00	$3,680.00	$3,900.00

Represented as a matrix, it looks like the following. Note the differences. A matrix contains only values, no labels. This matrix uses floating point values, hence the inclusion of decimal points.

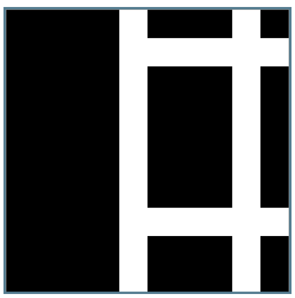
$$\begin{bmatrix} 115. & 120. & 125. \\ 45.0 & 45.0 & 45.0 \\ 100.0 & 100.0 & 100.0 \\ 2000. & 2000. & 2000. \\ 4325. & 4600. & 4875. \\ 865. & 920. & 975. \\ 3460. & 3680. & 3900. \end{bmatrix}$$

A matrix that represents an image contains only pixel values, whereas the barrel-making shop matrix represents seven kinds of variables: barrels produced, materials cost, sales price, rent, pretax earnings, taxes, and take home pay.

The simplest image to create is a bitmap, because that requires a matrix of zeros and ones. This is a matrix for a 10 pixel by 10 pixel image. Why use decimal points when this is obviously a matrix of integers? It turns out that when you use tools like Python to process matrices, you must be conscious of data types. Most of the Python methods we use for image operations expect floating points. A few expect integer types, but you'll see how to handle type conversion later, in the section on Python.

$$
\begin{bmatrix}
0. & 0. & 0. & 0. & 1. & 0. & 0. & 0. & 1. & 0 \\
0. & 0. & 0. & 0. & 1. & 1. & 1. & 1. & 1. & 1 \\
0. & 0. & 0. & 0. & 1. & 0. & 0. & 0. & 1. & 0 \\
0. & 0. & 0. & 0. & 1. & 0. & 0. & 0. & 1. & 0 \\
0. & 0. & 0. & 0. & 1. & 0. & 0. & 0. & 1. & 0 \\
0. & 0. & 0. & 0. & 1. & 0. & 0. & 0. & 1. & 0 \\
0. & 0. & 0. & 0. & 1. & 0. & 0. & 0. & 1. & 0 \\
0. & 0. & 0. & 0. & 1. & 1. & 1. & 1. & 1. & 1 \\
0. & 0. & 0. & 0. & 1. & 0. & 0. & 0. & 1. & 0 \\
0. & 0. & 0. & 0. & 1. & 0. & 0. & 0. & 1. & 0
\end{bmatrix}
$$

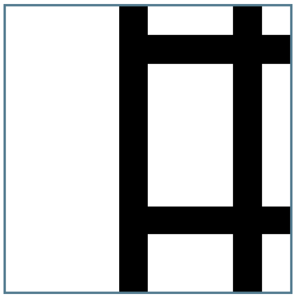
When converted to an image, it is very tiny. This is an enlarged version, so you can see the pattern.



We can create an inverse of this image by changing all the values in the matrix so that 0 becomes 1 and 1 becomes 0. (Technically this is not the way you would invert a matrix, as you will see in the Python section. For this example, you will get the same visual result, but when you start inverting matrices programmatically, you will learn the formal definition.)

$$
\begin{bmatrix}
1. & 1. & 1. & 1. & 0. & 1. & 1. & 1. & 0. & 1 \\
1. & 1. & 1. & 1. & 0. & 0. & 0. & 0. & 0. & 0 \\
1. & 1. & 1. & 1. & 0. & 1. & 1. & 1. & 0. & 1 \\
1. & 1. & 1. & 1. & 0. & 1. & 1. & 1. & 0. & 1 \\
1. & 1. & 1. & 1. & 0. & 1. & 1. & 1. & 0. & 1 \\
1. & 1. & 1. & 1. & 0. & 1. & 1. & 1. & 0. & 1 \\
1. & 1. & 1. & 1. & 0. & 1. & 1. & 1. & 0. & 1 \\
1. & 1. & 1. & 1. & 0. & 0. & 0. & 0. & 0. & 0 \\
1. & 1. & 1. & 1. & 0. & 1. & 1. & 1. & 0. & 1 \\
1. & 1. & 1. & 1. & 0. & 1. & 1. & 1. & 0. & 1
\end{bmatrix}
$$

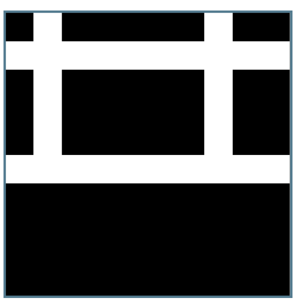
When converted to an image and enlarged, it looks like this:



Rotating the original matrix by 90 degrees gives this:

$$\begin{bmatrix} 0. & 1. & 0. & 0. & 0. & 0. & 0. & 1. & 0. & 0. \\ 1. & 1. & 1. & 1. & 1. & 1. & 1. & 1. & 1. & 1. \\ 0. & 1. & 0. & 0. & 0. & 0. & 0. & 1. & 0. & 0. \\ 0. & 1. & 0. & 0. & 0. & 0. & 0. & 1. & 0. & 0. \\ 0. & 1. & 0. & 0. & 0. & 0. & 0. & 1. & 0. & 0. \\ 1. & 1. & 1. & 1. & 1. & 1. & 1. & 1. & 1. & 1. \\ 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \end{bmatrix}$$

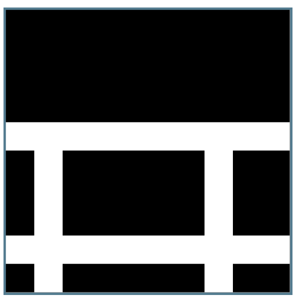
This is the resulting enlarged image:



You'll transpose many matrices in the upcoming pages. It requires swapping rows for columns.

$$\begin{bmatrix} 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\ 1. & 1. & 1. & 1. & 1. & 1. & 1. & 1. & 1. & 1. \\ 0. & 1. & 0. & 0. & 0. & 0. & 0. & 1. & 0. & 0. \\ 0. & 1. & 0. & 0. & 0. & 0. & 0. & 1. & 0. & 0. \\ 0. & 1. & 0. & 0. & 0. & 0. & 0. & 1. & 0. & 0. \\ 1. & 1. & 1. & 1. & 1. & 1. & 1. & 1. & 1. & 1. \\ 0. & 1. & 0. & 0. & 0. & 0. & 0. & 1. & 0. & 0. \end{bmatrix}$$

The resulting image looks like this:

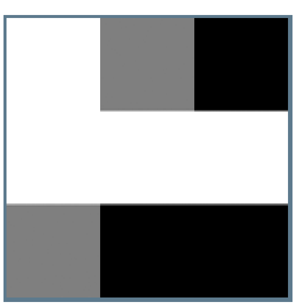


What about adding images? That fits the definition of a linear combination. Recall that grayscale images have values from 0 to 255. To make things simple, let's define two matrices with values ranging from 0.0 to 1.0. When we want a grayscale image, it is easy to multiply the matrix by 255.

Let's call this matrix f.

$$\begin{bmatrix} 1.0 & 0.5 & 0.0 \\ 1.0 & 1.0 & 1.0 \\ 0.5 & 0.0 & 0.0 \end{bmatrix}$$

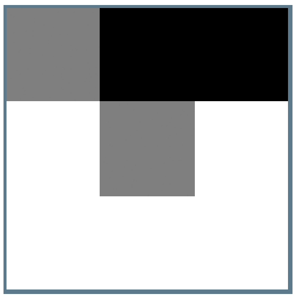
When multiplied by 255 and converted to a grayscale image:



Let's call this matrix g:

$$\begin{bmatrix} 0.5 & 0.0 & 0.0 \\ 1.0 & 0.5 & 1.0 \\ 1.0 & 1.0 & 1.0 \end{bmatrix}$$

When multiplied by 255 and converted to a grayscale image:



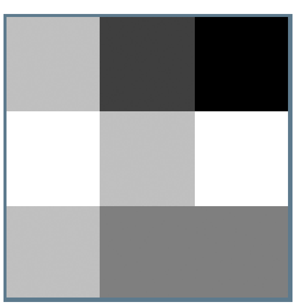
When we add f and g we get k:

$$\begin{bmatrix} 1.5 & 0.5 & 0.0 \\ 2.0 & 1.5 & 2.0 \\ 1.5 & 1.0 & 1.0 \end{bmatrix}$$

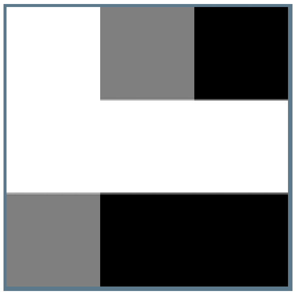
However, the values in k exceed the range of 0.0 to 1.0, so we normalize by dividing the matrix by 1.0

$$\begin{bmatrix} 0.75 & 0.25 & 0.00 \\ 1.00 & 0.75 & 1.00 \\ 0.75 & 0.50 & 0.50 \end{bmatrix}$$

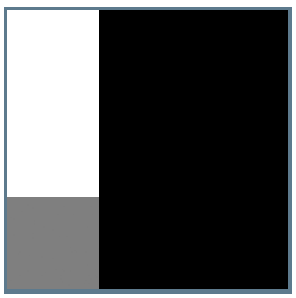
When multiplied by 255 and converted to grayscale, we get:



Let's go back to the first small grayscale image:



If you want to keep the pattern in the first column, you could multiply the matrix by a vector, [1.0 0.0 0.0]. The 1.0 will keep the values in the first column, but the 0.0 will knock out the other values because 0.0 times anything equals 0.0.



What do you think will happen if you use the vector [0.0 1.0 0.0] or [0.0 0.0 1.0]? You'll get a chance later to use Python to perform image operations.

All the operations we performed on these images satisfy the requirement for linear combinations: preserving addition and scalar multiplication.

2.5 Applications of Linear Algebra

So far you've seen how linear operations on matrices can process images by:

- multipling a matrix using a scalar (e.g., normalize, change the range)
- adding one matrix to another to get a composite image
- multipling two matrices to perform a transform (e.g., flipping)
- mulitpliing a matrix with a vector (isolating a channel)

Many areas in engineering and science rely on the matrix operations defines by linear algebra. Besides image processing, linear algebra is used for:

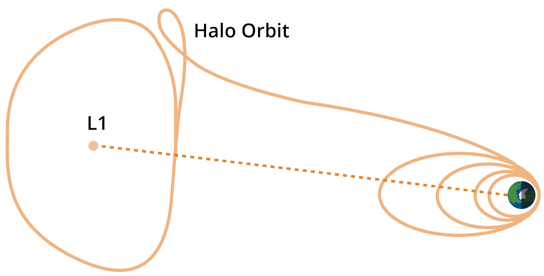
- Computer Graphics. When you play a video game or watch the latest CG animation, matrix operations transform objects in the scene to make them appear as if moving, getting closer, and so on. You can represent the vertices of objects as vectors, and then apply a transformation matrix.

- Data Analysis. We live in an era in which it's easy to collect so much data that it's difficult to make sense of the data by just looking at it. You can represent the data in matrix form and then find a solution vector. For example, scientists use this technique to figure out the effectiveness of drug treatments on disease.
- Economics. Take a look at financial section of any news organization and you'll see headlines such as "Economic Data Points to Faster Growth" or "Is the Inflation Battle Won?" Economists can use systems of linear equations to represent economic indicators, such as consumer consumption, government spending, investment rate, and gross national product. By using various methods that you'll learn about later, they can get a good idea of the state of the economy.
- Engineering. Engineers couldn't do without linear algebra. For example, the orbital dynamics of space travel relies on it. Engineers must predict and calculate the the motion of planetary bodies, satellites, and spacecraft. By solving systems of linear equations engineers can make sure that a spacecraft travels to its destination without running into a satellite or space rock.

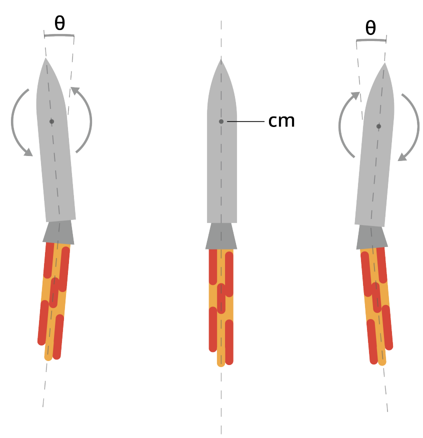
2.6 Let's Observe the Sun!

India recently sent the Aditya spacecraft on a mission to study the Sun. Without a thorough understanding of linear algebra (among other things), the engineers would not have accomplished the amazing feat of getting Aditya in a stable orbit around a Lagrange point. In previous chapters you learned about gravity and its effects.

A Lagrange point is a point in space between two bodies (e.g. Earth and Sun) where there is gravitational equilibrium. With the right trajectory, a spacecraft will orbit around a Lagrange point in a stable position that doesn't require much energy to maintain. That's called a Halo orbit. Because that there are no fueling stations in space, a Halo orbit will allow Aditya to maintain position for about 5 years. Pretty good mileage!



Aditya's engineers had to calculate a looping maneuver that would precisely inject the Aditya spacecraft into the Halo orbit. They determined the angles and burn times for the thrust engine. If they were wrong in one direction, the spacecraft would fly off to the sun. The other direction would send the spacecraft back in the direction of Earth. Their success is due to a solid understanding of vectors and linear algebra.



2.7 Images in Python

One of the wonderful things about python is the availability of libraries for specialized computation. The Python Imaging Library, PIL, is what you'll use to create images, read existing images from disk, and perform operation on images. To create and manipulate arrays, you will use NumPy.

Create a file called `image_creation.py` and enter this code:

```
# Import necessary modules
```

```
import numpy as np
import PIL
from PIL import Image
from PIL import ImageOps

# Create a 10 by 10 pixel bitmap Image.
# Using a decimal point ensure python see the values as floating point numbers
# Some image operations assume floats

bitmapArray = np.array([
[0., 0., 0., 0., 1., 0., 0., 0., 1., 0.],
[0., 0., 0., 0., 1., 1., 1., 1., 1., 1.],
[0., 0., 0., 0., 1., 0., 0., 0., 1., 0.],
[0., 0., 0., 0., 1., 0., 0., 0., 1., 0.],
[0., 0., 0., 0., 1., 0., 0., 0., 1., 0.],
[0., 0., 0., 0., 1., 0., 0., 0., 1., 0.],
[0., 0., 0., 0., 1., 0., 0., 0., 1., 0.],
[0., 0., 0., 0., 1., 1., 1., 1., 1., 1.],
[0., 0., 0., 0., 1., 0., 0., 0., 1., 0.],
[0., 0., 0., 0., 1., 0., 0., 0., 1., 0.]])

# Image.fromarray assumes a range of 0 to 255, so scale by 255

myImage = Image.fromarray(bitmapArray*255)
myImage.show()

# A window opens with an image so tiny you might think nothing is there
# Zoom in to see the pattern

# Transpose the array, create an image, and then show it.
# Note that you operate on the original array (not the image)
# Remember to zoom in to see the pattern

myImageTransposed = bitmapArray.transpose()
myImageTransposed.show()

# Invert the array. You'll use the NumPy invert method.
# The invert method assumes integer values. You need to convert the data type
# Numpy has a method for that

intBitmapArray = np.asarray(bitmapArray,dtype="int")
invertedArray = np.invert(intBitmapArray)

# Take a look at the array

invertedArray
```

```
# The values range from -2 to -1. Image values are positive.
# You need to change the range so the values are from 0 to 255
# Further you need to change back to floating point values because
# the PIL method requires them

invertedArray = (invertedArray + 2)*1.0
invertedImage = Image.fromarray(255*invertedArray)
invertedImage.show()

# Zoom in on the image and compare the pattern with the original
```

2.8 Exercise

Create a python program that creates matrix f and matrix g from the previous section, and then performs all the operations shown in that section. If you are not sure how to accomplish something, consult the online documentation for the PIL and NumPy python libraries.

Chapter 3

Vectors and Matrices

The last chapter provided an overview of linear algebra, using several image examples. In this chapter, we will focus primarily on vector-matrix multiplications. First, we will show how matrices can be used to represent a set of linear equations. Then, we will provide you with a general definition of vector-matrix multiplication, followed by a few examples. You will have an opportunity to solve a problem manually, then by using Python. In this chapter, we will use two-dimensional matrices for simplicity, but a matrix can have any number of dimensions.

3.1 Matrices

We've been looking at vectors, which are usually represented as a column of numbers. For example, while we may write $\mathbf{v} = [1, 2, 3]$ in line, the vector is really:

$$\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

A matrix can be made of many columns, like the 3×2 matrix shown below:

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 4 & 8 \\ 3 & 9 & 27 \end{bmatrix}$$

We describe the size and shape of matrices by saying *an* $m \times n$ *matrix*, where m is the number of rows and n is the number of columns. A vector is simply a one-column matrix. For example, the vectors $\mathbf{v}$ above is 3×1. Matrices aren't restricted to 2 dimensions: a matrix can be 3, 4, or any number of dimensions. For example, a $3 \times 2 \times 4$ matrix would be made of 4 stacked 3×2 matrices.

Exercise 12 Matrix Dimensions 1

Write the dimensions of the following matrices:

1. $\begin{bmatrix} -3 & 0 & 4 & -2 & -4 \\ -1 & 5 & 3 & 4 & -2 \\ -3 & 2 & 3 & -5 & 1 \end{bmatrix}$
2. $\begin{bmatrix} -3 & 1 \end{bmatrix}$
3. $\begin{bmatrix} -3 & 2 & -3 \\ 4 & 0 & -3 \\ -5 & -4 & 1 \\ 0 & -2 & 2 \end{bmatrix}$

Working Space

Answer on Page 85

Exercise 13 Matrix Dimensions 2

Create a matrix with the indicated dimensions.

1. 1×3
2. 2×4
3. 4×3

Working Space

Answer on Page 85

3.1.1 Zero Matrices

Recall that we can represent a generic zero vector as $\mathbf{0}$, which indicates a vector of any number of dimensions filled with zeros. Just like vectors, there are *zero matrices*, which can by of any number of dimensions, all filled with zeros. In two dimensions, zero matrices are denoted as $O_{m \times n}$, where the subscript is the dimension of the matrix. The subscript can be expanded to denote any number of dimensions.

3.2 Matrix Arithmetic

3.2.1 Adding and Subtracting Matrices

Matrices that are the same dimension can be added and subtracted. Just like vectors, to add matrices you add the elements in the same position:

$$\begin{bmatrix} -2 & -1 \\ 2 & 4 \end{bmatrix} + \begin{bmatrix} 5 & 2 \\ -1 & -4 \end{bmatrix} = \begin{bmatrix} -2+5 & -1+2 \\ 2+-1 & 4+-4 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & 0 \end{bmatrix}$$

And to subtract matrices, you subtract the elements in the same position:

$$\begin{bmatrix} -2 & -1 \\ 2 & 4 \end{bmatrix} - \begin{bmatrix} 5 & 2 \\ -1 & -4 \end{bmatrix} = \begin{bmatrix} -2-5 & -1-2 \\ 2-(-1) & 4-(-4) \end{bmatrix} = \begin{bmatrix} -7 & -3 \\ 3 & 8 \end{bmatrix}$$

Formally, for 2-dimensional matrices, we can say that:

Adding and Subtracting Matrics

For two $m \times n$ matrices, the sum of the matrices is the matrix of the sums of the elements in analogous positions:

$$\begin{bmatrix} x_{11} & x_{12} & x_{13} & \cdots & x_{1n} \\ x_{21} & x_{22} & x_{23} & \cdots & \vdots \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ x_{m1} & x_{m2} & x_{m3} & \cdots & x_{mn} \end{bmatrix} + \begin{bmatrix} y_{11} & y_{12} & y_{13} & \cdots & y_{1n} \\ y_{21} & y_{22} & y_{23} & \cdots & \vdots \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ y_{m1} & y_{m2} & y_{m3} & \cdots & y_{mn} \end{bmatrix} =$$

$$\begin{bmatrix} x_{11}+y_{11} & x_{12}+y_{12} & x_{13}+y_{13} & \cdots & x_{1n}+y_{1n} \\ x_{21}+y_{21} & x_{22}+y_{22} & x_{23}+y_{23} & \cdots & \vdots \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ x_{m1}+y_{m1} & x_{m2}+y_{m2} & x_{m3}+y_{m3} & \cdots & x_{mn}+y_{mn} \end{bmatrix}$$

To subtract matrices, simply add the negative of the second matrix (that is, $\boldsymbol{A}$ - $\boldsymbol{B}$ = $\boldsymbol{A}$ + - $\boldsymbol{B}$). Additionally, matrix addition is commutative ($\boldsymbol{A} + \boldsymbol{B} = \boldsymbol{B} + \boldsymbol{B}$).

Exercise 14 Adding and Subtracting Matrices

Find $A + B$, A - B, and B - A.

Working Space

1. $A = \begin{bmatrix} 0 & 4 & 0 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 3 & -2 & 5 \end{bmatrix}$
2. $A = \begin{bmatrix} 4 & -4 & -2 \\ 1 & -3 & 5 \\ -5 & 3 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 0 & -1 \\ -5 & -3 & -2 \\ -5 & 3 & -4 \end{bmatrix}$.
3. $A = \begin{bmatrix} -2 & -1 & -5 & -1 \\ 5 & -4 & 4 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} -5 & -2 & 3 & -5 \\ 0 & 5 & -4 & -3 \end{bmatrix}$.

Answer on Page 86

3.2.2 Multiplying Matrices

Surprisingly (it may be to you), matrix multiplication has dimension limits. We cannot multiply any two matrices: the first matrix must have the same number of columns as the second has number of rows. Let's examine the origin of the dimension limits on matrix multiplication. We begin with a review of the vector dot product.

Recall that in order to find the dot product of two vectors, they must be the same length (that is, the same number of dimensions). The result is always a scalar: one number. You can review finding the dot product of vectors and practice the dimension limits on the vector dot product in the next exercise.

Exercise 15 Vector Dot Product Review

Working Space

Find all possible pairs of vectors that can be used to find a dot product, then find the dot products.

1. $\mathbf{a} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$
2. $\mathbf{b} = \begin{bmatrix} -3 \\ 3 \\ 5 \\ -5 \end{bmatrix}$
3. $\mathbf{c} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$
4. $\mathbf{d} = \begin{bmatrix} -5 \\ -1 \end{bmatrix}$
5. $\mathbf{e} = \begin{bmatrix} 1 \\ -5 \\ 3 \\ 1 \end{bmatrix}$
6. $\mathbf{f} = \begin{bmatrix} 4 \\ 1 \\ -3 \end{bmatrix}$

Answer on Page 86

To multiply two matrices, it is helpful to think of the rows of the first matrix and the columns of the second matrix as vectors. Let's see how this shakes out for two 2×2 matrices:

Let's look at this more concretely. For two-dimensional matrices, it can be helpful to move your left index finger across the row and right index finger down the column, as shown in figure 3.2.

Since each entry in the product matrix is the dot product between a row of the first matrix and a column of the second matrix, the first matrix must have the same number of elements in each row as the second has in each column. Another way to say this is that

$$\begin{matrix} \mathbf{a}_1 \rightarrow \\ \mathbf{a}_2 \rightarrow \end{matrix} \begin{bmatrix} 5 & 4 \\ -5 & 1 \end{bmatrix} \cdot \overset{\mathbf{b}_1 \downarrow \quad \mathbf{b}_2 \downarrow}{\begin{bmatrix} -1 & -2 \\ -5 & -4 \end{bmatrix}} = \begin{bmatrix} \mathbf{a}_1 \cdot \mathbf{b}_1 & \mathbf{a}_1 \cdot \mathbf{b}_2 \\ \mathbf{a}_2 \cdot \mathbf{b}_1 & \mathbf{a}_2 \cdot \mathbf{b}_2 \end{bmatrix}$$

$$\boldsymbol{A} \quad \cdot \quad \boldsymbol{B}$$

Figure 3.1: Each entry in $\boldsymbol{C}$, c_{ij}, is the dot product of the i^{th} row of $\boldsymbol{A}$, a_i, and the j^{th} column of $\boldsymbol{B}$, b_j.

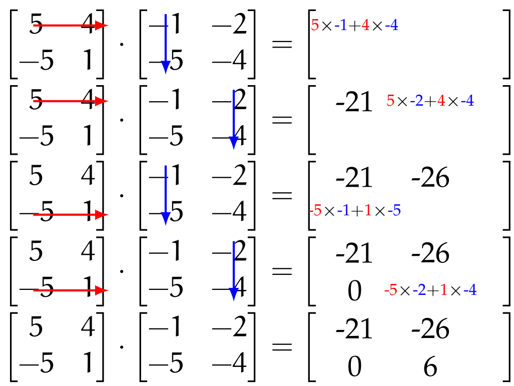


Figure 3.2: You can use your fingers to trace across matrix $\boldsymbol{A}$ and down matrix $\boldsymbol{B}$ to find $\boldsymbol{A} \cdot \boldsymbol{B}$.

the number of columns of the first matrix must match the number of rows in the second matrix.

Matrix Multiplication

For two-dimensional matrices, the inner dimensions must match in order to carry out matrix multiplication. That is, if we want to find $\boldsymbol{A} \cdot \boldsymbol{B}$, and $\boldsymbol{A}$ has dimensions $m \times n$, then $\boldsymbol{B}$ must have dimensions $n \times p$, where m, n, and p are integers. The resulting matrix will have dimensions $m \times p$ (m and p may be equal or unequal).

Exercise 16 Multiplying Matrices 1

Working Space

Multiply the matrices.

1. $\begin{bmatrix} -5 & -2 & 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} -3 & -1 & -5 \\ 3 & 0 & 3 \\ 4 & -1 & -4 \\ -1 & -4 & 2 \end{bmatrix}$

2. $\begin{bmatrix} 1 \\ 5 \\ -5 \\ 4 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 5 & 1 \end{bmatrix}$

3. $\begin{bmatrix} -1 & 4 & -4 \\ 5 & -3 & 5 \\ -1 & -4 & 4 \\ -4 & 1 & 4 \end{bmatrix} \cdot \begin{bmatrix} -3 & 5 & 1 \\ -3 & 0 & -3 \\ 0 & 3 & 0 \end{bmatrix}$

Answer on Page 86

Exercise 17 Multiplying Matrices 2

Find $A \cdot B$ and $B \cdot A$.

Working Space

1. $A = \begin{bmatrix} -2 \\ 2 \\ 1 \\ -2 \end{bmatrix}$ and $B = \begin{bmatrix} -4 & 3 & -5 & -2 \end{bmatrix}$

2. $A = \begin{bmatrix} -4 & -2 \\ 2 & 5 \\ -3 & -4 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & -2 & -4 \\ 1 & -4 & 0 \end{bmatrix}$

3. $A = \begin{bmatrix} 2 & 0 & 1 & 4 \\ -4 & 0 & -5 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & -3 \\ -4 & -1 \\ -2 & 3 \\ -5 & 1 \end{bmatrix}$

Answer on Page 86

What have you noticed about the results of $A \cdot B$ as compared to $B \cdot A$? You should have noticed that the product matrices are *different dimensions*. This leads us to the next unusual property of matrix multiplication: it is *non-commutative*. That is, the *order* in which you multiply matrices affects the result. This is very different from scalar values!

As you saw in the second matrix multiplication exercise, $\boldsymbol{A}$ is a 2×4 matrix and $\boldsymbol{B}$ is a 4×2 matrix, then $\boldsymbol{AB}$ is a 2×2 matrix, while $\boldsymbol{BA}$ is a 4×4 matrix. It is obvious, then, that $A \cdot B \neq B \cdot A$. What if $\boldsymbol{A}$ and $\boldsymbol{B}$ are square matrices?

Example: Find $A \cdot B$ and $B \cdot A$ if $A = \begin{bmatrix} -3 & 5 \\ -1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 1 \\ 4 & -3 \end{bmatrix}$.

Solution:

$$A \cdot B = \begin{bmatrix} -3 & 5 \\ -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} -1 & 1 \\ 4 & -3 \end{bmatrix} = \begin{bmatrix} -3(-1) + 5(4) & -3(1) + 5(-3) \\ -1(-1) + 0(4) & -1(1) + 0(-3) \end{bmatrix} = \begin{bmatrix} 23 & -18 \\ 1 & -1 \end{bmatrix}$$

$$B \cdot A = \begin{bmatrix} -1 & 1 \\ 4 & -3 \end{bmatrix} \cdot \begin{bmatrix} -3 & 5 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -1(-3) + 1(-1) & -1(5) + 1(0) \\ 4(-3) + -3(-1) & 4(5) + -3(0) \end{bmatrix} = \begin{bmatrix} 2 & -5 \\ -9 & 20 \end{bmatrix}$$

As you can see, even if $\boldsymbol{A}$ and $\boldsymbol{B}$ are square, matrix multiplication is still not commutative.

Non-Commutation of Matrix Multiplication

For two matrices $\boldsymbol{A}$ and $\boldsymbol{B}$, where neither is an identity matrix or a zero matrix:

$$A \cdot B \neq B \cdot A$$

Properties of the Zero Matrix

Just like the number 0, the zero matrix, $\boldsymbol{O}$ has unique mathematical properties:

Properties of the Zero Matrix

For a matrix, $\boldsymbol{A}$, and a zero matrix, $\boldsymbol{O}$

1. $A + O = A$
2. $A + -A = O$
3. $0 \cdot A = O$

The Identity Matrix

There is another special matrix, called the *identity matrix*, usually denoted with $\boldsymbol{I}$. An identity matrix is all zeroes except for a diagonal line of ones. A 3×3 identity matrix is shown below:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

All identity matrices are square (that is, they have the same number of rows as they do columns). The identity matrix has the special property that whenever a vector or matrix is multiplied by $\boldsymbol{I}$, it doesn't change. Let's look at some examples:

Example: If $\mathbf{x} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$, what is $I\mathbf{x}$? (Take $\boldsymbol{I}$ to be a 2×2 identity matrix.)

Solution:

$$I\mathbf{x} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 \cdot (2) + 0 \cdot (-3) \\ 0 \cdot (2) + 1 \cdot (-3) \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

Example: If $B = \begin{bmatrix} -2 & 5 \\ 3 & -4 \end{bmatrix}$, what is $I \cdot B$?

Solution:

$$I \cdot B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} -2 & 5 \\ 3 & -4 \end{bmatrix} = \begin{bmatrix} 1 \cdot (-2) + 0 \cdot (5) & 1 \cdot (5) + 0 \cdot (-4) \\ 0 \cdot (-2) + 1 \cdot (5) & 0 \cdot (5) + 1 \cdot (-4) \end{bmatrix} = \begin{bmatrix} -2 & 5 \\ 3 & -4 \end{bmatrix}$$

Properties of the Identity Matrix

An $n \times n$ identity matrix, $\boldsymbol{I}$, does not change any vectors or matrices it multiplies. That is:

1. $I \cdot \mathbf{x} = \mathbf{x}$
2. $I \cdot B = B$

where $\mathbf{x}$ is an $n \times 1$ vector and $\boldsymbol{B}$ is an $n \times p$ matrix (p may be, but is not necessarily, equal to n).

3.2.3 Can We Divide Matrices?

Matrices cannot be divided. Suppose we have a matrix, $\boldsymbol{A}$, a vector $\mathbf{x}$, and another vector $\mathbf{b}$ such that:

$$\boldsymbol{A} \cdot \mathbf{x} = \mathbf{b}$$

Now, if we know $\boldsymbol{A}$ and $\mathbf{x}$, it is easy to find $\mathbf{b}$. What if, on the other hand, we know $\boldsymbol{A}$ and $\mathbf{b}$ and want to find $\mathbf{x}$? We might be tempted to do something like this:

$$\mathbf{x} = \frac{\mathbf{b}}{\boldsymbol{A}}$$

While this would be correct if $\mathbf{x}$, $\mathbf{b}$, and $\boldsymbol{A}$ were scalars, but it is not for matrices. However, there is an analogy we can make. Instead of trying to divide by $\boldsymbol{A}$, we can multiply by its *inverse*:

Inverse Matrices

Given a matrix $\boldsymbol{A}$, and vectors $\mathbf{b}$ and $\mathbf{x}$, if

$$A \cdot \mathbf{x} = \mathbf{b}$$

Then,

$$\mathbf{x} = A^{-1} \cdot \mathbf{b}$$

A^{-1} is called the *inverse matrix*. We will explore inverse matrices and how to find them in the next chapter.

Chapter 4

Linear Combinations of Vectors

In the introductory linear algebra chapter, you learned that vectors and matrices can be rotated, inverted, and added. In this chapter, we will explore linear combinations of vectors and the span of group of vectors. The **span** of a group of vectors is the set of vectors that can be made with linear combinations of the original group of vectors. We will offer mathematical and visual explanations later in the chapter. First, let's examine linear combinations.

A linear combination is simply the addition of vectors with leading scalar multipliers. For example, $3\,[2,-1]+2\,[3,5]$ is a linear combination of the vectors $[2,-1]$ and $[3,5]$. Another way to say this is:

Linear Combination of Vectors

A linear combination of a list of n vectors, $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}$ takes the form:

$$a_1\mathbf{v_1} + a_2\mathbf{v_2} + \cdots + a_n\mathbf{v_n}$$

where $a_1, a_2, \cdots, a_n \in \mathbb{R}$

Example: Find a linear combination of $[2,1,-3]$ and $[1,-2,4]$ that gives the vector $[17,-4,2]$.

Solution: We are looking for a_1 and a_2 such that:

$$a_1\,[2,1,-3] + a_2\,[1,-2,4] = [17,-4,2]$$

Looking at each dimension separately, we get the system of equations:

$$2a_1 + 1a_2 = 17$$

$$1a_1 - 2a_2 = -4$$

$$-3a_1 + 4a_2 = 2$$

If we can solve this system of equations, we will find a_1 and a_2. Let's multiply the first equation by 2 and add it to the second equation:

$$2\,[2a_1 + a_2] + [a_1 - 2a_2] = 2\,(17) + -4$$

$$4a_1 + 2a_2 + a_1 - 2a_2 = 34 - 4$$
$$5a_1 = 30$$
$$a_1 = 6$$

Now we can take a_1 and substitute it back into any equation in our system to find a_2. Let's use the third equation:

$$-3(6) + 4a_2 = 2$$
$$-18 + 4a_2 = 2$$
$$4a_2 = 20$$
$$a_2 = 5$$

Since we used all 3 equations, we know $a_1 = 6$ and $a_2 = 5$ are solutions to all 3 equations. If we had only used the first two equations to find a_1 and a_2, we would want to substitute our values back into the third equation to make sure our solution holds for that equation also.

Therefore, $6[2, 1, -3] + 5[1, -2, 4] = [17, -4, 2]$.

Exercise 18 Linear Combinations

Find a linear combination of the first two vectors that yields the third vector.

1. $[1, 2]$, $[-3, 1]$, $[4, 5]$
2. $[9, 4]$, $[0, 1]$, $[-5, 3]$
3. $[7, -2]$, $[-8, 4]$, $[6, -2]$

Working Space

Answer on Page 87

Sometimes, a set of vectors cannot be combined to make a specific vector. Take the pair of vectors we have looked at before: $[2, 1, -3]$ and $[1, -2, 4]$. Can we find a combination to make vector $[17, -4, 5]$? Let's try. We define a_1 and a_2 such that:

$$a_1 [2, 1, -3] + a_2 [1, -2, 4] = [17, -4, 5]$$

Which creates the system of equations:

$$2a_1 + a_2 = 17$$

$$a_1 - 2a_2 = -4$$

$$-3a_1 + 4a_2 = 5$$

We have two variables (a_1 and a_2) and three equations. Let's use the first two to find a_1 and a_2, then check our answers by substituting our solutions into the third equation. First, we'll multiply the second equation by -2 and add that to the first equation:

$$2a_1 + a_2 + (-2)(a_1 - 2a_2) = 17 + (-2)(-4)$$

$$2a_1 + a_2 - 2a_1 + 4a_2 = 17 + 8$$
$$5a_2 = 25$$
$$a_2 = 5$$

Substituting for a_2 back into the first equation and solving for a_1:

$$2a_1 + 5 = 17$$
$$2a_1 = 12$$
$$a_1 = 6$$

Now, let's check if $a_1 = 6$, $a_2 = 5$ is a solution to the third equation:

$$-3(6) + 4(5) = 5$$
$$-18 + 20 = 2 \neq 5$$

Therefore, there is no linear combination of the vectors $[2, 1, -3]$ and $[1, -2, 4]$ that yields $[17, -4, 5]$.

4.1 Visualizing Linear Combinations

First, let's look at what vectors can be made from linear combinations of the 2-dimensional unit vectors $\mathbf{i} = [1, 0]$ and $\mathbf{j} = [0, 1]$. Suppose we are looking for a linear combination of $\mathbf{i}$ and $\mathbf{j}$ to create the vector $[3, -4]$. We can find such a linear combination:

$$3\mathbf{i} + (-4)\mathbf{j} = 3[1, 0] - 4[0, 1] = [3, -4]$$

In fact, with $\mathbf{i}$ and $\mathbf{j}$, we can create any 2-dimensional vector. To prove this, consider a generic vector, $\mathbf{z} = [a, b]$, where $a, b \in \mathbb{R}$. We are looking for a linear combination of $\mathbf{i}$ and $\mathbf{j}$ such that:

$$c_1\mathbf{i} + c_2\mathbf{j} = [a, b]$$

The above equation yields the system of equations:

$$c_1(1) + c_2(0) = a$$
$$c_1(0) + c_2(1) = b$$

And the solution to this system of equations is:

$$c_1 = a$$

$$c_2 = b$$

Therefore, using **i** and **j**, we can construct any vector in $\mathbb{R}^2$ (that is, any vector in the xy-plane). What about combinations of other vectors?

Let's consider linear combinations of two vectors: $\mathbf{u} = [1, 2]$ and $\mathbf{v} = [-2, 1]$. The vectors are shown in figure 4.1.

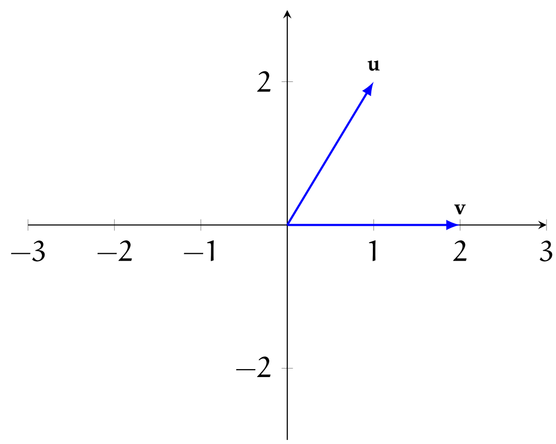


Figure 4.1: The vectors $\mathbf{u} = [1, 2]$ and $\mathbf{v} = [2, 0]$.

Suppose we want to construct the vector $[-1, 4]$. Since only **u** has value in the y-dimension, we can start by adding **u** vectors to reach $y = 4$ (see figure 4.2). Next, we can use **v** vectors to reach $[-1, 4]$ (see figure 4.3).

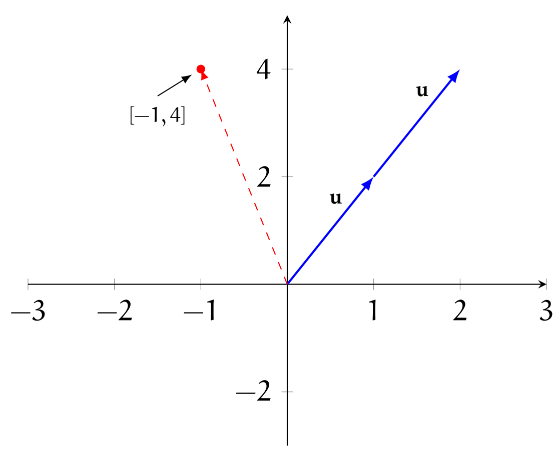


Figure 4.2: To create vector $[4, -2]$ with $\mathbf{u} = [1, 2]$ and $\mathbf{v} = [2, 0]$, we begin by adding two **u** vectors to reach a y-value of 4.

Using this method, we can imagine reaching any point in $\mathbb{R}^2$: we add or subtract as many

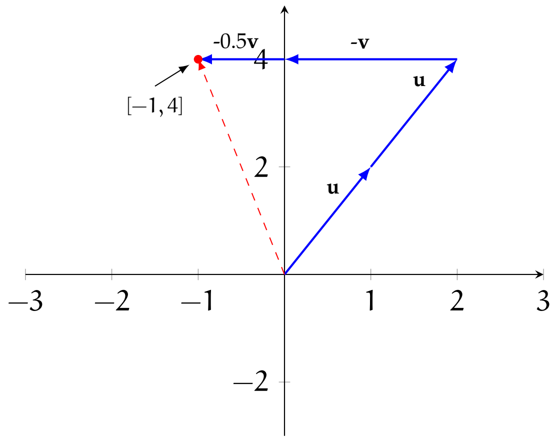


Figure 4.3: If $\mathbf{u} = [1, 2]$ and $\mathbf{v} = [2, 0]$, then $2\mathbf{u} - 1.5\mathbf{v} = [-1, 4]$.

u vectors as needed to reach the appropriate y-value, then add or subtract as many **v** vectors to reach the appropriate x-value.

Let's look at another pair of vectors: $\mathbf{p} = [2, 2]$ and $\mathbf{q} = [-1, -1]$ (see figure 4.4). Again, let's try to use **p** and **q** to construct the vector $[-1, 4]$. We begin by using **p** to reach the y-value of 4 (see figure 4.5).

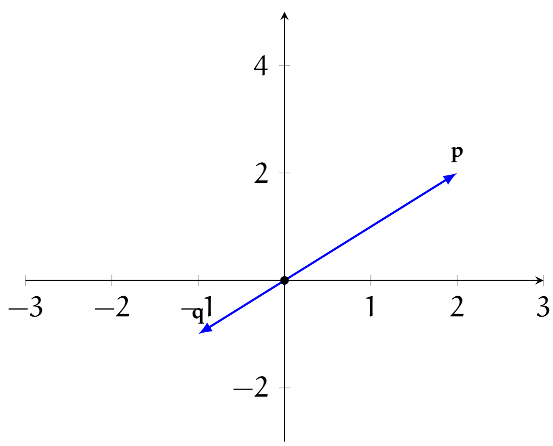


Figure 4.4: The vectors $\mathbf{p} = [2, 2]$ and $\mathbf{q} = [-1, -1]$.

But now we run into a problem: no matter how many **q** vectors we add or subtract, we just move along the **p** vector and never reach our goal of $[-1, 4]$ (see figure 4.6). Notice that **p** and **q** lie on the same line (for a better visualization, refer back to 4.4). When two vectors lie on the same line, we call them *linearly dependent*.

4.2 Linearly Dependent Vectors

Two vectors are linearly dependent if one is a multiple of the other. Mathematically,

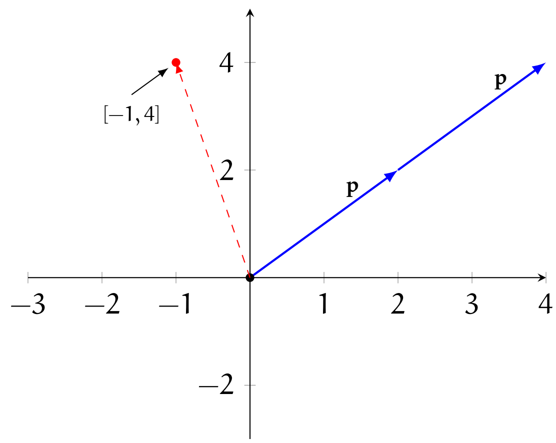


Figure 4.5: Adding 2 **p** vectors gets us to the appropriate y-value of 4.

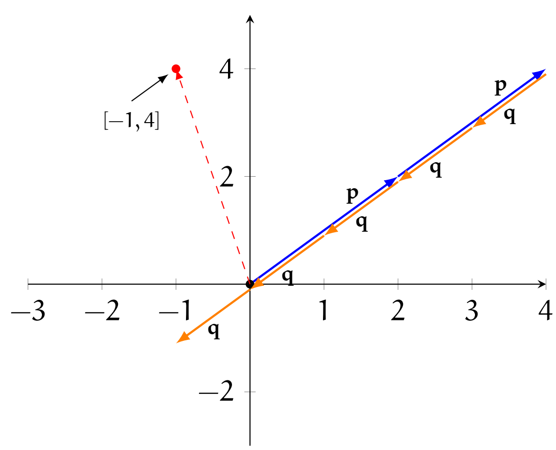


Figure 4.6: There is no linear combination of $\mathbf{p} = [2, 2]$ and $\mathbf{q} = [-1, -1]$ that yields the vector $[-1, 4]$.

Linearly dependent vectors

Vectors $\mathbf{u} = [u_1, u_2, \ldots, u_n]$ and $\mathbf{v} = [v_1, v_2, \ldots, v_n]$ are linearly dependent if

$$\mathbf{v} = a\mathbf{u}$$

Where $a \in \mathbb{R}$ is a constant.

If two vectors are linearly dependent, then linear combinations of those vectors can only create vectors that lie on the same line as the vectors. If two vectors are *not* linearly dependent, then linear combinations of those vectors can create any vector in $\mathbb{R}^2$ (for two dimensions, we will discuss higher dimensions in the next chapter).

Example: Which of the following 3 vectors are linearly dependent, if any? $\mathbf{u} = [1, 2, 3]$, $\mathbf{v} = [-3, 4, -1]$, $\mathbf{w} = [6, -8, 2]$.

Solution: Two vectors are linearly dependent if one is a scalar multiple of the other. Let's compare **u** and **v**. Since the first component of **u** is 1 and the first component of **v** is -3, let's multiply **u** by -3 to see if we get **v**:

$$-3\mathbf{u} = -3\,[1, 2, 3,] = [-3, -6, -9] \neq \mathbf{v}$$

Therefore, **u** and **v** are *not* linearly dependent. Now let's examine **v** and **w**. Again, we will use the first components: the first component of **w** is 6, so let's see if multiplying **v** by -2 yields **w**:

$$-2\mathbf{v} = -2\,[-3, 4, -1] = [6, -8, 2] = \mathbf{w}$$

Therefore, **v** and **w** are linearly dependent. Since we already know that **u** and **v** are not linearly dependent, we also know that **u** and **w** are also not linearly dependent.

Exercise 19 Linear Dependence

Identify which, if any, of the following vectors are linearly dependent:

1. $\mathbf{a} = [-4, 1, 4]$
2. $\mathbf{b} = [-4, 5, -3]$
3. $\mathbf{c} = [2, -4, 6]$
4. $\mathbf{d} = \left[1, -\frac{1}{4}, -1\right]$
5. $\mathbf{e} = [1, -2, 3]$
6. $\mathbf{f} = \left[-6, \frac{3}{2}, 6\right]$

Working Space

Answer on Page 89

Appendix A

Answers to Exercises

Answer to Exercise 1 (on page 7)

1. $f_x(x,y) = \frac{\partial}{\partial x}\left[3x^4 + 4x^2y^3\right] = 12x^3 + 8y^3$ and $f_y(x,y) = \frac{\partial}{\partial y}\left[3x^4 + 4x^2y^3\right] = 12x^2y^2$
2. $f_x(x,y) = \frac{\partial}{\partial x}\left(xe^{-y}\right) = e^{-y}$ and $f_y(x,y) = \frac{\partial}{\partial y}\left(xe^{-y}\right) = -xe^{-y}$
3. $f_x(x,y) = \frac{\partial}{\partial x}\sqrt{3x+4y^2} = \left(\frac{1}{2\sqrt{3x+4y^2}}\right)\left(\frac{\partial}{\partial x}\left(3x+4y^2\right)\right) = \frac{3}{2\sqrt{3x+4y^2}}$ and $f)y(x,y) = \frac{\partial}{\partial y}\sqrt{3x+4y^2} = \frac{1}{2\sqrt{3x+4y^2}}\left(\frac{\partial}{\partial y}\left(3x+4y^2\right)\right) = \frac{8y}{2\sqrt{3x+4y^2}} = \frac{4y}{\sqrt{3x+4y^2}}$
4. $f_x(x,y) = \frac{\partial}{\partial x}\sin\left(x^2y\right) = \cos\left(x^2y\right)\left(\frac{\partial}{\partial x}\left(x^2y\right)\right) = 2xy\cos\left(x^2y\right)$ and $f_y(x,y) = \frac{\partial}{\partial y}\sin\left(x^2y\right) = \cos\left(x^2y\right)\left(\frac{\partial}{\partial y}\left(x^2y\right)\right) = x^2\cos\left(x^2y\right)$
5. $f_x(x,y) = \frac{\partial}{\partial x}\ln\left(x^y\right) = \frac{\partial}{\partial x}\left(y\ln x\right) = \frac{y}{x}$ and $f_y(x,y) = \frac{\partial}{\partial y}\left(y\ln x\right) = \ln x$

Answer to Exercise 2 (on page 9)

1. Finding f_x:

$$f_x = \frac{\partial}{\partial x}\left[\sin\left(x^2-y^2\right)\cos\left(\sqrt{z}\right)\right] = \cos\left(x^2-y^2\right)\cos\left(\sqrt{z}\right)\left[\frac{\partial}{\partial x}\left(x^2-y^2\right)\right]$$

$$f_x = 2x\cos\left(x^2-y^2\right)\cos\left(\sqrt{z}\right)$$

Finding f_y:

$$f_y = \frac{\partial}{\partial y}\left[\sin\left(x^2-y^2\right)\cos\left(\sqrt{z}\right)\right] = \cos\left(x^2-y^2\right)\cos\left(\sqrt{z}\right)\left[\frac{\partial}{\partial y}\left(x^2-y^2\right)\right]$$

$$f_y = -2y\cos\left(x^2-y^2\right)\cos\left(\sqrt{z}\right)$$

Finding f_z:

$$f_z = \frac{\partial}{\partial z}\left[\sin\left(x^2-y^2\right)\cos\left(\sqrt{z}\right)\right] = \sin\left(x^2-y^2\right)\left(-\sin\sqrt{z}\right)\cdot\left(\frac{\partial}{\partial z}\sqrt{z}\right)$$

$$f_z = \frac{-\sin\left(x^2 - y^2\right)\sin\left(\sqrt{z}\right)}{2\sqrt{z}}$$

2. Finding q_t:

$$q_t = \frac{\partial}{\partial t}\sqrt[3]{t^3 + u^3\sin(5v)} = \frac{1}{3\left(t^3 + u^3\sin(5v)\right)^{2/3}}\left(\frac{\partial}{\partial t}\left(t^3 + u^3\sin(5v)\right)\right)$$

$$q_t = \frac{t^2}{\left(t^3 + u^3\sin(5v)\right)^{2/3}}$$

Finding q_u:

$$q_u = \frac{\partial}{\partial u}\sqrt[3]{t^3 + u^3\sin(5v)} = \frac{1}{3\left(t^3 + u^3\sin(5v)\right)^{2/3}}\left(\frac{\partial}{\partial u}\left(t^3 + u^3\sin(5v)\right)\right)$$

$$q_u = \frac{u^2\sin(5v)}{\left(t^3 + u^3\sin(5v)\right)^{2/3}}$$

Finding q_v:

$$q_v = \frac{\partial}{\partial v}\sqrt[3]{t^3 + u^3\sin(5v)} = \frac{1}{3\left(t^3 + u^3\sin(5v)\right)^{2/3}}\left(\frac{\partial}{\partial v}\left(t^3 + u^3\sin(5v)\right)\right)$$

$$q_v = \frac{u^3\cos(5v)}{3\left(t^3 + u^3\sin(5v)\right)^{2/3}}\left(\frac{\partial}{\partial v}(5v)\right) = \frac{5u^3\cos(5v)}{3\left(t^3 + u^3\sin(5v)\right)^{2/3}}$$

3. Finding w_x:

$$w_x = \frac{\partial}{\partial x}\left(x^z y^x\right) = \left(x^z\right)\cdot\left(\frac{\partial}{\partial x}y^x\right) + \left(y^x\right)\cdot\left(\frac{\partial}{\partial x}x^z\right)$$

$$w_x = \left(x^z\right)\left(\ln(y)y^x\right) + \left(y^x\right)\left(zx^{z-1}\right) = \left(x^{z-1}y^x\right)\left(x\ln(y) + z\right)$$

Finding w_y:

$$w_y = \frac{\partial}{\partial y}\left(x^z y^x\right) = \left(x^z\right)\left(\frac{\partial}{\partial y}y^x\right) = x^z\left(xy^{x-1}\right)$$

$$w_y = x^{z+1}y^{x-1}$$

Finding w_z:

$$w_z = \frac{\partial}{\partial z}\left(x^z y^x\right) = \left(y^x\right)\left(\frac{\partial}{\partial z}x^z\right) = \left(y^x\right)\left(\ln(x)x^z\right)$$

$$w_z = \ln(x)y^x x^z$$

Answer to Exercise 3 (on page 11)

1. $f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x} f(x,y)\right) = \frac{\partial}{\partial y}\left[\frac{\partial}{\partial x}\left(e^{2xy}\sin x\right)\right] = \frac{\partial}{\partial y}\left[\left(e^{2xy}\right)\left(\frac{\partial}{\partial x}\sin x\right) + (\sin x)\left(\frac{\partial}{\partial x}e^{2xy}\right)\right] = \frac{\partial}{\partial y}\left[e^{2xy}\cos x + 2ye^{2xy}\sin x\right] = \frac{\partial}{\partial y}\left(e^{2xy}\cos x\right) + \frac{\partial}{\partial y}\left(2ye^{2xy}\sin x\right) = 2xe^{2xy}\cos x + (2y)\left(\frac{\partial}{\partial y}e^{2xy}\sin x\right) + \left(e^{2xy}\sin x\right)\left(\frac{\partial}{\partial y}2y\right) = 2xe^{2xy}\cos x + 4xye^{2xy}\sin x + 2e^{2xy}\sin x$

   $f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y} f(x,y)\right) = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial y}\left(e^{2xy}\sin x\right)\right] = \frac{\partial}{\partial x}\left(2xe^{2xy}\sin x\right) = (2x)\left[\frac{\partial}{\partial x}\left(e^{2xy}\sin x\right)\right] + \left(e^{2xy}\sin x\right)\left(\frac{\partial}{\partial x}2x\right) = (2x)\left[\left(e^{2xy}\right)\left(\frac{\partial}{\partial x}\sin x\right) + (\sin x)\left(\frac{\partial}{\partial x}e^{2xy}\right)\right] + 2e^{2xy}\sin x = 2xe^{2xy}\cos x + 4xye^{2xy}\sin x + 2e^{2xy}\sin x = f_{xy}$

2. $f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x} f(x,y)\right) = \frac{\partial}{\partial y}\left[\frac{\partial}{\partial x}\left(\frac{x^2}{x+y}\right)\right] = \frac{\partial}{\partial y}\left[\frac{(x+y)(2x)-x^2(1)}{(x+y)^2}\right] = \frac{\partial}{\partial y}\left[\frac{x^2+2xy}{(x+y)^2}\right] = \frac{(x+y)^2(2x)-(x^2+2xy)(2(x+y))}{(x+y)^4} = \frac{(x^2+2xy+y^2)(2x)-(x^2+2xy)(2x+2y)}{(x+y)^4} = \frac{2x^3+4x^2y+2xy^2-2x^3-2x^2y-4x^2y-4xy^2}{(x+y)^4} = \frac{-2x^2y-2xy^2}{(x+y)^4}$

   $f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y} f(x,y)\right) = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial y}\left(\frac{x^2}{x+y}\right)\right] = \frac{\partial}{\partial x}\left[\frac{-x^2}{(x+y)^2}\right] = \frac{(x+y)^2(-2x)-(-x^2)(2(x+y))}{(x+y)^4} = \frac{(x^2+2xy+y^2)(-2x)+x^2(2x+2y)}{(x+y)^4} = \frac{-2x^3-4x^2y-2xy^2+2x^3+2x^2y}{(x+y)^4} = \frac{-2x^2y-2xy^2}{(x+y)^4} = f_{xy}$

3. $f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x} f(x,y)\right) = \frac{\partial}{\partial y}\left[\frac{\partial}{\partial x}\left(\ln(2x+3y)\right)\right] = \frac{\partial}{\partial y}\left[\frac{2}{2x+3y}\right] = \frac{-2(3)}{(2x+3y)^2} = \frac{-6}{(2x+3y)^2}$

   $f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y} f(x,y)\right) = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial y}\left(\ln(2x+3y)\right)\right] = \frac{\partial}{\partial x}\left(\frac{3}{2x+3y}\right) = \frac{-3(2)}{(2x+3y)^2} = \frac{-6}{(2x+3y)} = f_{xy}$

Answer to Exercise 4 (on page 12)

1. $f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x} f(x,y)\right) = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}\left(x^5y^2 - 3x^3y^2\right)\right] = \frac{\partial}{\partial x}\left(5x^4y^2 - 9x^2y^2\right) = 20x^3y^2 - 18xy^2.$

   $f_{xy} = f_{yx} = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x} f(x,y)\right) = \frac{\partial}{\partial y}\left[\frac{\partial}{\partial x}\left(x^5y^2 - 3x^3y^2\right)\right] = \frac{\partial}{\partial y}\left(5x^4y^2 - 9x^2y^2\right) = 10x^4y - 18x^2y.$

   $f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial y} f(x,y)\right) = \frac{\partial}{\partial y}\left[\frac{\partial}{\partial y}\left(x^5y^2 - 3x^3y^2\right)\right] = \frac{\partial}{\partial y}\left(2x^5y - 6x^3y\right) = 2x^5 - 6x^3.$

2. $v_{pp} = \frac{\partial}{\partial p}\left(\frac{\partial}{\partial p} v(p,q)\right) = \frac{\partial}{\partial p}\left[\frac{\partial}{\partial p}\left(\sin\left(p^3+q^2\right)\right)\right] = \frac{\partial}{\partial p}\left(\cos\left(p^3+q^2\right)\left(3p^2\right)\right) = \cos\left(p^3+q^2\right)\cdot \frac{\partial}{\partial p}\left(3p^2\right) + 3p^2 \cdot \frac{\partial}{\partial p}\left(\cos\left(p^3+q^2\right)\right)$

   $v_{pq} = v_{qp} = \frac{\partial}{\partial q}\left(\frac{\partial}{\partial p} v(p,q)\right) = \frac{\partial}{\partial q}\left[\frac{\partial}{\partial p}\left(\sin\left(p^3+q^2\right)\right)\right] = \frac{\partial}{\partial q}\left(\cos\left(p^3+q^2\right)\left(3p^2\right)\right) = \cos\left(p^3+q^2\right)\frac{\partial}{\partial q}\left(3p^2\right) + 3p^2\frac{\partial}{\partial q}\cos\left(p^3+q^2\right) = 0 + 3p^2\left(-\sin\left(p^3+q^2\right)\right)\left(\frac{\partial}{\partial q}\left(p^3+q^2\right)\right) = -6p^2q\sin\left(p^3+q^2\right)$

   $v_{qq} = \frac{\partial}{\partial q}\left(\frac{\partial}{\partial q} v(p,q)\right) = \frac{\partial}{\partial q}\left[\frac{\partial}{\partial q}\left(\sin\left(p^3+q^2\right)\right)\right] = \frac{\partial}{\partial q}\left[2q\cos\left(p^3+q^2\right)\right] = 2q\left[\frac{\partial}{\partial q}\cos\left(p^3+q^2\right)\right] + \cos\left(p^3+q^2\right)\left[\frac{\partial}{\partial q}(2q)\right] = (2q)\cdot\left[-2q\sin\left(p^3+q^2\right)\right] + 2\cos\left(p^3+q^2\right) = 2\cos\left(p^3+q^2\right) - 4q^2\sin\left(p^3+q^2\right)$

3. $T_{rr} = \frac{\partial}{\partial r}\left(\frac{\partial}{\partial r} T(r,\theta)\right) = \frac{\partial}{\partial r}\left[\frac{\partial}{\partial r}\left(e^{-3r}\cos\theta^2\right)\right] = \frac{\partial}{\partial r}\left(-3e^{-3r}\cos\theta^2\right) = 9e^{-3r}\cos\theta^2$

   $T_{\theta r} = T_{r\theta} = \frac{\partial}{\partial \theta}\left(\frac{\partial}{\partial r} T(r,\theta)\right) = \frac{\partial}{\partial \theta}\left[-3e^{-3r}\cos\theta^2\right] = 3re^{-3r}\sin\theta^2\left(\frac{\partial}{\partial \theta}\theta^2\right) = 6r\theta e^{-3r}\sin\theta^2$

   $T_{\theta\theta} = \frac{\partial}{\partial \theta}\left(\frac{\partial}{\partial \theta} T(r,\theta)\right) = \frac{\partial}{\partial \theta}\left[\frac{\partial}{\partial \theta}\left(e^{-3r}\cos\theta^2\right)\right] = \frac{\partial}{\partial \theta}\left[-e^{-3r}\sin\theta^2\left(\frac{\partial}{\partial \theta}\theta^2\right)\right] = \frac{\partial}{\partial \theta}\left(-2\theta e^{-3r}\sin\theta^2\right) =$

$\left(-2\theta e^{-3r}\right)\left(\frac{\partial}{\partial\theta}\sin\theta^2\right)+\left(\sin\theta^2\right)\left[\frac{\partial}{\partial\theta}\left(-2\theta e^{-3r}\right)\right] = \left(-2\theta e^{-3r}\right)\left(\cos\theta^2\right)\left(\frac{\partial}{\partial\theta}\theta^2\right)+\left(\sin\theta^2\right)\left(-2e^{-3r}\right) = -4\theta^2 e^{-3r}\cos\theta^2 - 2e^{-3r}\sin\theta^2$

Answer to Exercise 5 (on page 15)

1. We are looking for dz/dt only:

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$= \frac{\partial}{\partial x}[\sin x \cos y] \cdot \frac{d}{dt}\left[3\sqrt{t}\right] + \frac{\partial}{\partial y}[\sin x \cos y] \cdot \frac{d}{dt}[2/t]$$

$$= (\cos x \cos y) \cdot \left(\frac{3}{2\sqrt{t}}\right) + (-\sin x \sin y) \cdot \left(-\frac{2}{t^2}\right)$$

$$= \frac{3\cos x \cos y}{2\sqrt{t}} + \frac{2\sin x \sin y}{t^2}$$

Substituting for x and y:

$$\frac{dz}{dt} = \frac{3\cos\left(3\sqrt{t}\right)\cos(2/t)}{2\sqrt{t}} + \frac{2\sin\left(3\sqrt{t}\right)\sin(2/t)}{t^2}$$

2. We are looking for dz/dt only:

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$= \frac{\partial}{\partial x}\left[\sqrt{1+xy}\right] \cdot \frac{d}{dt}[\tan t] + \frac{\partial}{\partial y}\left[\sqrt{1+xy}\right] \cdot \frac{d}{dt}[\arctan t]$$

$$= \left(\frac{y}{2\sqrt{1+xy}}\right) \cdot \left(\sec^2 t\right) + \left(\frac{x}{2\sqrt{1+xy}}\right) \cdot \left(\frac{1}{t^2+1}\right)$$

Substituting for x and y:

$$\frac{dz}{dt} = \frac{\tan t \sec^2 t}{2\sqrt{1+\tan t \arctan t}} + \frac{\tan t}{2\sqrt{1+\tan t \arctan t}\,(t^2+1)}$$

$$= \frac{\tan t}{2\sqrt{1+\tan t \arctan t}}\left(\sec^2 t + \frac{1}{t^2+1}\right)$$

3. Finding $\partial z/\partial s$:

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

$$= \frac{\partial}{\partial x}\left[\arctan\left(x^2+y^2\right)\right] \cdot \frac{\partial}{\partial s}[t\ln s] + \frac{\partial}{\partial y}\left[\arctan\left(x^2+y^2\right)\right] \cdot \frac{\partial}{\partial s}\left[se^t\right]$$

$$\left(\frac{2x}{\left(x^2+y^2\right)^2+1}\right)\cdot\left(\frac{t}{s}\right)+\left(\frac{2y}{\left(x^2+y^2\right)^2+1}\right)\cdot\left(e^t\right)$$

Substituting for x and y:

$$\frac{\partial z}{\partial s}=\left(\frac{2\left(t\ln s\right)}{\left[\left(t\ln s\right)^2+\left(se^t\right)^2\right]^2+1}\right)\cdot\left(\frac{t}{s}\right)+\left(\frac{2\left(se^t\right)}{\left[\left(t\ln s\right)^2+\left(t\ln s\right)^2\right]^2+1}\right)\cdot\left(e^t\right)$$

$$=\left(\frac{2}{\left[t^2\left(\ln s\right)^2+se^{2t}\right]^2+1}\right)\cdot\left(\frac{t^2\ln s}{s}+se^{2t}\right)$$

Finding $\partial z/\partial t$:

$$\frac{\partial z}{\partial t}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

$$=\frac{\partial}{\partial x}\left(\arctan\left(x^2+y^2\right)\right)\cdot\frac{\partial}{\partial t}\left(t\ln s\right)+\frac{\partial}{\partial y}\left(\arctan\left(x^2+y^2\right)\right)\cdot\frac{\partial}{\partial t}\left(se^t\right)$$

$$=\left(\frac{2x}{\left(x^2+y^2\right)^2+1}\right)\cdot\left(\ln s\right)+\left(\frac{2y}{\left(x^2+y^2\right)^2+1}\right)\cdot\left(se^t\right)$$

Substituting for x and y:

$$\frac{\partial z}{\partial t}=\left(\frac{2}{\left[\left(t\ln s\right)^2+\left(se^t\right)^2\right]^2+1}\right)\cdot\left[t\left(\ln s\right)^2+s^2e^{2t}\right]$$

4. Finding $\partial z/\partial s$:

$$\frac{\partial z}{\partial s}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

$$=\frac{\partial}{\partial x}\left(\sqrt{x}e^{xy}\right)\cdot\frac{\partial}{\partial s}\left(1+st\right)+\frac{\partial}{\partial y}\left(\sqrt{x}e^{xy}\right)\cdot\frac{\partial}{\partial s}\left(s^2-t^2\right)$$

$$=\left[\frac{e^{xy}\left(2xy+1\right)}{2\sqrt{x}}\right]\cdot\left(t\right)+\left[x^{3/2}e^{xy}\right]\cdot\left(2s\right)$$

$$=\left[\frac{e^{xy}}{\sqrt{x}}\right]\cdot\left(\frac{\left(2xy+1\right)t}{2}+x^2\left(2s\right)\right)$$

Substituting for x and y:

$$\frac{\partial z}{\partial s}=\left[\frac{e^{(1+st)(s^2-t^2)}}{\sqrt{1+st}}\right]\cdot\left(\frac{\left(2(1+st)(s^2-t^2)+1\right)t}{2}+\left(1+st\right)^2\left(2s\right)\right)$$

Finding $\partial z/\partial t$:

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

$$= \frac{\partial}{\partial y}\left[\sqrt{x}e^{xy}\right] \cdot \frac{\partial}{\partial t}\left[1+st\right] + \frac{\partial}{\partial y}\left[\sqrt{x}e^{xy}\right] \cdot \frac{\partial}{\partial t}\left[s^2 - t^2\right]$$

$$= \left[\frac{e^{xy}\left(2xy+1\right)}{2\sqrt{x}}\right] \cdot (s) + \left[x^{3/2}e^{xy}\right] \cdot (-2t)$$

$$= \left[\frac{e^{xy}}{\sqrt{x}}\right] \cdot \left[\frac{\left(2xy+1\right)s}{2} - 2tx^2\right]$$

Substituting for x and y:

$$\frac{\partial z}{\partial t} = \left[\frac{e^{(1+st)\left(s^2-t^2\right)}}{\sqrt{(1+st)}}\right] \cdot \left[\frac{\left(2\left(1+st\right)\left(s^2-t^2\right)+1\right)s}{2} - 2t\left(1+st\right)^2\right]$$

Answer to Exercise 6 (on page 19)

1. $z(1,-1) = (1)^2 e(-1/1 = 1/e$. Therefore, we are looking for tangent lines through the point $(1,-1,1/e)$. Finding a tangent line parallel to the x-axis: $\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}\left(x^2 e^{y/x}\right) = x^2\left(\frac{\partial}{\partial x}e^{y/x}\right) + e^{y/x}\left(\frac{\partial}{\partial x}x^2\right) = x^2 e^{y/x}\left(\frac{\partial}{\partial x}\frac{y}{x}\right) + 2xe^{y/x} = x^2 e^{y/x}\left(\frac{-y}{x^2}\right) + 2xe^{y/x} = \left(2x - y\right)e^{y/x}$ and $z_x(1,-1) = \left(2(1) - (-1)\right)e^{-1/1} = (3)\,e^{-1} = 3/e$. So, the slope of a line tangent to the surface at $(1,-1,1/e)$ parallel to the x-axis is $3/e$ and an equation for that line is $z = 3/e\left(x-1\right) - 1/e$.

   Finding a tangent line parallel to the y-axis: $\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}\left(x^2 e^{y/x}\right) = xe^{y/x}$ and $z_y(1,-1) = (1)e^{-1/1} = 1/e$. So, the slope of a line tangent to the surface at $(1,-1,1/e)$ parallel to the y-axis is $1/e$ and an equation for that line is $z = 1/e\left(y+1\right) - 1/e$.

   The function is changing faster in the x-direction.

2. $z(\pi,\pi/2) = \cos\left(\pi\right) + \frac{\pi}{2}\sin\left(\pi/2\right) = \frac{\pi}{2} - 1$. Therefore, we are looking for tangent lines through the point $(\pi,\pi/2,\pi/2-1)$. Finding a tangent line parallel to the x-axis: $\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}\left(\cos x + y\sin y\right) = -\sin x$ and $z_x(\pi,\pi/2) = -\sin\pi = 0$. So, the slope of a line tangent to the surface at $(\pi,\pi/2,\pi/2-1)$ parallel to the x-axis is 0 and an equation for that line is $z = \pi/2 - 1$.

   Finding a tangent line parallel to the y-axis: $\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}\left(\cos x + y\sin y\right) = y\left(\frac{\partial}{\partial y}\sin y\right) + \sin y\left(\frac{\partial}{\partial y}y\right) = y\cos y + \sin y$ and $z_y(\pi,\pi/2) = \left(\frac{\pi}{2}\right)\cos\left(\frac{\pi}{2}\right) + \sin\left(\frac{\pi}{2}\right) = 1$. So, the slope of a line tangent to the surface at $(\pi,\pi/2,\pi/2-1)$ parallel to the y-axis is 1 and an equation for that line is $z = (y - \pi/2) - (\pi/2 - 1) = y - \pi + 1$.

   The function is changing faster in the y-direction.

3. $z(3,2) = 3^2(2) - 3(3)(2^2) = 18 - 36 = -18$. Therefore, we are looking for tangent lines through the point $(3,2,-18)$. Finding a tangent line parallel to the x-axis:

$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}\left(x^2y - 3xy^2\right) = 2xy - 3y^2$ and $z_x(3,2) = 2(3)(2) - 3(2)^2 = 0$. So, the slope of a line tangent to the surface at $(3,2,-18)$ is 0 and an equation for that line is $z = -18$

Finding a tangent line parallel to the y-axis: $\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}\left(x^2y - 3xy^2\right) = x^2 - 6xy$ and $z_y(3,2) = 3^2 - 6(3)(2) = 9 - 36 = -27$. So, the slope of a line tangent to the surface at $(3,2,-18)$ is -27 and an equation for that line is $z = -27(y-2) + -18 = -27y + 54 - 18 = 36 - 27y$.

The function is changing faster in the y-direction.

Answer to Exercise 7 (on page 27)

1. First, we define $\mathbf{u}$ such that $|\mathbf{u}| = 1$ and $\mathbf{u}$ is in the same direction as $\mathbf{v}$:

$$\mathbf{u} = k\mathbf{v} = [-3k, -4k]$$

$$\sqrt{(-3k)^2 + (4k)^2} = 1$$

$$\sqrt{9k^2 + 16k^2} = \sqrt{25k^2} = 5k = 1$$

$$k = \frac{1}{5}$$

Therefore, we define $\mathbf{u} = [-3/5, 4/5]$ and the directional derivative is given by:

$$D_u(x,y) = \left(\frac{-3}{5}\right)\frac{\partial}{\partial x}f(x,y) + \left(\frac{4}{5}\right)\frac{\partial}{\partial y}f(x,y)$$

$$= \left(\frac{-3}{5}\right)\frac{\partial}{\partial x}\left[e^{3x}\sin 2y\right] + \left(\frac{4}{5}\right)\frac{\partial}{\partial y}\left[e^{3x}\sin 2y\right]$$

$$= \left(\frac{-3}{5}\right)\left(3e^{3x}\sin 2y\right) + \left(\frac{4}{5}\right)\left(2e^{3x}\cos 2y\right)$$

And substituting for $(x,y) = (0, \pi/6)$:

$$D_u(0,\pi/6) = \left(\frac{-3}{5}\right)\cdot\left[3e^{3\cdot 0}\sin\left(\frac{\pi}{3}\right)\right] + \left(\frac{4}{5}\right)\cdot\left[2e^{3\cdot 0}\cos\left(\frac{\pi}{3}\right)\right]$$

$$D_u(0,\pi/6) = \left(\frac{-3}{5}\right)\cdot\left[3\cdot\frac{\sqrt{3}}{2}\right] + \left(\frac{4}{5}\right)\cdot\left[2\cdot\frac{1}{2}\right]$$

$$d_u(0,\pi/6) = \left(\frac{-3}{5}\right)\cdot\left(\frac{3\sqrt{3}}{2}\right) + \left(\frac{4}{5}\right)\cdot(1)$$

$$D_u(0,\pi/6) = \frac{-9\sqrt{3}}{10} + \frac{8}{10} = \frac{8 - 9\sqrt{3}}{10} \approx -0.759$$

2. We can express $\mathbf{v}$ as $\mathbf{v} = [2, -1]$. And we define $\mathbf{u}$ such that $|\mathbf{u}| = 1$ and $\mathbf{u}$ is in the same direction as $\mathbf{v}$:

$$\mathbf{u} = k\mathbf{v} = [2k, -k]$$
$$\sqrt{(2k)^2 + (-k)^2} = 1$$
$$\sqrt{4k^2 + k^2} = \sqrt{5}k = 1$$
$$k = \frac{1}{\sqrt{5}} = \frac{\sqrt{5}}{5}$$

Therefore, we define $\mathbf{u} = \left[2\sqrt{5}/5, -\sqrt{5}/5\right]$ and the directional derivative is given by:

$$D_u(x, y) = \left(\frac{2\sqrt{5}}{5}\right) \frac{\partial}{\partial x} f(x, y) + \left(\frac{-\sqrt{5}}{5}\right) \frac{\partial}{\partial y} f(x, y)$$
$$= \left(\frac{2\sqrt{5}}{5}\right) \frac{\partial}{\partial x} \left[x^2y + xy^3\right] + \left(\frac{-\sqrt{5}}{5}\right) \frac{\partial}{\partial y} \left[x^2y + xy^3\right]$$
$$= \left(\frac{2\sqrt{5}}{5}\right) \left[2xy + y^3\right] + \left(\frac{-\sqrt{5}}{5}\right) \left[x^2 + 3xy^2\right]$$

And substituting $(x, y) = (2, 4)$:

$$D_u(2, 4) = \left(\frac{2\sqrt{5}}{5}\right) \left[2(2)(4) + 4^3\right] + \left(\frac{-\sqrt{5}}{5}\right) \left[2^2 + 3(2)(4^2)\right]$$
$$D_u(2, 4) = \left(\frac{2\sqrt{5}}{5}\right) [80] + \left(\frac{-\sqrt{5}}{5}\right) [100]$$
$$D_u(2, 4) = 32\sqrt{5} - 20\sqrt{5} = 12\sqrt{5} \approx 26.833$$

3. We define $\mathbf{u}$ such that $|\mathbf{u}| = 1$ and $\mathbf{u}$ is in the same direction as $\mathbf{v}$:

$$\mathbf{u} = k\mathbf{v} = [k, k, k]$$
$$\sqrt{k^2 + k^2 + k^2} = 1$$
$$\sqrt{3}k = 1$$
$$k = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

Therefore, we let $\mathbf{u} = \left[\sqrt{3}/3, \sqrt{3}/3, \sqrt{3}/3\right]$ and the directional derivative is given by:

$$D_u(x, y, z) = \left(\frac{\sqrt{3}}{3}\right) \frac{\partial}{\partial x} f(x, y, z) + \left(\frac{\sqrt{3}}{3}\right) \frac{\partial}{\partial y} f(x, y, z) + \left(\frac{\sqrt{3}}{3}\right) \frac{\partial}{\partial z} f(x, y, z)$$

$$= \left(\frac{\sqrt{3}}{3}\right)\left[\frac{\partial}{\partial x}\ln\left(x^2+3y-z\right)+\frac{\partial}{\partial y}\ln\left(x^2+3y-z\right)+\frac{\partial}{\partial z}\ln\left(x^2+3y-z\right)\right]$$

$$= \left(\frac{\sqrt{3}}{3}\right)\left[\frac{2x}{x^2+3y-z}+\frac{3}{x^2+3y-z}+\frac{-1}{x^2+3y-z}\right]$$

$$= \left(\frac{\sqrt{3}}{3}\right)\left[\frac{2x+2}{x^2+3y-z}\right]=\frac{\sqrt{3}\,(2x+2)}{3\,(x^2+3y-z)}$$

And substituting $(x, y, z) = (2, 2, 1)$:

$$D_u(2,2,1)=\frac{\sqrt{3}\,(2(2)+2)}{3\,(2^2+3(2)-1)}=\frac{\sqrt{3}\,(6)}{3\,(9)}=\frac{2\sqrt{3}}{9}\approx 0.385$$

Answer to Exercise 8 (on page 30)

1. $z = f(50, 50) = 3000 - 0.01(50)^2 - 0.02(50)^2 = 2925$
2. A south-pointing unit vector is $\mathbf{u} = [0, -1]$. To find the rate of change, we find the directional derivative in the direction of $\mathbf{u}$ at $(50, 50)$:

$$D_u f(x,y) = (-1)\left[\frac{\partial}{\partial y}\left(3000-0.01x^2-0.02y^2\right)\right]$$

$$D_u f(x,y) = (-1)\,(-0.04y) = 0.04y$$

   And at $(50, 50)$, $D_u f(50, 50) = 0.04(50) = 2 > 0$. Therefore, if you walk south, you will ascend.

3. A northwest-pointing unit vector is $\mathbf{u} = \left[-\sqrt{2}/2, \sqrt{2}/2\right]$. To find the rate of change, we find the directional derivative at $(50, 50)$ in the direction of $\mathbf{u}$:

$$D_u f(x,y) = \left(\frac{-\sqrt{2}}{2}\right)\left[\frac{\partial}{\partial x}f(x,y)\right]+\left(\frac{\sqrt{2}}{2}\right)\left[\frac{\partial}{\partial y}f(x,y)\right]$$

$$D_u f(x,y) = \left(\frac{-\sqrt{2}}{2}\right)[-0.02x]+\left(\frac{\sqrt{2}}{2}\right)[-0.04y]$$

$$D_u f(x,y) = 0.01\sqrt{2}x - 0.02\sqrt{2}y$$

$$D_u f(50,50) = 0.01\sqrt{2}\,(50) - 0.02\sqrt{2}\,(50) = \frac{-\sqrt{2}}{2}\approx -0.707$$

   The rate of elevation change walking northwest is approximately -0.707, so you will descend and your rate of elevation change would be less than if you walked south.

4. To find the direction of maximum elevation gain, we find the direction the gradient vector points in:
$$\nabla f = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right]$$
$$\nabla f = [-0.02x, -0.04y]$$

And at $(50, 50)$,
$$\nabla f(50, 50) = [-0.02(50), -0.04(50)] = [-1, -2]$$

Therefore, the rate of greatest elevation change is in a south-by-southwest direction indicated by the vector $[-1, -2]$ and the rate of elevation change is $|\nabla f(50, 50)| = \sqrt{(-1)^2 + (-2)^2} = \sqrt{5}$. Notice this is greater than the other two rates of change we have found.

Answer to Exercise 9 (on page 32)

1.
$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}\left(x^2 - y^2\right)\right] + \frac{\partial}{\partial y}\left[\frac{\partial}{\partial y}\left(x^2 - y^2\right)\right]$$
$$= \frac{\partial}{\partial x}[2x] + \frac{\partial}{\partial y}[-2y] = 2 - 2 = 0$$

Therefore, $f(x, y) = x^2 - y^2$ is a solution to Laplace's equation.

2.
$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}(\sin x \cosh y + \cos x \sinh y)\right] + \frac{\partial}{\partial y}\left[\frac{\partial}{\partial y}(\sin x \cosh y + \cos x \sinh y)\right]$$
$$= \frac{\partial}{\partial x}[\cos x \cosh y - \sin x \sinh y] + \frac{\partial}{\partial y}[\sin x \sinh y + \cos x \cosh y]$$
$$= -\sin x \cosh y - \cos x \sinh y + \sin x \cosh y + \cos x \sinh y = 0$$

Therefore, $f(x, y) = \sin x \cosh y + \cos x \sinh y$ is a solution to Laplace's equation.

3.
$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}\left(e^{-x}\cos y - e^{-y}\cos x\right)\right] + \frac{\partial}{\partial y}\left[\frac{\partial}{\partial y}\left(e^{-x}\cos y - e^{-y}\cos x\right)\right]$$
$$= \frac{\partial}{\partial x}\left[-e^{-x}\cos y + e^{-y}\sin x\right] + \frac{\partial}{\partial y}\left[-e^{-x}\sin y + e^{-y}\cos x\right]$$
$$= e^{-x}\cos y + e^{-y}\cos x - e^{-x}\cos y - e^{-y}\cos x = 0$$

Therefore, $f(x, y) = e^{-x}\cos y - e^{-y}\cos x$.

Answer to Exercise 10 (on page 33)

(on page 33)

1. Finding the partial derivatives:

$$\frac{\partial^2 f}{\partial t^2} = \frac{\partial}{\partial t}\left[\frac{\partial}{\partial t}\left(\cos(kx)\cos(akt)\right)\right] = \frac{\partial}{\partial t}\left[-ak\cos(kx)\sin(akt)\right]$$

$$\frac{\partial^2 f}{\partial t^2} = -a^2k^2\cos(kx)\cos(akt)$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}\left(\cos(kx)\cos(akt)\right)\right] = \frac{\partial}{\partial x}\left[-k\sin(kx)\cos(akt)\right]$$

$$\frac{\partial^2 f}{\partial x^2} = -k^2\cos(kx)\cos(akt)$$

And we see that:

$$a^2\frac{\partial^2 f}{\partial x^2} = -a^2k^2\cos(kx)\cos(akt) = \frac{\partial^2 f}{\partial t^2}$$

Therefore, $f(x, t) = \cos(kx)\cos(akt)$ satisfies the wave equation.

2. Finding the partial derivatives:

$$\frac{\partial^2 f}{\partial t^2} = \frac{\partial}{\partial t}\left[\frac{\partial}{\partial t}\left(\sin(x - at) + \ln(x + at)\right)\right]$$

$$\frac{\partial^2 f}{\partial t^2} = \frac{\partial}{\partial t}\left[-a\cos(x - at) + \frac{a}{x + at}\right] = -a^2\sin(x - at) + \frac{-a^2}{(x + at)^2}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}\left(\sin(x - at) + \ln(x + at)\right)\right]$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left[\cos(x - at) + \frac{1}{x + at}\right] = -\sin(x - at) + \frac{-1}{(x + at)^2}$$

And we see that:

$$a^2\frac{\partial^2 f}{\partial x^2} = -a^2\sin(x - at) + \frac{-a^2}{(x + at)^2} = \frac{\partial^2 f}{\partial t^2}$$

Therefore, $f(x, t) = \sin(x - at) + \ln(x + at)$ satisfies the wave equation.

3. Finding the partial derivatives:

$$\frac{\partial^2 f}{\partial t^2} = \frac{\partial}{\partial t}\left[\frac{\partial}{\partial t}\left(\frac{t}{a^2t^2 - x^2}\right)\right] = \frac{\partial}{\partial t}\left[\frac{\left(a^2t^2 - x^2\right) - t\left(2a^2t\right)}{\left(a^2t^2 - x^2\right)^2}\right]$$

$$\frac{\partial^2 f}{\partial t^2} = \frac{\partial}{\partial t}\left[\frac{a^2t^2 - x^2 - 2a^2t^2}{\left(a^2t^2 - x^2\right)^2}\right] = \frac{\partial}{\partial t}\left[\frac{-a^2t^2 - x^2}{\left(a^2t^2 - x^2\right)^2}\right]$$

$$\frac{\partial^2 f}{\partial t^2} = \frac{\left(a^2t^2 - x^2\right)^2\left(-2a^2t\right) - \left(-a^2t^2 - x^2\right)\left(2\left(a^2t^2 - x^2\right)\left(2a^2t\right)\right)}{\left(a^2t^2 - x^2\right)^4}$$

$$\frac{\partial^2 f}{\partial t^2} = \frac{\left(a^2t^2 - x^2\right)\left(-2a^2t\right) - \left(-a^2t^2 - x^2\right)\left(4a^2t\right)}{\left(a^2t^2 - x^2\right)^3}$$

$$\frac{\partial^2 f}{\partial t^2} = \frac{-2a^4t^3 + 2a^2tx^2 + 4a^4t^3 + 4a^2tx^2}{\left(a^2t^2 - x^2\right)^3}$$

$$\frac{\partial^2 f}{\partial t^2} = \frac{2a^4t^3 + 6a^2tx^2}{\left(a^2t^2 - x^2\right)^3} = 2a^2t\left(\frac{a^2t^2 + 3x^2}{\left(a^2t^2 - x^2\right)^3}\right)$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}\left(\frac{t}{a^2t^2 - x^2}\right)\right] = \frac{\partial}{\partial x}\left[\frac{\left(a^2t^2 - x^2\right)(0) - t(-2x)}{\left(a^2t^2 - x^2\right)^2}\right]$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{2tx}{\left(a^2t^2 - x^2\right)^2}\right] = \frac{\left(a^2t^2 - x^2\right)^2(2t) - (2tx)\left(2\left(a^2t^2 - x^2\right)(-2x)\right)}{\left(a^2t^2 - x^2\right)^4}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\left(a^2t^2 - x^2\right)(2t) - (2tx)(2)(-2x)}{\left(a^2t^2 - x^2\right)^3}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{2a^2t^3 - 2tx^2 + 8tx^2}{\left(a^2t^2 - x^2\right)^3} = \frac{2a^2t^3 + 6tx^2}{\left(a^2t^2 - x^2\right)^3} = 2t\left(\frac{a^2t^2 + 3x^2}{\left(a^2t^2 - x^2\right)^3}\right)$$

And we see that:

$$a^2\frac{\partial^2 f}{\partial x^2} = 2a^2t\left(\frac{a^2t^2 + 3x^2}{\left(a^2t^2 - x^2\right)^3}\right) = \frac{\partial^2 f}{\partial t^2}$$

Therefore, $f(x, t) = \frac{t}{a^2t^2 - x^2}$ satisfies the wave equation.

Answer to Exercise 11 (on page 35)

1. The marginal utility of labor is given by $\partial P/\partial L$:

$$\frac{\partial P}{\partial L} = \frac{\partial}{\partial L}\left[1.01L^{0.75}K^{0.25}\right] = 0.7575L^{-0.25}K^{0.25}$$

2. The marginal utility of capital is given by $\partial P / \partial K$:

$$\frac{\partial P}{\partial K} = \frac{\partial}{\partial K}\left[1.01L^{0.75}K^{0.25}\right] = 0.2525L^{0.75}K^{-0.75}$$

3. Finding the marginal utility of labor in 1916:

$$\frac{\partial P}{\partial L} = 0.7575\,(382)^{-.25}\,(126)^{0.25} \approx 0.574$$

   And finding the marginal utility of capital in 1916:

$$\frac{\partial P}{\partial K} = 0.2525\,(382)^{0.75}\,(126)^{-0.75} \approx 0.580$$

4. Since the marginal utility of capital is greater, I would invest in capital. This would yield a greater increase in production than the same investment in labor.

Answer to Exercise 12 (on page 52)

1. 3×5
2. 1×2
3. 4×3

Answer to Exercise 13 (on page 52)

1. The matrix should have 1 row and 3 columns. For example,

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$$

2. The matrix should have 2 rows and 4 columns. For example,

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix}$$

3. The matrix should have 4 rows and 3 columns. For example,

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix}$$

Answer to Exercise 14 (on page 54)

1. $A + B = \begin{bmatrix} -2 & 7 & -2 & 10 \end{bmatrix}$. $A - B = \begin{bmatrix} 2 & 1 & 2 & 0 \end{bmatrix}$. $B - A = \begin{bmatrix} -2 & -1 & -2 & 0 \end{bmatrix}$
2. $A + B = \begin{bmatrix} 9 & -4 & -3 \\ -4 & -6 & 3 \\ -10 & 6 & -4 \end{bmatrix}$. $A - B = \begin{bmatrix} -1 & -4 & -1 \\ 6 & 0 & 7 \\ 0 & 0 & 4 \end{bmatrix}$. $B - A = \begin{bmatrix} 1 & 4 & 1 \\ -6 & 0 & -7 \\ 0 & 0 & -4 \end{bmatrix}$.
3. $A+B = \begin{bmatrix} -7 & -3 & -2 & -6 \\ 5 & 1 & 0 & 0 \end{bmatrix}$. $A-B = \begin{bmatrix} 3 & 1 & -8 & 4 \\ 5 & -9 & 8 & 6 \end{bmatrix}$. $B-A = \begin{bmatrix} -3 & -1 & 8 & -4 \\ -5 & 9 & -8 & -6 \end{bmatrix}$.

Answer to Exercise 15 (on page 55)

It is possible to compute $\mathbf{a} \cdot \mathbf{d}$, $\mathbf{b} \cdot \mathbf{e}$, and $\mathbf{c} \cdot \mathbf{f}$:

1. $\mathbf{a} \cdot \mathbf{d} = 1(-5) + 2(-1) = -5 + (-2) = -7$
2. $\mathbf{b} \cdot \mathbf{e} = -3(1) + 3(-5) + 5(3) + -5(1) = -3 + (-15) + 15 - 5 = -8$
3. $\mathbf{c} \cdot \mathbf{f} = 1(4) + 2(1) + -1(-3) = 4 + 2 + 3 = 9$

Answer to Exercise 16 (on page 57)

1. $\begin{bmatrix} -2 & -1 & 5 \end{bmatrix}$
2. $\begin{bmatrix} 0 & 5 & 1 \\ 0 & 25 & 5 \\ 0 & -25 & -5 \\ 0 & 20 & 4 \\ 0 & 5 & 1 \end{bmatrix}$
3. $\begin{bmatrix} -9 & -17 & 11 \\ -6 & 40 & -4 \\ 15 & 7 & -13 \\ 9 & -8 & -1 \end{bmatrix}$

Answer to Exercise 17 (on page 58)

1. $A \cdot B = \begin{bmatrix} 8 & -6 & 10 & 4 \\ -8 & 6 & -10 & -4 \\ -4 & 3 & -5 & -2 \\ 8 & -6+1- & 4 & \end{bmatrix}$ and $B \cdot A = [13]$

2. $A \cdot B = \begin{bmatrix} -2 & 16 & 16 \\ 5 & -24 & -8 \\ -4 & 22 & 12 \end{bmatrix}$ and $B \cdot A = \begin{bmatrix} 8 & 6 \\ -12 & -22 \end{bmatrix}$

3. $A \cdot B = \begin{bmatrix} -22 & 1 \\ 15 & -4 \end{bmatrix}$ and $B \cdot A = \begin{bmatrix} 12 & 0 & 15 & 3 \\ -4 & 0 & 1 & -15 \\ -16 & - & -17 & -11 \\ -14 & 0 & -10 & -21 \end{bmatrix}$

Answer to Exercise 18 (on page 65)

1. We are looking for a_1 and a_2 such that:

   $$a_1\,[1,2] + a_2\,[-3,1] = [4,5]$$

   Which creates the system of equations:

   $$a_1 - 3a_2 = 4$$

   $$2a_1 + a_2 = 5$$

   We can multiply the first equation by -2 and add it to the second to solve for a_2:

   $$-2\,(a_1 - 3a_2) + 2a_1 + a_2 = -2\,(4) + 5$$

   $$6a_2 + a_2 = -8 + 5$$
   $$7a_2 = -3$$
   $$a_2 = -\frac{3}{7}$$

   Substituting a_2 back into an equation and solving for a_1:

   $$a_1 - 3\left(-\frac{3}{7}\right) = 4$$

   $$a_1 + \frac{9}{7} = 4$$
   $$a_1 = \frac{19}{7}$$

   Therefore, $\frac{19}{7}\,[1,2] - \frac{3}{7}\,[-3,1] = [4,5]$.

2. We are looking for a_1 and a_2 such that:

$$a_1 [9, 4] + a_2 [0, 1] = [-5, 3]$$

Which creates the system of equations:

$$9a_1 = -5$$

$$4a_1 + a_2 = 3$$

We can find a_1 from the first equation:

$$a_1 = -\frac{5}{9}$$

Substituting for a_1 back into the second equation and solving for a_2:

$$4\left(-\frac{5}{9}\right) + a_2 = 3$$

$$a_2 - \frac{20}{9} = 3$$

$$a_2 = \frac{47}{9}$$

Therefore, $-\frac{5}{9}[9, 4] + \frac{47}{9}[0, 1] = [-5, 3]$.

3. We are looking for a_1 and a_2 such that:

$$a_1 [7, -2] + a_2 [-8, 4] = [6, -2]$$

Which yields the system of equations:

$$7a_1 - 8a_2 = 6$$

$$-2a_1 + 4a_2 = -2$$

Doubling the second equation and adding it to the first:

$$7a_1 - 8a_2 + 2(-2a_1 + 4a_2) = 6 + 2(-2)$$

$$7a_1 - 8a_2 - 4a_1 + 8a_2 = 6 - 4$$

$$3a_1 = 2$$

$$a_1 = \frac{2}{3}$$

Substituting for a_1 back into the second equation and solving for a_2:

$$-2\left(\frac{2}{3}\right) + 4a_2 = -2$$

$$-\frac{4}{3} + 4a_2 = -2$$
$$4a_2 = -\frac{2}{3}$$
$$a_2 = -\frac{1}{6}$$

Therefore, $\frac{2}{3}[7, -2] - \frac{1}{6}[-8, 4] = [6, -2]$

Answer to Exercise 19 (on page 71)

We see that $\frac{\mathbf{a}}{-4} = -\frac{1}{4}[-4, 1, 4] = \left[1, -\frac{1}{4}, -1\right] = \mathbf{d}$. Additionally, $\frac{3}{2}\mathbf{a} = \frac{3}{2}[-4, 1, 4] = \left[-6, \frac{3}{2}, 6\right] = \mathbf{f}$. Therefore, vectors **a**, **d**, and **f** are linearly dependent.

We also see that $\frac{1}{2}\mathbf{c} = \frac{1}{2}[2, -4, 6] = [1, -2, 3] = \mathbf{e}$. Therefore, vectors **c** and **e** are linearly dependent. Vector **b** is not linearly dependent to any of the other vectors.

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