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Chapter 1

Optimization

Optimization is a branch of mathematics that involves finding the best solution from all feasible solutions. In the field of operations research, optimization plays a crucial role. Whether it is minimizing costs, maximizing profits, or reducing the time taken to perform a task, optimization techniques are employed to make decisions effectively and efficiently.

In this chapter, the primary goal is to find the maximum or minimum value of a function, often referred to as the objective function, by finding the point on the graph where the derivative is equal to zero. This is usually referred to as the critical point!

At a critical point where the derivative is zero, the graph of the function has a horizontal tangent line. At such a point, the function may be reaching a peak or a valley.

Not every critical point corresponds to a maximum or minimum. Some critical points may instead be points of inflection. This is when the second derivative of the point is also equal to zero! You will see this in a few examples!

1.1 Extrema

There are two types of extrema: relative extrema and absolute extrema.

A relative maximum or minimum occurs when the function reaches a peak or valley compared to nearby values. This does not guarantee that the value is the highest or lowest value of the function overall.

An absolute maximum or minimum is the greatest or least value that the function takes on a given interval.

An absolute maximum or minimum can occur either at a critical point or at an endpoint of the interval.

For this reason, when solving optimization problems on a restricted domain, it is important to test both critical points and endpoints.

1.2 The Second Derivative Test

Once a critical point has been found, we often want to determine whether it corresponds to a maximum or a minimum. One method for doing this is the Second Derivative Test.

If a function has a critical point at $x = c$, then:

- if the second derivative is positive at $x = c$, the function is concave up and the point is a relative minimum;
- if the second derivative is negative at $x = c$, the function is concave down and the point is a relative maximum;
- if the second derivative is zero at $x = c$, the test is inconclusive.

If the second derivative is always positive or always negative on an interval, then any critical point in that interval must be an absolute minimum or absolute maximum, respectively.

1.3 Solving Optimization Problems

Optimization problems apply the ideas of derivatives, critical points, and extrema to real situations. Although the context of each problem may be different, the method used to solve them follows the same general pattern.

1.4 General Strategy for Solving Optimization Problems

Most calculus optimization problems follow these steps:

1. Identify the quantity to be optimized.
2. Write an equation for the objective function. Sometimes, you might be given an equation already. In that case, you will skip this step. For word problems, you will typically be required to write the equations yourself.
3. Use the given constraints to rewrite the function using one variable.
4. Determine the appropriate domain.
5. Find the derivative of the function.
6. Find critical points by setting the derivative equal to zero.
7. Interpret the result in the context of the problem.

When solving optimization problems, some common mistakes include forgetting to define variables, not using the constraints of the problem appropriately, misinterpreting or failing to interpret the final answer, and giving answers without appropriate units. For word problems, be very vigilant baout how your answer relates back to the question given! It serves as a small check for accuracy.

1.4.1 Example

Consider the function

$$f(x) = x^2 - 4x + 1.$$

We want to find the maximum or minimum value of this function.

First, take the derivative of the function:

$$f'(x) = 2x - 4.$$

Next, set the derivative equal to zero to find the critical point:

$$2x - 4 = 0.$$

Solving for x gives

$$x = 2.$$

Now determine whether this critical point corresponds to a maximum or a minimum by using the second derivative.

The second derivative is

$$f''(x) = 2.$$

Because the second derivative is positive, the function is concave up, and the critical point corresponds to a minimum.

To find the minimum value of the function, substitute $x = 2$ back into the original function:

$$f(2) = 2^2 - 4(2) + 1 = -3.$$

Therefore, the function has a minimum value of -3 at $x = 2$.

Exercise 1

Consider the function

$$f(x) = x^2 - 6x + 5.$$

Find the maximum or minimum value of the function using calculus.

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Now, let's look at a word example:

Example

The cost, in dollars, of producing x units of a product is given by the function

$$C(x) = 2x^2 - 24x + 100.$$

Find the number of units that should be produced in order to minimize the cost, and determine the minimum cost.

To minimize the cost, take the derivative of the cost function:

$$C'(x) = 4x - 24.$$

Set the derivative equal to zero to find the critical point:

$$4x - 24 = 0.$$

Solving for x gives

$$x = 6.$$

Next, take the second derivative:

$$C''(x) = 4.$$

Since the second derivative is positive, the cost function is concave up, and the critical point corresponds to a minimum.

Substitute $x = 6$ back into the original cost function:

$$C(6) = 2(6)^2 - 24(6) + 100 = 28.$$

Therefore, the cost is minimized when 6 units are produced, and the minimum cost is $28.

Exercise 2

A rectangular enclosure is to be built using 60 units of fencing. Three sides of the enclosure require fencing, while the fourth side is along a wall and does not require fencing.

Let x represent the length of the side parallel to the wall and y represent the width of the enclosure.

Find the dimensions of the enclosure that minimize the amount of fencing used.

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1.5 Helpful Table for Optimization Problems

Condition at a point $x = c$	**What it tells you**	**What kind of point it could be**
$f'(c) \neq 0$	The graph has a non-horizontal tangent	Not a maximum or minimum
$f'(c) = 0$	The graph has a horizontal tangent	Possible maximum, minimum, or neither
$f'(c)$ does not exist	The graph may have a corner, cusp, or vertical tangent	Possible maximum or minimum so run second derivative test to check!
$f'(c) = 0$ and $f''(c) > 0$	Graph is concave up at c	Local minimum
$f'(c) = 0$ and $f''(c) < 0$	Graph is concave down at c	Local maximum
$f'(c) = 0$ and $f''(c) = 0$	Second derivative test fails	Could be max, min, or neither

1.6 Using Python to Visualize Your Optimization Problem

You may be familiar with the use of Python to find the derivative of functions. We are going to use Python to supplement your knowledge of Optimization problems! Below is a script that can be used to visualize what your world problems may be asking you to do! This script will help you with finding the first derivative, performing the second derivative test, and graphs of your equations showing these points.

```
import sympy as sp
import numpy as np
import matplotlib.pyplot as plt

# 1) Define the variable
x = sp.symbols('x')

# 2) Define the objective function (STUDENTS EDIT THIS)
f = 20*x - x**2

# 3) Derivatives
f_prime = sp.diff(f, x)
f_double_prime = sp.diff(f_prime, x)

# 4) Solve f'(x) = 0
critical_point = sp.solve(f_prime, x)[0]

# 5) Classify using second derivative
second_derivative_value = f_double_prime.subs(x, critical_point)

if second_derivative_value > 0:
    classification = "minimum"
elif second_derivative_value < 0:
    classification = "maximum"
```

```
else:
    classification = "inconclusive"

print("f(x) =", f)
print("f'(x) =", f_prime)
print("f''(x) =", f_double_prime)
print("Critical point:", critical_point)
print("Classification:", classification)

# 6) Plot
f_num = sp.lambdify(x, f, "numpy")

X = np.linspace(0, 20, 400)
Y = f_num(X)

xc = float(critical_point)
yc = float(f_num(xc))

plt.plot(X, Y)
plt.scatter([xc], [yc])
plt.title("Optimization")
plt.xlabel("x")
plt.ylabel("f(x)")
plt.grid(True)

plt.annotate(
    classification,
    (xc, yc),
    xytext=(xc + 1, yc),
    arrowprops=dict(arrowstyle="->")
)

plt.show()
```

Exercise 3

The Python script below uses derivatives to locate and classify a critical point of a function.

1. Run the script using the function

$$f(x) = 20x - x^2.$$

   Record the critical point and whether it is classified as a maximum or minimum.

2. Change the objective function in the script to

$$f(x) = x^2 - 6x + 5.$$

   Run the script again. How does the classification change? Explain why this happens using the second derivative test.

3. Change the objective function to

$$f(x) = x^3 - 3x.$$

   Run the script. What does the script report for the classification? Explain why the second derivative test is inconclusive in this case.

4. Finally, choose your own quadratic function and predict whether the critical point will be a maximum or minimum before running the script. Check your prediction using both calculus and the graph.

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CHAPTER 2

Implicit Differentiation

Implicit differentiation is a technique in calculus for finding the derivative of a relation defined implicitly (that is, a relation between variables x and y that is not explicitly solved for one variable in terms of the other).

2.1 Implicit Differentiation Procedure

Consider an equation that defines a relationship between x and y:

$$F(x, y) = 0$$

To find the derivative of y with respect to x, we differentiate both sides of this equation with respect to x, treating y as an implicit function of x:[1]

$$\frac{d}{dx}F(x, y) = \frac{d}{dx}0$$

Applying the chain rule during the differentiation on the left side of the equation gives:

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0$$

Finally, we solve for $\frac{dy}{dx}$ to find the derivative of y with respect to x:

$$\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}$$

This result is obtained using the implicit differentiation method.

[1]This $\frac{d}{dx}$ form of the derivative is the same as y' said as taking the derivative of y with respect to x

2.2 Example

Consider the equation of a circle with radius r:

$$x^2 + y^2 = r^2$$

First, we will find $\frac{dy}{dx}$ without implicit differentiation. Newxt, we will apply implicit differentiation to get the same result.

2.2.1 Without Implicit Differentiation

First, we need to rearrange the equation to solve for y:

$$y^2 = r^2 - x^2$$

$$y = \pm\sqrt{r^2 - x^2}$$

We take the derivative of y by applying the Chain Rule:

$$\frac{dy}{dx} = \frac{1}{2 \pm \sqrt{r^2 - x^2}} \cdot (-2x) = \frac{-x}{\pm\sqrt{r^2 - x^2}}$$

Notice the denominator of this fraction is the same as the solution we found for y, $y = \pm\sqrt{r^2 - x^2}$. So, we can also represent this as:

$$\frac{dy}{dx} = \frac{-x}{y}$$

2.2.2 With Implicit Differentiation

With implicit differentiation, we assume y is a function of x and apply the Chain Rule.

$$\frac{d}{dx}[x^2 + y^2] = \frac{d}{dx}[r^2]$$

For x^2 and r^2, we take the derivative as we normally would.[2] For y^2, we apply the Chain Rule, as outlined above.[3]

$$2x + 2y\frac{dy}{dx} = 0$$

Solving for $\frac{dy}{dx}$, we find

$$\frac{dy}{dx} = \frac{-x}{y}$$

, which is the same result as we found without implicit differentiation.

2.3 Folium of Descartes

It was relatively easy to rearrange the equation for a circle to solve for y, but that is not always the case. To help you understand this better, we will consider a famous curve known as the *Folium of Descartes*, given by the equation,

$$x^3 + y^3 = 3xy.$$

The word *folium* comes from the Latin word for "leaf," describing the curve's distinctive looped shape. The curve is named after René Descartes, a French mathematician who originally presented the curve as a challenge to fellow mathematician Pierre de Fermat, asking him to find the tangent line to the curve. Fermat was able to solve the problem with ease!

The Folium of Descartes (seen in Figure 2.1) is historically significant because unlike simpler curves such as circles or parabolas, it exhibits more complex behavior, including a self-intersection at the origin (the curve crosses itself!). This complexity makes it difficult to describe the curve using a single equation of the form $y = f(x)$.

To explicitly solve the equation $x^3 + y^3 = 3xy$ for y requires multiple expressions to fully capture the entire curve. As a result, the familiar techniques used for functions defined explicitly in terms of x are no longer sufficient. Instead, to find the slope of the tangent line at a point on the folium, we must use *implicit differentiation*.

[2] The $\frac{d}{dx}$ part disappears when taking the derivative of x, as the derivative of x with respect to x is just regular differentiation.

[3] Applying the chain rule is only allowed as y is not the variable we were taking the derivative with respect to.

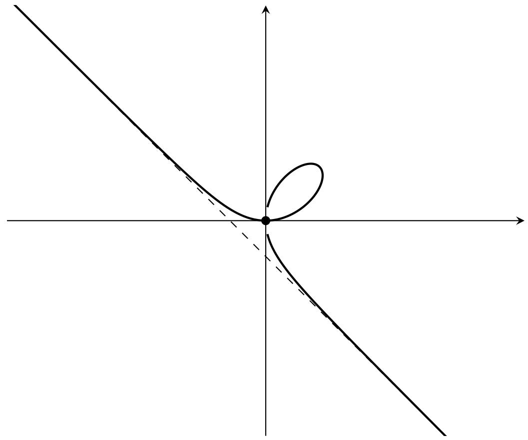

Figure 2.1: The Folium of Descartes $x^3 + y^3 = 3xy$ with asymptote $x + y + 1 = 0$.

2.3.1 Example: Tangent to Folium of Descartes

In this example, we will use implicit differentiation to easily find the tangent line at a point on the folium.

(a) Find $\frac{dy}{dx}$ if $x^3 + y^3 = 6xy$

(b) Find the tangent to the folium $x^3 + y^3 = 3xy$ at the point $(2, 2)$

(c) Is there any place in the first quadrant where the tangent line is horizontal? If so, state the point(s).

Solution:

(a) $\frac{d}{dx}[x^3 + y^3] = \frac{d}{dx}[3xy]$

$$3x^2 + 3y^2\frac{dy}{dx} = 3x\frac{dy}{dx} + 3y$$

$$x^2 + y^2\frac{dy}{dx} = x\frac{dy}{dx} + y$$

Rearranging to solve for $\frac{dy}{dx}$:

$$\frac{dy}{dx}(y^2 - x) = y - x^2$$

$$\frac{dy}{dx} = \frac{y - x^2}{y^2 - x}$$

(b) We already have the coordinate point, $(2, 2)$, so to write an equation for the tangent line, all we need is the slope. Substituting $x = 2$ and $y = 2$ into our result from part (a):

$$\frac{dy}{dx} = \frac{2 - 2^2}{2^2 - 2} = \frac{-2}{2} = -1$$

This is the slope, m. Using the point-slope form of a line, our tangent line is $y - 2 = -(x - 2)$.

(c) Recall that in the first quadrant, $x > 0$ and $y > 0$. We will set our solution for $\frac{dy}{dx}$ equal to 0:

$$\frac{y - x^2}{y + 2 - x} = 0$$

which implies that

$$y - x^2 = 0$$

Substituting $y = x^2$ into the original equation:

$$x^3 + (x^2)^3 = 3(x)(x^2)$$

$$x^3 + x^6 = 3x^3$$

Which simplifies to

$$x^6 = 2x^3$$

Since we have excluded $x = 0$ by restricting our search to the first quadrant, we can divide both sides by x^3:

$$x^3 = 2$$

$$x = \sqrt[3]{2} \approx 1.26$$

Substituting $x \approx 1.26$ into our equation for y:

$$y \approx 1.26^2 = 1.59$$

Therefore, the folium has a horizontal tangent line at the point $(1.26, 1.59)$.

2.4 Practice

Exercise 4

[This problem was originally presented as a no-calculator, multiple-choice question on the 2012 AP Calculus BC Exam.] If $\arcsin x = \ln y$, what is $\frac{dy}{dx}$?

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Exercise 5

[This problem was originally presented as a no-calculator, multiple-choice question on the 2012 AP Calculus BC Exam.] The points $(-1, -1)$ and $(1, -5)$ are on the graph of a function $y = f(x)$ that satisfies the differential equation $\frac{dy}{dx} = x^2 + y$. Use implicit differentiation to find $\frac{d^2y}{dx^2}$. Determine if each point is a local minimum, local maximum, or inflection point by substituting the x and y values of the coordinates into $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

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Chapter 3

Related Rates

In calculus, related rates problems involve finding a rate at which a quantity changes by relating that quantity to other quantities whose rates of change are known. The technique used to solve these problems is known as ”related rates” because one rate is related to another rate.

3.1 Steps to solve related rates problems

Step 1: Understand the problem. First, read the problem carefully. Understand what rates are given and what rate you need to find.

Step 2: Draw a diagram. For most problems, especially geometry problems, drawing a diagram can be very helpful.

Step 3: Write down what you know. Write down the rates that you know and the rate that you need to find.

Step 4: Write an equation. Write an equation that relates the quantities in the problem. This equation will be your main tool to solve the problem. Some cases may require an implict equation to be constructed.

Step 5: Differentiate both sides of the equation. Now you can use calculus. Differentiate both sides of the equation with respect to time.

Step 6: Substitute the known rates and solve for the unknown. Now that you have an equation that relates the rates, substitute the known rates into the equation and solve for the unknown rate.

Step 7: Create a concluding statement/ Write a concluding statement that sums up what you have a calculated in a concise manner.

3.2 Example

Here is an example of a related rates problem:

A balloon is rising at a constant rate of 5 m/s. A boy is cycling towards the balloon along a straight path at 15 m/s. If the balloon is 100 m above the ground, find the rate at which the distance from *the boy to the balloon is changing when the boy is 40 m from the point on the ground directly beneath the balloon.*

The problem can be modeled with a right triangle where the vertical side is the height of the balloon, the horizontal side is the distance of the boy from the point on the ground directly beneath the balloon, and the hypotenuse is the distance from the boy to the balloon.

Let x be the distance of the boy from the point on the ground directly beneath the balloon, y the height of the balloon above the ground, and z the distance from the boy to the balloon. From the Pythagorean theorem, we have

$$z^2 = x^2 + y^2 \tag{3.1}$$

Differentiating both sides with respect to time t gives

$$2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt} \tag{3.2}$$

Given that $\frac{dx}{dt} = -15$ m/s (the boy is moving towards the point beneath the balloon), $\frac{dy}{dt} = 5$ m/s (the balloon is rising), $x = 40$ m, $y = 100$ m, we can substitute these into the equation and solve for $\frac{dz}{dt}$.

All related rates problems are different, so it is important to continually do them so that you encounter many different examples. For more examples and practice, work through the problems included in your digital resources!

Chapter 4

Multivariate Functions and Partial Derivatives

A real-valued multivariate function is a function that takes multiple real variables as input and produces a single real output. We generally denote such a function as $f : \mathbb{R}^n \to \mathbb{R}$, where $\mathbb{R}^n$ is the domain and $\mathbb{R}$ is the co-domain, (ie. $\mathbb{R}$ is the domain of one variable and $\mathbb{R}^2$ is the domain of a 2 variable function)

For example, consider a function f that takes two variables, x and y:

$$f(x, y) = x^2 + y^2$$

Here, $f : \mathbb{R}^2 \to \mathbb{R}$ takes an ordered pair (x, y) from the 2-dimensional real coordinate space, squares each, and adds them to produce a real number.

In a similar way, a function $g : \mathbb{R}^3 \to \mathbb{R}$ could take three variables, x, y, and z, and might be defined as:

$$g(x, y, z) = x^2 + y^2 + z^2$$

FIXME graphic of this showing area splotch mapping to line segment

Here, the function squares each of the input variables, then adds them to produce a real number.

These functions are "real-valued" because their outputs are real numbers, and "multivariate" because they take multiple variables as inputs.

The concepts of limits, continuity, differentiability, and integrability can all be extended to multivariate functions, although they become more complex because we now have to consider different directions in which we approach a point, not just from the left or right, as in the univariate case. FIXME expand this?

For example, the partial derivative is the derivative of the function with respect to one variable, holding the others constant. It is one of the basic concepts in the calculus of multivariate functions.

For example, given the function $f(x, y) = x^2 + y^2$, the partial derivatives of f are computed as:

$$\frac{\partial f}{\partial x}(x, y) = 2x$$

$$\frac{\partial f}{\partial y}(x, y) = 2y$$

We will expand on these partial derivatives in the next chapter.

Chapter 5

Partial Derivatives and Gradients

This chapter will introduce you to partial derivatives and gradients, equipping you with the tools to study functions of multiple variables. We will explore how these concepts provide valuable insights into optimization, vector calculus, and various fields of science and engineering.

Partial derivatives come into play when dealing with functions that depend on multiple variables. Unlike ordinary derivatives that consider changes along a single variable, partial derivatives focus on how a function changes concerning each individual variable while holding the others constant. In essence, partial derivatives measure the rate of change of a function with respect to one variable, while keeping the other variables fixed.

The notation for a partial derivative of a function $f(x, y, \ldots)$ with respect to a specific variable, say x, is denoted as $\frac{\partial f}{\partial x}$. Similarly, $\frac{\partial f}{\partial y}$ represents the partial derivative with respect to y, and so on. It is essential to remember that when taking partial derivatives, we treat the other variables as constants during the differentiation process.

The gradient is a vector that combines the partial derivatives of a function. It provides a concise representation of the direction and magnitude of the steepest ascent or descent of the function. The gradient vector points in the direction of the greatest rate of increase of the function. By understanding the gradient, we gain insights into optimizing functions and finding critical points where the function reaches maximum or minimum values.

Throughout this chapter, we will explore the following key topics related to partial derivatives and gradients:

- Calculating partial derivatives: We will delve into the techniques and rules for computing partial derivatives of various functions, including polynomials, exponential functions, and trigonometric functions. We will also explore higher-order partial derivatives and mixed partial derivatives.
- Interpreting partial derivatives: Understanding the geometric and physical interpretations of partial derivatives is essential. We will discuss the notion of tangent planes, directional derivatives, and the relationship between partial derivatives and local linearity.
- Gradient vectors and their properties: We will introduce this concept, including it connection to the direction of steepest ascent, its relationship with partial derivatives,

and how it relates to level curves and level surfaces.

- Applications of partial derivatives and gradients: We will explore various applications of these concepts, including optimization problems, constrained optimization, tangent planes, linear approximations, and their relevance in fields like physics, economics, and engineering.

By grasping the concepts of partial derivatives and gradients, you will unlock a powerful mathematical framework for analyzing and optimizing functions of multiple variables. These tools will equip you to tackle advanced calculus problems and gain deeper insights into the behavior of functions in diverse fields.

5.1 Calculating Partial Derivatives

For a function of two variables, $f(x, y)$, we can take the derivative with respect to x or with respect to y. These are called the *partial derivatives* of f. Formally, the partial derivatives are defined as:

Limit Definition of Partial Derivatives

$$f_x(x, y) = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

$$f_y(x, y) = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}$$

Let's consider a polynomial function of two variables: $f(x, y) = 3x^2 + y^3 + 4xy$. We will use the limit definition to find the partial derivative with respect to x, then compare this to what we already know about derivatives of single-variable functions. Recall that if we can describe a function as a sum of two other functions, the derivative of the original function is the same as the sum of the derivatives of the other functions. That is,

$$\text{if } f(x) = g(x) + h(x)$$

$$\text{then } f'(x) = g'(x) + h'(x)$$

Let's then define $r(x, y) = 3x^2$, $s(x, y) = y^3$, and $t(x, y) = 4xy$. And so $f(x, y) = r(x, y) + s(x, y) + t(x, y)$, which means $f_x(x, y) = r_x(x, y) + s_x(x, y) + t_x(x, y)$. Then,

$$\begin{aligned} f_x(x, y) &= \lim_{h \to 0} \frac{r(x+h, y) - r(x, y)}{h} + \lim_{h \to 0} \frac{s(x+h, y) - s(x, y)}{h} + \lim_{h \to 0} \frac{t(x+h, y) - t(x, y)}{h} \\ &= \lim_{h \to 0} \frac{3(x+h)^2 - 3x^2}{h} + \lim_{h \to 0} \frac{y^3 - y^3}{h} + \lim_{h \to 0} \frac{4(x+h)y - 4xy}{h} \end{aligned}$$

$$= \lim_{h \to 0} \frac{3x^2 + 6xh + h^2 - 3x^2}{h} + 0 + \lim_{h \to 0} \frac{4xy + 4hy - 4xy}{h}$$

Notice that $s_x(x, y) = 0$. This term only had y, and its derivative with respect to x is zero. Continuing,

$$f_x(x, y) = \lim_{h \to 0} \frac{6xh + h^2}{h} + \lim_{h \to 0} \frac{4hy}{h} = \lim_{h \to 0} 6x + h + \lim_{h \to 0} 4y$$
$$= 6x + 4y$$

As you can see, $r_x(x, y) = 6x$ and $t_x(x, y) = 4y$. Recall the polynomial rule for single derivatives. The derivative of $3x^2$ is $6x$, which is also what we see with the partial derivative in this case. What about the other term, $4xy$? Well, we know the derivative of bx, where b is a constant, is b. The partial derivative of $4xy$ with respect to x being $4y$ suggests the rule for determining partial derivatives:

Rule for Finding Partial Derivatives of $f(x, y)$

1. To find the partial derivative with respect to x, f_x, treat y as a constant and differentiate with respect to x.
2. To find the partial derivative with respect to y, f_y, treat x as a constant and differentiate with respect to y.

Let's check this by predicting f_y, then using the limit definition to confirm our prediction. Applying the polynomial rule, we predict that f_y is:

$$f_y(x, y) = 3y^2 + 4x$$

Which we found by treating x as a constant and taking the derivative of each term with respect to y. Let's see if we get the same result using the limit definition of the derivative with respect to y:

$$f_y(x, y) = \lim_{h \to 0} \frac{f(x, y + h) - f(x, y)}{h}$$
$$= \lim_{h \to 0} \frac{\left[3x^2 + (y + h)^3 + 4x(y + h)\right] - \left[3x^2 + y^3 + 4xy\right]}{h}$$
$$= \lim_{h \to 0} \frac{3x^2 + y^3 + 3y^2h + 3yh^2 + h^3 + 4xy + 4xh - 3x^2 - y^3 - 4xy}{h}$$
$$= \lim_{h \to 0} \frac{3y^2h + 3yh^2 + h^3 + 4xh}{h} = \lim_{h \to 0} 3y^2 + 3yh + h^2 + 4x = 3y^2 + 4x$$

Which is our expected result. In summary, you find the partial derivative with respect to a particular variable by treating all the other variables as constants and differentiating with respect to the particular variable, applying the rules of differentiation you've already learned.

5.1.1 Partial Derivative Notation

There are many ways to denote a partial derivative. We've already seen one way, f_x and f_y. Another common notation uses a lowercase Greek letter delta, and a further uses capital D. They are shown below:

Partial Derivative Notations

$$f_x(x,y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x,y) = D_x f$$
$$f_y(x,y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x,y) = D_y f$$

Exercise 6 First Partial Derivatives

Find f_x and f_y for the following functions.

1. $f(x, y) = 3x^4 + 4x^2y^3$
2. $f(x, y) = xe^{-y}$
3. $f(x, y) = \sqrt{3x + 4y^2}$
4. $f(x, y) = \sin x^2y$
5. $f(x, y) = \ln\left(x^y\right)$

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5.1.2 Partial Derivatives of Functions of More than Two Variables

The above method of determining partial derivatives applies to functions with three, four, or any number of variables.

Example: Find all the first derivatives of the function $f(x, y, z) = y\cos\left(x^2 + 3z\right)$.

Solution:

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left[y\cos\left(x^2 + 3z\right)\right] = -y\sin\left(x^2 + 3z\right)\left(\frac{\partial}{\partial x}\left(x^2 + 3z\right)\right)$$

$$\frac{\partial f}{\partial x} = -2xy\sin\left(x^2 + 3z\right)$$

And

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left[y\cos\left(x^2+3z\right)\right]$$
$$\frac{\partial f}{\partial y} = \cos\left(x^2+3z\right)$$

And

$$\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}\left[y\cos\left(x^2+3z\right)\right] = -y\sin\left(x^2+3z\right)\left(\frac{\partial}{\partial z}\left(x^2+3z\right)\right)$$
$$\frac{\partial f}{\partial z} = -3y\sin\left(x^2+3z\right)$$

Exercise 7 Partial Derivatives with 3 or More Variables

Find all first partial derivatives of the following functions.

1. $f = \sin\left(x^2 - y^2\right)\cos\left(\sqrt{z}\right)$
2. $q = \sqrt[3]{t^3 + u^3 \sin(5v)}$
3. $w = x^z y^x$

Working Space

Answer on Page 60

5.1.3 Higher Order Partial Derivatives

Just like with single-variable equations, we can take the partial derivative more than once. There are also several notations for second partial derivatives.

Second Partial Derivative Notation

$$(f_x)_x = f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}$$

$$(f_x)_y = f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}$$

$$(f_y)_x = f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}$$

$$(f_y)_y = f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}$$

Notice that for $\left(\partial^2 f/\partial y \partial x\right)$, we first take the derivative with respect to x, then with respect to y.

Example: Find all the second order partial derivatives of $f(x, y) = 2x^2 - x^3y^2 + y^3$.

Solution: We begin by finding f_x and f_y:

$$f_x(x, y) = 4x - 3x^2y^2$$

$$f_y(x, y) = -2x^3y + 3y^2$$

We then take another partial derivative to find all the second order partial derivatives:

$$f_{xx}(x, y) = \frac{\partial}{\partial x} f_x(x, y) = \frac{\partial}{\partial x}\left(4x - 3x^2y^2\right) = 4 - 6xy^2$$

$$f_{xy}(x, y) = \frac{\partial}{\partial y} f_x(x, y) = \frac{\partial}{\partial y}\left(4x - 3x^2y^2\right) = -6x^2y$$

$$f_{yx}(x, y) = \frac{\partial}{\partial x} f_y(x, y) = \frac{\partial}{\partial x}\left(-2x^3y + 3y^2\right) = -6x^2y$$

$$f_{yy}(x, y) = \frac{\partial}{\partial y} f_y(x, y) = \frac{\partial}{\partial y}\left(-2x^3y + 3y^2\right) = -2x^3 + 6y$$

What do you notice about f_{xy} and f_{yx}? They are the same! This is not a coincidence of the particular function used in the example. For most functions, $f_{xy} = f_{yx}$, as stated by Clairaut's theorem.

Clairaut's Theorem

If f is defined on a disk D and f_{xy} and f_{yx} are both continuous on D, then $f_{xy} = f_{yx}$ on D.

This is also true for third, fourth, and higher-order derivatives.

Exercise 8 Clairaut's Theorem

Show that Clairaut's theorem holds for the following functions (show that $f_{xy} = f_{yx}$).

1. $f(x, y) = e^{2xy} \sin x$
2. $f(x, y) = \frac{x^2}{x+y}$
3. $f(x, y) = \ln(2x + 3y)$

Working Space

Answer on Page 62

Exercise 9 Second Order Partial Derivatives

Find all second order partial derivatives of the function.

1. $f(x, y) = x^5y^2 - 3x^3y^2$
2. $v = \sin\left(p^3 + q^2\right)$
3. $T = e^{-3r}\cos\theta^2$

Working Space

Answer on Page 62

5.1.4 The Chain Rule

For single-variable functions, where $y = f(x)$ and $x = g(t)$, we have seen that:

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$$

Which is the Chain Rule for single-variable functions. For multi-variable functions, there are several versions of the Chain Rule, depending on how the variables and functions are defined. First, we consider the case where $z = f(x, y)$ and $x = g(t)$ and $y = h(t)$ (i.e. f is a multi-variable function of x and y, while x and y are single-variable functions of t).

This means that z is an indirect function of t:

$$z = f(x, y) = f\left(g(t), h(t)\right)$$

Then the derivative of z with respect to t is given by:

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

Example: If $z = xy^2 + 3x^4y$, where $x = 2\sin(t)$ and $y = \cos(3t)$, find dz/dt when $t = \pi/2$.

Solution: First, we apply the Chain Rule to z:

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial x}\frac{dy}{dt}$$

$$= \frac{\partial}{\partial x}\left[xy^2 + 3x^4y\right] \cdot \frac{d}{dt}[2\sin(t)] + \frac{\partial}{\partial y}\left[xy^2 + 3x^4y\right] \cdot \frac{d}{dt}[\cos(3t)]$$

$$= \left(y^2 + 12x^3y\right) \cdot (2\cos(t)) + \left(2xy + 3x^4\right) \cdot (-3\sin(3t))$$

When $t = \pi/2$, $\cos(t) = 0$, $\sin(3t) = -1$, $x = 2$, and $y = 0$. Substituting:

$$\frac{dz}{dt} = (0 + 0) \cdot (0) + \left(0 + 3(2)^4\right) \cdot (-3 \cdot -1)$$

$$= 3(2)^4 \cdot 3 = 144$$

Another case is where x and y are also multi-variable functions. Consider $z = f(x, y)$, $x = g(s, t)$, and $y = h(s, t)$. This means z is an indirect function of s and t:

$$z = f(x, y) = f\left(g(s, t), h(s, t)\right)$$

In this case, there are two partial derivatives of z:

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

Example: Find $\partial z/\partial s$ and $\partial z/\partial t$ if $z = e^{2x}\cos y$, $x = s^2t$, and $y = st^2$.

Solution: First, let's find $\partial z/\partial s$:

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

$$= \frac{\partial}{\partial x}\left[e^{2x}\cos y\right] \cdot \frac{\partial}{\partial s}\left[s^2 t\right] + \frac{\partial}{\partial y}\left[e^{2x}\cos y\right] \cdot \frac{\partial}{\partial s}\left[st^2\right]$$

$$= \left(2e^{2x}\cos y\right) \cdot (2st) + \left(-e^{2x}\sin y\right) \cdot \left(t^2\right)$$

Substituting for x and y:

$$\frac{\partial z}{\partial s} = 4ste^{2s^2 t}\cos\left(st^2\right) - t^2 e^{2s^2 t}\sin\left(st^2\right)$$

And finding $\partial z/\partial t$:

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

$$= \frac{\partial}{\partial x}\left[e^{2x}\cos y\right] \cdot \frac{\partial}{\partial t}\left[s^2 t\right] + \frac{\partial}{\partial y}\left[e^{2x}\cos y\right] \cdot \frac{\partial}{\partial t}\left[st^2\right]$$

$$= \left(2e^{2x}\cos y\right) \cdot \left(s^2\right) + \left(-e^{2x}\sin y\right) \cdot (2st)$$

Substituting for x and y:

$$\frac{\partial z}{\partial t} = 2s^2 e^{2s^2 t}\cos\left(st^2\right) - 2ste^{2s^2 t}\sin\left(st^2\right)$$

Exercise 10 The Chain Rule for Multivariable Functions

Working Space

Find dz/dt or $\partial z/\partial s$ and $\partial z/\partial t$.

1. $z = \sin x \cos y$, $x = 3\sqrt{t}$, $y = 2/t$
2. $z = \sqrt{1 + xy}$, $x = \tan t$, $y = \arctan t$
3. $z = \arctan\left(x^2 + y^2\right)$, $x = t \ln s$, $y = se^t$
4. $z = \sqrt{x}e^{xy}$, $x = 1 + st$, $y = s^2 - t^2$

Answer on Page 63

5.2 Interpreting Partial Derivatives

What is the meaning of a partial derivative? Recall that $z = f(x, y)$ plots a surface, S. Consider the function $z = \cos y - x^2$, shown in figure 5.1.

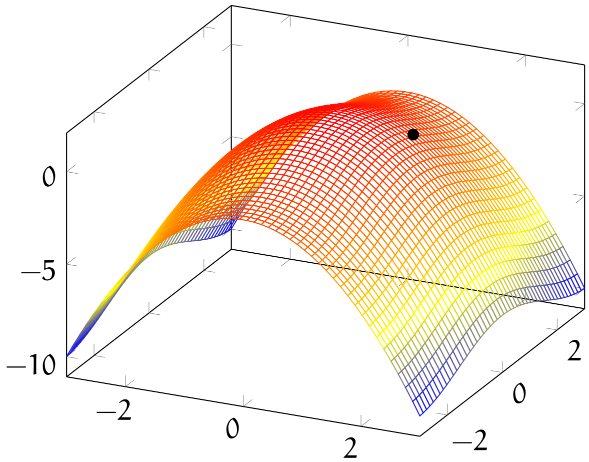


Figure 5.1: The surface $z = \cos y - x^2$

We can see that $f(1, \pi/3) = -1/2$; therefore, the point $(1, \pi/3, -1/2)$ lies on the surface $z = \cos y - x^2$ (the black dot shown in figure **??**). If we fix y such that $y = \pi/3$, we are looking at the intersection between the surface and the plane $y = \pi/3$ (see figure **??**).

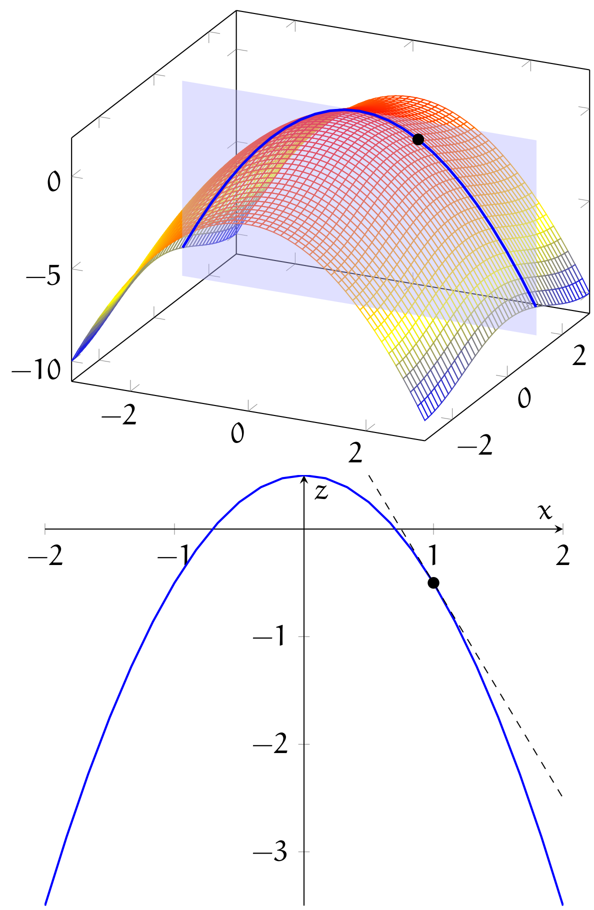
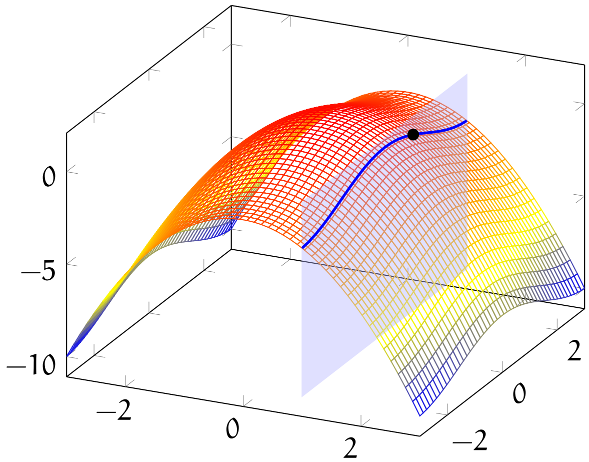


Figure 5.2: The intersection between the surface $z = \cos y - x^2$ and $y = \pi/3$ is the parabola $z(x) = 1/2 - x^2$

We can describe this intersection as $g(x) = f(x, \pi/3)$, so the slope of a tangent line to this intersection is given by $g'(x) = f_x(x, \pi/3)$. This means, geometrically, $f_x(1, \pi/3)$ is the slope of the line that lies tangent to $z = f(x, y)$ at the point $(1, \pi/3, -1/2)$ and in the plane $y = \pi/3$ (see figure 5.2). Alternatively, you could think of f_x as the slope of the tangent line to the surface that is parallel to the x-axis.

Similarly, we can fix $x = 1$ and look at the intersection between the surface $z = \cos y - x^2$ and the plane $x = 1$ (see figure 5.3. Just like before, we can describe this intersection as $h(y) = f(1, y)$, which means the slope of a line tangent to the intersection is given by $h'(y) = f_y(1, y)$. Therefore, as with f_x, $f_y(a, b)$ gives the slope of a line tangent to the point $(a, b, f(a, b))$ and parallel to the y-axis.



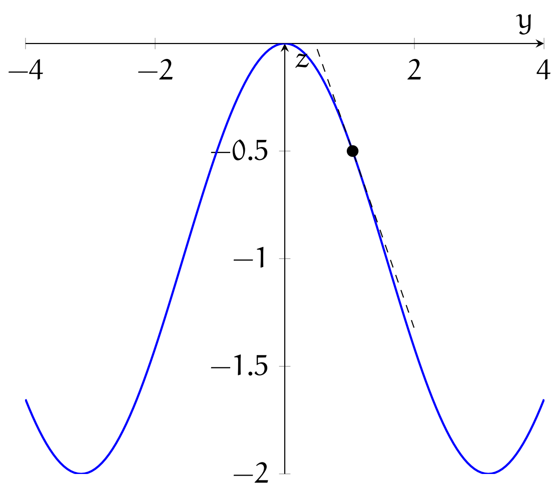


Figure 5.3: The intersection between the surface $z = \cos y - x^2$ and $x = 1$ is the trigonometric function $z = \cos y - 1$

Example: The density of bacterial growth at a point (x, y) on a flat agar plate is given by $D = 45/\left(2 + x^2 + y^2\right)$. Find the rate of change of bacterial density at the point $(1, 3)$ (a) in the x-direction and (b) in the y-direction. Interpret the meaning of your results.

Solution: The rate of change of a two-variable function in the x-direction is given by the partial derivative with respect to x:

$$D_x = \frac{\partial}{\partial x}\frac{45}{2 + x^2 + y^2} = \frac{-45\,(\partial/\partial x)\left(2 + x^2 + y^2\right)}{(2 + x^2 + y^2)^2}$$

$$= \frac{-90x}{(2 + x^2 + y^2)^2}$$

The rate of change in the x-direction at $(x, y) = (1, 3)$ is given by:

$$D_x(1,3) = \frac{-90(1)}{(2+1^2+3^2)^2} = \frac{-90}{(12)^2} = \frac{-90}{144} = -\frac{5}{8}$$

This means that at $(1, 3)$, the density of bacteria is decreasing as you move away $x = 0$ along the line $y = 3$.

Similarly, the rate of change in the y-direction is given by the partial derivative with respect to y:

$$\begin{aligned} D_y = \frac{\partial}{\partial y}\frac{45}{2+x^2+y^2} &= \frac{-45\,(\partial/\partial y)\left(2+x^2+y^2\right)}{(2+x^2+y^2)^2} \\ &= \frac{-90y}{(2+x^2+y^2)^2} \end{aligned}$$

The rate of change in the y-direction at $(x, y) = (1, 3)$ is given by:

$$D_y(1,3) = \frac{-90(3)}{(2+1^2+3^2)^2} = \frac{-270}{144} = -\frac{15}{8}$$

This means that at $(1, 3)$ the density of bacteria is decreasing faster along the y-direction than along the x-direction.

Exercise 11 Using partial derivatives to find tangent lines

Find equations for tangent lines to the surface at the given xy-coordinate. In which direction is the function changing the fastest?

1. $z = x^2 e^{y/x}$, $(1, -1)$
2. $z = \cos x + y \sin y$, $(\pi, \pi/2)$
3. $z = x^2 y - 3xy^2$, $(3, 2)$

Working Space

Answer on Page 65

5.3 Gradient Vectors

The gradient vector is used to find the direction of the maximum rate of change of a surface (for example, the steepest part of a mountain). In order to understand the gradient, we must first discuss directional derivatives. Recall that the partial derivatives, f_x and f_y, can be used to define a plane tangent to the surface $z = f(x, y)$ (see figure 5.4). Directional derivatives allow us to find the slope of the tangent plane in directions other than the x- and y-directions.

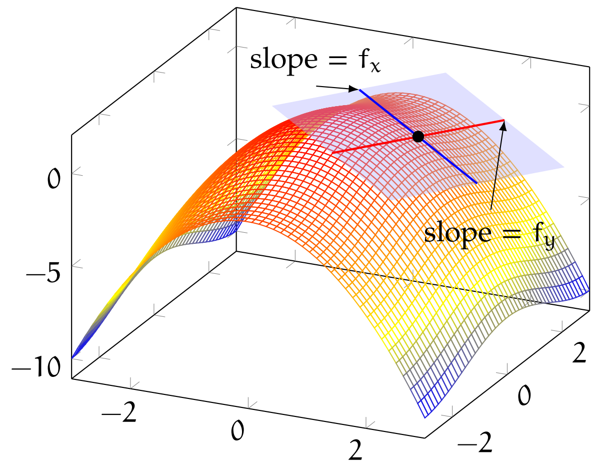


Figure 5.4: The directional derivatives, f_x and f_y define a tangent plane

5.3.1 Directional Derivatives

The contour map in figure 5.5 shows the elevation, $f(x, y)$ for a mountain. You already know that you can use the partial derivatives, f_x and f_y to find the rate of change in elevation going east-west or north-south. But what about other directions? Suppose the hiking path you're on goes north-east. How can you predict the steepness (i.e. the rate of elevation change) along this path? The directional derivative allows us to find the rate of change in any direction.

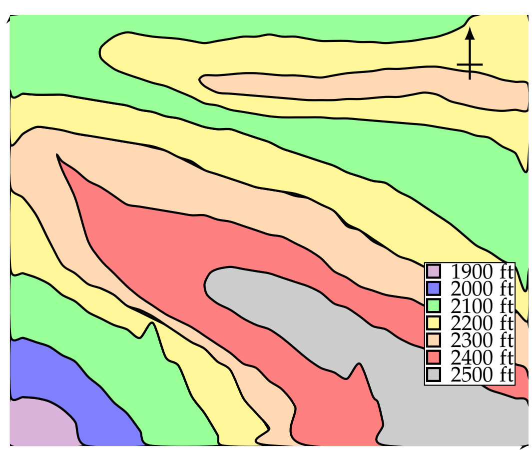


Figure 5.5: The contour plot shows the elevation of a mountain. f_x gives the slope going east, while f_y gives the slope going north

At some point, (x_o, y_o), the partial derivatives $f_x(x_o, y_o)$ and $f_y(x_o, y_o)$ give the rate of change of elevation in the east-west and north-south directions, respectively (see figure 5.6).

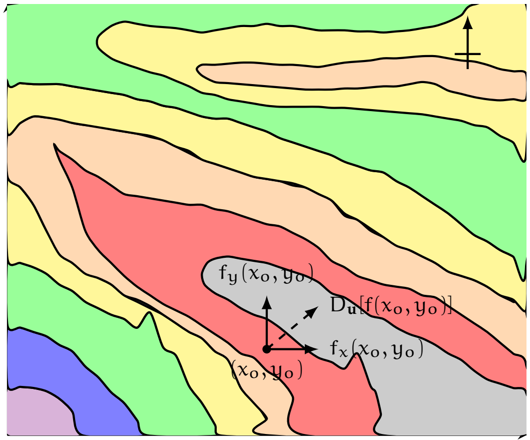


Figure 5.6: If **u** points north-east, then the directional derivative of $f(x, y)$ at (x_o, y_o), $D_{\mathbf{u}}[f(x_o, y_o)]$, tells the rate of change going north-east

To find the rate of change at (x_o, y_o), in the direction of some arbitrary unit vector, $\mathbf{u} = [a, b] = a\mathbf{i} + b\mathbf{j}$, we first note that the point $Q = (x_o, y_o, z_o)$, where $z_o = f(x_o, y_o)$, lies on the surface defined by $z = f(x, y)$. There is a vertical plane, P, that passes through Q and points in the direction of **u**. This intersection defines curve C, which lies on the surface, and the slope of this curve at $Q = (x_o, y_o, z_o)$ is the directional derivative of H in the direction of **u** (see figure 5.7).

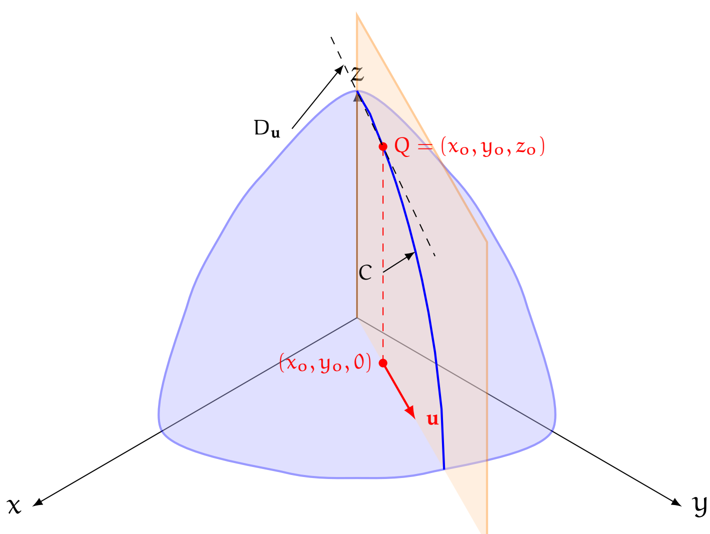


Figure 5.7: The slope of the curve formed between the plane parallel to **u** and the surface $z = f(x, y)$ is the directional derivative, $D_{\mathbf{u}}$

We can choose another point, $R = (x, y, z)$, the is h units away from Q along **u** (see 5.8). Then the change in x is $x - x_o = ha$ and the change is y is $y - y_o = hb$. And the slope from Q to R is given by:

$$\frac{\delta z}{h} = \frac{f(x, y) - f(x_o, y_o)}{h}$$

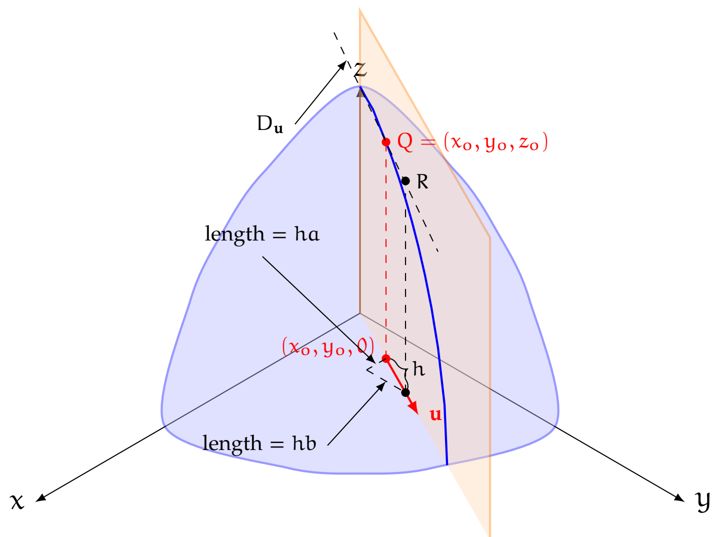


Figure 5.8: A second point, R, along $\mathbf{u}$ is h units away along $\mathbf{u}$

We find the directional derivative by substituting for x and y and taking the limit as h goes to zero:

$$D_{\mathbf{u}}f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_o + ha, y_o + hb) - f(x_o, y_o)}{h}$$

How is this related to f_x and f_y? Let's define $g(h)$ such that $g(h) = f(x_o + ha, y_o + hb)$. Then

$$g'(0) = \lim_{h \to 0} \frac{g(h) - g(0)}{h}$$

$$= \lim_{h \to 0} = \frac{f(x_o + ha, y_o + hb) - f(x_o, y_o)}{h} = D_{\mathbf{u}}f(x_o, y_o)$$

We can also apply the Chain Rule to $g(h)$:

$$g'(h) = \frac{\partial f}{\partial x}\frac{dx}{dh} + \frac{\partial f}{\partial y}\frac{dy}{dh} = f_x(x, y)a + f_y(x, y)b$$

Substituting $h = 0$, $x = x_o$, and $y = y_o$, we see that:

$$g'(0) = f_x(x_o, y_o)a + f_y(x_o, y_o)b$$

Which means that:

$$D_{\mathbf{u}}f(x_o, y_o) = f_x(x_o, y_o)a + f_y(x_o, y_o)b$$

So a directional derivative is:

The Directional Derivative

Let f be a differentiable function and $\mathbf{u}$ be a unit vector, $\mathbf{u} = [a, b]$. Then the directional derivative in the direction of $\mathbf{u}$ is:

$$D_{\mathbf{u}}f(x, y) = f_x(x, y)a + f_y(x, y)b = \mathbf{u}_x \left[\frac{\partial}{\partial x} f(x, y)\right] + \mathbf{u}_y \left[\frac{\partial}{\partial y} f(x, y)\right] \tag{5.1}$$

Where $\mathbf{u}_x$ and $\mathbf{u}_y$ are the x- and y-components of $\mathbf{u}$, respectively.

Example: Find the directional derivative $D_{\mathbf{u}}f(x, y)$ if $f(x, y) = y^3 - 3xy + 4x^2$ and $\mathbf{u}$ is the unit vector given by the angle $\theta = \pi/3$. What is the rate of change in the direction of $\mathbf{u}$ at $(1, 2)$?

Solution: We can describe $\mathbf{u}$ thusly:

$$\mathbf{u} = \left[\cos\frac{\pi}{3}, \sin\frac{\pi}{3}\right] = \left[\frac{1}{2}, \frac{\sqrt{3}}{2}\right]$$

And therefore:

$$D_{\mathbf{u}}f(x, y) = f_x(x, y)\left(\frac{1}{2}\right) + f_y(x, y)\left(\frac{\sqrt{3}}{2}\right)$$

$$= \frac{\partial}{\partial x}\left(y^3 - 3xy + 4x^2\right)\left(\frac{1}{2}\right) + \frac{\partial}{\partial y}\left(y^3 - 3xy + 4x^2\right)\left(\frac{\sqrt{3}}{2}\right)$$

$$= \frac{1}{2}(-3y + 8x) + \frac{\sqrt{3}}{2}\left(3y^2 - 3x\right)$$

$$= \frac{-3}{2}y + 4x + \frac{3\sqrt{3}}{2}y^2 - \frac{3\sqrt{3}}{2}x = \frac{3\sqrt{3}}{2}y^2 + \frac{8 - 3\sqrt{3}}{2}x - \frac{3}{2}y$$

And therefore $D_{\mathbf{u}}f(1, 2)$ is:

$$= \frac{3\sqrt{3}}{2}(2)^2 + \frac{8 - 3\sqrt{3}}{2}(1) - \frac{3}{2}(2) = 6\sqrt{3} + 4 - \frac{3\sqrt{3}}{2} - 3$$

$$= 1 + \frac{9\sqrt{3}}{2}$$

5.3.2 Unit Vectors in Two Dimensions

What if the given vector is not a unit vector? We can scale the given vector to find a unit vector in the same direction:

Example: Find the directional derivative of $f(x, y) = 3x\sqrt{y}$ at $(1, 4)$ in the direction of $\mathbf{v} = [2, 1]$.

Solution: First, we need to find a unit vector in the same direction as $\mathbf{v}$. There are several ways to do this. In two dimensions, a unit vector in the same direction as $\mathbf{v}$ can be found using trigonometry (see figure 5.9 for an illustration).

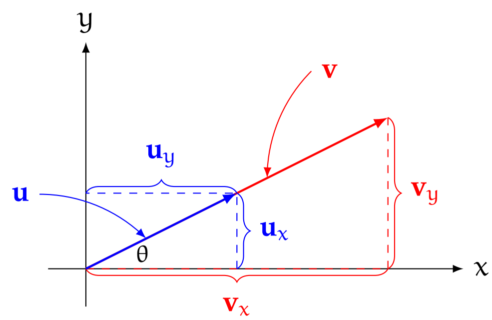


Figure 5.9: $\mathbf{u}$ is a unit vector in the same direction as $\mathbf{v}$

We know that $\theta = \arctan(\mathbf{v}_y/\mathbf{v}_x)$. Therefore, the x-component of the unit vector, $\mathbf{u}$, is given by:

$$\mathbf{u}_x = |\mathbf{u}| \cos\theta = \cos\left(\arctan\frac{\mathbf{v}_y}{\mathbf{v}_x}\right)$$

Similarly, we know that:

$$\mathbf{u}_y = |\mathbf{u}| \sin\theta = \sin\left(\arctan\frac{\mathbf{v}_y}{\mathbf{v}_x}\right)$$

(Recall that since $\mathbf{u}$ is a unit vector, $|\mathbf{u}| = 1$).

Let's use this method to find a unit vector, $\mathbf{u}$, in the same direction as $\mathbf{v} = [2, 1]$:

$$\mathbf{u}_x = \cos\left(\arctan\frac{1}{2}\right) \approx \cos(0.464) = \frac{2}{\sqrt{5}}$$

$$\mathbf{u}_y = \sin\left(\arctan\frac{1}{2}\right) \approx \sin(0.464) = \frac{1}{\sqrt{5}}$$

Therefore, a unit vector in the same direction as $\mathbf{v}$ is $\mathbf{u} = \left[2/\sqrt{5}, 1/\sqrt{5}\right]$.

And we can find the directional derivative:

$$D_{\mathbf{u}}(x, y) = \mathbf{u}_x \left[\frac{\partial}{\partial x} f(x, y)\right] + \mathbf{u}_y \left[\frac{\partial}{\partial y} f(x, y)\right]$$

$$D_{\mathbf{u}}(x, y) = \left(\frac{2}{\sqrt{5}}\right) \left[\frac{\partial}{\partial x} (3x\sqrt{y})\right] + \left(\frac{1}{\sqrt{5}}\right) \left[\frac{\partial}{\partial y} (3x\sqrt{y})\right]$$

$$D_{\mathbf{u}}(x, y) = \left(\frac{2}{\sqrt{5}}\right) (3\sqrt{y}) + \left(\frac{1}{\sqrt{5}}\right) \left(\frac{3x}{2\sqrt{y}}\right)$$

$$D_{\mathbf{u}}(x, y) = \frac{12y + 3x}{2\sqrt{5y}}$$

To find the magnitude of the directional derivative at $(1, 4)$, we substitute for x and y:

$$D_{\mathbf{u}}(1, 4) = \frac{12(4) + 3(1)}{2\sqrt{5(4)}} = \frac{51}{4\sqrt{5}} \approx 5.702$$

5.3.3 Unit Vectors in Higher Dimensions

The trigonometric explanation for finding unit vectors is more difficult to visualize in higher dimensions. However, there is another method that works well in 2, 3, and higher dimensions. Recall that the magnitude of a vector, $\mathbf{v} = [\mathbf{v}_x, \mathbf{v}_y]$ is given by $|\mathbf{v}| = \sqrt{(\mathbf{v}_x)^2 + (\mathbf{v}_y)^2}$. For a vector with n dimensions, $\mathbf{v} = [\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n]$, the magnitude is given by $|\mathbf{v}| = \sqrt{(\mathbf{v}_1)^2 + (\mathbf{v}_2)^2 + \cdots + (\mathbf{v}_n)^2}$.

To find a unit vector, $\mathbf{u}$, in the same direction as $\mathbf{v}$, we can scale $\mathbf{v}$ up or down so that its magnitude is 1. We can do this by dividing by $\mathbf{v}$'s magnitude. Consider the two-dimensional vector used in the last example, $\mathbf{v} = [2, 1]$. Its magnitude is:

$$|\mathbf{v}| = \sqrt{(2)^2 + (1)^2} = \sqrt{5}$$

Let's check if $\mathbf{v}/|\mathbf{v}|$ is a unit vector:

$$\frac{\mathbf{v}}{|\mathbf{v}|} = \left(\frac{1}{\sqrt{5}}\right) [2, 1] = \left[\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right]$$

And the magnitude of this scaled vector is:

$$\left|\frac{\mathbf{v}}{|\mathbf{v}|}\right| = \sqrt{\left(\frac{2}{\sqrt{5}}\right)^2 + \left(\frac{1}{\sqrt{5}}\right)^2} = \sqrt{\frac{4}{5} + \frac{1}{5}} = \sqrt{1} = 1$$

Notice our unit vector is the same as we found using the trigonometric method above.

Another way to think of the question is: what factor, k, can we multiply $\mathbf{v}$ by to yield a vector with a magnitude of 1? Let's see this method for the 3-dimensional vector $\mathbf{v} = [3, 2, 1]$. We are looking for a k such that:

$$|k\mathbf{v}| = 1$$

$$\begin{aligned} |k\mathbf{v}| = |[3k, 2k, 1k]| &= \sqrt{(3k)^2 + (2k)^2 + (1k)^2} \\ &= \sqrt{9k^2 + 4k^2 + k^2} = k\sqrt{14} = 1 \end{aligned}$$

Which implies that $k = 1/\sqrt{14}$, which is $1/|\mathbf{v}|$. And therefore a unit vector in the same direction as $\mathbf{v} = [3, 2, 1]$ is:

$$\mathbf{u} = \frac{1}{\sqrt{14}}[3, 2, 1] = \left[\frac{3}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{1}{\sqrt{14}}\right]$$

Exercise 12 Finding Directional Derivatives

Find the directional derivative of the function at the given point in the direction of the given vector.

1. $f(x, y) = e^{3x} \sin 2y$, $(0, \pi/6)$, $\mathbf{v} = [-3, 4]$
2. $f(x, y) = x^2y + xy^3$, $(2, 4)$, $\mathbf{v} = 2\mathbf{i} - \mathbf{j}$
3. $f(x, y, z) = \ln\left(x^2 + 3y - z\right)$, $(2, 2, 1)$, $\mathbf{v} = [1, 1, 1]$

Working Space

Answer on Page 66

5.3.4 Maximizing the Gradient

The directional derivative can be written as the dot product of two vectors:

$$D_u f(x,y) = a f_x(x,y) + b f_y(x,y) = [f_x(x,y), f_y(x,y)] \cdot \mathbf{u}$$

The first vector, $[f_x(x,y), f_y(x,y)]$, is called *the gradient of f*, and is noted as ∇f.

The Gradient

For a two-variable function, $f(x,y)$, the gradient of f is the vector:

$$\nabla f = [f_x(x,y), f_y(x,y)] = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

Where $\mathbf{i}$ and $\mathbf{j}$ are the unit vectors in the x- and y-directions, respectively.

Think back to the elevation example we opened the chapter with. What if we wanted to complete our ascent as quickly as possible? We would want to know the direction in which the elevation is changing the fastest. This occurs when the direction we are going is the same direction as the gradient vector, ∇f.

Recall that the dot product is defined as:

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}|\,|\mathbf{v}| \cos\theta$$

Where θ is the angle between the vectors $\mathbf{u}$ and $\mathbf{v}$. Applying this to the directional derivative, we see that:

$$D_u f = \nabla f \cdot \mathbf{u} = |\nabla f|\,|\mathbf{u}| \cos\theta = |\nabla f| \cos\theta$$

Which is at its maximum when ∇f and $\mathbf{u}$ point in the same direction (because $\cos(0) = 1$). Therefore, the gradient vector points in the direction of maximum change and the magnitude of that vector is the rate of maximum change.

Example: Find the maximum rate of change of $f(x,y) = 4y\sqrt{x}$ at $(4,1)$. In what direction does the maximum change occur?

Solution: We begin by finding ∇f:

$$\nabla f = \left[\frac{\partial}{\partial x}\left(4y\sqrt{x}\right), \frac{\partial}{\partial y}\left(4y\sqrt{x}\right)\right]$$

$$\nabla f = \left[\frac{2y}{\sqrt{x}}, 4\sqrt{x}\right]$$

And thus,

$$\nabla f(4,1) = \left[\frac{2(1)}{\sqrt{4}}, 4\sqrt{4}\right] = [1,8]$$

Therefore, the maximum value of ∇f at $(4,1)$ is:

$$|\nabla f| = \sqrt{1^2 + 8^2} = \sqrt{65}$$

in the direction of the vector $[1,8]$.

Exercise 13 Using the Gradient to find Maximum Change

Suppose you are climbing a mountain whose elevation is described by $z = 3000 - 0.01x^2 - 0.02y^2$. Take the positive x-direction to be east and the positive y-direction to be north.

1. If you are at $(x, y) = (50, 50)$, what is your elevation?
2. If you walk south, will you ascend or descend?
3. If you walk northwest, will you ascend or descend? Will the rate of elevation change be greater or less than if you walked south?
4. In what direction should you walk for the steepest ascent? What will your ascension rate be?

Working Space

Answer on Page 68

5.3.5 Conclusion

in summary, we have learned that the gradient of some multivariable function is the point of steepest increase of the function, represented as:

$$\nabla f = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right] = \langle\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\rangle$$

The directional derivative is the gradient at some point a dotted by the unit vector:

$$D_{\mathbf{u}} f(\mathbf{a}) = \nabla f(\mathbf{a}) \cdot \mathbf{u}$$

It is best to think of the gradient as an arrow pointing in the steepest uphill direction (vector), and the directional derivative as the slope of any specific uphill direction (scalar).

5.4 Applications of Partial Derivatives and Gradients

5.4.1 Laplace's Equation

A partial differential equation that has applications in fluid dynamics and electronics is Laplace's Equation. Solutions to Laplace's Equation are called *harmonic functions*.

Laplace's Equation

Consider a twice-differentiable function, f. In two dimensions, Laplace's Equation is given by:

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

And in three dimensions,

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$$

Another way to represent Laplace's Equation is:

$$\delta f = \nabla^2 f = \nabla \cdot \nabla f = 0$$

Where $\nabla^2 = \delta$ is called the *Laplace operator*.

Example: Determine whether or not $f = x^2 + y^2$ is a solution to Laplace's Equation.

Solution: We are checking to see if $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$ for $f(x, y) = x^2 + y^2$. Finding $\partial^2 f / \partial x^2$:

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}\left(x^2 + y^2\right)\right]$$
$$= \frac{\partial}{\partial x}(2x) = 2$$

And finding $\partial^2 f / \partial y^2$:

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left[\frac{\partial}{\partial y}\left(x^2 + y^2\right)\right]$$
$$= \frac{\partial}{\partial y}(2x) = 2$$

Then $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 2 + 2 = 4 \neq 0$. Therefore, $f(x, y) = x^2 + y^2$ is not a solution to Laplace's Equation.

Exercise 14 Solutions to Laplace's Equation

Determine whether the function is a solution to Laplace's Equation.

1. $f(x, y) = x^2 - y^2$
2. $f(x, y) = \sin x \cosh y + \cos x \sinh y$
3. $f(x, y) = e^{-x} \cos y - e^{-y} \cos x$

Working Space

Answer on Page 69

5.4.2 The Wave Equation

Another useful equation with partial derivatives is the Wave Equation:

$$\frac{\partial^2 f}{\partial t^2} = a^2 \frac{\partial^2 f}{\partial x^2}$$

Where f is a function of x and t and a is a constant. This equation describes waves, such as a vibrating string, light waves, or sound waves.

Example: Show that $f(x,t) = \sin(x - at)$ satisfies the Wave Equation.

Solution: First, we find the second partial derivatives:

$$\frac{\partial^2 f}{\partial t^2} = \frac{\partial}{\partial t}\left[\frac{\partial}{\partial t}(\sin(x - at))\right] = \frac{\partial}{\partial t}[-a\cos(x - at)] = -a^2 \sin(x - at)$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}(\sin(x - at))\right] = \frac{\partial}{\partial x}[\cos(x - at)] = -\sin(x - at)$$

And we see that:

$$a^2 \frac{\partial^2 f}{\partial x^2} = -a^2 \sin(x - at) = \frac{\partial^2 f}{\partial t^2}$$

Therefore, this function satisfies the Wave eEuation.

Exercise 15 The Wave Equation

Show that the following functions satisfy the Wave Equation:

1. $f(x,t) = \cos(kx)\cos(akt)$
2. $f(x,t) = \sin(x - at) + \ln(x + at)$
3. $f(x,t) = \frac{t}{a^2t^2 - x^2}$

Working Space

Answer on Page 70

5.4.3 Cobb-Douglas Production Function

The Cobb-Douglas function describes the marginal utility of capital and labor as theorized by the economists Charles Cobb and Paul Douglas. Capital investments are things like new machinery, expanded factories, or raw materials. Labor investments involve hiring more workers or improving working conditions to improve work rates. We can describe total production, P, as a function of labor, L, and capital, K. Cobb and Douglas posit three conditions:

1. Without either labor or capital, production will cease.
2. The marginal utility of labor is proportional to the amount of production per unit of labor.
3. The marginal utility of capital is proportional to the amount of production per unit of capital.

The marginal utility of labor is given by the partial derivative, $\partial P/\partial L$ and the production per unit of labor is given by P/L. Therefore, statement 2 says that:

$$\frac{\partial P}{\partial L} = \alpha \frac{P}{L}$$

where α is some constant. Keeping K constant at $K = K_0$, we have the differential equation:

$$\frac{dP}{dL} = \alpha \frac{P}{L}$$

Solving, we find that:

$$P(L, K_0) = C_1 (K_0) L^{\alpha}$$

We make C_1 a function of K_0 because it could depend on K_0. In a similar manner to above, we can write statement 3 as a mathematical statement:

$$\frac{\partial P}{\partial K} = \beta \frac{P}{K}$$

where β is also a constant. Keeping $L = L_0$ and solving, we see that:

$$P(L_0, K) = C_2(L_0) K^{\beta}$$

again, we assume C_2 is a function of the fixed labor, L_0. Combining these equations, we get:

$$P(L, K) = bL^{\alpha}K^{\beta}$$

where b is a constant independent of capital and labor. Additionally, from statement 1, we know that $\alpha > 0$ and $\beta > 0$. What happens if both labor and capital are increased by a factor of n? Let's examine the effect on P:

$$P(nL, nK) = b (nL)^{\alpha} (nK)^{\beta}$$

$$P(nL, nK) = n^{\alpha+\beta} bL^{\alpha}K^{beta} = n^{\alpha+\beta} P(L, K)$$

Cobb and Douglas noted that if $\alpha + \beta = 1$, then $P(nL, nK) = nP(L, K)$, and therefore increasing labor and capital by a factor of n increases production by a factor of n as well. Therefore, the Cobb-Douglas equation assumes $\alpha + \beta = 1$ and can be written as:

$$P(L, K) = bL^{\alpha}K^{1-\alpha}$$

Exercise 16 Cobb-Douglas Production Model

Cobb and Douglas modeled production in the US from 1900 to 1922 with the equation $P(L, K) = 1.01L^{0.75}K^{0.75}$.

1. Express the marginal utility of labor as a function of L and K.
2. Express the marginal utility of capital as a function of L and K.
3. In 1916, $L = 382$ and $K = 126$ (compared to initial values of 100 in 1900). What is the marginal utility of labor in 1916? Of capital?
4. Based on your answer to the previous question, would you invest in capital or labor if you owned a factory in 1916? Why?

Working Space

Answer on Page 72

Appendix A

Answers to Exercises

Answer to Exercise 1 (on page 6)

First, take the derivative of the function:

$$f'(x) = 2x - 6.$$

Set the derivative equal to zero to find the critical point:

$$2x - 6 = 0.$$

Solving this equation gives

$$x = 3.$$

Next, take the second derivative:

$$f''(x) = 2.$$

Since the second derivative is positive, the function is concave up, and the critical point corresponds to a minimum.

Finally, substitute $x = 3$ back into the original function:

$$f(3) = 3^2 - 6(3) + 5 = -4.$$

Therefore, the function has a minimum value of -4 at $x = 3$.

Answer to Exercise 2 (on page 7)

Because only three sides require fencing, the total amount of fencing is given by

$$x + 2y = 60.$$

Solving this equation for x gives

$$x = 60 - 2y.$$

The area of the enclosure is

$$A = xy.$$

Substituting for x yields

$$A(y) = y(60 - 2y) = 60y - 2y^2.$$

Take the derivative:

$$A'(y) = 60 - 4y.$$

Set the derivative equal to zero:

$$60 - 4y = 0.$$

Solving for y gives

$$y = 15.$$

The second derivative is

$$A''(y) = -4.$$

Since the second derivative is negative, this critical point corresponds to a maximum area.

Substituting $y = 15$ into the constraint gives

$$x = 60 - 2(15) = 30.$$

Therefore, the enclosure has dimensions 30 units by 15 units.

Answer to Exercise 3 (on page 10)

1. For $f(x) = 20x - x^2$,

   $$f'(x) = 20 - 2x.$$

   Setting $f'(x) = 0$ gives $20 - 2x = 0$, so $x = 10$.

   $$f''(x) = -2.$$

   Since $f''(10) < 0$, the function is concave down at the critical point, so the critical point is a maximum.

2. For $f(x) = x^2 - 6x + 5$,

$$f'(x) = 2x - 6.$$

Setting $f'(x) = 0$ gives $2x - 6 = 0$, so $x = 3$.

$$f''(x) = 2.$$

Since $f''(3) > 0$, the function is concave up at the critical point, so the critical point is a minimum.

3. For $f(x) = x^3 - 3x$,

$$f'(x) = 3x^2 - 3 = 3(x^2 - 1).$$

Setting $f'(x) = 0$ gives $x^2 - 1 = 0$, so $x = -1$ and $x = 1$.

$$f''(x) = 6x.$$

At $x = -1$, $f''(-1) = -6 < 0$, so there is a local maximum at $x = -1$. At $x = 1$, $f''(1) = 6 > 0$, so there is a local minimum at $x = 1$.

The second derivative test is not inconclusive for this function at the critical points, because $f''(-1)$ and $f''(1)$ are not zero.

4. For a quadratic function of the form $f(x) = ax^2 + bx + c$ with $a \neq 0$, there is exactly one critical point at

$$x = -\frac{b}{2a}.$$

Also,

$$f''(x) = 2a.$$

If $a > 0$, then $f''(x) > 0$ and the critical point is a minimum. If $a < 0$, then $f''(x) < 0$ and the critical point is a maximum.

Answer to Exercise 4 (on page 16)

Using implicit differentiation, we see that:

$$\frac{d}{dx}\arcsin x = \frac{d}{dx}\ln y$$

$$\frac{1}{\sqrt{1-x^2}} = \frac{1}{y}\frac{dy}{dx}$$

Multiplying both sides by y to isolate $\frac{dy}{dx}$, we find that:

$$\frac{dy}{dx} = \frac{y}{\sqrt{1-x^2}}$$

Answer to Exercise 5 (on page 16)

First, we need to find $\frac{d^y}{dx^2}$:

$$\frac{d}{dx}\frac{dy}{dx} = \frac{d}{dx}x^2 + \frac{d}{dx}y$$
$$= 2x + \frac{dy}{dx} = 2x + x^2 + y$$

At $(-1,-1)$, $\frac{dy}{dx} = (-1)^2 + (-1) = 0$ and $\frac{d^2y}{dx^2} = 2(-1) + (-1)^2 + (-1) = -2 < 0$. Since the slope of y is zero and the graph of y is concave down, $(-1,-1)$ is a local maximum. At $(1,-5)$, $\frac{dy}{dx} = 1^2 + -5 = -4 \neq 0$ and $\frac{d^2y}{dx^2} = 2(1) + 1^2 + (-5) = -2 \neq 0$. Since neither the first nor second derivative of y are zero, $(1,-5)$ is neither a local extrema nor an inflection point.

Answer to Exercise 6 (on page 25)

1. $f_x(x,y) = \frac{\partial}{\partial x}\left[3x^4 + 4x^2y^3\right] = 12x^3 + 8y^3$ and $f_y(x,y) = \frac{\partial}{\partial y}\left[3x^4 + 4x^2y^3\right] = 12x^2y^2$
2. $f_x(x,y) = \frac{\partial}{\partial x}\left(xe^{-y}\right) = e^{-y}$ and $f_y(x,y) = \frac{\partial}{\partial y}\left(xe^{-y}\right) = -xe^{-y}$
3. $f_x(x,y) = \frac{\partial}{\partial x}\sqrt{3x+4y^2} = \left(\frac{1}{2\sqrt{3x+4y^2}}\right)\left(\frac{\partial}{\partial x}\left(3x+4y^2\right)\right) = \frac{3}{2\sqrt{3x+4y^2}}$ and $f)y(x,y) = \frac{\partial}{\partial y}\sqrt{3x+4y^2} = \frac{1}{2\sqrt{3x+4y^2}}\left(\frac{\partial}{\partial y}\left(3x+4y^2\right)\right) = \frac{8y}{2\sqrt{3x+4y^2}} = \frac{4y}{\sqrt{3x+4y^2}}$
4. $f_x(x,y) = \frac{\partial}{\partial x}\sin\left(x^2y\right) = \cos\left(x^2y\right)\left(\frac{\partial}{\partial x}\left(x^2y\right)\right) = 2xy\cos\left(x^2y\right)$ and $f_y(x,y) = \frac{\partial}{\partial y}\sin\left(x^2y\right) = \cos\left(x^2y\right)\left(\frac{\partial}{\partial y}\left(x^2y\right)\right) = x^2\cos\left(x^2y\right)$
5. $f_x(x,y) = \frac{\partial}{\partial x}\ln\left(x^y\right) = \frac{\partial}{\partial x}\left(y\ln x\right) = \frac{y}{x}$ and $f_y(x,y) = \frac{\partial}{\partial y}\left(y\ln x\right) = \ln x$

Answer to Exercise 7 (on page 27)

1. Finding f_x:

$$f_x = \frac{\partial}{\partial x}\left[\sin\left(x^2 - y^2\right)\cos\left(\sqrt{z}\right)\right] = \cos\left(x^2 - y^2\right)\cos\left(\sqrt{z}\right)\left[\frac{\partial}{\partial x}\left(x^2 - y^2\right)\right]$$

$$f_x = 2x\cos\left(x^2 - y^2\right)\cos\left(\sqrt{z}\right)$$

Finding f_y:

$$f_y = \frac{\partial}{\partial y}\left[\sin\left(x^2 - y^2\right)\cos\left(\sqrt{z}\right)\right] = \cos\left(x^2 - y^2\right)\cos\left(\sqrt{z}\right)\left[\frac{\partial}{\partial y}\left(x^2 - y^2\right)\right]$$

$$f_y = -2y\cos\left(x^2 - y^2\right)\cos\left(\sqrt{z}\right)$$

Finding f_z:

$$f_z = \frac{\partial}{\partial z}\left[\sin\left(x^2 - y^2\right)\cos\left(\sqrt{z}\right)\right] = \sin\left(x^2 - y^2\right)\left(-\sin\sqrt{z}\right)\cdot\left(\frac{\partial}{\partial z}\sqrt{z}\right)$$

$$f_z = \frac{-\sin\left(x^2 - y^2\right)\sin\left(\sqrt{z}\right)}{2\sqrt{z}}$$

2. Finding q_t:

$$q_t = \frac{\partial}{\partial t}\sqrt[3]{t^3 + u^3\sin(5v)} = \frac{1}{3\left(t^3 + u^3\sin(5v)\right)^{2/3}}\left(\frac{\partial}{\partial t}\left(t^3 + u^3\sin(5v)\right)\right)$$

$$q_t = \frac{t^2}{\left(t^3 + u^3\sin(5v)\right)^{2/3}}$$

Finding q_u:

$$q_u = \frac{\partial}{\partial u}\sqrt[3]{t^3 + u^3\sin(5v)} = \frac{1}{3\left(t^3 + u^3\sin(5v)\right)^{2/3}}\left(\frac{\partial}{\partial u}\left(t^3 + u^3\sin(5v)\right)\right)$$

$$q_u = \frac{u^2\sin(5v)}{\left(t^3 + u^3\sin(5v)\right)^{2/3}}$$

Finding q_v:

$$q_v = \frac{\partial}{\partial v}\sqrt[3]{t^3 + u^3\sin(5v)} = \frac{1}{3\left(t^3 + u^3\sin(5v)\right)^{2/3}}\left(\frac{\partial}{\partial v}\left(t^3 + u^3\sin(5v)\right)\right)$$

$$q_v = \frac{u^3\cos(5v)}{3\left(t^3 + u^3\sin(5v)\right)^{2/3}}\left(\frac{\partial}{\partial v}(5v)\right) = \frac{5u^3\cos(5v)}{3\left(t^3 + u^3\sin(5v)\right)^{2/3}}$$

3. Finding w_x:

$$w_x = \frac{\partial}{\partial x}\left(x^z y^x\right) = \left(x^z\right)\cdot\left(\frac{\partial}{\partial x}y^x\right) + \left(y^x\right)\cdot\left(\frac{\partial}{\partial x}x^z\right)$$

$$w_x = \left(x^z\right)\left(\ln(y)y^x\right) + \left(y^x\right)\left(zx^{z-1}\right) = \left(x^{z-1}y^x\right)\left(x\ln(y)+z\right)$$

Finding w_y:

$$w_y = \frac{\partial}{\partial y}\left(x^z y^x\right) = \left(x^z\right)\left(\frac{\partial}{\partial y}y^x\right) = x^z\left(xy^{x-1}\right)$$

$$w_y = x^{z+1}y^{x-1}$$

Finding w_z:

$$w_z = \frac{\partial}{\partial z}\left(x^z y^x\right) = \left(y^x\right)\left(\frac{\partial}{\partial z}x^z\right) = \left(y^x\right)\left(\ln(x)x^z\right)$$

$$w_z = \ln(x)y^x x^z$$

Answer to Exercise 8 (on page 29)

1. $f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}f(x,y)\right) = \frac{\partial}{\partial y}\left[\frac{\partial}{\partial x}\left(e^{2xy}\sin x\right)\right] = \frac{\partial}{\partial y}\left[\left(e^{2xy}\right)\left(\frac{\partial}{\partial x}\sin x\right) + \left(\sin x\right)\left(\frac{\partial}{\partial x}e^{2xy}\right)\right] = \frac{\partial}{\partial y}\left[e^{2xy}\cos x + 2ye^{2xy}\sin x\right] = \frac{\partial}{\partial y}\left(e^{2xy}\cos x\right) + \frac{\partial}{\partial y}\left(2ye^{2xy}\sin x\right) = 2xe^{2xy}\cos x + (2y)\left(\frac{\partial}{\partial y}e^{2xy}\sin x\right) + \left(e^{2xy}\sin x\right)\left(\frac{\partial}{\partial y}2y\right) = 2xe^{2xy}\cos x + 4xye^{2xy}\sin x + 2e^{2xy}\sin x$

   $f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}f(x,y)\right) = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial y}\left(e^{2xy}\sin x\right)\right] = \frac{\partial}{\partial x}\left(2xe^{2xy}\sin x\right) = (2x)\left[\frac{\partial}{\partial x}\left(e^{2xy}\sin x\right)\right] + \left(e^{2xy}\sin x\right)\left(\frac{\partial}{\partial x}2x\right) = (2x)\left[\left(e^{2xy}\right)\left(\frac{\partial}{\partial x}\sin x\right) + \left(\sin x\right)\left(\frac{\partial}{\partial x}e^{2xy}\right)\right] + 2e^{2xy}\sin x = 2xe^{2xy}\cos x + 4xye^{2xy}\sin x + 2e^{2xy}\sin x = f_{xy}$

2. $f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}f(x,y)\right) = \frac{\partial}{\partial y}\left[\frac{\partial}{\partial x}\left(\frac{x^2}{x+y}\right)\right] = \frac{\partial}{\partial y}\left[\frac{(x+y)(2x)-x^2(1)}{(x+y)^2}\right] = \frac{\partial}{\partial y}\left[\frac{x^2+2xy}{(x+y)^2}\right] = \frac{(x+y)^2(2x)-\left(x^2+2xy\right)(2(x+y))}{(x+y)^4} = \frac{\left(x^2+2xy+y^2\right)(2x)-\left(x^2+2xy\right)(2x+2y)}{(x+y)^4} = \frac{2x^3+4x^2y+2xy^2-2x^3-2x^2y-4x^2y-4xy^2}{(x+y)^4} = \frac{-2x^2y-2xy^2}{(x+y)^4}$

   $f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}f(x,y)\right) = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial y}\left(\frac{x^2}{x+y}\right)\right] = \frac{\partial}{\partial x}\left[\frac{-x^2}{(x+y)^2}\right] = \frac{(x+y)^2(-2x)-\left(-x^2\right)(2(x+y))}{(x+y)^4} = \frac{\left(x^2+2xy+y^2\right)(-2x)+x^2(2x+2y)}{(x+y)^4} = \frac{-2x^3-4x^2y-2xy^2+2x^3+2x^2y}{(x+y)^4} = \frac{-2x^2y-2xy^2}{(x+y)^4} = f_{xy}$

3. $f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}f(x,y)\right) = \frac{\partial}{\partial y}\left[\frac{\partial}{\partial x}\left(\ln(2x+3y)\right)\right] = \frac{\partial}{\partial y}\left[\frac{2}{2x+3y}\right] = \frac{-2(3)}{(2x+3y)^2} = \frac{-6}{(2x+3y)^2}$

   $f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}f(x,y)\right) = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial y}\left(\ln(2x+3y)\right)\right] = \frac{\partial}{\partial x}\left(\frac{3}{2x+3y}\right) = \frac{-3(2)}{(2x+3y)^2} = \frac{-6}{(2x+3y)} = f_{xy}$

Answer to Exercise 9 (on page 30)

1. $f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x} f(x,y)\right) = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}\left(x^5y^2 - 3x^3y^2\right)\right] = \frac{\partial}{\partial x}\left(5x^4y^2 - 9x^2y^2\right) = 20x^3y^2 - 18xy^2.$
   $f_{xy} = f_{yx} = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x} f(x,y)\right) = \frac{\partial}{\partial y}\left[\frac{\partial}{\partial x}\left(x^5y^2 - 3x^3y^2\right)\right] = \frac{\partial}{\partial y}\left(5x^4y^2 - 9x^2y^2\right) = 10x^4y - 18x^2y.$
   $f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial y} f(x,y)\right) = \frac{\partial}{\partial y}\left[\frac{\partial}{\partial y}\left(x^5y^2 - 3x^3y^2\right)\right] = \frac{\partial}{\partial y}\left(2x^5y - 6x^3y\right) = 2x^5 - 6x^3.$
2. $v_{pp} = \frac{\partial}{\partial p}\left(\frac{\partial}{\partial p} v(p,q)\right) = \frac{\partial}{\partial p}\left[\frac{\partial}{\partial p}\left(\sin\left(p^3 + q^2\right)\right)\right] = \frac{\partial}{\partial p}\left(\cos\left(p^3 + q^2\right)\left(3p^2\right)\right) = \cos\left(p^3 + q^2\right) \cdot \frac{\partial}{\partial p}\left(3p^2\right) + 3p^2 \cdot \frac{\partial}{\partial p}\left(\cos\left(p^3 + q^2\right)\right)$
   $v_{pq} = v_{qp} = \frac{\partial}{\partial q}\left(\frac{\partial}{\partial p} v(p,q)\right) = \frac{\partial}{\partial q}\left[\frac{\partial}{\partial p}\left(\sin\left(p^3 + q^2\right)\right)\right] = \frac{\partial}{\partial q}\left(\cos\left(p^3 + q^2\right)\left(3p^2\right)\right) = \cos\left(p^3 + q^2\right)\frac{\partial}{\partial q}\left(3p^2\right) + 3p^2\frac{\partial}{\partial q}\cos\left(p^3 + q^2\right) = 0 + 3p^2\left(-\sin\left(p^3 + q^2\right)\right)\left(\frac{\partial}{\partial q}\left(p^3 + q^2\right)\right) = -6p^2q\sin\left(p^3 + q^2\right)$
   $v_{qq} = \frac{\partial}{\partial q}\left(\frac{\partial}{\partial q} v(p,q)\right) = \frac{\partial}{\partial q}\left[\frac{\partial}{\partial q}\left(\sin\left(p^3 + q^2\right)\right)\right] = \frac{\partial}{\partial q}\left[2q\cos\left(p^3 + q^2\right)\right] = 2q\left[\frac{\partial}{\partial q}\cos\left(p^3 + q^2\right)\right] + \cos\left(p^3 + q^2\right)\left[\frac{\partial}{\partial q}(2q)\right] = (2q) \cdot \left[-2q\sin\left(p^3 + q^2\right)\right] + 2\cos\left(p^3 + q^2\right) = 2\cos\left(p^3 + q^2\right) - 4q^2\sin\left(p^3 + q^2\right)$
3. $T_{rr} = \frac{\partial}{\partial r}\left(\frac{\partial}{\partial r} T(r,\theta)\right) = \frac{\partial}{\partial r}\left[\frac{\partial}{\partial r}\left(e^{-3r}\cos\theta^2\right)\right] = \frac{\partial}{\partial r}\left(-3e^{-3r}\cos\theta^2\right) = 9e^{-3r}\cos\theta^2$
   $T_{\theta r} = T_{r\theta} = \frac{\partial}{\partial \theta}\left(\frac{\partial}{\partial r} T(r,\theta)\right) = \frac{\partial}{\partial \theta}\left[-3e^{-3r}\cos\theta^2\right] = 3re^{-3r}\sin\theta^2\left(\frac{\partial}{\partial \theta}\theta^2\right) = 6r\theta e^{-3r}\sin\theta^2$
   $T_{\theta\theta} = \frac{\partial}{\partial \theta}\left(\frac{\partial}{\partial \theta} T(r,\theta)\right) = \frac{\partial}{\partial \theta}\left[\frac{\partial}{\partial \theta}\left(e^{-3r}\cos\theta^2\right)\right] = \frac{\partial}{\partial \theta}\left[-e^{-3r}\sin\theta^2\left(\frac{\partial}{\partial \theta}\theta^2\right)\right] = \frac{\partial}{\partial \theta}\left(-2\theta e^{-3r}\sin\theta^2\right) = \left(-2\theta e^{-3r}\right)\left(\frac{\partial}{\partial \theta}\sin\theta^2\right) + \left(\sin\theta^2\right)\left[\frac{\partial}{\partial \theta}\left(-2\theta e^{-3r}\right)\right] = \left(-2\theta e^{-3r}\right)\left(\cos\theta^2\right)\left(\frac{\partial}{\partial \theta}\theta^2\right) + \left(\sin\theta^2\right)\left(-2e^{-3r}\right) = -4\theta^2e^{-3r}\cos\theta^2 - 2e^{-3r}\sin\theta^2$

Answer to Exercise 10 (on page 33)

1. We are looking for dz/dt only:

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$= \frac{\partial}{\partial x}\left[\sin x \cos y\right] \cdot \frac{d}{dt}\left[3\sqrt{t}\right] + \frac{\partial}{\partial y}\left[\sin x \cos y\right] \cdot \frac{d}{dt}\left[2/t\right]$$

$$= (\cos x \cos y) \cdot \left(\frac{3}{2\sqrt{t}}\right) + (-\sin x \sin y) \cdot \left(-\frac{2}{t^2}\right)$$

$$= \frac{3\cos x \cos y}{2\sqrt{t}} + \frac{2\sin x \sin y}{t^2}$$

   Substituting for x and y:

$$\frac{dz}{dt} = \frac{3\cos\left(3\sqrt{t}\right)\cos\left(2/t\right)}{2\sqrt{t}} + \frac{2\sin\left(3\sqrt{t}\right)\sin\left(2/t\right)}{t^2}$$

2. We are looking for dz/dt only:

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$= \frac{\partial}{\partial x}\left[\sqrt{1+xy}\right] \cdot \frac{d}{dt}[\tan t] + \frac{\partial}{\partial y}\left[\sqrt{1+xy}\right] \cdot \frac{d}{dt}[\arctan t]$$

$$= \left(\frac{y}{2\sqrt{1+xy}}\right) \cdot \left(\sec^2 t\right) + \left(\frac{x}{2\sqrt{1+xy}}\right) \cdot \left(\frac{1}{t^2+1}\right)$$

Substituting for x and y:

$$\frac{dz}{dt} = \frac{\tan t \sec^2 t}{2\sqrt{1+\tan t \arctan t}} + \frac{\tan t}{2\sqrt{1+\tan t \arctan t}\,(t^2+1)}$$

$$= \frac{\tan t}{2\sqrt{1+\tan t \arctan t}}\left(\sec^2 t + \frac{1}{t^2+1}\right)$$

3. Finding $\partial z/\partial s$:

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

$$= \frac{\partial}{\partial x}\left[\arctan\left(x^2+y^2\right)\right] \cdot \frac{\partial}{\partial s}[t \ln s] + \frac{\partial}{\partial y}\left[\arctan\left(x^2+y^2\right)\right] \cdot \frac{\partial}{\partial s}\left[se^t\right]$$

$$\left(\frac{2x}{(x^2+y^2)^2+1}\right) \cdot \left(\frac{t}{s}\right) + \left(\frac{2y}{(x^2+y^2)^2+1}\right) \cdot \left(e^t\right)$$

Substituting for x and y:

$$\frac{\partial z}{\partial s} = \left(\frac{2\,(t \ln s)}{\left[(t\ln s)^2 + (se^t)^2\right]^2+1}\right) \cdot \left(\frac{t}{s}\right) + \left(\frac{2\left(se^t\right)}{\left[(t\ln s)^2 + (t\ln s)^2\right]^2+1}\right) \cdot \left(e^t\right)$$

$$= \left(\frac{2}{\left[t^2(\ln s)^2 + se^{2t}\right]^2+1}\right) \cdot \left(\frac{t^2 \ln s}{s} + se^{2t}\right)$$

Finding $\partial z/\partial t$:

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

$$= \frac{\partial}{\partial x}\left(\arctan\left(x^2+y^2\right)\right) \cdot \frac{\partial}{\partial t}(t \ln s) + \frac{\partial}{\partial y}\left(\arctan\left(x^2+y^2\right)\right) \cdot \frac{\partial}{\partial t}\left(se^t\right)$$

$$= \left(\frac{2x}{(x^2+y^2)^2+1}\right) \cdot (\ln s) + \left(\frac{2y}{(x^2+y^2)^2+1}\right) \cdot \left(se^t\right)$$

Substituting for x and y:

$$\frac{\partial z}{\partial t} = \left(\frac{2}{\left[(t \ln s)^2 + (se^t)^2\right]^2 + 1} \right) \cdot \left[t (\ln s)^2 + s^2 e^{2t} \right]$$

4. Finding $\partial z / \partial s$:

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

$$= \frac{\partial}{\partial x}\left(\sqrt{x}e^{xy}\right) \cdot \frac{\partial}{\partial s}(1 + st) + \frac{\partial}{\partial y}\left(\sqrt{x}e^{xy}\right) \cdot \frac{\partial}{\partial s}\left(s^2 - t^2\right)$$

$$= \left[\frac{e^{xy}(2xy + 1)}{2\sqrt{x}}\right] \cdot (t) + \left[x^{3/2}e^{xy}\right] \cdot (2s)$$

$$= \left[\frac{e^{xy}}{\sqrt{x}}\right] \cdot \left(\frac{(2xy + 1)\, t}{2} + x^2 (2s)\right)$$

Substituting for x and y:

$$\frac{\partial z}{\partial s} = \left[\frac{e^{(1+st)(s^2-t^2)}}{\sqrt{1 + st}}\right] \cdot \left(\frac{(2(1 + st)(s^2 - t^2) + 1)\, t}{2} + (1 + st)^2 (2s)\right)$$

Finding $\partial z / \partial t$:

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

$$= \frac{\partial}{\partial y}\left[\sqrt{x}e^{xy}\right] \cdot \frac{\partial}{\partial t}[1 + st] + \frac{\partial}{\partial y}\left[\sqrt{x}e^{xy}\right] \cdot \frac{\partial}{\partial t}\left[s^2 - t^2\right]$$

$$= \left[\frac{e^{xy}(2xy + 1)}{2\sqrt{x}}\right] \cdot (s) + \left[x^{3/2}e^{xy}\right] \cdot (-2t)$$

$$= \left[\frac{e^{xy}}{\sqrt{x}}\right] \cdot \left[\frac{(2xy + 1)\, s}{2} - 2tx^2\right]$$

Substituting for x and y:

$$\frac{\partial z}{\partial t} = \left[\frac{e^{(1+st)(s^2-t^2)}}{\sqrt{(1 + st)}}\right] \cdot \left[\frac{\left(2(1 + st)\left(s^2 - t^2\right) + 1\right) s}{2} - 2t(1 + st)^2\right]$$

Answer to Exercise 11 (on page 38)

1. $z(1, -1) = (1)^2 e(-1/1 = 1/e$. Therefore, we are looking for tangent lines through the point $(1, -1, 1/e)$. Finding a tangent line parallel to the x-axis: $\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}\left(x^2 e^{y/x}\right) = x^2\left(\frac{\partial}{\partial x}e^{y/x}\right) + e^{y/x}\left(\frac{\partial}{\partial x}x^2\right) = x^2 e^{y/x}\left(\frac{\partial}{\partial x}\frac{y}{x}\right) + 2xe^{y/x} = x^2 e^{y/x}\left(\frac{-y}{x^2}\right) + 2xe^{y/x} = (2x - y)\, e^{y/x}$

and $z_x(1,-1) = (2(1) - (-1))\, e^{-1/1} = (3)\, e^{-1} = 3/e$. So, the slope of a line tangent to the surface at $(1,-1,1/e)$ parallel to the x-axis is $3/e$ and an equation for that line is $z = 3/e\,(x-1) - 1/e$.

Finding a tangent line parallel to the y-axis: $\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}\left(x^2 e^{y/x}\right) = xe^{y/x}$ and $z_y(1,-1) = (1)e^{-1/1} = 1/e$. So, the slope of a line tangent to the surface at $(1,-1,1/e)$ parallel to the y-axis is $1/e$ and an equation for that line is $z = 1/e\,(y+1) - 1/e$.

The function is changing faster in the x-direction.

2. $z(\pi,\pi/2) = \cos(\pi) + \frac{\pi}{2}\sin(\pi/2) = \frac{\pi}{2} - 1$. Therefore, we are looking for tangent lines through the point $(\pi,\pi/2,\pi/2-1)$. Finding a tangent line parallel to the x-axis: $\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(\cos x + y\sin y) = -\sin x$ and $z_x(\pi,\pi/2) = -\sin\pi = 0$. So, the slope of a line tangent to the surface at $(\pi,\pi/2,\pi/2-1)$ parallel to the x-axis is 0 and an equation for that line is $z = \pi/2 - 1$.

   Finding a tangent line parallel to the y-axis: $\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(\cos x + y\sin y) = y\left(\frac{\partial}{\partial y}\sin y\right) + \sin y\left(\frac{\partial}{\partial y}y\right) = y\cos y + \sin y$ and $z_y(\pi,\pi/2) = \left(\frac{\pi}{2}\right)\cos\left(\frac{\pi}{2}\right) + \sin\left(\frac{\pi}{2}\right) = 1$. So, the slope of a line tangent to the surface at $(\pi,\pi/2,\pi/2-1)$ parallel to the y-axis is 1 and an equation for that line is $z = (y - \pi/2) - (\pi/2 - 1) = y - \pi + 1$.

   The function is changing faster in the y-direction.

3. $z(3,2) = 3^2(2) - 3(3)(2^2) = 18 - 36 = -18$. Therefore, we are looking for tangent lines through the point $(3,2,-18)$. Finding a tangent line parallel to the x-axis: $\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}\left(x^2y - 3xy^2\right) = 2xy - 3y^2$ and $z_x(3,2) = 2(3)(2) - 3(2)^2 = 0$. So, the slope of a line tangent to the surface at $(3,2,-18)$ is 0 and an equation for that line is $z = -18$

   Finding a tangent line parallel to the y-axis: $\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}\left(x^2y - 3xy^2\right) = x^2 - 6xy$ and $z_y(3,2) = 3^2 - 6(3)(2) = 9 - 36 = -27$. So, the slope of a line tangent to the surface at $(3,2,-18)$ is -27 and an equation for that line is $z = -27(y-2) + -18 = -27y + 54 - 18 = 36 - 27y$.

   The function is changing faster in the y-direction.

Answer to Exercise 12 (on page 46)

1. First, we define $\mathbf{u}$ such that $|\mathbf{u}| = 1$ and $\mathbf{u}$ is in the same direction as $\mathbf{v}$:

$$\mathbf{u} = k\mathbf{v} = [-3k, -4k]$$

$$\sqrt{(-3k)^2 + (4k)^2} = 1$$

$$\sqrt{9k^2 + 16k^2} = \sqrt{25k^2} = 5k = 1$$

$$k = \frac{1}{5}$$

Therefore, we define $\mathbf{u} = [-3/5, 4/5]$ and the directional derivative is given by:

$$D_u(x,y) = \left(\frac{-3}{5}\right)\frac{\partial}{\partial x}f(x,y) + \left(\frac{4}{5}\right)\frac{\partial}{\partial y}f(x,y)$$

$$= \left(\frac{-3}{5}\right)\frac{\partial}{\partial x}\left[e^{3x}\sin 2y\right] + \left(\frac{4}{5}\right)\frac{\partial}{\partial y}\left[e^{3x}\sin 2y\right]$$

$$= \left(\frac{-3}{5}\right)\left(3e^{3x}\sin 2y\right) + \left(\frac{4}{5}\right)\left(2e^{3x}\cos 2y\right)$$

And substituting for $(x,y) = (0, \pi/6)$:

$$D_u(0,\pi/6) = \left(\frac{-3}{5}\right)\cdot\left[3e^{3\cdot 0}\sin\left(\frac{\pi}{3}\right)\right] + \left(\frac{4}{5}\right)\cdot\left[2e^{3\cdot 0}\cos\left(\frac{\pi}{3}\right)\right]$$

$$D_u(0,\pi/6) = \left(\frac{-3}{5}\right)\cdot\left[3\cdot\frac{\sqrt{3}}{2}\right] + \left(\frac{4}{5}\right)\cdot\left[2\cdot\frac{1}{2}\right]$$

$$d_u(0,\pi/6) = \left(\frac{-3}{5}\right)\cdot\left(\frac{3\sqrt{3}}{2}\right) + \left(\frac{4}{5}\right)\cdot(1)$$

$$D_u(0,\pi/6) = \frac{-9\sqrt{3}}{10} + \frac{8}{10} = \frac{8-9\sqrt{3}}{10} \approx -0.759$$

2. We can express $\mathbf{v}$ as $\mathbf{v} = [2, -1]$. And we define $\mathbf{u}$ such that $|\mathbf{u}| = 1$ and $\mathbf{u}$ is in the same direction as $\mathbf{v}$:

$$\mathbf{u} = k\mathbf{v} = [2k, -k]$$

$$\sqrt{(2k)^2 + (-k)^2} = 1$$

$$\sqrt{4k^2 + k^2} = \sqrt{5}k = 1$$

$$k = \frac{1}{\sqrt{5}} = \frac{\sqrt{5}}{5}$$

Therefore, we define $\mathbf{u} = \left[2\sqrt{5}/5, -\sqrt{5}/5\right]$ and the directional derivative is given by:

$$D_u(x,y) = \left(\frac{2\sqrt{5}}{5}\right)\frac{\partial}{\partial x}f(x,y) + \left(\frac{-\sqrt{5}}{5}\right)\frac{\partial}{\partial y}f(x,y)$$

$$= \left(\frac{2\sqrt{5}}{5}\right)\frac{\partial}{\partial x}\left[x^2y + xy^3\right] + \left(\frac{-\sqrt{5}}{5}\right)\frac{\partial}{\partial y}\left[x^2y + xy^3\right]$$

$$= \left(\frac{2\sqrt{5}}{5}\right)\left[2xy + y^3\right] + \left(\frac{-\sqrt{5}}{5}\right)\left[x^2 + 3xy^2\right]$$

And substituting $(x, y) = (2, 4)$:

$$D_u(2,4) = \left(\frac{2\sqrt{5}}{5}\right)\left[2(2)(4) + 4^3\right] + \left(\frac{-\sqrt{5}}{5}\right)\left[2^2 + 3(2)(4^2)\right]$$

$$D_u(2,4) = \left(\frac{2\sqrt{5}}{5}\right)[80] + \left(\frac{-\sqrt{5}}{5}\right)[100]$$

$$D_u(2,4) = 32\sqrt{5} - 20\sqrt{5} = 12\sqrt{5} \approx 26.833$$

3. We define $\mathbf{u}$ such that $|\mathbf{u}| = 1$ and $\mathbf{u}$ is in the same direction as $\mathbf{v}$:

$$\mathbf{u} = k\mathbf{v} = [k, k, k]$$

$$\sqrt{k^2 + k^2 + k^2} = 1$$

$$\sqrt{3}k = 1$$

$$k = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

Therefore, we let $\mathbf{u} = \left[\sqrt{3}/3, \sqrt{3}/3, \sqrt{3}/3\right]$ and the directional derivative is given by:

$$D_u(x,y,z) = \left(\frac{\sqrt{3}}{3}\right)\frac{\partial}{\partial x}f(x,y,z) + \left(\frac{\sqrt{3}}{3}\right)\frac{\partial}{\partial y}f(x,y,z) + \left(\frac{\sqrt{3}}{3}\right)\frac{\partial}{\partial z}f(x,y,z)$$

$$= \left(\frac{\sqrt{3}}{3}\right)\left[\frac{\partial}{\partial x}\ln\left(x^2 + 3y - z\right) + \frac{\partial}{\partial y}\ln\left(x^2 + 3y - z\right) + \frac{\partial}{\partial z}\ln\left(x^2 + 3y - z\right)\right]$$

$$= \left(\frac{\sqrt{3}}{3}\right)\left[\frac{2x}{x^2 + 3y - z} + \frac{3}{x^2 + 3y - z} + \frac{-1}{x^2 + 3y - z}\right]$$

$$= \left(\frac{\sqrt{3}}{3}\right)\left[\frac{2x + 2}{x^2 + 3y - z}\right] = \frac{\sqrt{3}\,(2x + 2)}{3\,(x^2 + 3y - z)}$$

And substituting $(x, y, z) = (2, 2, 1)$:

$$D_u(2,2,1) = \frac{\sqrt{3}\,(2(2) + 2)}{3\,(2^2 + 3(2) - 1)} = \frac{\sqrt{3}\,(6)}{3\,(9)} = \frac{2\sqrt{3}}{9} \approx 0.385$$

Answer to Exercise 13 (on page 49)

1. $z = f(50, 50) = 3000 - 0.01(50)^2 - 0.02(50)^2 = 2925$
2. A south-pointing unit vector is $\mathbf{u} = [0, -1]$. To find the rate of change, we find the

directional derivative in the direction of **u** at $(50, 50)$:

$$D_u f(x, y) = (-1)\left[\frac{\partial}{\partial y}\left(3000 - 0.01x^2 - 0.02y^2\right)\right]$$

$$D_u f(x, y) = (-1)(-0.04y) = 0.04y$$

And at $(50, 50)$, $D_u f(50, 50) = 0.04(50) = 2 > 0$. Therefore, if you walk south, you will ascend.

3. A northwest-pointing unit vector is $\mathbf{u} = \left[-\sqrt{2}/2, \sqrt{2}/2\right]$. To find the rate of change, we find the directional derivative at $(50, 50)$ in the direction of **u**:

$$D_u f(x, y) = \left(\frac{-\sqrt{2}}{2}\right)\left[\frac{\partial}{\partial x} f(x, y)\right] + \left(\frac{\sqrt{2}}{2}\right)\left[\frac{\partial}{\partial y} f(x, y)\right]$$

$$D_u f(x, y) = \left(\frac{-\sqrt{2}}{2}\right)[-0.02x] + \left(\frac{\sqrt{2}}{2}\right)[-0.04y]$$

$$D_u f(x, y) = 0.01\sqrt{2}x - 0.02\sqrt{2}y$$

$$D_u f(50, 50) = 0.01\sqrt{2}(50) - 0.02\sqrt{2}(50) = \frac{-\sqrt{2}}{2} \approx -0.707$$

The rate of elevation change walking northwest is approximately -0.707, so you will descend and your rate of elevation change would be less than if you walked south.

4. To find the direction of maximum elevation gain, we find the direction the gradient vector points in:

$$\nabla f = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right]$$

$$\nabla f = [-0.02x, -0.04y]$$

And at $(50, 50)$,

$$\nabla f(50, 50) = [-0.02(50), -0.04(50)] = [-1, -2]$$

Therefore, the rate of greatest elevation change is in a south-by-southwest direction indicated by the vector $[-1, -2]$ and the rate of elevation change is $|\nabla f(50, 50)| = \sqrt{(-1)^2 + (-2)^2} = \sqrt{5}$. Notice this is greater than the other two rates of change we have found.

Answer to Exercise 14 (on page 51)

1.
$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}\left(x^2 - y^2\right)\right] + \frac{\partial}{\partial y}\left[\frac{\partial}{\partial y}\left(x^2 - y^2\right)\right]$$
$$= \frac{\partial}{\partial x}[2x] + \frac{\partial}{\partial y}[-2y] = 2 - 2 = 0$$

Therefore, $f(x, y) = x^2 - y^2$ is a solution to Laplace's Equation.

2.
$$\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2} = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}(\sin x \cosh y + \cos x \sinh y)\right]+\frac{\partial}{\partial y}\left[\frac{\partial}{\partial y}(\sin x \cosh y + \cos x \sinh y)\right]$$
$$= \frac{\partial}{\partial x}[\cos x \cosh y - \sin x \sinh y] + \frac{\partial}{\partial y}[\sin x \sinh y + \cos x \cosh y]$$
$$= -\sin x \cosh y - \cos x \sinh y + \sin x \cosh y + \cos x \sinh y = 0$$

Therefore, $f(x, y) = \sin x \cosh y + \cos x \sinh y$ is a solution to Laplace's Equation.

3.
$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}\left(e^{-x}\cos y - e^{-y}\cos x\right)\right] + \frac{\partial}{\partial y}\left[\frac{\partial}{\partial y}\left(e^{-x}\cos y - e^{-y}\cos x\right)\right]$$
$$= \frac{\partial}{\partial x}\left[-e^{-x}\cos y + e^{-y}\sin x\right] + \frac{\partial}{\partial y}\left[-e^{-x}\sin y + e^{-y}\cos x\right]$$
$$= e^{-x}\cos y + e^{-y}\cos x - e^{-x}\cos y - e^{-y}\cos x = 0$$

Therefore, $f(x, y) = e^{-x}\cos y - e^{-y}\cos x$.

Answer to Exercise 15 (on page 53)

1. Finding the partial derivatives:

$$\frac{\partial^2 f}{\partial t^2} = \frac{\partial}{\partial t}\left[\frac{\partial}{\partial t}(\cos(kx)\cos(akt))\right] = \frac{\partial}{\partial t}[-ak\cos(kx)\sin(akt)]$$
$$\frac{\partial^2 f}{\partial t^2} = -a^2k^2\cos(kx)\cos(akt)$$
$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}(\cos(kx)\cos(akt))\right] = \frac{\partial}{\partial x}[-k\sin(kx)\cos(akt)]$$
$$\frac{\partial^2 f}{\partial x^2} = -k^2\cos(kx)\cos(akt)$$

And we see that:

$$a^2 \frac{\partial^2 f}{\partial x^2} = -a^2k^2 \cos(kx)\cos(akt) = \frac{\partial^2 f}{\partial t^2}$$

Therefore, $f(x, t) = \cos(kx)\cos(akt)$ satisfies the Wave Equation.

2. Finding the partial derivatives:

$$\frac{\partial^2 f}{\partial t^2} = \frac{\partial}{\partial t}\left[\frac{\partial}{\partial t}\left(\sin(x - at) + \ln(x + at)\right)\right]$$

$$\frac{\partial^2 f}{\partial t^2} = \frac{\partial}{\partial t}\left[-a\cos(x - at) + \frac{a}{x + at}\right] = -a^2 \sin(x - at) + \frac{-a^2}{(x + at)^2}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}\left(\sin(x - at) + \ln(x + at)\right)\right]$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left[\cos(x - at) + \frac{1}{x + at}\right] = -\sin(x - at) + \frac{-1}{(x + at)^2}$$

And we see that:

$$a^2 \frac{\partial^2 f}{\partial x^2} = -a^2 \sin(x - at) + \frac{-a^2}{(x + at)^2} = \frac{\partial^2 f}{\partial t^2}$$

Therefore, $f(x, t) = \sin(x - at) + \ln(x + at)$ satisfies the Wave Equation.

3. Finding the partial derivatives:

$$\frac{\partial^2 f}{\partial t^2} = \frac{\partial}{\partial t}\left[\frac{\partial}{\partial t}\left(\frac{t}{a^2t^2 - x^2}\right)\right] = \frac{\partial}{\partial t}\left[\frac{\left(a^2t^2 - x^2\right) - t\left(2a^2t\right)}{\left(a^2t^2 - x^2\right)^2}\right]$$

$$\frac{\partial^2 f}{\partial t^2} = \frac{\partial}{\partial t}\left[\frac{a^2t^2 - x^2 - 2a^2t^2}{\left(a^2t^2 - x^2\right)^2}\right] = \frac{\partial}{\partial t}\left[\frac{-a^2t^2 - x^2}{\left(a^2t^2 - x^2\right)^2}\right]$$

$$\frac{\partial^2 f}{\partial t^2} = \frac{\left(a^2t^2 - x^2\right)^2\left(-2a^2t\right) - \left(-a^2t^2 - x^2\right)\left(2\left(a^2t^2 - x^2\right)\left(2a^2t\right)\right)}{\left(a^2t^2 - x^2\right)^4}$$

$$\frac{\partial^2 f}{\partial t^2} = \frac{\left(a^2t^2 - x^2\right)\left(-2a^2t\right) - \left(-a^2t^2 - x^2\right)\left(4a^2t\right)}{\left(a^2t^2 - x^2\right)^3}$$

$$\frac{\partial^2 f}{\partial t^2} = \frac{-2a^4t^3 + 2a^2tx^2 + 4a^4t^3 + 4a^2tx^2}{\left(a^2t^2 - x^2\right)^3}$$

$$\frac{\partial^2 f}{\partial t^2} = \frac{2a^4t^3 + 6a^2tx^2}{\left(a^2t^2 - x^2\right)^3} = 2a^2t\left(\frac{a^2t^2 + 3x^2}{\left(a^2t^2 - x^2\right)^3}\right)$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}\left(\frac{t}{a^2t^2 - x^2}\right)\right] = \frac{\partial}{\partial x}\left[\frac{\left(a^2t^2 - x^2\right)(0) - t(-2x)}{\left(a^2t^2 - x^2\right)^2}\right]$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{2tx}{\left(a^2t^2 - x^2\right)^2}\right] = \frac{\left(a^2t^2 - x^2\right)^2(2t) - (2tx)\left(2\left(a^2t^2 - x^2\right)(-2x)\right)}{\left(a^2t^2 - x^2\right)^4}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\left(a^2t^2 - x^2\right)(2t) - (2tx)(2)(-2x)}{\left(a^2t^2 - x^2\right)^3}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{2a^2t^3 - 2tx^2 + 8tx^2}{\left(a^2t^2 - x^2\right)^3} = \frac{2a^2t^3 + 6tx^2}{\left(a^2t^2 - x^2\right)^3} = 2t\left(\frac{a^2t^2 + 3x^2}{\left(a^2t^2 - x^2\right)^3}\right)$$

And we see that:

$$a^2\frac{\partial^2 f}{\partial x^2} = 2a^2t\left(\frac{a^2t^2 + 3x^2}{\left(a^2t^2 - x^2\right)^3}\right) = \frac{\partial^2 f}{\partial t^2}$$

Therefore, $f(x, t) = \frac{t}{a^2t^2 - x^2}$ satisfies the Wave Equation.

Answer to Exercise 16 (on page 55)

1. The marginal utility of labor is given by $\partial P/\partial L$:

$$\frac{\partial P}{\partial L} = \frac{\partial}{\partial L}\left[1.01L^{0.75}K^{0.25}\right] = 0.7575L^{-0.25}K^{0.25}$$

2. The marginal utility of capital is given by $\partial P/\partial K$:

$$\frac{\partial P}{\partial K} = \frac{\partial}{\partial K}\left[1.01L^{0.75}K^{0.25}\right] = 0.2525L^{0.75}K^{-0.75}$$

3. Finding the marginal utility of labor in 1916:

$$\frac{\partial P}{\partial L} = 0.7575(382)^{-.25}(126)^{0.25} \approx 0.574$$

   And finding the marginal utility of capital in 1916:

$$\frac{\partial P}{\partial K} = 0.2525(382)^{0.75}(126)^{-0.75} \approx 0.580$$

4. Since the marginal utility of capital is greater, I would invest in capital. This would yield a greater increase in production than the same investment in labor.

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