CONTENTS

Chapter 1

Differentiation

We have done some differentiation, but you haven't been given the real definition yet, because it is based on limits.

The idea is that we can find the slope between two points on the graph a and b like this:

$$m = \frac{f(b) - f(a)}{b - a}$$

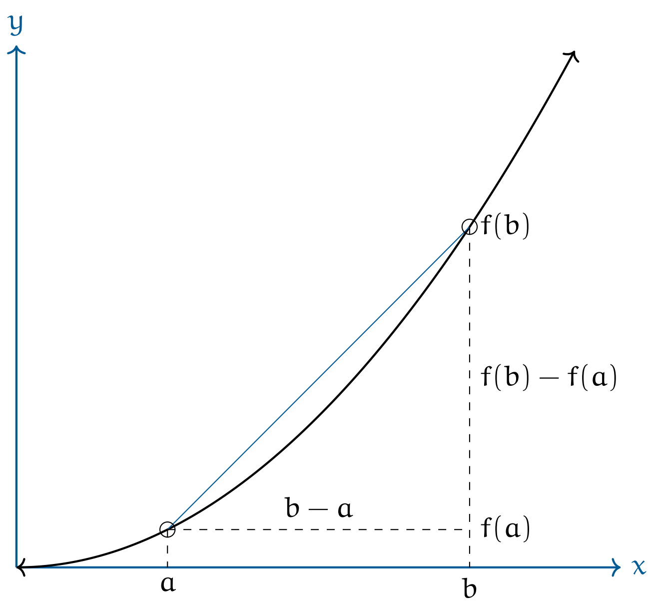


Figure 1.1: The slope of point a using points a and b.

If we want to find the slope at a, we take the limit of this as the b goes to a:

$$f'(a) = \lim_{b \to b} \frac{f(b) - f(a)}{b - a}$$

This idea is usually expressed using h as the difference between b and a:

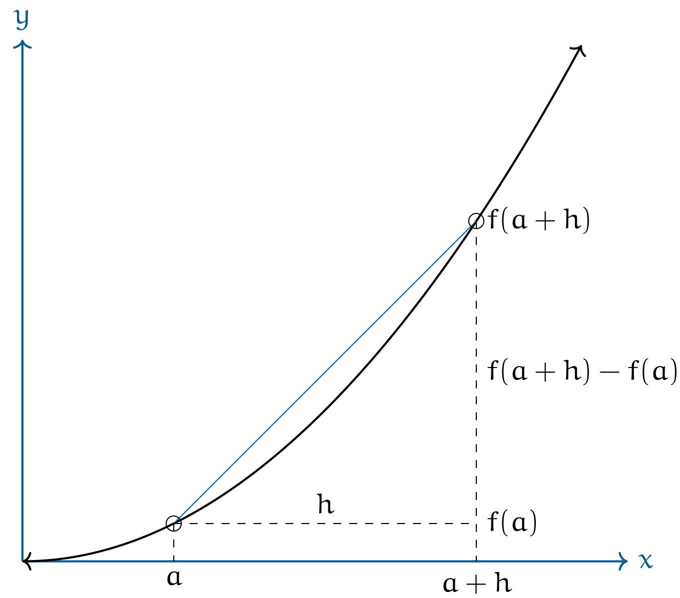


Figure 1.2: Limit of a point using h difference.

The formula then becomes:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

As h decreases to very close to zero, almost infinitesimal, what we are finding is the sloe of the tangent line to point a.

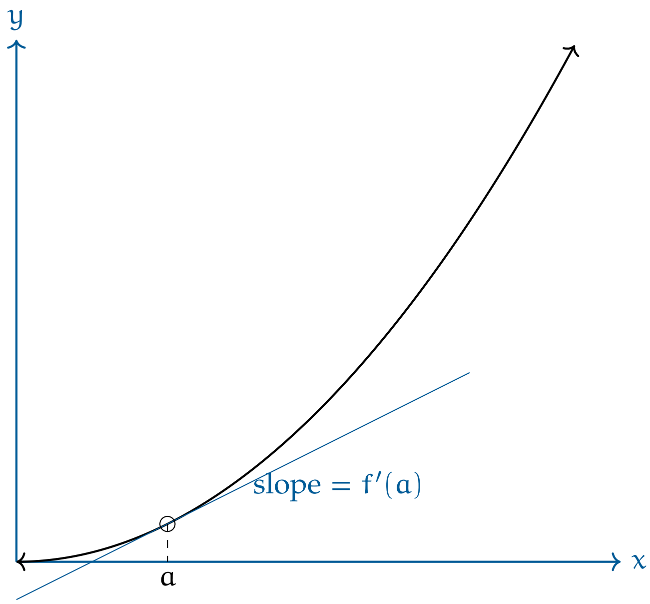


Figure 1.3: Slope at a as h approaches 0.

1.1 Differentiability

Warning: Not every function is differentiable everywhere. For example, if $f(x) = |x|$, you get a corner at zero.

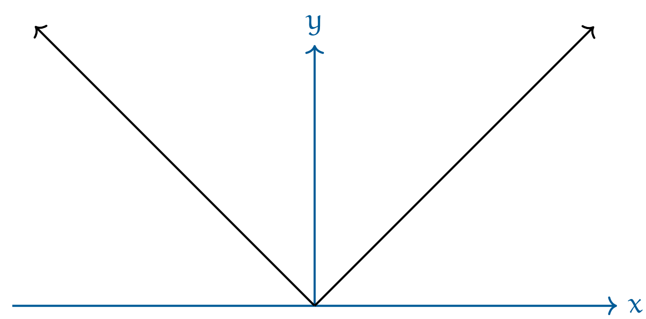


Figure 1.4: Absolute value function.

To the left of zero, the slope is -1. To the right of zero, the slope is 1. At zero? The derivative is not defined because the slopes are two different numbers.

If a function has a derivative everywhere, it is said to be *differentiable*. Generally, you can think of differentiable functions as smooth — their graphs have no corners or sharp turns. Another place where a function is not differentiable is at a vertical tangent, as the slope reaches $\pm\infty$.

It is important to note: if f is differentiable at a, it *must* also be continuous at a.

Likewise, if f is not continuous at a it is not differentiable. An example of this is a jump discontinuity. FIXME diagram here of jump discontinuity.

Exercise 1

[This problem was originally presented as a no-calculator, multiple-choice question on the 2012 AP Calculus BC exam.] Let f be the function defined by $f(x) = \sqrt{|x-2|}$ for all x. Classify each of the following statements as true or false.

1. f is continuous at $x = 2$.
2. f is differentiable at $x = 2$.
3. $\lim_{x \to 2} f(x) = 0$.
4. $x = 2$ is a vertical asymptote of the graph of $f(x)$.

Working Space

Answer on Page 49

1.2 Using the definition of derivative

Let's say that you want to know the slope of $f(x) = -3x^2$ at $x = 2$. Using the definition of the derivative, that would be:

$$f'(2) = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h} = \lim_{h \to 0} \frac{-3(2+h)^2 - \left(-3(2)^2\right)}{h}$$

$$= \lim_{h \to 0} \frac{-12 - 12h + -3(h)^2 + 12}{h} = -12$$

Chapter 2

Derivatives

In calculus, the derivative of a function represents the rate at which the function is changing at a particular point. It is a fundamental concept that has vast applications in various fields, including physics.

2.1 Definition

The derivative of a function $f(x)$ at a point x is defined as the limit:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \tag{2.1}$$

provided this limit exists. In words, the derivative of f at x is the limit of the rate of change of f at x as the change in x approaches zero. The derivative of a function is equal to the slope of the function. The derivative of a function, $f(x)$, is denoted as $f'(x)$ (read out loud as "f prime of x") or df/dx. The origin of this definition was shown in the previous chapter, Differentiation.

2.1.1 Estimating the Derivative

Consider the function $f(x) = x^2$. Suppose we want to write an equation for a line that is tangent to the curve at $x = 2$ (see figure 2.1). We already have a point that the line passes through: $(2, 4)$. To write an equation for the tangent line, we would need to know its slope, m.

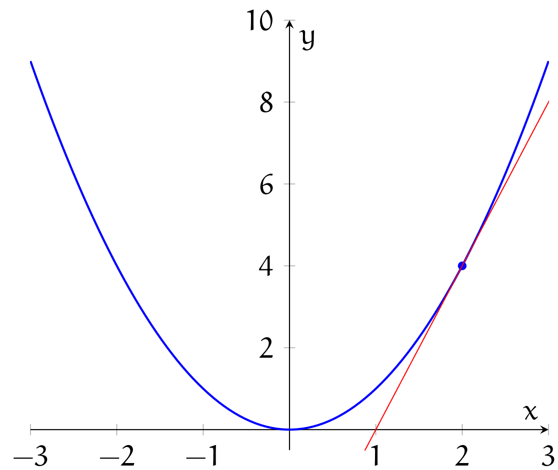


Figure 2.1: The red line is tangent to $f(x) = x^2$ at the point $(2, 4)$

We can estimate the slope by choosing points on either side of P, drawing a line through those points, and calculating the slope of that secant line (it is a secant line because it intersects the curve more than once). See figure 2.2 for a visualization.

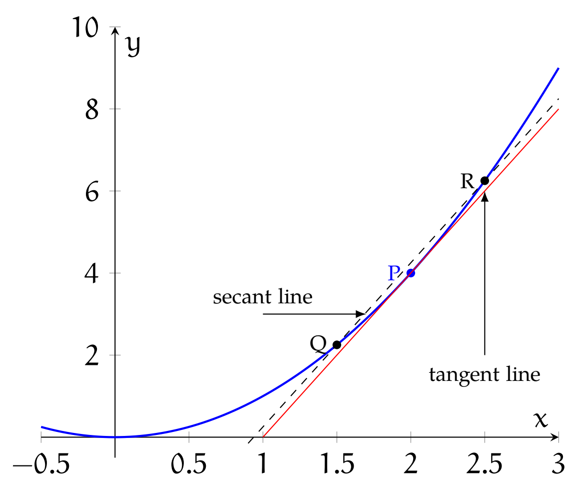


Figure 2.2: The slope of the secant line is approximately the slope of the tangent line

As the points Q and R get closer to P, the better the estimate becomes.

Much scientific data is not described as continuous functions, but rather as discrete data points. Consider the following data of a falling object:

time (seconds)	height (m)
0	50
0.5	48.775
1	45.1
1.5	38.975
2	30.4
2.5	19.375
3	5.9

A graph of the data is shown in figure 2.3.

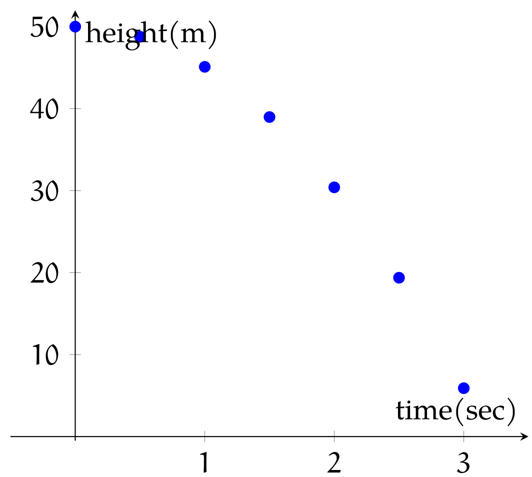


Figure 2.3: The height of a falling object over time

Suppose we wanted to estimate the velocity of the falling object at $t = 1.5s$. Recall that velocity is given by the change in position divided by the change in time. We can select data points on either side of $t = 1.5s$ and use them to find the average velocity from $t = 1s$ and $t = 2s$ (see figure 2.4):

$$v = \frac{h_2 - h_2}{t_2 - t_1} = \frac{30.4m - 45.1m}{2s - 1s} = -14.7\frac{m}{s}$$

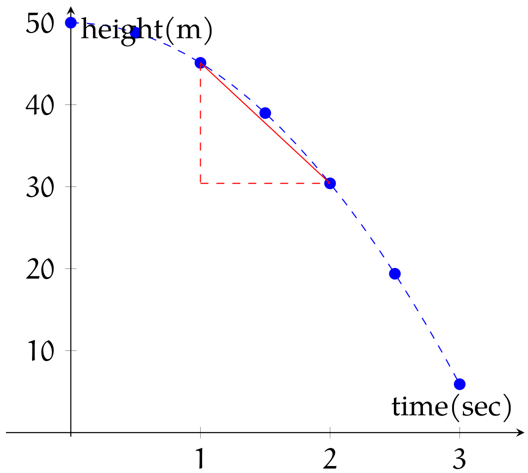


Figure 2.4: The slope of the line connecting the data points on either side of $t = 1.5$s is approximately the velocity of the falling object at $t =$ s

Example: A 1000-gallon tank drains from the bottom in 30 minutes. The volume left in the tank is recorded every 5 minutes, as shown in the data table below. Use the data to estimate $V'(15)$ and $V'(25)$, including appropriate units. At which time is the tank draining faster?

t (min)	V (gal)
5	694
10	444
15	250
20	111
25	28
30	0

Solution: To estimate $V'(15)$, we find the slope of the line connecting the data points on either side of $t = 15$:

$$V'(15) \approx \frac{111\text{gal} - 444 textgal}{20\text{min} - 10\text{min}}$$

$$V'(15) \approx \frac{-333\text{gal}}{10\text{min}}$$

$$V'(15) \approx -33.3\frac{\text{gal}}{\text{min}}$$

And we can use the data at $t = 20$ and $t = 30$ to estimate $V'(25)$:

$$V'(25) \approx \frac{0\text{gal} - 111\text{gal}}{30\text{min} - 20\text{min}}$$

$$V'(25) \approx \frac{-111\text{gal}}{10\text{min}}$$

$$V'(25) \approx -11.1\frac{\text{gal}}{\text{min}}$$

Both answers are negative because the tank is emptying, and the tank is draining faster at $t = 15$ than at $t = 25$.

Exercise 2

[This question was originally presented as a free-response, calculator-allowed question on the 2012 AP Calculus BC Exam.] The temperature of water in a tub at time t is modeled by a function, W, where $W(t)$ is measured in degrees Fahrenheit and t is measured in minutes. Values of $W(t)$ at selected times for the first 20 minutes are given in the table. Use the data in the table to estimate $W'(12)$. Show the computations that lead to your answer. Using correct units, interpret the meaning of your answer in the context of the problem.

t (minutes)	$W(t)$ (degrees Fahrenheit)
0	55.0
4	57.1
9	61.8
15	67.9
20	71.0

Working Space

Answer on Page 49

2.2 The Derivative as a Function

We have seen how to estimate the value of a derivative at a specific point on a graph. Suppose we wanted to describe the slope of a graph everywhere. That is: can we find a function, $g(x)$ that describes the slope of another function, $f(x)$, over the domain of f? Using the definition of a derivative, we can. You have already seen an algorithm to find the derivatives of polynomial functions (see chapter Differentiating Polynomials FIXME can this be linked). Recall that for a function, $f(x) = x^n$, the derivative is $f'(x) = nx^{n-1}$. Here is the proof:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$
$$f'(x) = \lim_{h \to 0} \frac{(x+h)^n - x^n}{h}$$

In order to expand the polynomial, $(x+h)^n$, we'll need to apply the Binomial Theorem, which tells us that:

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}b^3 + \cdots + nab^{n-1} + b^n$$

Substituting this into our limit definition of a derivative, we see that:

$$f'(x) = \lim_{h \to 0} \frac{x^n + nx^{n-1}h + \frac{n(n-1)}{2!}x^{n-2}h^2 + \cdots nxh^{n-1} + h^n - x^n}{h}$$
$$f'(x) = \lim_{h \to 0} \frac{nx^{n-1}h + \frac{n(n-1)}{2!}x^{n-2}h^2 + \cdots nxh^{n-1} + h^n}{h}$$
$$f'(x) = \lim_{h \to 0} nx^{n-1} + \frac{n(n-1)}{2!}x^{n-2}h + \cdots nxh^{n-2} + h^{n-1}$$
$$f'(x) = nx^{n-1}$$

Example: Use the limit definition of a derivative to find $f'(x)$ if $f(x) = 2x^3 - x^2$.

Solution: According to the limit definition, f' is:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \frac{\left[2(x+h)^3 - (x+h)^2\right] - \left[2x^3 - x^2\right]}{h}$$
$$f'(x) = \lim_{h \to 0} \frac{\left[2\left(x^3 + 3hx^2 + 3h^2x + h^3\right) - \left(x^2 + 2xh + h^2\right)\right] - 2x^3 + x^2}{h}$$
$$f'(x) = \lim_{h \to 0} \frac{2x^3 - 2x^3 + 6hx^2 + 6h^2x + 6h^3 - x^2 + x^2 - 2xh - h^2}{h}$$
$$f'(x) = \lim_{h \to 0} \frac{6hx^2 + 6h^2x + 6h^3 - 2xh - h^2}{h}$$
$$f'(x) = \lim_{h \to 0} 6x^2 + 6hx + 6h^2 - 2x - h = 6x^2 - 2x$$

Therefore, if $f(x) = 2x^3 - x^2$, then $f'(x) = 6x^2 - 2x$.

Exercise 3 Finding Functions for Derivatives

Use the limit definition of a derivative to find an equation for $f'(x)$.

1. $f(x) = mx + b$
2. $f(x) = \sqrt{16 - x}$
3. $f(x) = \frac{x^2 - 1}{2x - 3}$

Working Space

Answer on Page 49

2.3 Applications in Mathematics

2.3.1 l'Hôpital's Rule

Consider the function $h(x) = \frac{\ln x}{x-1}$ and suppose we are interested in the behavior of $h(x)$ around $x = 1$. As x approaches 0 and the denominator $x - 1$ also approaches 0. This gives the an indeterminate result:

$$\lim_{x \to 1} \frac{\ln x}{x - 1} = \frac{0}{0}$$

This is exactly the situation where L'Hôpital's Rule applies.

In Chapter 3, we will learn about Quotient Rule, which when directly applied will give this indeterminate form! Keep an eye out for that.

Now, looking at the graph of $h(x)$ (see figure 2.5), we can guess that $\lim_{x \to 1} \frac{\ln x}{x-1} = 1$.

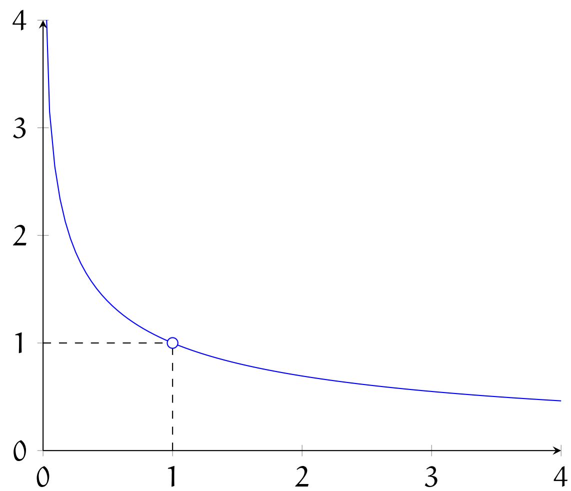


Figure 2.5: $h(x) = \frac{\ln x}{x-1}$

Let's examine the numerator and denominator separately: we'll define $f(x) = \ln x$ and $g(x) = x - 1$ (see figure 2.6).

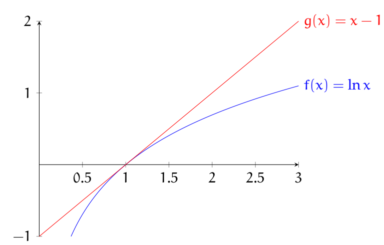


Figure 2.6: Examining each part of $\frac{\ln x}{x-1}$ separately

If we zoom in very far around $x = 1$, the graphs begin to look linear (see figure 2.7):

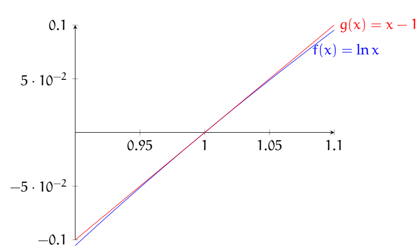


Figure 2.7: As we zoom in, the graph of $\ln x$ appears linear

We can approximate these graphs as linear functions with slopes m_1 and m_2, so that the blue curve is approximated as $y = m_1(x - 1)$ and the red curve is approximated as

$y = m_2(x-1)$. The ratio of the functions would then be

$$\frac{m_1(x-1)}{m_2(x-1)} = \frac{m_1}{m_2}$$

which is the same as the ratio of the derivatives of our linear approximations. This suggests l'Hopital's rule, that the limit of a ratio is the same as the limit of the ratio of the derivatives for certain indeterminate forms:

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

.

Let's apply l'Hopital's rule to our limit of $h(x)$:

$$\lim_{x\to 1} \frac{\ln x}{x-1} = \lim_{x\to 1} \frac{\frac{d}{dx}\ln x}{\frac{d}{dx}(x-1)} = \lim_{x\to 1} \frac{\frac{1}{x}}{1} = 1$$

Notice our result with l'Hôpital's rule matches our guess based on the graph of $h(x) = \frac{\ln x}{x-1}$.

L'Hospital's rule also applies to the indeterminate result $\frac{\pm\infty}{\pm\infty}$. For a limit of the form $\lim_{x\to a} \frac{f(x)}{g(x)}$, l'Hôpital's rule applies if:

1. the original limit is of the indeterminate form $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$
2. f and g are differentiable on an interval containing a (but possibly not differentiable at a)
3. $g'(x) \neq 0$ on said interval

Example: Determine $\lim_{x\to\infty} \frac{e^x}{x^2}$.

Solution: We begin by evaluating the limit:

$$\lim_{x\to\infty} \frac{e^x}{x^2} = \frac{e^\infty}{\infty^2} = \frac{\infty}{\infty}$$

This is an indeterminate form that we can apply l'Hopital's rule to:

$$= \lim_{x\to\infty} \frac{\frac{d}{dx}e^x}{\frac{d}{dx}x^2} = \lim_{x\to\infty} \frac{e^x}{2x}$$

Evaluating this limit, we get another indeterminate form:

$$= \frac{e^{\infty}}{2 \cdot \infty} = \frac{\infty}{\infty}$$

Don't panic! We can apply l'Hôpital's rule again (in fact, we can apply l'Hôpital's rule as many times as needed to evaluate a limit, as long as we keep getting $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$):

$$= \lim_{x \to \infty} \frac{\frac{d}{dx} e^x}{\frac{d}{dx} 2x} = \lim_{x \to \infty} \frac{e^x}{2} = \frac{\infty}{2} = \infty$$

and therefore, $\lim_{x \to \infty} \frac{e^x}{x^2} = \infty$.

Exercise 4

What is $\lim_{x \to 0} \frac{\tan x - x}{x^3}$?

Working Space

Answer on *Page 50*

Note that this rule only applies if it is in *indeterminate form*. The most common indeterminate forms are

$$\frac{0}{0} \quad \text{and} \quad \frac{\infty}{\infty},$$

which you may end up with when evaluating quotients!

However, indeterminate forms can also appear in other ways. Some common examples include:

$$0 \cdot \infty, \quad \infty - \infty, \quad 0^0, \quad 1^{\infty}, \quad \infty^0.$$

These expressions are indeterminate because their values cannot be determined without evaluating further. In many cases, such limits can be rewritten into a quotient that results in either

$$\frac{0}{0} \quad \text{or} \quad \frac{\infty}{\infty}.$$

This allows l'Hôpital's Rule to be applied!

Example: Evaluate

$$\lim_{x \to 0^+} x \ln x.$$

Solution: As $x \to 0^+$, we have $x \to 0$ and $\ln x \to -\infty$, producing the indeterminate form

$$0 \cdot (-\infty).$$

Rewrite the expression as a quotient:

$$x \ln x = \frac{\ln x}{1/x}.$$

Now the limit has the form

$$\frac{-\infty}{\infty},$$

so l'Hôpital's Rule applies. Differentiate the numerator and denominator:

$$\lim_{x \to 0^+} \frac{\ln x}{1/x} = \lim_{x \to 0^+} \frac{1/x}{-1/x^2}.$$

Simplifying,

$$\lim_{x \to 0^+} (-x) = 0.$$

Therefore,

$$\lim_{x \to 0^+} x \ln x = 0.$$

Exercise 5

Evaluate each of the following limits, using l'Hôpital's rule where needed.

1. $\lim_{x \to 3} \frac{x-3}{x^2-9}$
2. $\lim_{x \to 1/2} \frac{6x^2+5x-4}{4x^2+16x-9}$
3. $\lim_{x \to 0^+} \frac{\ln x}{\sqrt{x}}$
4. $\lim_{x \to \pi} \frac{1+\cos x}{1-\cos x}$
5. $\lim_{x \to 1} \frac{x \sin x - 1}{2x^2-x-1}$

Working Space

Answer on Page 51

2.3.2 Mean Value Theorem

The Mean Value Theorem (MVT) states that on an interval $[a, b]$ where a continuous function f is differentiable on an open interval (a, b), there is at least one point where the tangent line to f has the same slope as a line connecting the points $(a, f(a))$ and $(b, f(b))$. Consider the graph of $f(x) = x^2$ (see figure 2.8). The line connecting the points $(-1, 1)$ and $(2, 4)$ has a slope of $\frac{1}{2}$. MVT tells us there must be *at least one* point, c, on the interval $x \in (-1, 2)$ where $f'(c) = \frac{1}{2}$. We can find this point by setting $f'(x)$ equal to $\frac{1}{2}$:

$$2x = \frac{1}{2} \rightarrow x = \frac{1}{4}$$

Examining the figure 2.8, you can see that the tangent at $f(\frac{1}{4})$ (the black line) is parallel to the red line connecting $(-1, f(-1))$ and $(2, f(2))$.

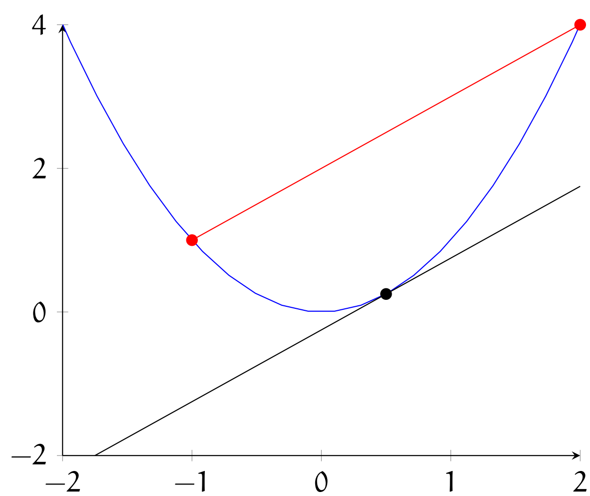


Figure 2.8: $f(x) = x^2$

Note that MVT doesn't tell us *where* $f'(x)$ is parallel to the line connecting $(a, f(a))$ and $(b, f(b))$, just that some value c exists that satisfies the condition.

Example: Consider a hammer thrown upwards at 5 $\frac{m}{s^2}$ on Earth (where the acceleration due to gravity is approximately $-9.8\frac{m}{s^2}$).

Solution: We can use the MVT to show that there must be some point in the hammer's path upwards where the velocity of the hammer is exactly equal to its average velocity as it flies through the air.

The hammer's rise can be described with the function $y(t) = 5t - 4.9t^2$. The hammer reaches its peak at approximately $t = 0.51$. So, we are looking for some value, c, such that

$$y'(c) = \frac{y(0.51) - y(0)}{0.51 - 0} = \frac{5(0.51) - 4.9(0.51^2)}{0.51} = \frac{1.2755}{0.51} = 2.5$$

Solving $y'(t) = 5 - 9.8t = 2.5$, we find that the c that satisfies the MVT is approximately 0.255. This result is illustrated in figure 2.9:

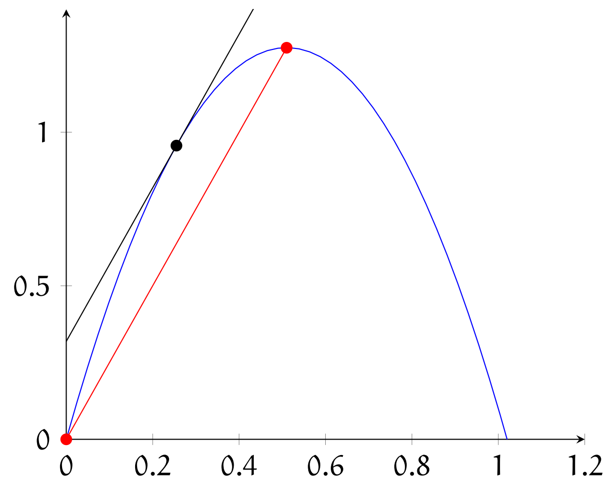


Figure 2.9: The height of a hammer tossed upwards at $5\frac{m}{s}$

MVT Practice

Exercise 6

AT 3:30 PM, a car's speedometer reads $30\frac{mi}{hr}$. At 3:40 PM, it reads $50\frac{mi}{hr}$. Show that at some time between 3:30 and 3:340 PM, the car's acceleration is exactly $120\frac{mi}{hr^2}$.

Working Space

Answer on Page 52

Exercise 7

Find the number c that satisfies the MVT on the given interval.

(a) $f(x) = \sqrt{x}$, $[0, 4]$

(b) $f(x) = e^{-x}$, $[0, 2]$

(c) $f(x) = \ln x$, $[1, 4]$

Working Space

Answer on Page 52

2.4 Applications in Physics

In physics, derivatives play a vital role in describing how quantities change with respect to one another.

2.4.1 Velocity and Acceleration

In kinematics, the derivative of the position function with respect to time gives the velocity function, and further taking the derivative of the velocity function gives the acceleration function. For example, if $s(t)$ represents the position of an object at time t, then the velocity $v(t)$ and acceleration $a(t)$ are given by:

$$v(t) = \frac{ds}{dt} \quad \text{and} \quad a(t) = \frac{dv}{dt} = \frac{d^2s}{dt^2} \tag{2.2}$$

Practice

A particle's motion is described by $s(t) = t^3 - 6t^2 + 6t$, where t is measured in seconds and s is measured in meters. Answer the following questions about the particle's motion:

Exercise 8

Find the velocity at time t.

Working Space

Answer on Page 53

Exercise 9

What is the velocity after 2s? After 4s?

Working Space

Answer on Page 53

Exercise 10

When is the particle at rest?

Working Space

Answer on Page 54

2.4.2 Force and Momentum

In mechanics, the derivative of the momentum of an object with respect to time gives the net force acting on the object, as stated by Newton's second law of motion:

$$F = \frac{dp}{dt} \tag{2.3}$$

where F is the force, p is the momentum, and t is the time.

Chapter 3

Rules for Finding Derivatives

Derivatives play a key role in calculus, providing us with a means of calculating rates of change and the slopes of curves. In this chapter, we present some common rules used to calculate derivatives.

3.1 Constant Rule

The derivative of a constant is zero. If c is a constant and x is a variable, then:

$$\frac{d}{dx}c = 0 \tag{3.1}$$

3.2 Power Rule

For any real number n, the derivative of x^n is:

$$\frac{d}{dx}x^n = nx^{n-1} \tag{3.2}$$

3.3 Product Rule

The derivative of the product of two functions is:

$$\frac{d}{dx}(fg) = f'g + fg' \tag{3.3}$$

where f' and g' denote the derivatives of f and g, respectively.

3.4 Quotient Rule

The derivative of the quotient of two functions is:

$$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{f'g - fg'}{g^2} \tag{3.4}$$

3.5 Chain Rule

The derivative of a composition of functions is:

$$\frac{d}{dx}(f(g(x))) = f'(g(x)) \cdot g'(x) \tag{3.5}$$

3.6 Practice

Exercise 11

If f is the function given, find f'.

1. $f(x) = x \sin x$
2. $f(x) = (x^3 - \cos x)^5$
3. $f(x) = \sin^3 x$

Working Space

Answer on Page 54

Exercise 12

Let $f(x) = 7x - 3 + \ln x$. Find $f'(x)$ and $f'(1)$

Working Space

Answer on Page 54

Exercise 13

[This question was originally presented as a multiple-choice, no-calculator question on the 2012 AP Calculus BC exam.] The position of a particle in the xy-plane is given by the parametric equations $x(t) = t^3 - 3t^2$ and $y(t) = 12t - 3t^2$. State a coordinate point (x, y) at which the particle is at rest.

Working Space

Answer on Page 54

Exercise 14

Let $f(x) = \sqrt{x^2 - 4}$ and $g(x) = 3x - 2$. Find the derivative of $f(g(x))$ at $x = 3$.

Working Space

Answer on Page 55

Exercise 15

The particle's position on the x-axis is given by $x(t) = (t - a)(t - b)$, where a and b are constants and $a \neq b$. At what time(s) is the particle at rest?

Working Space

Answer on Page 55

Exercise 16

[This question was originally presented as a multiple-choice, no-calculator question on the 2012 AP Calculus BC exam.] Let $f(x) = \frac{x}{x+2}$. At what values of x does f have the property that the line tangent to f has a slope of $\frac{1}{2}$?

Working Space

Answer on Page 55

Exercise 17

For $t \geq 0$, the position of a particle moving along the x-axis is given by $x(t) = \sin t - \cos t$. (a) When does the velocity first equal 0? (b) What is the acceleration at the time when the velocity first equals 0?

Working Space

Answer on Page 56

Exercise 18

The graph of $y = e^{(}\tan x) - 2$ crosses the x-axis at one point on the interval $[0, 1]$. What is the slope of the graph at this point?

Working Space

Answer on Page 56

Exercise 19

The function f is defined by $f(x) = \sqrt{25 - x^2}$ for $-5 \leq x \leq 5$.
(a) Find $f'(x)$.
(b) Write an equation for the line tangent to the graph at $x = -3$.

Working Space

Answer on Page 56

Exercise 20

For $0 \leq t \leq 12$, a particle moves along the x-axis. The velocity of the particle at a time t is given by $v(t) = \cos \frac{\pi}{6} t$. What is the acceleration of the particle at time $t = 4$?

Working Space

Answer on Page 57

Exercise 21

[This question was originally presented as a multiple-choice, calculator-allowed question on the 2012 AP Calculus BC exam.] Let f and g be the functions given by $f(x) = e^x$ and $g(x) = x^4$. On what intervals is the rate of change of $f(x)$ greater than the rate of change of $g(x)$?

Working Space

Answer on Page 57

3.7 Conclusion

These rules form the basis for calculating derivatives in calculus. Many more complex rules and techniques are built upon these fundamental rules.

Chapter 4

First and Second Derivatives and the Shape of a Function

4.1 Using first derivatives to describe a function

4.1.1 Critical Values

Let's re-examine our graph showing the height of a hammer tossed in the air:

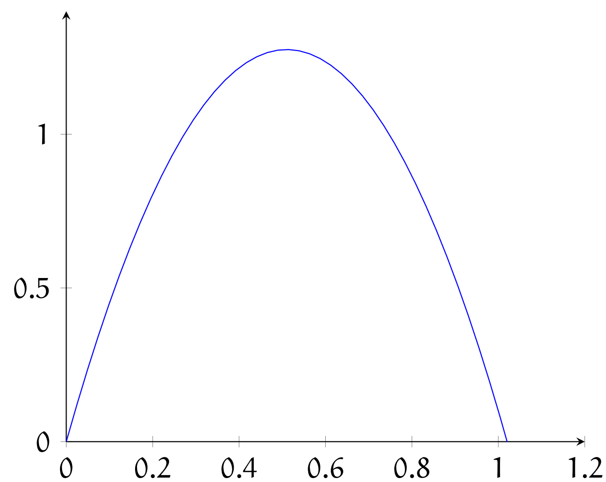


Figure 4.1: Height of a hammer over time

As you can see, the hammer reaches its peak around $t \approx 0.5s$ (see figure 4.1). Let's add tangent lines just before and after the peak of the hammer's path, so we can more easily examine how the slope of the graph changes:

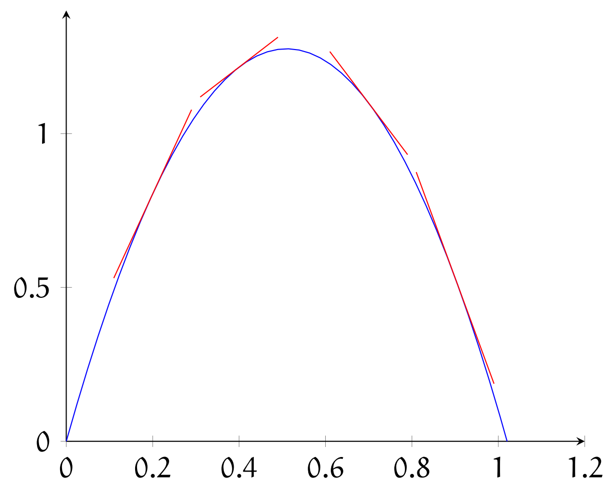


Figure 4.2: height of a hammer over time

In figure 4.2, we see that the slope changes from positive to negative as t increases. That implies that $f'(t)$ also changes from positive to negative. In fact, at the highest point of the hammer's flight, the slope (and therefore $f'(t)$) is exactly zero! In general,

1. If $f'(x) > 0$ (positive slope) on an interval, then $f(x)$ is increasing on that interval.
2. If $f'(x) < 0$ (negative slope) on an interval, then $f(x)$ is decreasing on that interval.

Example 1: Find where the function $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ is increasing.

Solution: We want to find the intervals where $f'(x) > 0$. First, we take the derivative to find $f'(x)$:

$$f'(x) = 12x^3 - 12x^2 - 24x$$

It will be easier to analyze the value of $f'(x)$ if we factor it so:

$$f'(x) = 12x(x-2)(x+1)$$

To determine where $f'(x) > 0$, we start by finding where $f'(x) = 0$ (in this case, this is true when $x = -1, 0, 2$). These values of x are called *critical values*, and we will use them to divide $f'(x)$ into intervals. (Critical values are also called critical numbers, and we will use both in this text.) On each of these intervals, $f'(x)$ must be always positive or always negative. This is shown in the graph below:

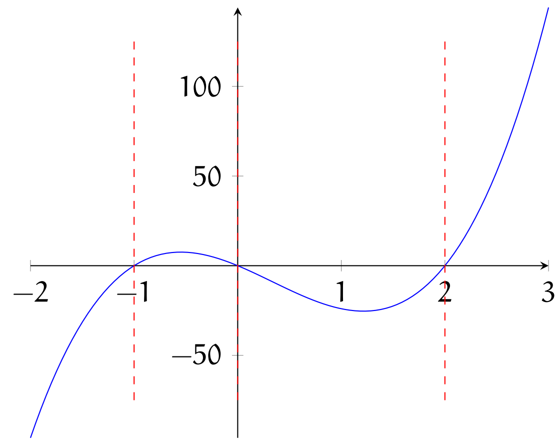


Figure 4.3: $f'(x)$ with critical values

As you can see in figure 4.3, $f'(x) > 0$ on two intervals: $x \in (-1, 0)$ and $x \in (2, \infty)$. These are open intervals because $f'(x) = 0$ at $x = -1$, $x = 0$, and $x = 2$. But what if we had a more complex function, or didn't have the resources to graph it?

We can use a table to help us analyze the value of $f'(x)$ (and therefore the behavior of $f(x)$). For each interval around the critical values, we can determine if $f'(x)$ is positive or negative by noting the value of the factors of $f'(x)$, which are $12x$, $x - 2$, and $x + 1$ in this case. For example, for $x < -1$, $12x < 0$, $(x - 2) < 0$, and $(x + 1) < 0$. Three negatives multiplied together is also negative. Therefore, for $x < -1$, $f'(x)$ is negative and $f(x)$ is decreasing. We can analyze all of the intervals similarly and log the results in a table:

x	$12x$	$x - 2$	$x + 1$	$f'(x)$	$f(x)$
$x < -1$	negative	negative	negative	negative	decreasing
$-1 < x < 0$	negative	negative	positive	positive	increasing
$0 < x < 2$	positive	negative	positive	negative	decreasing
$2 < x$	positive	positive	positive	positive	increasing

Notice the table method yields the same result as examining the graph: $f(x)$ is increasing for $x \in (0, -1)$ and $x \in (2, \infty)$, which can also be written as $x \in (-1, 0) \cup (2, \infty)$.

Exercise 22

Let g be the function given by $g(x) = x^2e^{kx}$, where k is a constant. For what value(s) of k does g have a critical value at $x = \frac{2}{3}$?

Working Space

Answer on Page 57

4.1.2 Local Extrema

Examine the graphs of x^2, $\sin x$, and $y = \sqrt{4 - x^2}$ below. Each has a dot at a local extreme (either a local minimum or local maximum). Sketch what you think the tangent line to the graph would be at each local extreme. Use this to estimate the value of the derivative at that point.

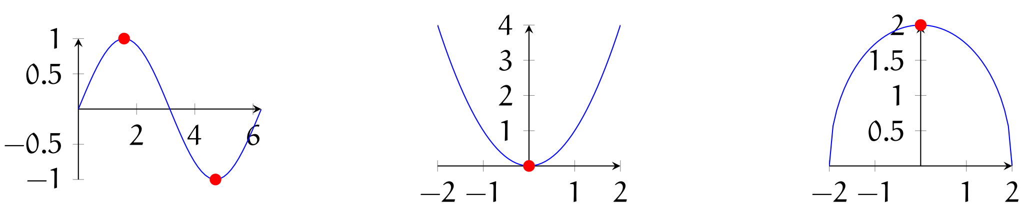


Figure 4.4: Three functions with highlighted points.

You should notice that all of the tangent lines are horizontal. Since the tangent lines at these local extrema have a slope of 0, that tells us $f'(x) = 0$ at these points as well. In fact, for *all* local minima and maxima, the value of the derivative is zero at that point. However, the converse statement is not necessarily true; just because the derivative is zero at some $x = c$, it does not mean there is a local extrema at $f(c)$. Consider $f(x) = x^3 + 3$, shown in figure 4.5:

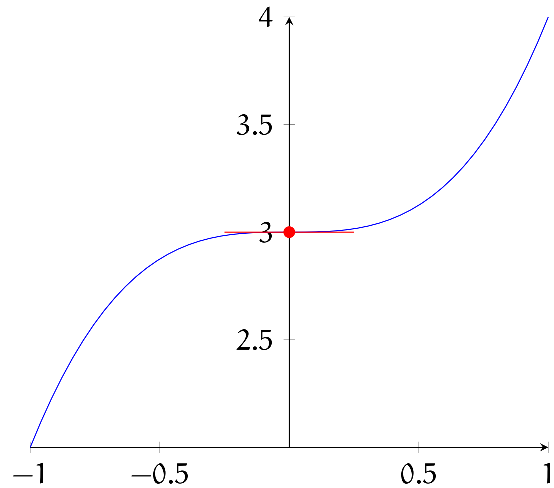


Figure 4.5: $f(x) = x^3 + 3$

At $x = 0$, $f'(x) = 0$, but there is not a local extreme. For a local extreme to exist, the graph of $f(x)$ must change from increasing to decreasing, or vice versa. Look closely at figure 4.5: the function is increasing for $x < 0$ and $x > 0$. Another way of saying this is to note that the graph of $f'(x)$ touches but does not cross the x-axis in this case:

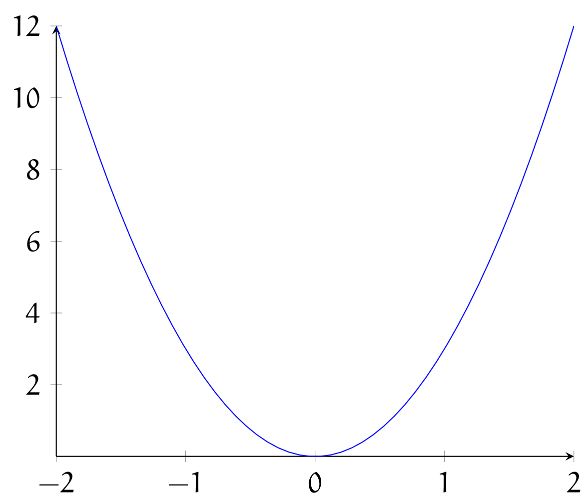


Figure 4.6: $f'(x) = 3x^2$

If $f(x)$ changes from increasing to decreasing, then $f'(x)$ is changing from positive to negative (i.e. crossing the x-axis). Look at the derivative of $f(x) = \sin x$, $f'(x) = \cos x$, presented in figure 4.7. The x-values where local extrema exist on $f(x)$ are marked in red (recall $\sin x = \pm 1$ when $x = \frac{n\pi}{2}$):

As you can see, local extrema are indicated when $f'(x)$ crosses the x-axis. If $f'(x)$ is negative to the left of $x = c$ and positive to the right, then $f(x)$ has a local minimum at $x = c$. On the other hand, if $f'(x)$ is positive to the left of $x = c$ and negative to the right, then $f(x)$ has a local maximum at $x = c$. Any value of $x = c$ where $f'(c) = 0$ is called a **critical number** or a **critical value**. Values where $f(c)$ does not exist are also a critical numbers.

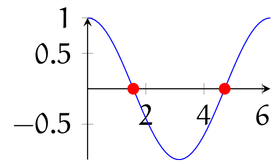


Figure 4.7: $f'(x) = \cos x$

4.1.3 Practice: Interval of Increasing and Decreasing, Local Extrema

Exercise 23

Let f be the function given by $f(x) = 300x - x^3$. On which of the following intervals is f increasing?

Working Space

Answer on Page 58

Exercise 24

Find the intervals on which $f(x) = x^3 - 3x^2 - 9x + 4$ is increasing or decreasing. Then, find all local minimum and/or maximum values of $f(x)$.

Working Space

Answer on Page 59

4.1.4 Global Extrema

Now that we've learned how to identify local minima and maxima, let's expand the discussion to include global extrema. A global extreme is an absolute minimum or maximum value of a function over a particular interval or the entire domain of the function. Let's examine the graph of $f(x) = x^4 - 5x^3 + 6x^2$ over the domain $x \in [-1, 4]$.

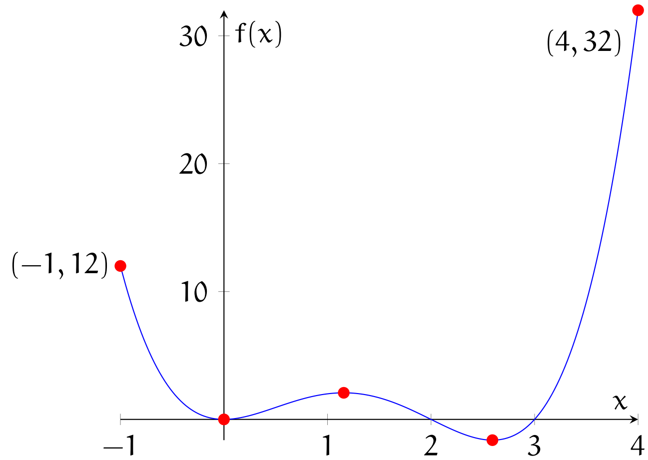


Figure 4.8: Graph of $f(x) = x^4 - 5x^3 + 6x^2$

As you can see in figure 4.8, $f(x)$ has two local minima and one local maximum. Additionally, the endpoints are labeled. To determine the *global* extrema, we need to examine the any local extrema (identified here graphically, but you can also identify them mathematically using that you learned in the "Local Extrema" subsection) **and** the endpoints of the domain (or the function's behavior at $\pm\infty$, if you are Notet restricted to a specific domain).

In the case of $f(x) = x^4 - 5x^3 + 6x^2, \text{ for} x \in [-1, 4]$, the global maximum value is 32 at $x = 4$ and the global minimum is -1.623 at $x = 2.593$.

If a function is continuous on an interval, then there must exist a global maximum and global minimum on that interval. These global extrema may also be local extrema (as is the case for $f(2.593)$ in the example above) or not (as is the case for $f(4)$). Applying the Closed Interval Method is a straightforward way to identify global (absolute) extrema.

To find the global extrema of a continuous function, f, on a closed interval $[a, b]$:

1. Find the values of f at the critical numbers of f in (a, b).
2. Find the values of f at the endpoints of the interval.
3. The largest of the values from steps 1 and 2 is the absolute maximum; the smallest of the values is the absolute minimum.

Let's use the Closed Interval Method to determine the global extrema for the function $g(x) = x - 3\sin x$ on the interval $x \in [0, 2\pi]$.

To find the value of g at any critical numbers, we must first identify the critical numbers. Recall that critical numbers are values where the first derivative of the function is 0 or does not exist. To find critical numbers, we set g' equal to 0:

$$g'(x) = 1 - 3\cos x = 0$$

$$3\cos x = 1$$
$$\cos x = \frac{1}{3}$$
$$x = 1.23, 5.052$$

Now, we substitute these critical numbers back into $g(x)$:

$$g(1.23) \approx -1.60$$

$$g(5.052) = 7.881$$

Now we need to check the endpoints:

$$g(0) = 0 - 3 * 0 = 0$$

$$g(2\pi) = 2\pi - 3 * 0 = 2\pi \approx 6.28$$

The results are presented in the table below:

x	$g(x)$
0	0
1.23	-1.60
5.052	7.881
6.28	6.28

Therefore, for $g(x) = x - 3\sin x$ on the interval $x \in [0, 2\pi]$, the global maximum is $g(5.052) = 7.881$ and the global minimum is $g(1.23) = -1.60$.

4.1.5 Practice: Global Extrema

Exercise 25

Let f be the function defined by $f(x) = \frac{\ln x}{x}$. What is the absolute maximum value of f?

Working Space

Answer on Page 59

Exercise 26

Find the global minimum and maximum values on the stated interval.

1. $f(x) = 12 + 4x - x^2$, $[0, 5]$
2. $f(t) = \frac{\sqrt{t}}{1+t^2}$, $[0, 2]$
3. $f(t) = 2\cos t + \sin 2t$, $[0, \frac{\pi}{2}]$
4. $f(x) = \ln x^2 + x + 1$, $[-1, 1]$

Working Space

Answer on Page 60

4.2 Sketching f from f'

Now that we know how the shape of f is related to the value of f', we can predict the shape of f if we are given f'. Take the example $f'(x) = -(x-1)(x-5)$, shown in figure 4.9:

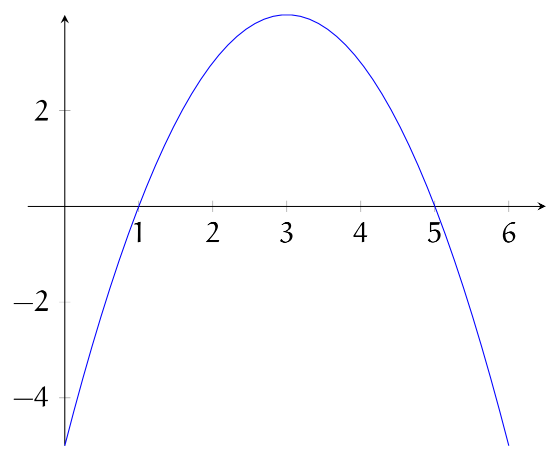


Figure 4.9: Graph of $f' = -(x-1)(x-5)$

Using the graph of f', we can construct an approximate sketch of f. First, let's identify the critical numbers. Where does $f' = 0$? Take a second to examine the graph of f' above and jot down what you think the critical numbers are.

You should recall that critical numbers are x-values where $f' = 0$. Examining the graph of f', we see that $f' = 0$ at $x = 1$ and $x = 5$. We can now use a table to describe the behavior of f:

x	$x-1$	$x-5$	f'	behavior of f
$x < 1$	negative	negative	negative	decreasing
$x = 1$	zero	negative	zero	local minimum
$1 < x < 5$	positive	negative	positive	increasing
$x = 5$	positive	zero	zero	local maximum
$x > 5$	positive	positive	negative	decreasing

We can use this information to sketch a possible graph of f. We start by noting the location of local extrema:

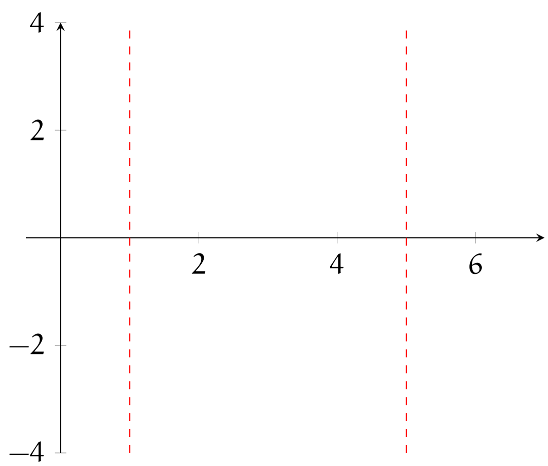


Figure 4.10: Possible graph of f

We know there is a local minimum at $x = 1$ and a local maximum at $x = 5$. We can add sketches around these values to indicate what we know about f:

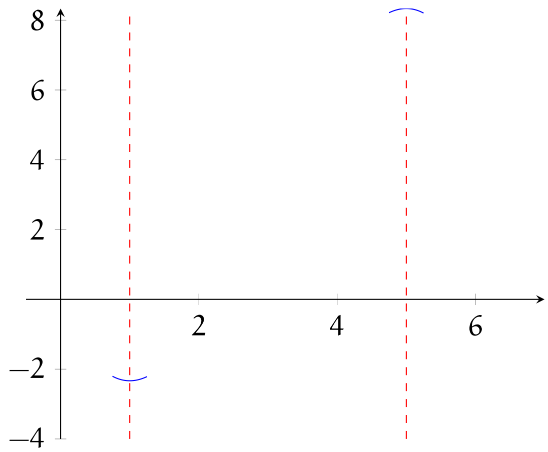


Figure 4.11: Possible graph of f

Lastly, we know f is increasing on $1 < x < 5$ and decreasing everywhere else, so we fill in the space between our local extrema:

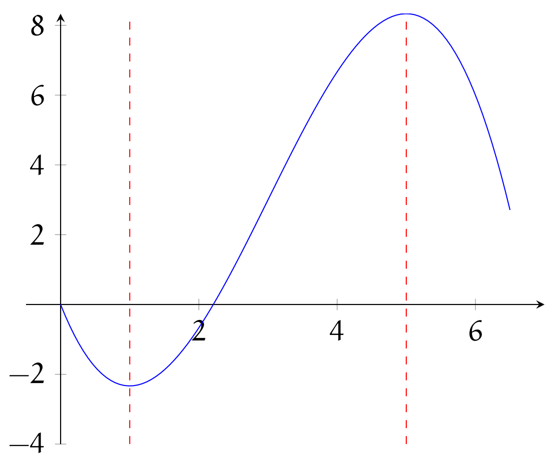


Figure 4.12: Possible graph of f

However, figure 4.12 is only a *possible* graph of f. Analyzing f' reveals the shape of f, but not how high or low it is on the y-axis. Recall that the derivative of a constant is zero. Therefore, any $+c$ (where c is a constant) is lost when taking the derivative. So, there are many sketches of f that fulfill the behavior of f indicated by f'. You can see several of the possible sketches for f in figure 4.13.

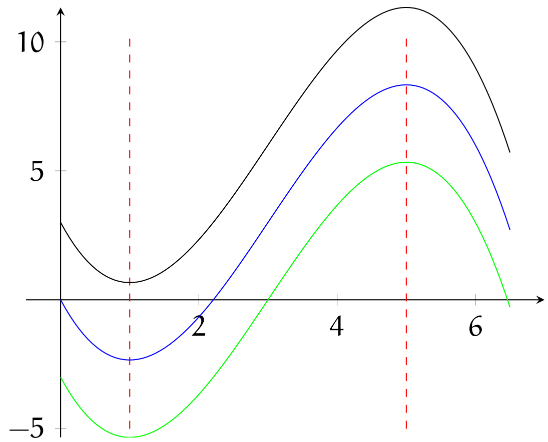


Figure 4.13: Possible graphs of f

4.2.1 Practice Sketching f from f′

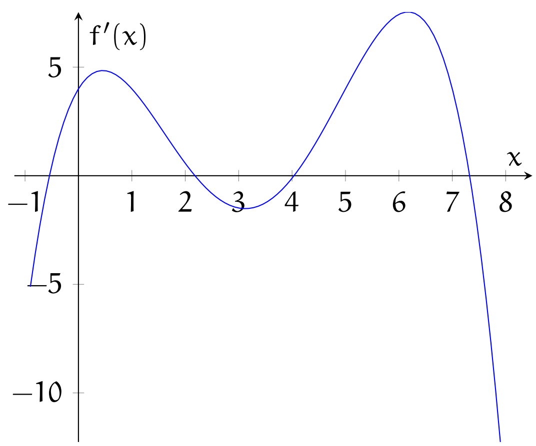


Figure 4.14: Graph of $f'(x)$

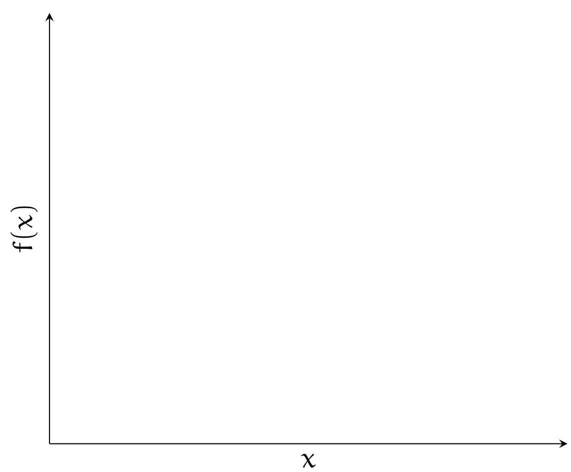
Exercise 27

Use figure 4.14 to answer the following questions:

1. On what approximate intervals is f increasing or decreasing?
2. At what approximate values of x does f have a local maximum or minimum?
3. Sketch a possible graph of f in the space below:

Working Space

Answer on Page 62

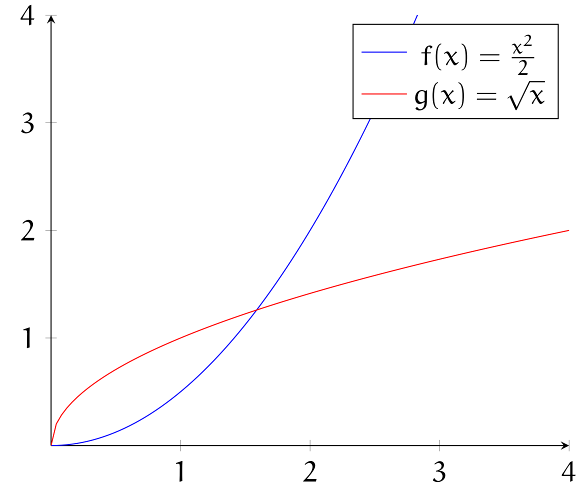


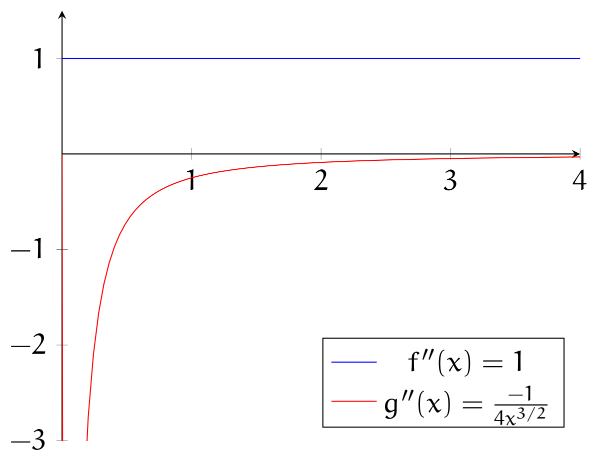
4.3 Using second derivatives to describe a function

4.3.1 Concavity

Let's examine two increasing functions, $f(x) = \frac{x^2}{2}$ and $g(x) = \sqrt{x}$:

Even though both of these functions are increasing, they have different shapes. $f(x)$ looks like a bowl. On the other hand, $g(x)$ looks like an upside-down bowl. These shapes are called *concave up* (in the case of $f(x)$) and *concave down* (in the case of $g(x)$). Both functions





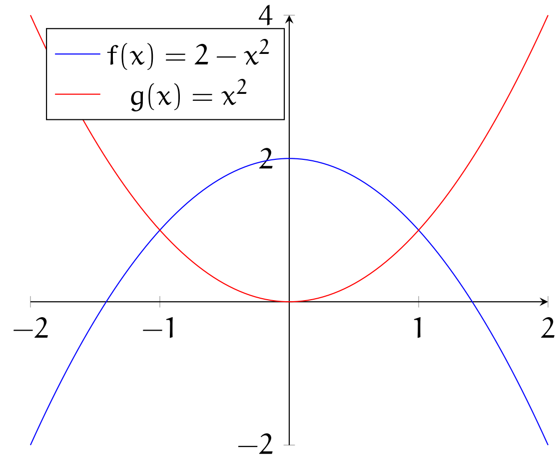
are increasing on the interval $x \in [0, 4]$, and therefore both $f'(x)$ and $g'(x)$ are positive on the stated interval. Let's look at their second derivatives, $f''(x)$ and $g''(x)$:

As you can see, $f''(x) > 0$ and $g''(x) < 0$. The second derivative tells us if a function is concave up or concave down. In general:

1. If $f''(x) > 0$ for all x in a given interval, then the graph of f is concave up on the interval.
2. If $f''(x) < 0$ for all x in a given interval, then the graph of f is concave down on the interval.

Additionally, the second derivative can help us determine if there is a local minimum or maximum at critical numbers. Look at the graphs of $f(x) = 2 - x^2$ and $g(x) = x^2$, which both have first derivatives equal to 0 at $x = 0$:

When the graph is concave up, there is a local minimum where the first derivative equals



0. When the graph is concave down, there is a local maximum where the first derivative equals 0. This is summarized with the Second Derivative Test:

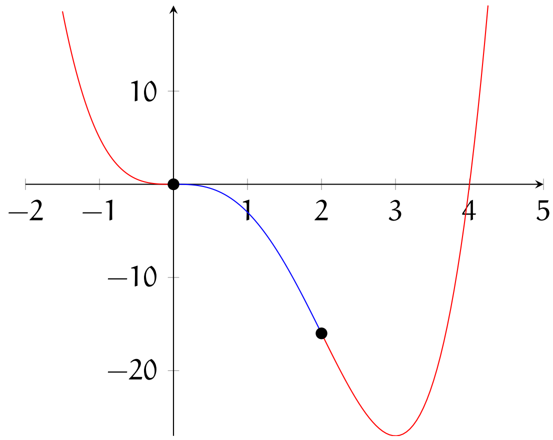
Suppose f'' is continuous near c. Then,

1. If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.
2. If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.

4.3.2 Inflection Points

If f is concave up when $f'' > 0$ and concave down when $f'' < 0$, what about when $f'' = 0$? This is the value at which f changes from concave up to concave down (or vice versa), which is called an *inflection point*. Similar to local extrema with f', if there is an inflection point at $x = c$, then $f''(c) = 0$, but the converse is not necessarily true. To check if $x = c$ is an inflection point, then f'' should change signs on either side of $x = c$ (either from positive to negative to from negative to positive).

Look at the graph of $f(x) = x^4 - 4x^3$. The concave up areas are shown in red, and the concave down in blue:



Let's examine f'' to confirm the inflection points are at $(0, 0)$ and $(2, -16)$. First, we note that $f''(x) = 12x^2 - 24x$. Factoring, we see that $f''(x) = 12x(x - 2)$, which has zeroes at $x = 0$ and $x = 2$. For $x < 0$, $f'' > 0$, and for $0 < x < 2$, $f'' < 0$; therefore, there is an inflection point in f at $(0, 0)$.

Exercise 28

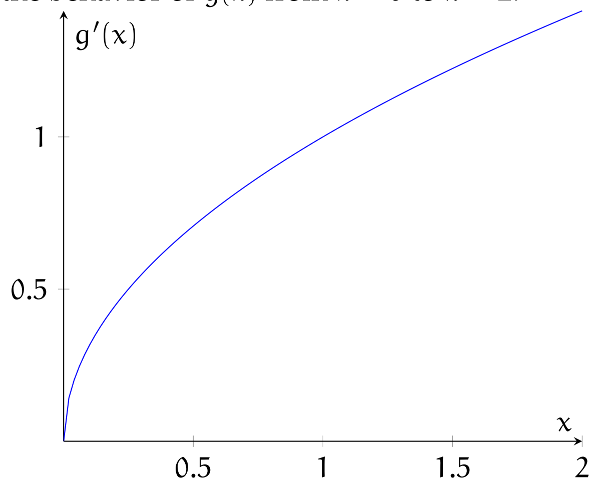
Prove that the other inflection point for $f(x) = x^4 - 4x^3$ is $(2, -16)$.

Working Space

Answer on *Page 63*

Exercise 29

The graph below shows $g'(x)$. Describe the behavior of $g(x)$ from $x = 0$ to $x = 2$.



Working Space

Answer on Page 63

Exercise 30

[This question was originally presented as a calculator-allowed, multiple-choice problem on the 2012 AP Calculus BC exam.] For $-1.5 < x < 1.5$, let f be a function with first derivative given by $f'(x) = e^{(x^4-2x^2+1)} - 2$. State the interval(s) (to three decimal places) for which f is concave down.

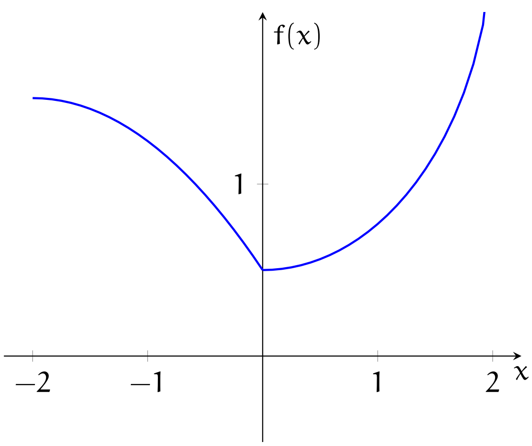
Working Space

Answer on Page 63

Exercise 31

[The following problem was originally presented as a calculator-allowed, multiple-choice question on the 2012 AP Calculus BC exam.] Consider the function, f, whose graph is shown below. Classify each of the following statements as true or false and explain.

1. $f' > 0$ for $x \in (-2, 0)$.
2. f is differentiable at $x = 0$.
3. $f'' > 0$ for $x \in (0, 2)$
4. f has a critical value at $x = 0$



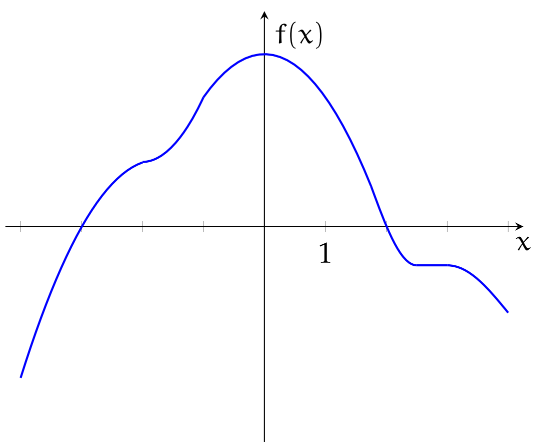
Working Space

Answer on Page 64

Exercise 32

[The following problem was originally presented as a calculator-allowed, multiple-choice question on the 2012 AP Calculus BC exam.] The graph of f', the derivative of f, is shown below. Classify each of the following statements as true or false and explain your answer.

1. f has a relative minimum at $x = -3$.
2. The graph of f has a point of inflection at $x = -2$.
3. The graph of f is concave down for $0 < x < 4$.



Working Space

Answer on Page 64

Exercise 33

[The following problem was originally presented as a calculator-allowed, multiple-choice question on the 2012 AP Calculus BC exam.] Let f be a function that is twice differentiable on $-2 < x < 2$ and satisfies the conditions in the table below. If $f(x) = f(-x)$, what are the x-coordinates of the points of inflection of the graph of f on $-2 < x < 2$?

	$0 < x < 1$	$1 < x < 2$
$f(x)$	Positive	Negative
$f'(x)$	Negative	Negative
$f''(x)$	Negative	Positive

Working Space

Answer on Page 64

Appendix A

Answers to Exercises

Answer to Exercise 1 (on page 6)

1. True. $f(2)$ exists and $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^-} f(x) = f(2) = 0$.
2. False. Because of the absolute value, there is a corner in the graph of f at $x = 2$. $\lim_{x\to 2^+} f'(x) < 0$ and $\lim_{x\to 2^-} f'(x) < 0$. Therefore there is a discontinuity in $f'(x)$ at $x = 2$ and $f(x)$ is not differentiable at $x = 2$.
3. True. $\sqrt{|2-2|} = \sqrt{0} = 0$.
4. False. $f(2)$ is defined at $x = 2$.

Answer to Exercise 2 (on page 11)

To estimate the slope at $t = 12$, we can use the data at $t = 9$ and $t = 15$. The slope of the line connecting those points is approximate of the slope at $t = 12$.

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{67.9 - 61.8}{15 - 9} = \frac{6.1}{6} = 1.017$$

The units for the numerator are degrees Fahrenheit and for the denominator are minutes. Therefore, the estimated slope has units of degrees Fahrenheit per minute. This represents the change in temperature of the water in the tub. When $t = 12$, the water in the tub is increasing in temperature at about 1 degree Fahrenheit per minute.

Answer to Exercise 3 (on page 13)

1.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{[m(x+h)+b] - [mx+b]}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{mx + mh + b - mx - b}{h} = \lim_{h\to 0} \frac{mh}{h} = \lim_{h\to 0} m = m$$

2.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{16-x-h} - \sqrt{16-x}}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{\sqrt{16-x-h} - \sqrt{16-x}}{h} \cdot \frac{\sqrt{16-x-h} + \sqrt{16-x}}{\sqrt{16-x-h} + \sqrt{16-x}}$$

$$f'(x) = \lim_{h\to 0} \frac{(16-x-h) - (16-x)}{h\left(\sqrt{16-x-h} + \sqrt{16-x}\right)}$$

$$f'(x) = \lim_{h\to 0} \frac{-h}{h\left(\sqrt{16-x-h} + \sqrt{16-x}\right)} = \lim_{h\to 0} \frac{-1}{\sqrt{16-x-h} + \sqrt{16-x}}$$

$$f'(x) = \frac{-1}{\sqrt{16-x} + \sqrt{16-x}} = \frac{-1}{2\sqrt{16-x}}$$

3.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\frac{(x+h)^2-1}{2(x+h)-3} - \frac{x^2-1}{2x-3}}{h}$$

$$f'(x) = \lim_{h\to 0} \left(\frac{1}{h}\right) \left[\frac{x^2+2xh+h^2-1}{2x+2h-3} - \frac{x^2-1}{2x-3}\right]$$

$$f'(x) = \lim_{h\to 0} \left(\frac{1}{h}\right) \left[\frac{x^2+2xh+h^2-1}{2x+2h-3}\left(\frac{2x-3}{2x-3}\right) - \frac{x^2-1}{2x-3}\left(\frac{2x+2h-3}{2x+2h-3}\right)\right]$$

$$f'(x) = \lim_{h\to 0} \left(\frac{1}{h}\right) \left[\frac{\left(x^2+2xh+h^2-1\right)(2x-3) - \left(x^2-1\right)(2x+2h-3)}{(2x-3)(2x+2h-3)}\right]$$

$$f'(x) = \left(\frac{1}{h}\right) \left[\frac{2x^3+4x^2h+2xh^2-2x-3x^2-6xh-3h^2+3\left(2x^3+2x^2h-3x^2-2x-2h+3\right)}{(2x-3)(2x+2h-3)}\right]$$

$$f'(x) = \lim_{h\to 0} \left(\frac{1}{h}\right) \left[\frac{2x^2h+2xh^2-6xh-3h^2+2h}{(2x-3)(2x+2h-3)}\right]$$

$$f'(x) = \lim_{h\to 0} \frac{2x^2+2xh-6x-3h+2}{(2x-3)(2x+2h-3)} = \frac{2x^2-6x+2}{(2x-3)^2}$$

Answer to Exercise 4 (on page 16)

First, let's confirm that l'Hôpital's rule applies here:

$$\lim_{x\to 0} \frac{\tan x - x}{x^3} = \frac{0-0}{0} = \frac{0}{0}$$

Therefore, we can apply l'Hôpital's rule:

$$\lim_{x\to 0} \frac{\tan x - x}{x^3} = \lim_{x\to 0} \frac{\frac{d}{dx}(\tan x - x)}{\frac{d}{dx}x^3}$$

$$= \lim_{x\to 0} \frac{\sec^2 x - 1}{3x^2} = \frac{1-1}{0} = \frac{0}{0}$$

which is an indeterminate form. We apply l'Hôpital's rule again:

$$\lim_{x\to 0} \frac{\tan x - x}{x^3} = \lim_{x\to 0} \frac{\frac{d}{dx}(\sec^2 x - 1)}{\frac{d}{dx}3x^2}$$

$$= \lim_{x\to 0} \frac{2\tan x \sec^2 x}{6x} = \frac{2(0)(1^2)}{6\cdot 0} = \frac{0}{0}$$

which is also an indeterminate form. We apply l'Hôpital's rule again:

$$\lim_{x\to 0} \frac{\tan x - x}{x^3} = \lim_{x\to 0} \frac{\frac{d}{dx}(2\tan x \sec^2 x)}{\frac{d}{dx}6x}$$

$$= \lim_{x\to 0} \frac{2\sec^2 x[2\tan^2 x + \sec^2 x]}{6} = \frac{2\cdot 1\cdot [2\cdot 0 + 1]}{6}$$

$$= \frac{2}{6} = \frac{1}{3}$$

Answer to Exercise 5 (on page 17)

1. $\lim_{x\to 3} \frac{x-3}{x^2-9} = \frac{0}{0}$, so we apply l'Hôpital's rule. $\lim_{x\to 3} \frac{x-3}{x^2-9} = \lim_{x\to 3} \frac{1}{2x} = \frac{1}{6}$
2. $\lim_{x\to 1/2} \frac{6x^2+5x-4}{4x^2+16x-9} = \frac{0}{0}$, so we apply l'Hôpital's rule. $\lim_{x\to 1/2} \frac{6x^2+5x-4}{4x^2+16x-9} = \lim_{x\to 1/2} \frac{12x+5}{8x+16} = \frac{11}{20}$
3. $\lim_{x\to 0^+} \frac{\ln x}{\sqrt{x}} = \frac{-\infty}{0} = -\infty$. This limit does not require l'Hôpital's rule because it is evaluable
4. $\lim_{x\to 1} \frac{x\sin x-1}{2x^2-x-1} = \frac{1\cdot \sin 1-1}{2(1)^2-1-1} = \frac{0}{0}$, so we apply l'Hospital's rule: $\lim_{x\to 1} \frac{x\sin x-1}{2x^2-x-1} = \lim_{x\to 1} \frac{\frac{d}{dx}(x\sin x-1}{\frac{d}{dx}(2x^2-x-1} = \lim_{x\to 1} \frac{x\cdot\cos x-1+\sin x-1}{4x-1} = \frac{1\cdot\cos 0+\sin 0}{4-1} = \frac{1\cdot 1+0}{-3} = \frac{-1}{3}$.

Answer to Exercise 6 (on page 19)

The speed of a car must a continuous, differentiable function, since your car can't "jump" from one speed to another: it must smoothly accelerate from one speed to another. Therefore, the Mean Value Theorem applies. The average acceleration from 3:30 PM to 3:40 PM is given by:

$$\frac{\text{change in speed}}{\text{change in time}} = \frac{50\frac{mi}{hr} - 30\frac{mi}{hr}}{3:40PM - 3:30PM}$$

Simplifying and converting minutes to hours, we see the average acceleration is:

$$\frac{20\frac{mi}{hr}}{\frac{1}{6}hr} = 120\frac{mi}{hr^2}$$

Therefore, by MVT, there must be some time between 3:30 and 3:40 PM where the car's acceleration is exactly $120\frac{mi}{hr^2}$.

Answer to Exercise 7 (on page 20)

(a) For the domain given, $f(x)$ is defined and differentiable. Finding the slope of the secant line connecting the endpoints:

$$\frac{f(b) - f(a)}{b - a} = \frac{\sqrt{4} - \sqrt{0}}{4 - 0} = \frac{2}{4} = \frac{1}{2}$$

So we are looking for some number c such that $f'(c) = \frac{1}{2}$. Let's find $f'(x)$:

$$f'(x) = \frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}$$

Setting this equal to $\frac{1}{2}$ to find c:

$$f'(c) = \frac{1}{2\sqrt{c}} = \frac{1}{2}$$
$$\sqrt{c} = 1$$
$$c = 1$$

(b)For the domain given, $f(x)$ is defined and differentiable. Finding the slope of the secant line connecting the endpoints:

$$\frac{f(2) - f(0)}{2 - 0} = \frac{e^{-2} - e^{0}}{2} = \frac{1 - e^2}{2e^2} \approx -0.432$$

And find $f'(x)$:

$$f'(x) = -e^{-x}$$

According to MVT, there must be some c such that $f'(c) \approx -0.432$:

$$-e^{-c} \approx -0.432$$

$$e^{-c} \approx 0.432$$

$$-c \approx \ln 0.432$$

$$c \approx -\ln 0.432 \approx 0.839$$

(c) For the domain given, $f(x)$ is defined and differentiable. Finding the secant line connecting the endpoints:

$$\frac{f(b) - f(a)}{b - a} = \frac{\ln 4 - \ln 1}{4 - 1} = \frac{\ln 4}{3} \approx 0.462$$

And find $f'(x)$:

$$f'(x) = \frac{1}{x}$$

According to MVT, there must be some c such that $f'(c) \approx 0.462$

$$f'(c) = \frac{1}{c} \approx 0.462$$

$$c \approx \frac{1}{0.462} = 2.164$$

Answer to Exercise 8 (on page 21)

Velocity is the derivative of position. Therefore, $v(t) = s'(t) = 3t^2 - 12t + 6$.

Answer to Exercise 9 (on page 21)

$$v(2) = 3(2)^2 - 12(2) + 6 = -6\frac{m}{s}$$

$$v(4) = 3(4)^2 - 12(4) + 6 = 6\frac{m}{s}$$

Answer to Exercise 10 (on page 21)

When the particle is at rest, $v(t) = 0$.

$$3t^2 - 12t + 6 = 0$$

$$3(t^2 - 4t + 2) = 0$$

$$t^2 - 4t + 2 = 0$$

This is not easily factorable, so we will use the quadratic formula:

$$t = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(2)}}{2(1)}$$

$$x = \frac{4 \pm \sqrt{16-8}}{2} = 2 \pm \sqrt{2} \approx 0.586, 3.414$$

Therefore, the particle is at rest at 0.586s and 3.414s.

Answer to Exercise 11 (on page 24)

1. $\frac{dy}{dx} = \frac{d}{dx}[x \sin x] = x \frac{d}{dx} \sin x + \sin x \frac{d}{dx} x = x(-\cos x) + \sin x(1) = \sin x - x \cos x$
2. By the chain rule, $f'(x) = 5(x^3 - \cos x)^4 \cdot \frac{d}{dx}(x^3 - \cos x) = 5(x^3 - \cos x)^4 \cdot (3x^2 + \sin x)$
3. By the chain rule, $f'(x) = \frac{d}{d(\sin x)}[\sin^3 x] \times \frac{d}{dx} \sin x = 3 \sin^2 x \cdot \cos x$

Answer to Exercise 12 (on page 24)

$f'(x) = \frac{d}{dx}(7x) - \frac{d}{dx}(3) + \frac{d}{dx}(\ln x) = 7 - 0 + \frac{1}{x} = 7 - \frac{1}{x}$ and $f'(1) = 7 - \frac{1}{1} = 6$

Answer to Exercise 13 (on page 25)

The particle is at rest when $x'(t) = y'(t) = 0$. First, we find each of the derivatives:

$$x'(t) = 3t^2 - 6t$$

$$y'(t) = 12 - 6t$$

We can solve $y' = 0$ for t and find that the y-velocity is 0 when $t = 2$. Substituting $t = 2$ into our expression for x', we find $x'(2) = 3(2)^2 - 6(2) = 0$. Therefore, the particle is at rest when $t = 0$. To find the xy-coordinate, we substitute $t = 2$ into $x(t)$ and $y(t)$:

$$x(2) = (2)^3 - 3(2)^2 = 8 - 12 = -4$$
$$y(2) = 12(2) - 6(2) = 24 - 12 = 12$$

Therefore, the particle is at rest when it is located at $(-4, 12)$.

Answer to Exercise 14 (on page 25)

$f(g(x)) = \sqrt{(3x-2)^2 - 4} = \sqrt{9x^2 - 12x}$ and $\frac{d}{dx}f(g(x)) = \frac{18x-12}{2\sqrt{9x^2-12x}}$. Substituting $x = 3$, we find $f'(g(x)) = \frac{18(3)-12}{2\sqrt{9(3)^2-12(3)}} = \frac{42}{2\sqrt{45}} = \frac{21}{3\sqrt{5}} = \frac{7}{\sqrt{5}}$

Answer to Exercise 15 (on page 25)

First, recall that the velocity of a particle is the derivative of its position function. Therefore, $v(t) = x'(t) = \frac{d}{dt}[(t-a)(t-b)]$. Applying the Product Rule for derivatives, we see that $v(t) = (t-a)(1) + (t-b)(1) = 2t - a - b$. To find the time(s) when the particle is at rest, we set $v(t) = 0$ and solve for t.

$$0 = 2t - a - b$$
$$2t = a + b$$
$$t = \frac{a+b}{2}$$

Answer to Exercise 16 (on page 26)

The question is asking when the derivative of f is $\frac{1}{2}$. We will take the derivative and set it equal to $\frac{1}{2}$.

$$f'(x) = \frac{(x+2)(1) - x(1)}{(x+2)^2} = \frac{2}{(x+2)^2}$$
$$\frac{2}{(x+2)^2} = \frac{1}{2}$$
$$4 = (x+2)^2$$
$$\pm 2 = x + 2$$
$$x = 2 - 2 = 0 \text{ and } x = -2 - 2 = -4$$

Answer to Exercise 17 (on page 26)

(a) Let t_0 be the time at which the particle is first at rest. The velocity of the particle is given by $v(t) = x'(t) = \cos t + \sin t$. Setting $v(t) = 0$, we find:

$$\cos t = -\sin t$$

which is true for $t = \frac{3\pi + 4n}{4}$, where n is an integer. Therefore, the first time the velocity is 0 is $t_0 = \frac{3\pi}{4}$.
(b) To find the acceleration at $t = \frac{3\pi}{4}$, we take the derivative of the velocity function to yield the acceleration function.

$$a(t) = v'(t) = -\sin t + \cos t$$

. Substituting $t = \frac{3\pi}{4}$, we find the acceleration is $-\sin \frac{3\pi}{4} + \cos \frac{3\pi}{4} = \frac{-\sqrt{2}}{2} - \frac{\sqrt{2}}{2} = -\sqrt{2}$

Answer to Exercise 18 (on page 26)

First, we find the x such that $y = 0$

$$0 = e^{\tan x} - 2$$

$$2 = e^{\tan x}$$

$$\ln 2 = \tan x$$

$$x = \arctan(\ln 2) = \arctan 0.693 \approx 0.606$$

Then, we find the slope of the function at $x = 0.606$ by finding $y'(0.606)$

$$y' = e^{\tan x}(\sec x)^2 = \frac{e^{\tan x}}{(\cos x)^2}$$

$$y'(0.606) = \frac{e^{\tan 0.606}}{(\cos 0.606)^2} = 2.961$$

Answer to Exercise 19 (on page 27)

(a) Apply the chain rule to find $f'(x)$

$$f'(x) = \frac{1}{2\sqrt{25 - x^2}} \cdot (-2x) = \frac{-x}{\sqrt{25 - x^2}}$$

.

(b) First, substitute $x = -3$ into $f'(x)$

$$f'(-3) = \frac{-(-3)}{\sqrt{25-(-3)^2}} = \frac{3}{\sqrt{16}} = \frac{3}{4}$$

This is the slope of the line. To complete an equation for the tangent line, we need a point. We know the tangent line touches $f(x)$ at $x = -3$, so the tangent line must pass through the point $(-3, f(-3))$.

$$f(-3) = \sqrt{25-(-3)^2} = 4$$

We use $m = \frac{3}{4}$ and the coordinate point $(x_1, y_1) = (-3, 16)$ to complete the equation $y - y_1 = m(x - x_1)$

$$y - 16 = \frac{3}{4}(x+3)$$

Answer to Exercise 20 (on page 27)

$$a(t) = v'(t) = -\frac{\pi}{6}\sin\frac{\pi}{6}t$$

$$a(4) = -\frac{\pi}{6}\sin\frac{2\pi}{3} = -\frac{\pi}{6}\cdot\frac{\sqrt{3}}{2} = -\frac{\pi\sqrt{3}}{12}$$

Answer to Exercise 21 (on page 27)

Recall that the rate of change of a function is given by the derivative of that function. Therefore, we are looking for the interval(s) where $f'(x) > g'(x)$. First, we find each derivative:

$$f'(x) = e^x$$

$$g'(x) = 4x^3$$

We are looking for x-values, such that $e^x > 4x^3$. This inequality can be restated as $e^x - 4x^3 > 0$. Using a calculator, you should find that $e^x - 4x^3 = 0$ when $x \approx 0.831$ and $x \approx 7.384$. We will check values on either side of and in the interval $x \in (0.831, 7.384)$ to determine the sign value of $e^x - 4x^3$. We know that when $x = 0$, $e^x - 4x^3 > 0$, when $x = 5$, $e^x - 4x^3 < 0$, and when $x = 10$, $e^x - 4x^3 > 0$. Therefore, $f'(x)$ is greater than $g'(x)$ on the open intervals $x \in (-\infty, 0.831) \cup (7.384, \infty)$.

Answer to Exercise 22 (on page 32)

Recall that critical values are values of x where $g'(x) = 0$ or is undefined. We need to find an expression for $g'(x)$, set it equal to zero when $x = \frac{2}{3}$, and solve for k.

$$g'(x) = x^(2)[k * \exp kx] + \exp kx[2x]$$

$$g'(\frac{2}{3}) = (\frac{2}{3})^2[k * \exp \frac{2k}{3}] + \frac{4}{3} \exp \frac{2k}{3} = 0$$

$$\frac{4k}{9} e^{\frac{2k}{3}} + \frac{4}{3} e^{\frac{2k}{3}} = 0$$

$$(\frac{4k}{9} + \frac{4}{3}) e^{\frac{2k}{3}} = 0$$

There are no real values of k such that $e^{\frac{2k}{3}} = 0$, therefore, we will examine the other factor:

$$\frac{4k}{9} + \frac{4}{3} = 0$$

$$\frac{4k}{9} = \frac{-4}{3}$$

$$\frac{k}{3} = -1$$

$$k = -3$$

Therefore, $g(x)$ has a critical value at $x = \frac{2}{3}$ when $k = -3$.

Answer to Exercise 23 (on page 34)

First, we will find f' and set it equal to zero:

$$f'(x) = 300 - 3x^2 = 0$$

$$300 = 3x^2 \to x = \pm\sqrt{100} = \pm 10$$

(Note: $f'(x) = 3(10 - x)(10 + x)$, which implies roots at $x = \pm 10$. Now, we will evaluate the value of $f'(x)$ for $x < -10$, $-10 < x < 10$, and $x > 10$.

Value of x	(10-x)	(10+x)	$f'(x)$	$f(x)$ behavior
$x < -10$	positive	negative	negative	decreasing
$-10 < x < 10$	positive	positive	positive	increasing
$x > 10$	negative	positive	negative	decreasing

Therefore, the function is increasing on the interval $x \in [-10, 10]$ because $f'(x) > 0$ for $x \in [-10, 10]$.

Answer to Exercise 24 (on page 34)

Given $f(x) = x^3 - 3x^2 - 9x + 4$, it follows that $f'(x) = 3x^2 - 6x - 9$. Factoring, we find that $f'(x) = 9(x-3)(x+1)$ and $f'(x) = 0$ when $x = 3$ and $x = -1$. We construct our table to help us analyze the value of $f'(x)$ and behavior of $f(x)$ on the whole domain of the function:

Value of x	$(x-3)$	$(x+1)$	$f'(x)$	$f(x)$ behavior
$x < -1$	negative	negative	positive	increasing
$-1 < x < 3$	negative	positive	negative	decreasing
$x > 3$	positive	positive	positive	increasing

So, $f(x)$ is increasing for $x \in (-\infty, -1) \cup (3, \infty)$ and decreasing for $x \in (-1, 3)$. Since $f'(-1) = 0$ and changes from positive to negative, $f(x)$ has a local maximum at $x = -1$. And since $f'(3) = 0$ and changes from negative to positive, $f(x)$ has a local minimum at $x = 3$.

Answer to Exercise 25 (on page 36)

First, we identify any critical numbers:

$$f'(x) = \frac{x * (\frac{1}{x}) - \ln x * 1}{x^2} = \frac{1 - \ln x}{x^2}$$

Recall that critical numbers are values where $f'(x) = 0$ or does not exist. We might identify $x = 0$ as a critical number, but the presence of $\ln x$ limits the domain of the function to $x \in (0, \infty)$, excluding $x = 0$. For all $x \in (0, \infty)$, $f'(x)$ exists. So, we look for values where $f'(x) = 0$.

$$\frac{1 - \ln x}{x^2} = 0$$

$$1 - \ln x = 0$$

$$1 = \ln x$$

$$x = e$$

Finding the value of $f(x)$ at $x = e$:

$$f(e) = \frac{\ln e}{e} = \frac{1}{e}$$

Because the domain of $f(x)$ is on an *open interval*, instead of checking the endpoints directly, we'll take the limits as x approaches 0 and ∞.

$$\lim_{x\to 0} \frac{\ln x}{x} = -\infty < \frac{1}{e}$$

$$\lim_{x\to \infty} \frac{\ln x}{x} = 0 < \frac{1}{e}$$

Therefore, the absolute maximum values of $f(x) = \frac{\ln x}{x}$ is $\frac{1}{e}$ at $x = e$.

Answer to Exercise 26 (on page 37)

1. $f'(x) = 4 - 2x$ and to find the critical numbers, we set $f'(x) = 0$:

   $$4 - 2x = 0$$

   $$x = 2$$

   We evaluate $f(x)$ at $x = 0, 2, 5$:

   $$f(0) = 12 + 4(0) - 0^2 = 12$$

   $$f(2) = 12 + 4(2) - 2^2 = 12 + 8 - 4 = 16$$
   $$f(5) = 12 + 4(5) - 5^2 = 12 + 20 - 25 = 7$$

   Therefore, the global maximum is $f(2) = 16$ and the global minimum is $f(5) = 7$.

2. $f(t) = \dfrac{\sqrt{t}}{1+t^2}$, $[0, 2]$.

   We write $f(t) = t^{1/2}(1+t^2)^{-1}$ and differentiate:

   $$f'(t) = \frac{1-3t^2}{2\sqrt{t}(1+t^2)^2}.$$

   Critical points occur when the numerator is zero (and t is in the interval):

   $$1 - 3t^2 = 0 \quad \Rightarrow \quad t = \frac{1}{\sqrt{3}}.$$

   We evaluate $f(t)$ at $t = 0, \frac{1}{\sqrt{3}}, 2$:

   $$f(0) = 0, \qquad f\left(\frac{1}{\sqrt{3}}\right) = \frac{3^{3/4}}{4}, \qquad f(2) = \frac{\sqrt{2}}{5}.$$

Since $\dfrac{3^{3/4}}{4}$ is the largest of these and 0 is the smallest, the global maximum is

$$f\left(\frac{1}{\sqrt{3}}\right) = \frac{3^{3/4}}{4},$$

and the global minimum is

$$f(0) = 0.$$

3. $f(t) = 2\cos t + \sin(2t)$, $\left[0, \frac{\pi}{2}\right]$.

   We differentiate:

$$f'(t) = -2\sin t + 2\cos(2t).$$

   Set $f'(t) = 0$:

$$-2\sin t + 2\cos(2t) = 0 \quad \Rightarrow \quad \cos(2t) = \sin t.$$

   Use $\cos(2t) = 1 - 2\sin^2 t$:

$$1 - 2\sin^2 t = \sin t \quad \Rightarrow \quad 2\sin^2 t + \sin t - 1 = 0.$$

   Let $u = \sin t$:

$$2u^2 + u - 1 = 0 \quad \Rightarrow \quad u = \frac{1}{2} \text{ or } u = -1.$$

   On $\left[0, \frac{\pi}{2}\right]$, $\sin t = \frac{1}{2}$ gives $t = \frac{\pi}{6}$ (and $\sin t = -1$ is not in this interval).
   Evaluate $f(t)$ at $t = 0, \frac{\pi}{6}, \frac{\pi}{2}$:

$$f(0) = 2\cos 0 + \sin 0 = 2,$$

$$f\left(\frac{\pi}{6}\right) = 2\cos\left(\frac{\pi}{6}\right) + \sin\left(\frac{\pi}{3}\right) = 2 \cdot \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{2},$$
$$f\left(\frac{\pi}{2}\right) = 2\cos\left(\frac{\pi}{2}\right) + \sin(\pi) = 0 + 0 = 0.$$

   Hence the global maximum is

$$f\left(\frac{\pi}{6}\right) = \frac{3\sqrt{3}}{2},$$

   and the global minimum is

$$f\left(\frac{\pi}{2}\right) = 0.$$

4. $f(x) = \ln(x^2 + x + 1)$, $[-1, 1]$.

   Differentiate:

$$f'(x) = \frac{2x+1}{x^2+x+1}.$$

   To find critical points, set the numerator equal to 0:

$$2x + 1 = 0 \quad \Rightarrow \quad x = -\frac{1}{2}.$$

Evaluate $f(x)$ at $x = -1, -\frac{1}{2}, 1$:

$$f(-1) = \ln(1 - 1 + 1) = \ln 1 = 0,$$
$$f\left(-\frac{1}{2}\right) = \ln\left(\left(-\frac{1}{2}\right)^2 - \frac{1}{2} + 1\right) = \ln\left(\frac{1}{4} - \frac{1}{2} + 1\right) = \ln\left(\frac{3}{4}\right),$$
$$f(1) = \ln(1 + 1 + 1) = \ln 3.$$

Since $\ln\left(\frac{3}{4}\right) < 0 < \ln 3$, the global minimum is

$$f\left(-\frac{1}{2}\right) = \ln\left(\frac{3}{4}\right),$$

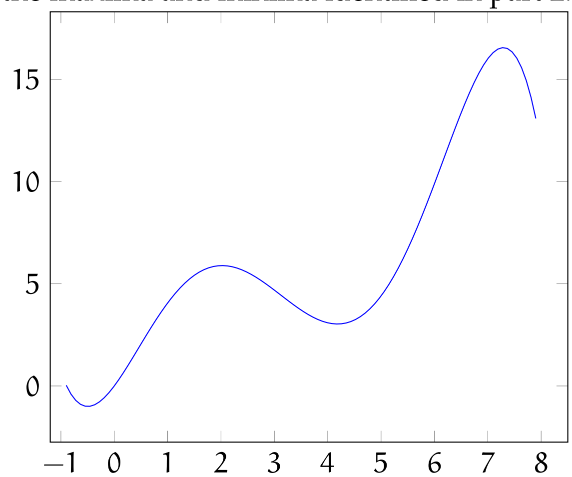
and the global maximum is

$$f(1) = \ln 3.$$

Answer to Exercise 27 (on page 40)

[Your answers are meant to be estimates; anything within ± 0.1 of the given answers are reasonable estimates.]

1. $f(x)$ is increasing on the intervals $x \in (-0.5, 2.2) \cup (4, 7.3)$. $f(x)$ is decreasing on the intervals $x \in (-\infty, -0.5) \cup (2.2, 4) \cup (7.3, \infty)$.
2. $f(x)$ has local maxima at $x = 2.2, 7.3$ and local minima at $x = -0.5, 4$.
3. Your sketch should show the maxima and minima identified in part 2. One possible solution is shown below.



Answer to Exercise 28 (on page 43)

Noting that $f''(2) = 0$, we examine the value of f'' around $x = 2$. For $0 < x < 2$, $f'' < 0$, which indicates f is concave down in the domain $x \in (0, 2)$. For $x > 2$, $f'' > 0$, which indicates f is concave up. Therefore, there is an inflection point at $x = 2$ for f. Recalling that $f(x) = x^4 - 4x^3$, we find the coordinate of the inflection point by substituting $x = 2$:

$$f(2) = 2^4 - 4 * 2^3 = 16 - 4 * 8 = 16 - 32 = -16$$

Therefore, $f(x)$ has an inflection point at $(2, -16)$.

Answer to Exercise 29 (on page 44)

According to the graph, g' is positive and increasing. Therefore, g is increasing (because g' is positive) and concave up (because g' is increasing, and therefore g'' is positive).

Answer to Exercise 30 (on page 44)

Since the question asks about concavity, we need to examine the second derivative:

$$f''(x) = \frac{d}{dx} f'(x) = \frac{d}{dx}\left[e^{(x^4-2x^2+1)} - 2\right]$$

$$f''(x) = \left(x^4 - 2x^2 + 1\right) e^{(x^4-2x^2+1)} \left(4x^3 - 4x\right)$$

The second derivative equals zero when $x^4 - 2x^2 + 1 = (x^2 - 1)^2 = 0$ or $4x^3 - 4x = 4(x)(x^2 - 1) = 0$, which gives roots $x = 0$, $x = 1$, and $x = -1$. So the intervals we need to test are $(-1.5, -1)$, $(-1, 0)$, $(0, 1)$, and $(1, 1.5)$. To test $x \in (-1.5, -1)$, we will substitute $x = -1.25$ into $f''(x)$:

$$f''(-1.25) = -3.85928 < 0$$

Therefore, $f(x)$ is concave down on the interval $x \in (-1.5, -1)$. Next, we test $x \in (-1, 0)$:

$$f''(-0.5) = 2.63258 > 0$$

So, we eliminate $x \in (-1, 0)$. Next, we test $x \in (0, 1)$:

$$f''(0.5) = -2.63258 < 0$$

And $f(x)$ is concave down on the interval $x \in (0, 1)$. Finally, we test the interval $x \in (1, 1.5)$:

$$f''(1.25) = 3.85928 > 0$$

Which eliminates that interval. Therefore, $f(x)$ is concave down on the intervals $x \in (-1.5, -1)$ and $x \in (0, 1)$.

Answer to Exercise 31 (on page 45)

1. False. For $x \in (-2, 0)$, the slope of $f(x)$ is negative, which implies that $f'(x) < 0$ for $x \in (-2, 0)$.
2. False. The graph comes to a point at $x = 0$, therefore $\lim_{x \to 0^+} f'(x) \neq \lim_{x \to 0^-} f'(x)$, which means the limit does not exist and f is not differentiable at $x = 0$.
3. True. The graph of $f(x)$ is concave up for $x \in (0, 2)$, which means the second derivative is positive.
4. True. Recall that critical values are where derivatives equal 0 or do not exist. Since we have established that $f(x)$ does not exist at $x = 0$, then there is a critical value at $x = 0$.

Answer to Exercise 32 (on page 46)

1. True. $f'(3) = 0$ and f' has a positive slope, which means there is a local extreme and f is concave up at $x = 3$. Therefore, there is a local minimum at $x = 3$.
2. False. Though it appears that $f'' = 0$ at $x = -2$, the slope of f' is positive before and after. Therefore, f'' does not cross the x-axis and there is not an inflection point at $x = -2$.
3. True. For $0 < x < 4$, the slope of f' is negative, which means f'' is negative, which means f is concave down.

Answer to Exercise 33 (on page 47)

The graph of f has inflection points at $x = -1$ and $x = 1$. Since $f(x) = f(-x)$, we can expand the table to include the entire window we are investigating:

	$-2 < x < -1$	$-1 < x < 0$	$0 < x < 1$	$1 <$
$f(x)$	Negative	Positive	Positive	Ne
$f'(x)$	Negative	Negative	Negative	Ne
$f''(x)$	Positive	Negative	Negative	Po

Recall that inflection points occur when f'' changes from positive to negative or from negative to positive. Examining the table, we see that the sign of f'' changes at $x = -1$ and $x = 1$.

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