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Optimization

What ticket price brings in the most money for the school dance? What shape of soda can uses the least aluminum? How should you fence a garden to enclose the most area? Each of these asks for the *best* value of something, and finding it is called *optimization*.

The quantity you are trying to make as large or as small as possible is the *objective function*. The key idea is one you met in the last volume: at a smooth peak or valley, the derivative is zero. So we look for the critical points, where $f'(x) = 0$.

At a critical point where the derivative is zero, the graph of the function has a horizontal tangent line. At such a point, the function may be reaching a peak or a valley (Figure 1.1).

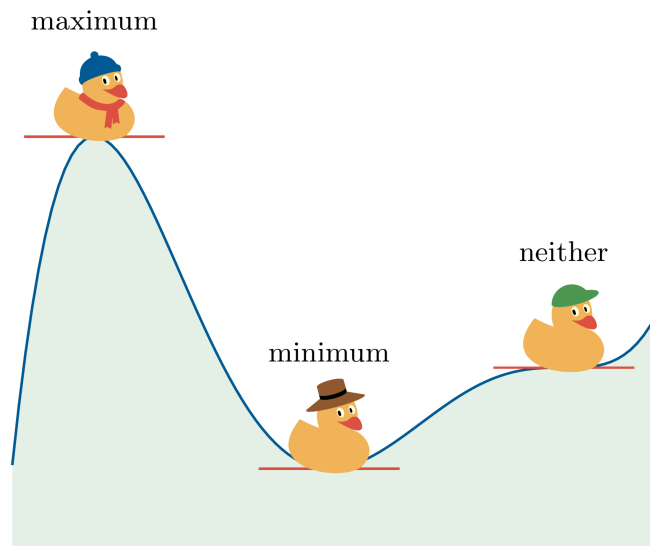


Figure 1.1: Each duck sits where $f'(x) = 0$: at a maximum, at a minimum, and at a point that is neither.

Not every critical point is a peak or a valley. At some, like $x = 0$ on $y = x^3$, the graph flattens out for a moment and keeps going: a horizontal point of inflection (Figure 1.2).

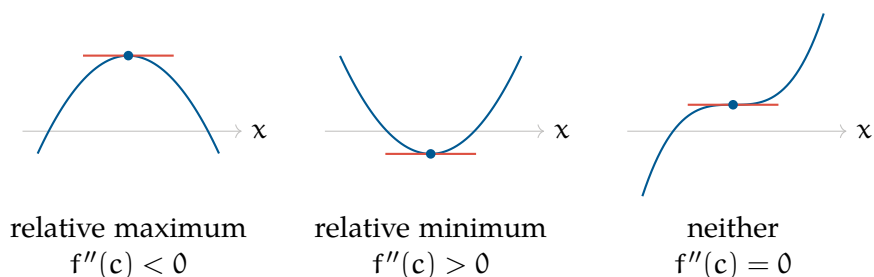


Figure 1.2: At a critical point $x = c$ the tangent line is horizontal, and the graph may have a maximum, a minimum, or neither.

1.1 Extrema

There are two types of extrema: relative extrema and absolute extrema.

A relative maximum or minimum occurs when the function reaches a peak or valley compared to nearby values. This does not guarantee that the value is the highest or lowest value of the function overall.

An absolute maximum or minimum is the greatest or least value that the function takes on a given interval.

An absolute maximum or minimum can occur either at a critical point or at an endpoint of the interval.

For this reason, when solving optimization problems on a restricted domain, it is important to test both critical points and endpoints (Figure 1.3).

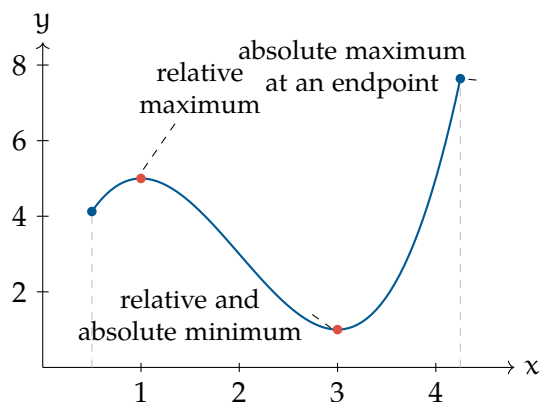


Figure 1.3: On the interval $0.5 \leq x \leq 4.25$, the absolute maximum occurs at an endpoint and the absolute minimum occurs at a critical point.

1.2 The Second Derivative Test

Once a critical point has been found, we often want to determine whether it corresponds to a maximum or a minimum. One method for doing this is the Second Derivative Test.

If a function has a critical point at $x = c$, then:

- if the second derivative is positive at $x = c$, the function is concave up and the point is a relative minimum;
- if the second derivative is negative at $x = c$, the function is concave down and the point is a relative maximum;
- if the second derivative is zero at $x = c$, the test is inconclusive.

If the second derivative is always positive or always negative on an interval, then any critical point in that interval must be an absolute minimum or absolute maximum, respectively.

1.3 A Strategy for Optimization Problems

The stories change, but the steps do not:

1. Identify the quantity to be optimized.
2. Write an equation for the objective function. In word problems, this is usually the hard part, and a sketch helps.
3. Use the given constraints to rewrite the function using one variable.
4. Determine the appropriate domain.
5. Find the derivative of the function.
6. Find critical points by setting the derivative equal to zero.
7. Check that each one is a maximum or minimum (second derivative test), and check the endpoints of the domain.
8. Interpret the result in the context of the problem.

When solving optimization problems, some common mistakes include forgetting to define variables, not using the constraints of the problem appropriately, misinterpreting or failing to interpret the final answer, and giving answers without units. Ask yourself whether the answer makes sense: a fence with negative length means something went wrong.

1.3.1 Example

Consider the function

$$f(x) = x^2 - 4x + 1.$$

We want to find the maximum or minimum value of this function.

First, take the derivative of the function:

$$f'(x) = 2x - 4.$$

Next, set the derivative equal to zero to find the critical point:

$$2x - 4 = 0.$$

Solving for x gives

$$x = 2.$$

Now determine whether this critical point corresponds to a maximum or a minimum by using the second derivative.

The second derivative is

$$f''(x) = 2.$$

Because the second derivative is positive, the function is concave up, and the critical point corresponds to a minimum.

To find the minimum value of the function, substitute $x = 2$ back into the original function:

$$f(2) = 2^2 - 4(2) + 1 = -3.$$

Therefore, the function has a minimum value of -3 at $x = 2$.

Exercise 1

Consider the function

$$f(x) = x^2 - 6x + 5.$$

Working Space

Find the maximum or minimum value

of the function using calculus.

Answer on Page 61

Now, let us look at a word example:

Example

The cost, in dollars, of producing x units of a product is given by the function

$$C(x) = 2x^2 - 24x + 100.$$

Find the number of units that should be produced in order to minimize the cost, and determine the minimum cost.

To minimize the cost, take the derivative of the cost function:

$$C'(x) = 4x - 24.$$

Set the derivative equal to zero to find the critical point:

$$4x - 24 = 0.$$

Solving for x gives

$$x = 6.$$

Next, take the second derivative:

$$C''(x) = 4.$$

Since the second derivative is positive, the cost function is concave up, and the critical point corresponds to a minimum.

Substitute $x = 6$ back into the original cost function:

$$C(6) = 2(6)^2 - 24(6) + 100 = 28.$$

Therefore, the cost is minimized when 6 units are produced, and the minimum cost is \$28

(Figure 1.4).

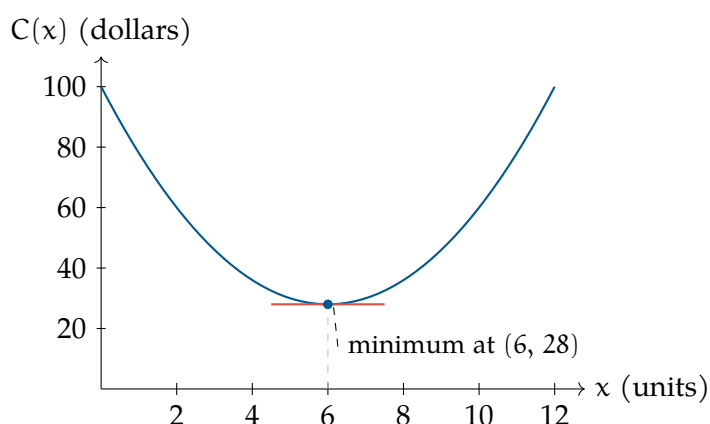


Figure 1.4: The cost function $C(x) = 2x^2 - 24x + 100$ reaches its minimum value of \$28 at $x = 6$ units.

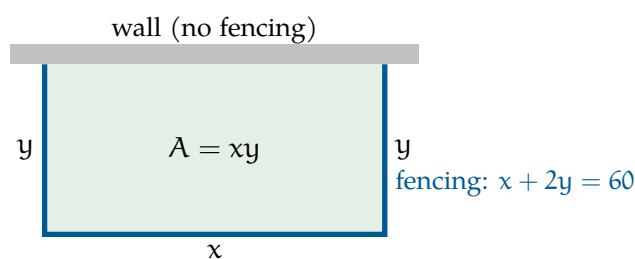


Figure 1.5: The enclosure uses the wall as one side, so fencing covers one side of length x and two sides of length y .

Exercise 2

A rectangular enclosure is to be built using 60 units of fencing. Three sides of the enclosure require fencing, while the fourth side is along a wall and does not require fencing.

Let x represent the length of the side parallel to the wall and y represent the width of the enclosure (Figure 1.5).

Find the dimensions of the enclosure

Working Space

that maximize the enclosed area.

Answer on Page 61

Example: The Soda Can

A soda can holds 355 cm^3 . What radius and height use the least aluminum?

Solution: We want to minimize the surface area of a cylinder with radius r and height h : two circular ends plus the side.

$$A = 2\pi r^2 + 2\pi r h$$

The constraint is the volume: $\pi r^2 h = 355$, so $h = \frac{355}{\pi r^2}$. Substituting,

$$A(r) = 2\pi r^2 + 2\pi r \cdot \frac{355}{\pi r^2} = 2\pi r^2 + \frac{710}{r}$$

The domain is $r > 0$. Differentiate and set equal to zero:

$$A'(r) = 4\pi r - \frac{710}{r^2} = 0 \quad \Rightarrow \quad r^3 = \frac{710}{4\pi} \quad \Rightarrow \quad r \approx 3.84 \text{ cm}$$

$A''(r) = 4\pi + \frac{1420}{r^3} > 0$, so this is a minimum. The height is $h = \frac{355}{\pi(3.84)^2} \approx 7.67 \text{ cm}$, exactly twice the radius: the best can is as tall as it is wide.

Real soda cans are about 6.6 cm wide and 12 cm tall. They use about 2% more aluminum than the ideal, but they are easier to hold, and their thick ends are cheaper to make small.

Exercise 3 The Open Box

You have a 30 cm by 30 cm square of cardboard. You cut a square of side x from each corner and fold up the sides to make an open box. What value of x gives the largest volume? What is that volume?

Working Space

Answer on Page 62

Exercise 4 The School Dance

At \$10 a ticket, 300 students come to the school dance. For every \$1 the price goes up, 20 fewer students come. What ticket price brings in the most money? How many students come, and what is the revenue?

Working Space

Answer on Page 63

1.4 Helpful Table for Optimization Problems

Condition at a point $x = c$	What it tells you	What kind of point it could be
$f'(c) \neq 0$	The graph has a non-horizontal tangent	Not a maximum or minimum (unless c is an endpoint)
$f'(c) = 0$	The graph has a horizontal tangent	Possible maximum, minimum, or neither
$f'(c)$ does not exist	The graph may have a corner, cusp, or vertical tangent	Possible maximum or minimum; check the sign of f' on each side
$f'(c) = 0$ and $f''(c) > 0$	Graph is concave up at c	Local minimum
$f'(c) = 0$ and $f''(c) < 0$	Graph is concave down at c	Local maximum
$f'(c) = 0$ and $f''(c) = 0$	Second derivative test fails	Could be max, min, or neither

1.5 Using Python to Visualize Your Optimization Problem

The sympy library does algebra and calculus symbolically, the way you do on paper. The script below takes the derivative, solves $f'(x) = 0$, runs the second derivative test, and plots the result.

```

1 import sympy as sp
2 import numpy as np
3 import matplotlib.pyplot as plt

```

```
4
5 # 1) Define the variable
6 x = sp.symbols('x')
7
8 # 2) Define the objective function (STUDENTS EDIT THIS)
9 f = 20*x - x**2
10
11 # 3) Derivatives
12 f_prime = sp.diff(f, x)
13 f_double_prime = sp.diff(f_prime, x)
14
15 # 4) Solve  $f'(x) = 0$ 
16 critical_point = sp.solve(f_prime, x)[0]
17
18 # 5) Classify using second derivative
19 second_derivative_value = f_double_prime.subs(x, critical_point)
20
21 if second_derivative_value > 0:
22     classification = "minimum"
23 elif second_derivative_value < 0:
24     classification = "maximum"
25 else:
26     classification = "inconclusive"
27
28 print("f(x) =", f)
29 print("f'(x) =", f_prime)
30 print("f''(x) =", f_double_prime)
31 print("Critical point:", critical_point)
32 print("Classification:", classification)
33
34 # 6) Plot
35 f_num = sp.lambdify(x, f, "numpy")
36
37 X = np.linspace(0, 20, 400)
38 Y = f_num(X)
39
40 xc = float(critical_point)
41 yc = float(f_num(xc))
42
43 plt.plot(X, Y)
44 plt.scatter([xc], [yc])
45 plt.title("Optimization")
46 plt.xlabel("x")
47 plt.ylabel("f(x)")
48 plt.grid(True)
49
50 plt.annotate(
51     classification,
52     (xc, yc),
53     xytext=(xc + 1, yc),
54     arrowprops=dict(arrowstyle="->")
55 )
56
```

```
57 plt.show()
```

Exercise 5

The Python script above uses derivatives to locate and classify a critical point of a function.

Working Space

1. Run the script using the function

$$f(x) = 20x - x^2.$$

Record the critical point and whether it is classified as a maximum or minimum.

2. Change the objective function in the script to

$$f(x) = x^2 - 6x + 5.$$

Run the script again. How does the classification change? Explain why this happens using the second derivative test.

3. Change the objective function to

$$f(x) = x^3.$$

Run the script. What does the script report for the classification? Explain why the second derivative test is inconclusive in this case.

4. Finally, choose your own quadratic function and predict whether the critical point will be a maximum or minimum before running the script. Check your prediction using both calculus and the graph.

Answer on Page 63

Implicit Differentiation

The equation $x^2 + y^2 = 25$ draws a perfectly good circle, but it is not a function: most values of x give two values of y . You could split it into two functions, $y = \sqrt{25 - x^2}$ and $y = -\sqrt{25 - x^2}$, and differentiate each. But some curves, like the leaf-shaped $x^3 + y^3 = 3xy$ later in this chapter, are nearly impossible to solve for y . *Implicit differentiation* finds slopes without solving for y at all.

2.1 Implicit Differentiation Procedure

The trick: pretend y is some function of x , even if you do not know which one. Then differentiate both sides of the equation with respect to x . Terms with only x work as usual. Whenever you differentiate something with y in it, the chain rule tacks on a factor of $\frac{dy}{dx}$. For example,

$$\frac{d}{dx}y^2 = 2y \frac{dy}{dx}$$

just as $\frac{d}{dx}(g(x))^2 = 2g(x)g'(x)$. Finally, solve the resulting equation for $\frac{dy}{dx}$.

Worked example: a slope on a circle. Differentiate both sides of $x^2 + y^2 = 25$ with respect to x :

$$2x + 2y \frac{dy}{dx} = 0 \implies \frac{dy}{dx} = -\frac{x}{y}$$

At the point $(3, 4)$, the slope is $\frac{dy}{dx} = -\frac{3}{4}$. The answer depends on both x and y , which makes sense: at $(3, 4)$ and $(3, -4)$ the circle has different slopes.

2.2 Example

Consider the equation of a circle with radius r :

$$x^2 + y^2 = r^2$$

First, we will find $\frac{dy}{dx}$ without implicit differentiation. Next, we will apply implicit differentiation to get the same result.

2.2.1 Without Implicit Differentiation

First, we need to rearrange the equation to solve for y :

$$y^2 = r^2 - x^2$$
$$y = \pm\sqrt{r^2 - x^2}$$

We take the derivative of y by applying the Chain Rule:

$$\frac{dy}{dx} = \frac{1}{2(\pm\sqrt{r^2 - x^2})} \cdot (-2x) = \frac{-x}{\pm\sqrt{r^2 - x^2}}$$

Notice the denominator of this fraction is the same as the solution we found for y , $y = \pm\sqrt{r^2 - x^2}$. So, we can also represent this as:

$$\frac{dy}{dx} = \frac{-x}{y}$$

2.2.2 With Implicit Differentiation

With implicit differentiation, we assume y is a function of x and apply the Chain Rule.

$$\frac{d}{dx}[x^2 + y^2] = \frac{d}{dx}[r^2]$$

For x^2 and r^2 (a constant), we take the derivative as we normally would. For y^2 , the chain rule adds a $\frac{dy}{dx}$:

$$2x + 2y \frac{dy}{dx} = 0$$

Solving for $\frac{dy}{dx}$, we find

$$\frac{dy}{dx} = \frac{-x}{y}$$

which is the same result as we found without implicit differentiation.

2.3 Folium of Descartes

It was relatively easy to rearrange the equation for a circle to solve for y , but that is not always the case. To help you understand this better, we will consider a famous curve known as the *Folium of Descartes*, given by the equation,

$$x^3 + y^3 = 3xy.$$

The word *folium* comes from the Latin word for “leaf,” describing the curve’s distinctive looped shape. The curve is named after René Descartes, a French mathematician who originally presented the curve as a challenge to fellow mathematician Pierre de Fermat, asking him to find the tangent line to the curve. Fermat was able to solve the problem with ease!

The Folium of Descartes (seen in Figure 2.1) is historically significant because unlike simpler curves such as circles or parabolas, it exhibits more complex behavior, including a self-intersection at the origin (the curve crosses itself!). This complexity makes it difficult to describe the curve using a single equation of the form $y = f(x)$.

To explicitly solve the equation $x^3 + y^3 = 3xy$ for y requires multiple expressions to fully capture the entire curve. As a result, the familiar techniques used for functions defined explicitly in terms of x are no longer sufficient. Instead, to find the slope of the tangent line at a point on the folium, we must use *implicit differentiation*.

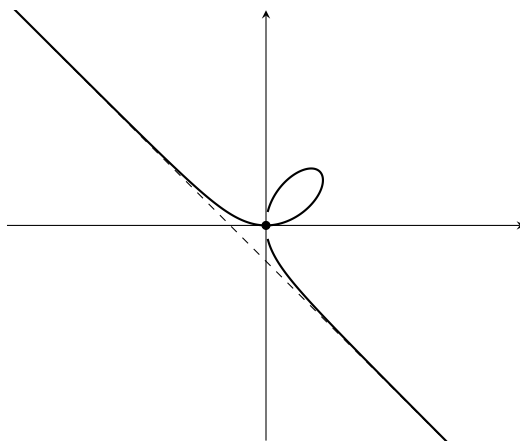


Figure 2.1: The Folium of Descartes $x^3 + y^3 = 3xy$ with asymptote $x + y + 1 = 0$.

2.3.1 Example: Tangent to Folium of Descartes

In this example, we will use implicit differentiation to easily find the tangent line at a point on the folium.

- Find $\frac{dy}{dx}$ if $x^3 + y^3 = 3xy$
- Find the tangent to the folium $x^3 + y^3 = 3xy$ at the point $(\frac{3}{2}, \frac{3}{2})$. (Check: $\frac{27}{8} + \frac{27}{8} = \frac{27}{4} = 3 \cdot \frac{3}{2} \cdot \frac{3}{2}$.)
- Is there any place in the first quadrant where the tangent line is horizontal? If so, state the point(s).

Solution:

(a) $\frac{d}{dx}[x^3 + y^3] = \frac{d}{dx}[3xy]$

$$3x^2 + 3y^2 \frac{dy}{dx} = 3x \frac{dy}{dx} + 3y$$

$$x^2 + y^2 \frac{dy}{dx} = x \frac{dy}{dx} + y$$

Rearranging to solve for $\frac{dy}{dx}$:

$$\frac{dy}{dx}(y^2 - x) = y - x^2$$

$$\frac{dy}{dx} = \frac{y - x^2}{y^2 - x}$$

(b) We already have the point, $(\frac{3}{2}, \frac{3}{2})$, so to write an equation for the tangent line, all we need is the slope. Substituting $x = y = \frac{3}{2}$ into our result from part (a):

$$\frac{dy}{dx} = \frac{\frac{3}{2} - \frac{9}{4}}{\frac{9}{4} - \frac{3}{2}} = \frac{-3/4}{3/4} = -1$$

This is the slope, m . Using the point-slope form of a line, our tangent line is $y - \frac{3}{2} = -(x - \frac{3}{2})$, or $y = 3 - x$.

(c) Recall that in the first quadrant, $x > 0$ and $y > 0$. We will set our solution for $\frac{dy}{dx}$ equal to 0:

$$\frac{y - x^2}{y^2 - x} = 0$$

which implies that

$$y - x^2 = 0$$

Substituting $y = x^2$ into the original equation:

$$x^3 + (x^2)^3 = 3(x)(x^2)$$

$$x^3 + x^6 = 3x^3$$

Which simplifies to

$$x^6 = 2x^3$$

Since we have excluded $x = 0$ by restricting our search to the first quadrant, we can divide both sides by x^3 :

$$x^3 = 2$$

$$x = \sqrt[3]{2} \approx 1.26$$

Substituting $x \approx 1.26$ into our equation for y :

$$y = \sqrt[3]{4} \approx 1.59$$

(The denominator $y^2 - x$ is not zero there, so the slope really is 0.)

Therefore, the folium has a horizontal tangent line at the point (1.26, 1.59).

2.4 Practice

Exercise 6

[This problem was originally presented as a no-calculator, multiple-choice question on the 2012 AP Calculus BC Exam.]
If $\arcsin x = \ln y$, what is $\frac{dy}{dx}$?

Working Space

Answer on Page 64

Exercise 7

[This problem was originally presented as a no-calculator, multiple-choice question on the 2012 AP Calculus BC Exam.]
The points $(-1, -1)$ and $(1, -5)$ are on the graph of a function $y = f(x)$ that satisfies the differential equation $\frac{dy}{dx} = x^2 + y$. Use implicit differentiation to find $\frac{d^2y}{dx^2}$. Determine if each point is a local minimum, local maximum, or inflection point by substituting the x and y values of the coordinates into $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

Working Space

Answer on Page 64

Exercise 8 Slope on a Circle

For $x^2 + y^2 = 25$, find $\frac{dy}{dx}$ at the point

Working Space

$(-4, 3)$.

Answer on Page 64

Exercise 9 Tangent to an Ellipse

Find the equation of the line tangent to the ellipse $x^2 + 4y^2 = 8$ at the point $(2, 1)$.

Working Space

Answer on Page 65

2.5 Derivatives of Parametric Functions

In the parametric functions chapter, you described curves with a pair of equations $x(t)$ and $y(t)$. Now that you can take derivatives, you can find the slope of such a curve. You could eliminate t and differentiate the result, but there is an easier way. By the chain rule,

$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$, so:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}.$$

The tangent is horizontal where $\frac{dy}{dt} = 0$ (and $\frac{dx}{dt} \neq 0$), and vertical where $\frac{dx}{dt} = 0$ (and $\frac{dy}{dt} \neq 0$).

Example

Find $\frac{dy}{dx}$ for the parametric equations

$$x = t^2 + 1, \quad y = 3t - 2.$$

Solution

First compute the derivatives with respect to t :

$$\frac{dx}{dt} = 2t, \quad \frac{dy}{dt} = 3.$$

Now divide:

$$\frac{dy}{dx} = \frac{3}{2t}.$$

As a check, eliminate the parameter. From $x = t^2 + 1$, we get (for $t > 0$)

$$t = \sqrt{x - 1}.$$

Substitute into $y = 3t - 2$:

$$y = 3\sqrt{x - 1} - 2.$$

Differentiating gives

$$\frac{dy}{dx} = \frac{3}{2\sqrt{x - 1}},$$

which matches the parametric result.

Exercise 10

Working Space

Consider the parametric equations

$$x = t^3 - 3t, \quad y = t^2.$$

- (a) Compute $\frac{dx}{dt}$ and $\frac{dy}{dt}$.
- (b) Find the values of t for which the curve has horizontal tangents.
- (c) Find the values of t for which the curve has vertical tangents.
- (d) Identify the points on the curve where these tangents occur.

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Exercise 11

Working Space

Consider the parametric equations

$$x = 2t + 1, \quad y = t^2.$$

- (a) Find $\frac{dy}{dx}$ in terms of t .
- (b) Evaluate $\frac{dy}{dx}$ at $t = 1$.
- (c) Interpret your result as the slope of the tangent line at the corresponding point.

Answer on Page 65

CHAPTER 3

Related Rates

Drop a pebble in a pond and a ring of ripples spreads out. The radius grows at a steady 30 cm per second. Does the area inside the ring also grow at a steady rate? No: the bigger the ring, the faster its area grows. The two rates are linked by the formula $A = \pi r^2$, and calculus tells you exactly how.

In a *related rates* problem, you know how fast one quantity is changing and want to know how fast another one is. The secret is that every quantity is a function of time t , so you can differentiate an equation that links them with respect to t , using the chain rule.

Worked example: the ripple. When the radius is 50 cm, how fast is the area growing?

Solution. Differentiate $A = \pi r^2$ with respect to t :

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt} = 2\pi(50)(30) \approx 9,425 \text{ cm}^2/\text{s}$$

At $r = 100$ cm, it would be twice as fast.

3.1 Steps to Solve Related Rates Problems

1. **Draw a diagram** and label the quantities that change with variables.
2. **Write down what you know and what you want**, as rates: $\frac{dx}{dt} = \dots$, find $\frac{dy}{dt}$.
3. **Write an equation** that relates the quantities. (Geometry is your friend: the Pythagorean theorem, similar triangles, volume formulas.)
4. **Differentiate both sides with respect to t .**
5. **Substitute and solve.** Plug in the numbers only *after* differentiating. If you plug in $x = 6$ first, x becomes a constant and its rate disappears.
6. **Answer in words**, with units.

3.2 Example: The Balloon and the Cyclist

A balloon rises at a constant 5 m/s. A cyclist rides toward the spot directly under it at 15 m/s. How fast is the distance between the cyclist and the balloon changing when the balloon is 100 m up and the cyclist is 40 m from the spot beneath it?

Let x be the cyclist's distance from the spot under the balloon, y the balloon's height, and z the distance from the cyclist to the balloon. They form a right triangle, so by the Pythagorean theorem

$$z^2 = x^2 + y^2 \quad (3.1)$$

Differentiating both sides with respect to time t gives

$$2z \frac{dz}{dt} = 2x \frac{dx}{dt} + 2y \frac{dy}{dt} \quad (3.2)$$

We know $\frac{dx}{dt} = -15$ m/s (negative, because x is shrinking), $\frac{dy}{dt} = 5$ m/s, $x = 40$ m, and $y = 100$ m. Then $z = \sqrt{40^2 + 100^2} = \sqrt{11600} \approx 107.7$ m, so

$$2(107.7) \frac{dz}{dt} = 2(40)(-15) + 2(100)(5) = -200$$

Therefore $\frac{dz}{dt} \approx -0.93$ m/s. The distance between them is shrinking at about 0.93 m/s. The cyclist closes in fast, but the rising balloon nearly cancels it out.

3.3 Example: Filling a Cone

A water tank is an upside-down cone 4 m tall with a radius of 2 m at the top. Water is pumped in at $2 \text{ m}^3/\text{min}$. How fast is the water level rising when the water is 3 m deep?

The volume of water at depth h is $V = \frac{1}{3}\pi r^2 h$, where r is the radius of the water's surface. That is two changing variables, but similar triangles link them: the cone is half as wide as it is tall, so $r = \frac{h}{2}$. Then

$$V = \frac{1}{3}\pi \left(\frac{h}{2}\right)^2 h = \frac{\pi h^3}{12}$$

Differentiate with respect to t :

$$\frac{dV}{dt} = \frac{\pi h^2}{4} \frac{dh}{dt}$$

Substitute $\frac{dV}{dt} = 2$ and $h = 3$:

$$2 = \frac{9\pi}{4} \frac{dh}{dt} \Rightarrow \frac{dh}{dt} = \frac{8}{9\pi} \approx 0.28 \text{ m/min}$$

The level rises about 28 cm per minute. It slows down as the tank fills, because the same volume spreads over a wider surface.

Exercise 12 A Sliding Ladder

A 10 m ladder leans against a wall. Its base slides away from the wall at 1 m/s. When the base is 6 m from the wall, how fast is the top of the ladder sliding down the wall?

*Working Space**Answer on Page 66***Exercise 13 Your Shadow**

You are 1.8 m tall and walk away from a 6 m streetlight at 1.5 m/s.

1. How fast is your shadow getting longer?
2. How fast is the tip of your shadow moving?

*Working Space**Answer on Page 66***Exercise 14 The Snowball**

A spherical snowball melts so that its volume shrinks at $10 \text{ cm}^3/\text{min}$. How fast is its radius shrinking when the radius is 5 cm? (The volume of a sphere is $V = \frac{4}{3}\pi r^3$.)

*Working Space**Answer on Page 66*

Multivariate Functions

In a video game, the ground is not flat. Wherever your character stands, at some east-west position x and north-south position y , the game needs to know how high the terrain is. That height is a function of *two* inputs: $h(x, y)$.

A real-valued *multivariate function* takes several real numbers as input and produces a single real number as output. Mathematicians write this as $f : \mathbb{R}^n \rightarrow \mathbb{R}$: the inputs come from $\mathbb{R}^n$ (lists of n real numbers) and the output is in $\mathbb{R}$.

You have used these for years without the name:

- The area of a rectangle, $A(w, \ell) = w\ell$, depends on two inputs.
- Your grade might be $G(h, q, t) = 0.2h + 0.3q + 0.5t$: a weighted average of homework, quizzes, and tests.
- The temperature in your classroom depends on position and time: $T(x, y, z, t)$.

Worked example. Let $f(x, y) = x^2 + y^2$. Find $f(3, 4)$ and $f(-1, 2)$.

Solution. $f(3, 4) = 9 + 16 = 25$ and $f(-1, 2) = 1 + 4 = 5$. Notice that $f(x, y)$ is the square of the distance from (x, y) to the origin.

4.1 Graphs as Surfaces

To graph a one-variable function, we plot the points $(x, f(x))$ and get a curve. To graph a two-variable function, we plot the points $(x, y, f(x, y))$ in three dimensions and get a surface. Think of $f(x, y)$ as the height of the ground above the point (x, y) on a map.

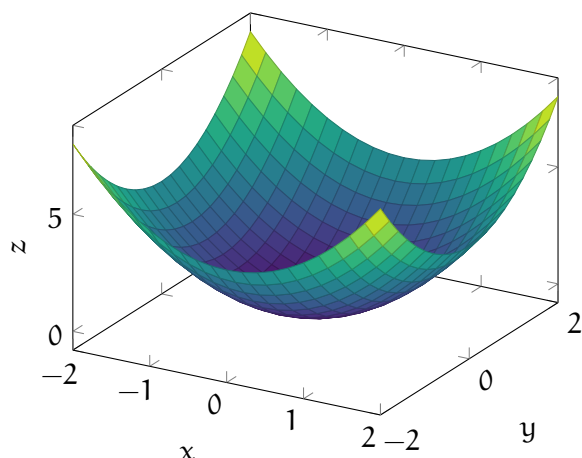


Figure 4.1: The graph of $f(x, y) = x^2 + y^2$ is a bowl called a paraboloid.

A function of three variables, like $g(x, y, z) = x^2 + y^2 + z^2$, would need four dimensions to graph. We cannot draw it, but we can still understand it with level sets.

4.2 Level Curves and Contour Maps

Hikers have a clever way to draw mountains on flat paper: a *topographic map*. Each curve on the map connects points at the same elevation. Where the curves bunch together, the ground is steep; where they spread apart, it is gentle.

For a function $f(x, y)$, the curve where $f(x, y) = c$ is called a *level curve*. A collection of level curves is a *contour map*. For $f(x, y) = x^2 + y^2$, the level curve $f = c$ is the circle $x^2 + y^2 = c$, with radius $\sqrt{c}$:

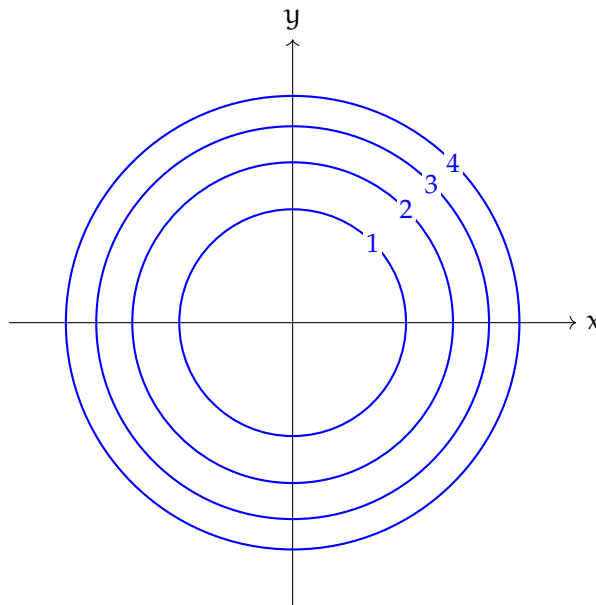


Figure 4.2: Level curves of $f(x, y) = x^2 + y^2$ for $c = 1, 2, 3, 4$. They get closer together as you move outward, because the bowl gets steeper.

For a function of three variables, $g(x, y, z) = c$ is a *level surface*. The level surfaces of $g(x, y, z) = x^2 + y^2 + z^2$ are spheres centered at the origin.

You can draw contour maps in Python with matplotlib:

```

1 import numpy as np
2 import matplotlib.pyplot as plt
3
4 x = np.linspace(-2, 2, 200)
5 y = np.linspace(-2, 2, 200)
6 X, Y = np.meshgrid(x, y)      # every (x, y) pair on a grid
7 Z = X**2 + Y**2
8
9 contours = plt.contour(X, Y, Z, levels=[1, 2, 3, 4])
10 plt.clabel(contours)
11 plt.gca().set_aspect('equal')
12 plt.show()

```

4.3 Limits and Partial Derivatives

Limits, continuity, and derivatives all extend to multivariate functions, with a twist. On a number line, you can approach a point only from the left or the right. On a plane, you can approach (a, b) from any direction, along any path, and the limit exists only if every path gives the same answer.

The simplest kind of derivative of a multivariate function is the *partial derivative*: the derivative with respect to one variable, holding the others constant. Standing on the bowl $f(x, y) = x^2 + y^2$, you could walk due east (changing only x) or due north (changing only y):

$$\frac{\partial f}{\partial x} = 2x \quad \frac{\partial f}{\partial y} = 2y$$

The next chapter explores partial derivatives in detail.

Exercise 15 Evaluating

Let $f(x, y) = 3x - y^2$ and $V(r, h) = \pi r^2 h$.

1. Find $f(2, 1)$ and $f(0, -3)$.
2. V gives the volume of a cylinder. Find $V(3, 10)$ and explain what it means.
3. Does doubling r or doubling h change V more?

Working Space

Answer on Page 66

Exercise 16 Level Curves

Describe the level curves of each function.

1. $f(x, y) = x + y$
2. $g(x, y) = y - x^2$

Working Space

Answer on Page 67

Exercise 17 Reading a Topographic Map

On a hiking map, the contour lines are 10 m apart in elevation. On the east side

Working Space

of a hill, the lines are 5 mm apart on the map; on the west side, they are 20 mm apart. Which side would you rather climb, and why?

Answer on Page 67

Partial Derivatives

Imagine standing on a hillside. Walk east and the ground might rise steeply; walk north and it might stay nearly level. A hill does not have one slope at a point; it has a slope in every direction. If the height is $f(x, y)$, the two simplest slopes are the one you feel walking due east (changing x only) and the one you feel walking due north (changing y only). These are the *partial derivatives* of f .

To find one, you hold every other variable constant and differentiate as usual. We write $\frac{\partial f}{\partial x}$ for the partial derivative with respect to x , using a curly ∂ (read “partial”) instead of d as a reminder that other variables are being held still. In the next chapter, we combine the partial derivatives into a vector called the gradient, which points straight uphill.

For a function of two variables, $f(x, y)$, we can take the derivative with respect to x or with respect to y . These are called the *partial derivatives* of f . Formally, the partial derivatives are defined as:

Limit Definition of Partial Derivatives

$$f_x(x, y) = \lim_{h \rightarrow 0} \frac{f(x + h, y) - f(x, y)}{h}$$

$$f_y(x, y) = \lim_{h \rightarrow 0} \frac{f(x, y + h) - f(x, y)}{h}$$

Let us consider a polynomial function of two variables: $f(x, y) = 3x^2 + y^3 + 4xy$. We will use the limit definition to find the partial derivative with respect to x , then compare this to what we already know about derivatives of single-variable functions. Recall that if we can describe a function as a sum of two other functions, the derivative of the original function is the same as the sum of the derivatives of the other functions. That is,

$$\text{if } f(x) = g(x) + h(x)$$

$$\text{then } f'(x) = g'(x) + h'(x)$$

Let us then define $r(x, y) = 3x^2$, $s(x, y) = y^3$, and $t(x, y) = 4xy$. And so $f(x, y) = r(x, y) + s(x, y) + t(x, y)$, which means $f_x(x, y) = r_x(x, y) + s_x(x, y) + t_x(x, y)$. Then,

$$f_x(x, y) = \lim_{h \rightarrow 0} \frac{r(x + h, y) - r(x, y)}{h} + \lim_{h \rightarrow 0} \frac{s(x + h, y) - s(x, y)}{h} + \lim_{h \rightarrow 0} \frac{t(x + h, y) - t(x, y)}{h}$$

$$\begin{aligned}
&= \lim_{h \rightarrow 0} \frac{3(x+h)^2 - 3x^2}{h} + \lim_{h \rightarrow 0} \frac{y^3 - y^3}{h} + \lim_{h \rightarrow 0} \frac{4(x+h)y - 4xy}{h} \\
&= \lim_{h \rightarrow 0} \frac{3x^2 + 6xh + 3h^2 - 3x^2}{h} + 0 + \lim_{h \rightarrow 0} \frac{4xy + 4hy - 4xy}{h}
\end{aligned}$$

Notice that $s_x(x, y) = 0$. This term only had y , and its derivative with respect to x is zero. Continuing,

$$\begin{aligned}
f_x(x, y) &= \lim_{h \rightarrow 0} \frac{6xh + 3h^2}{h} + \lim_{h \rightarrow 0} \frac{4hy}{h} = \lim_{h \rightarrow 0} (6x + 3h) + \lim_{h \rightarrow 0} 4y \\
&= 6x + 4y
\end{aligned}$$

As you can see, $r_x(x, y) = 6x$ and $t_x(x, y) = 4y$. Recall the polynomial rule for single derivatives. The derivative of $3x^2$ is $6x$, which is also what we see with the partial derivative in this case. What about the other term, $4xy$? Well, we know the derivative of bx , where b is a constant, is b . The partial derivative of $4xy$ with respect to x being $4y$ suggests the rule for determining partial derivatives:

Rule for Finding Partial Derivatives of $f(x, y)$

1. To find the partial derivative with respect to x , f_x , treat y as a constant and differentiate with respect to x .
2. To find the partial derivative with respect to y , f_y , treat x as a constant and differentiate with respect to y .

Let us check this by predicting f_y , then using the limit definition to confirm our prediction. Applying the polynomial rule, we predict that f_y is:

$$f_y(x, y) = 3y^2 + 4x$$

Which we found by treating x as a constant and taking the derivative of each term with respect to y . Let us see if we get the same result using the limit definition of the derivative with respect to y :

$$\begin{aligned}
f_y(x, y) &= \lim_{h \rightarrow 0} \frac{f(x, y+h) - f(x, y)}{h} \\
&= \lim_{h \rightarrow 0} \frac{[3x^2 + (y+h)^3 + 4x(y+h)] - [3x^2 + y^3 + 4xy]}{h} \\
&= \lim_{h \rightarrow 0} \frac{3x^2 + y^3 + 3y^2h + 3yh^2 + h^3 + 4xy + 4xh - 3x^2 - y^3 - 4xy}{h} \\
&= \lim_{h \rightarrow 0} \frac{3y^2h + 3yh^2 + h^3 + 4xh}{h} = \lim_{h \rightarrow 0} 3y^2 + 3yh + h^2 + 4x = 3y^2 + 4x
\end{aligned}$$

Which is our expected result. In summary, you find the partial derivative with respect to a particular variable by treating all the other variables as constants and differentiating with respect to the particular variable, applying the rules of differentiation you have already learned.

5.1 Partial Derivative Notation

There are many ways to denote a partial derivative. We have already seen one way, f_x and f_y . Another common notation uses the curly ∂ , and another uses capital D. They are shown below:

Partial Derivative Notations

$$f_x(x, y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x, y) = D_x f$$

$$f_y(x, y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x, y) = D_y f$$

Exercise 18 First Partial Derivatives

Find f_x and f_y for the following functions.

Working Space

1. $f(x, y) = 3x^4 + 4x^2y^3$
2. $f(x, y) = xe^{-y}$
3. $f(x, y) = \sqrt{3x + 4y^2}$
4. $f(x, y) = \sin x^2y$
5. $f(x, y) = \ln(x^y)$

Answer on Page 67

5.2 Partial Derivatives of Functions of More than Two Variables

The above method of determining partial derivatives applies to functions with three, four, or any number of variables.

Example: Find all the first derivatives of the function $f(x, y, z) = y \cos(x^2 + 3z)$.

Solution:

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x} [y \cos(x^2 + 3z)] = -y \sin(x^2 + 3z) \left(\frac{\partial}{\partial x} (x^2 + 3z) \right)$$

$$\frac{\partial f}{\partial x} = -2xy \sin(x^2 + 3z)$$

And

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y} [y \cos(x^2 + 3z)]$$

$$\frac{\partial f}{\partial y} = \cos(x^2 + 3z)$$

And

$$\frac{\partial f}{\partial z} = \frac{\partial}{\partial z} [y \cos(x^2 + 3z)] = -y \sin(x^2 + 3z) \left(\frac{\partial}{\partial z} (x^2 + 3z) \right)$$

$$\frac{\partial f}{\partial z} = -3y \sin(x^2 + 3z)$$

Exercise 19 Partial Derivatives with 3 or More Variables

Find all first partial derivatives of the following functions.

Working Space

1. $f = \sin(x^2 - y^2) \cos(\sqrt{z})$

2. $q = \sqrt[3]{t^3 + u^3} \sin(5v)$

3. $w = x^z y^x$

Answer on Page 67

5.3 Higher Order Partial Derivatives

Just like with single-variable equations, we can take the partial derivative more than once. There are also several notations for second partial derivatives.

Second Partial Derivative Notation

$$(f_x)_x = f_{xx} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right) = \frac{\partial^2 f}{\partial x^2}$$

$$(f_x)_y = f_{xy} = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right) = \frac{\partial^2 f}{\partial y \partial x}$$

$$(f_y)_x = f_{yx} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right) = \frac{\partial^2 f}{\partial x \partial y}$$

$$(f_y)_y = f_{yy} = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial y} \right) = \frac{\partial^2 f}{\partial y^2}$$

Notice that for $(\partial^2 f / \partial y \partial x)$, we first take the derivative with respect to x , then with respect to y .

Example: Find all the second order partial derivatives of $f(x, y) = 2x^2 - x^3y^2 + y^3$.

Solution: We begin by finding f_x and f_y :

$$f_x(x, y) = 4x - 3x^2y^2$$

$$f_y(x, y) = -2x^3y + 3y^2$$

We then take another partial derivative to find all the second order partial derivatives:

$$f_{xx}(x, y) = \frac{\partial}{\partial x} f_x(x, y) = \frac{\partial}{\partial x} (4x - 3x^2y^2) = 4 - 6xy^2$$

$$f_{xy}(x, y) = \frac{\partial}{\partial y} f_x(x, y) = \frac{\partial}{\partial y} (4x - 3x^2y^2) = -6x^2y$$

$$f_{yx}(x, y) = \frac{\partial}{\partial x} f_y(x, y) = \frac{\partial}{\partial x} (-2x^3y + 3y^2) = -6x^2y$$

$$f_{yy}(x, y) = \frac{\partial}{\partial y} f_y(x, y) = \frac{\partial}{\partial y} (-2x^3y + 3y^2) = -2x^3 + 6y$$

What do you notice about f_{xy} and f_{yx} ? They are the same! This is not a coincidence of the particular function used in the example. For most functions, $f_{xy} = f_{yx}$, as stated by Clairaut's theorem.

Clairaut's Theorem

If f is defined on a disk D and f_{xy} and f_{yx} are both continuous on D , then $f_{xy} = f_{yx}$ on D .

This is also true for third, fourth, and higher-order derivatives.

Exercise 20 Clairaut's Theorem

Show that Clairaut's theorem holds for the following functions (show that $f_{xy} = f_{yx}$).

1. $f(x, y) = e^{2xy} \sin x$

2. $f(x, y) = \frac{x^2}{x+y}$

3. $f(x, y) = \ln(2x + 3y)$

Working Space

Answer on Page 69

Exercise 21 Second Order Partial Derivatives

Find all second order partial derivatives of the function.

1. $f(x, y) = x^5y^2 - 3x^3y^2$

2. $v = \sin(p^3 + q^2)$

3. $T = e^{-3r} \cos \theta^2$

Working Space

Answer on Page 69

5.4 The Chain Rule

For single-variable functions, where $y = f(x)$ and $x = g(t)$, we have seen that:

$$\frac{dy}{dt} = \frac{dy}{dx} \frac{dx}{dt}$$

Which is the Chain Rule for single-variable functions. For multi-variable functions, there

are several versions of the Chain Rule, depending on how the variables and functions are defined. First, we consider the case where $z = f(x, y)$ and $x = g(t)$ and $y = h(t)$ (i.e. f is a multi-variable function of x and y , while x and y are single-variable functions of t). This means that z is an indirect function of t :

$$z = f(x, y) = f(g(t), h(t))$$

Then the derivative of z with respect to t is given by:

$$\frac{dz}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}$$

Example: If $z = xy^2 + 3x^4y$, where $x = 2 \sin(t)$ and $y = \cos(3t)$, find dz/dt when $t = \pi/2$.

Solution: First, we apply the Chain Rule to z :

$$\begin{aligned} \frac{dz}{dt} &= \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt} \\ &= \frac{\partial}{\partial x} [xy^2 + 3x^4y] \cdot \frac{d}{dt} [2 \sin(t)] + \frac{\partial}{\partial y} [xy^2 + 3x^4y] \cdot \frac{d}{dt} [\cos(3t)] \\ &= (y^2 + 12x^3y) \cdot (2 \cos(t)) + (2xy + 3x^4) \cdot (-3 \sin(3t)) \end{aligned}$$

When $t = \pi/2$, $\cos(t) = 0$, $\sin(3t) = -1$, $x = 2$, and $y = 0$. Substituting:

$$\begin{aligned} \frac{dz}{dt} &= (0 + 0) \cdot (0) + (0 + 3(2)^4) \cdot (-3 \cdot -1) \\ &= 3(2)^4 \cdot 3 = 144 \end{aligned}$$

Another case is where x and y are also multi-variable functions. Consider $z = f(x, y)$, $x = g(s, t)$, and $y = h(s, t)$. This means z is an indirect function of s and t :

$$z = f(x, y) = f(g(s, t), h(s, t))$$

In this case, there are two partial derivatives of z :

$$\begin{aligned} \frac{\partial z}{\partial s} &= \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s} \\ \frac{\partial z}{\partial t} &= \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t} \end{aligned}$$

Example: Find $\partial z/\partial s$ and $\partial z/\partial t$ if $z = e^{2x} \cos y$, $x = s^2t$, and $y = st^2$.

Solution: First, let us find $\partial z/\partial s$:

$$\begin{aligned}\frac{\partial z}{\partial s} &= \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s} \\ &= \frac{\partial}{\partial x} [e^{2x} \cos y] \cdot \frac{\partial}{\partial s} [s^2 t] + \frac{\partial}{\partial y} [e^{2x} \cos y] \cdot \frac{\partial}{\partial s} [st^2] \\ &= (2e^{2x} \cos y) \cdot (2st) + (-e^{2x} \sin y) \cdot (t^2)\end{aligned}$$

Substituting for x and y :

$$\frac{\partial z}{\partial s} = 4ste^{2s^2t} \cos(st^2) - t^2 e^{2s^2t} \sin(st^2)$$

And finding $\partial z/\partial t$:

$$\begin{aligned}\frac{\partial z}{\partial t} &= \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t} \\ &= \frac{\partial}{\partial x} [e^{2x} \cos y] \cdot \frac{\partial}{\partial t} [s^2 t] + \frac{\partial}{\partial y} [e^{2x} \cos y] \cdot \frac{\partial}{\partial t} [st^2] \\ &= (2e^{2x} \cos y) \cdot (s^2) + (-e^{2x} \sin y) \cdot (2st)\end{aligned}$$

Substituting for x and y :

$$\frac{\partial z}{\partial t} = 2s^2 e^{2s^2t} \cos(st^2) - 2ste^{2s^2t} \sin(st^2)$$

Exercise 22 The Chain Rule for Multivariable Functions

Find dz/dt or $\partial z/\partial s$ and $\partial z/\partial t$.

Working Space

1. $z = \sin x \cos y$, $x = 3\sqrt{t}$, $y = 2/t$
2. $z = \sqrt{1 + xy}$, $x = \tan t$, $y = \arctan t$
3. $z = \arctan(x^2 + y^2)$, $x = t \ln s$, $y = se^t$
4. $z = \sqrt{x}e^{xy}$, $x = 1 + st$, $y = s^2 - t^2$

Answer on Page 70

5.5 Interpreting Partial Derivatives

What is the meaning of a partial derivative? Recall that $z = f(x, y)$ plots a surface, S . Consider the function $z = \cos y - x^2$, shown in figure 5.1.

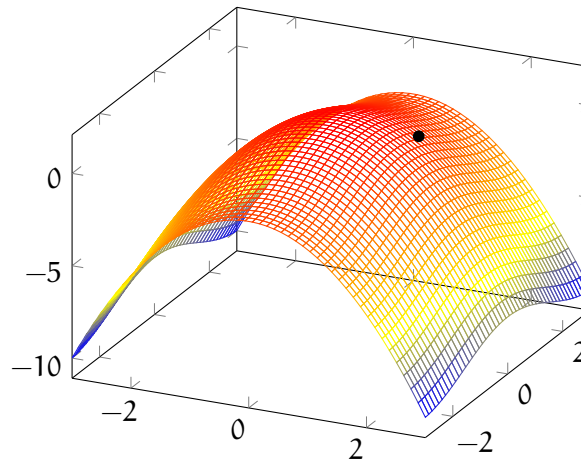


Figure 5.1: The surface $z = \cos y - x^2$

We can see that $f(1, \pi/3) = -1/2$; therefore, the point $(1, \pi/3, -1/2)$ lies on the surface $z = \cos y - x^2$ (the black dot shown in figure ??). If we fix y such that $y = \pi/3$, we are looking at the intersection between the surface and the plane $y = \pi/3$ (see figure ??).

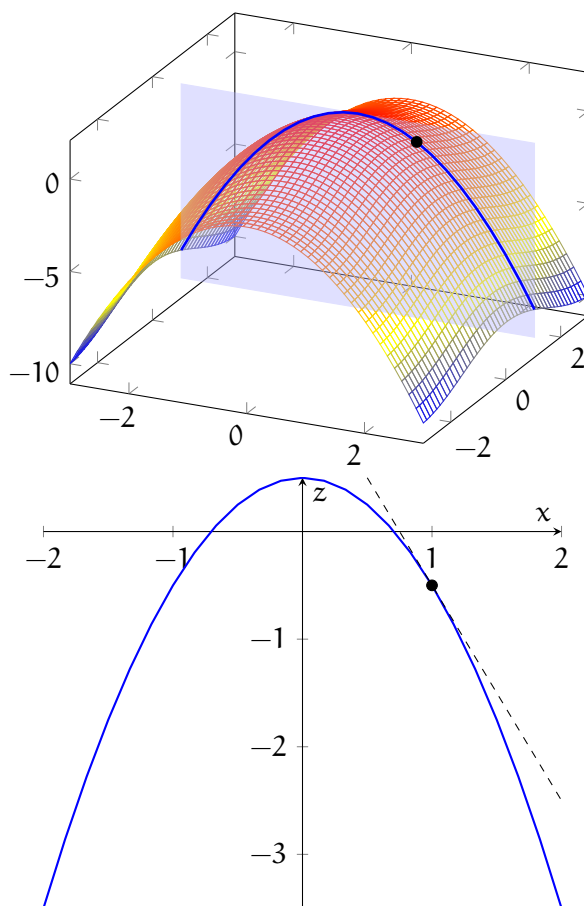


Figure 5.2: The intersection between the surface $z = \cos y - x^2$ and $y = \pi/3$ is the parabola $z(x) = 1/2 - x^2$

We can describe this intersection as $g(x) = f(x, \pi/3)$, so the slope of a tangent line to this intersection is given by $g'(x) = f_x(x, \pi/3)$. This means, geometrically, $f_x(1, \pi/3)$ is the slope of the line that lies tangent to $z = f(x, y)$ at the point $(1, \pi/3, -1/2)$ and in the plane $y = \pi/3$ (see figure 5.2). Alternatively, you could think of f_x as the slope of the tangent line to the surface that is parallel to the x -axis.

Similarly, we can fix $x = 1$ and look at the intersection between the surface $z = \cos y - x^2$ and the plane $x = 1$ (see figure 5.3). Just like before, we can describe this intersection as $h(y) = f(1, y)$, which means the slope of a line tangent to the intersection is given by $h'(y) = f_y(1, y)$. Therefore, as with f_x , $f_y(a, b)$ gives the slope of a line tangent to the point $(a, b, f(a, b))$ and parallel to the y -axis.

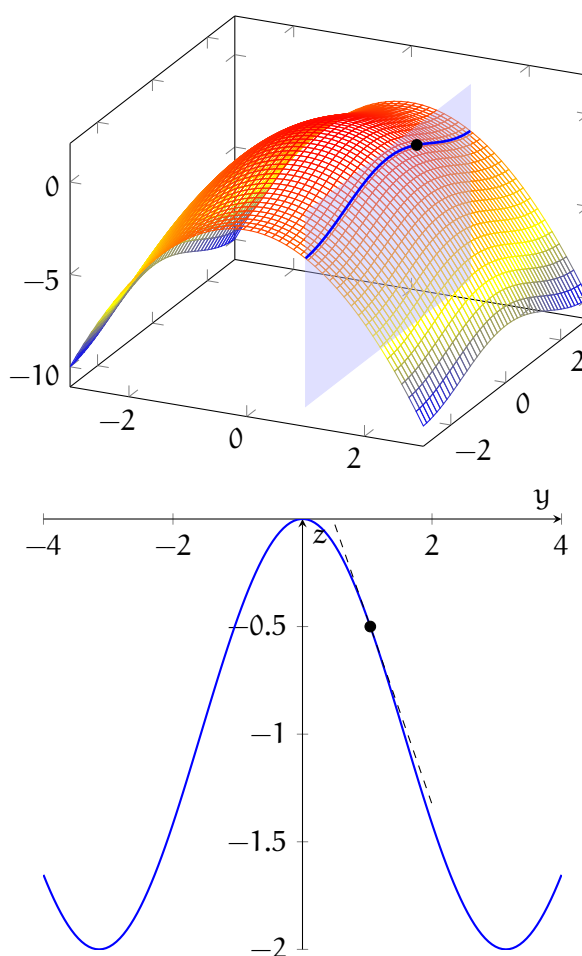


Figure 5.3: The intersection between the surface $z = \cos y - x^2$ and $x = 1$ is the trigonometric function $z = \cos y - 1$

Example: The density of bacterial growth at a point (x, y) on a flat agar plate is given by $D = 45 / (2 + x^2 + y^2)$. Find the rate of change of bacterial density at the point $(1, 3)$ (a) in the x -direction and (b) in the y -direction. Interpret the meaning of your results.

Solution: The rate of change of a two-variable function in the x -direction is given by the partial derivative with respect to x :

$$\begin{aligned} D_x &= \frac{\partial}{\partial x} \frac{45}{2 + x^2 + y^2} = \frac{-45 (\partial/\partial x) (2 + x^2 + y^2)}{(2 + x^2 + y^2)^2} \\ &= \frac{-90x}{(2 + x^2 + y^2)^2} \end{aligned}$$

The rate of change in the x -direction at $(x, y) = (1, 3)$ is given by:

$$D_x(1, 3) = \frac{-90(1)}{(2 + 1^2 + 3^2)^2} = \frac{-90}{(12)^2} = \frac{-90}{144} = -\frac{5}{8}$$

This means that at $(1, 3)$, the density of bacteria is decreasing as you move away from $x = 0$ along the line $y = 3$.

Similarly, the rate of change in the y -direction is given by the partial derivative with respect to y :

$$\begin{aligned} D_y &= \frac{\partial}{\partial y} \frac{45}{2 + x^2 + y^2} = \frac{-45 (\partial/\partial y) (2 + x^2 + y^2)}{(2 + x^2 + y^2)^2} \\ &= \frac{-90y}{(2 + x^2 + y^2)^2} \end{aligned}$$

The rate of change in the y -direction at $(x, y) = (1, 3)$ is given by:

$$D_y(1, 3) = \frac{-90(3)}{(2 + 1^2 + 3^2)^2} = \frac{-270}{144} = -\frac{15}{8}$$

This means that at $(1, 3)$ the density of bacteria is decreasing faster along the y -direction than along the x -direction.

Exercise 23 Using partial derivatives to find tangent lines

Find equations for tangent lines to the surface at the given xy -coordinate. In which direction is the function changing the fastest?

Working Space

1. $z = x^2 e^{y/x}, (1, -1)$
2. $z = \cos x + y \sin y, (\pi, \pi/2)$
3. $z = x^2 y - 3xy^2, (3, 2)$

Answer on Page 72

Gradients

You are halfway up a mountain in thick fog and want to reach the top as fast as possible. Which way should you walk? The *gradient* answers exactly that question: it is a vector that points in the direction of steepest ascent. In order to understand the gradient, we must first discuss directional derivatives. Recall that the partial derivatives, f_x and f_y , can be used to define a plane tangent to the surface $z = f(x, y)$ (see figure 6.1). Directional derivatives allow us to find the slope of the tangent plane in directions other than the x - and y -directions.

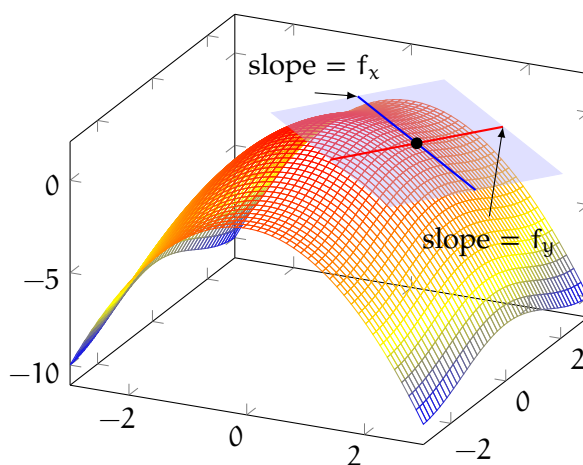


Figure 6.1: The partial derivatives f_x and f_y define a tangent plane

6.1 Directional Derivatives

The contour map in figure 6.2 shows the elevation, $f(x, y)$ for a mountain. You already know that you can use the partial derivatives, f_x and f_y to find the rate of change in elevation going east-west or north-south. But what about other directions? Suppose the hiking path you are on goes north-east. How can you predict the steepness (i.e. the rate of elevation change) along this path? The directional derivative allows us to find the rate of change in any direction.

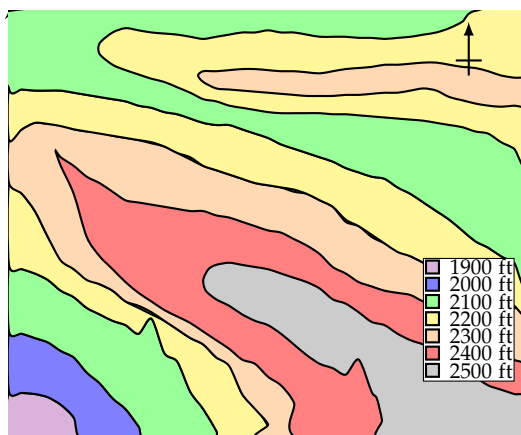


Figure 6.2: The contour plot shows the elevation of a mountain. f_x gives the slope going east, while f_y gives the slope going north

At some point, (x_0, y_0) , the partial derivatives $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ give the rate of change of elevation in the east-west and north-south directions, respectively (see figure 6.3).

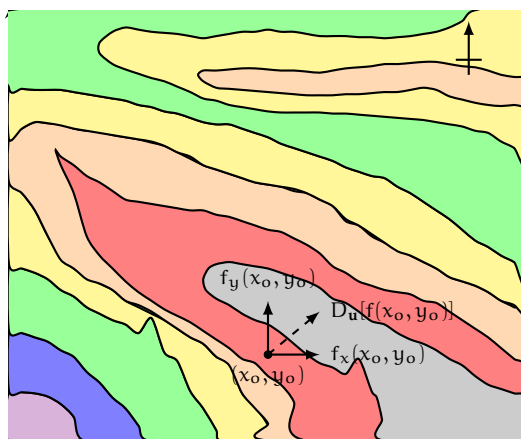


Figure 6.3: If $\mathbf{u}$ points north-east, then the directional derivative of $f(x, y)$ at (x_0, y_0) , $D_{\mathbf{u}}[f(x_0, y_0)]$, tells the rate of change going north-east

To find the rate of change at (x_0, y_0) , in the direction of some arbitrary unit vector, $\mathbf{u} = [a, b] = a\mathbf{i} + b\mathbf{j}$, we first note that the point $Q = (x_0, y_0, z_0)$, where $z_0 = f(x_0, y_0)$, lies on the surface defined by $z = f(x, y)$. There is a vertical plane, P , that passes through Q and points in the direction of $\mathbf{u}$. This intersection defines curve C , which lies on the surface, and the slope of this curve at $Q = (x_0, y_0, z_0)$ is the directional derivative of f in the direction of $\mathbf{u}$ (see figure 6.4).

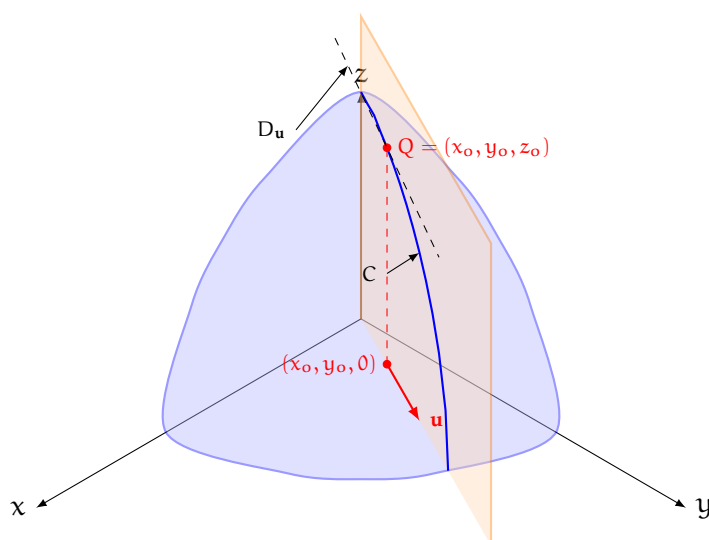


Figure 6.4: The slope of the curve formed between the plane parallel to $\mathbf{u}$ and the surface $z = f(x, y)$ is the directional derivative, $D_{\mathbf{u}}$

We can choose another point, $R = (x, y, z)$, that is h units away from Q along $\mathbf{u}$ (see 6.5). Then the change in x is $x - x_0 = ha$ and the change in y is $y - y_0 = hb$. And the slope from Q to R is given by:

$$\frac{\Delta z}{h} = \frac{f(x, y) - f(x_0, y_0)}{h}$$

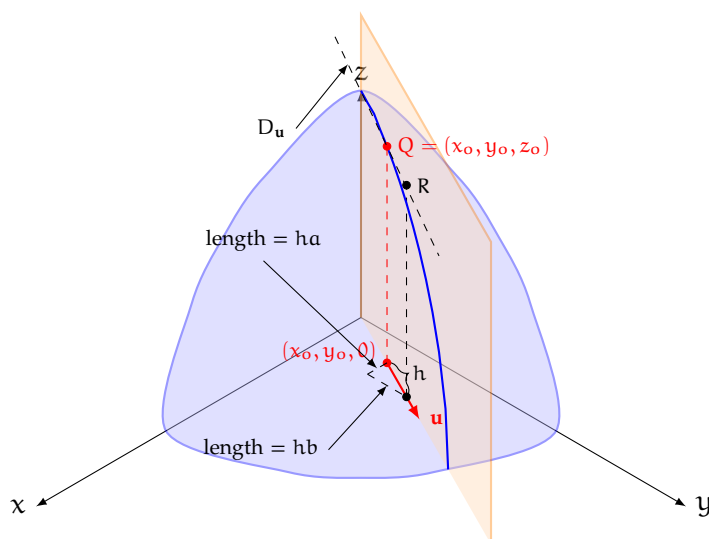


Figure 6.5: A second point, R , along $\mathbf{u}$ is h units away along $\mathbf{u}$

We find the directional derivative by substituting for x and y and taking the limit as h

goes to zero:

$$D_{\mathbf{u}}f(x_0, y_0) = \lim_{h \rightarrow 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

How is this related to f_x and f_y ? Let us define $g(h)$ such that $g(h) = f(x_0 + ha, y_0 + hb)$. Then

$$\begin{aligned} g'(0) &= \lim_{h \rightarrow 0} \frac{g(h) - g(0)}{h} \\ &= \lim_{h \rightarrow 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h} = D_{\mathbf{u}}f(x_0, y_0) \end{aligned}$$

We can also apply the Chain Rule to $g(h)$:

$$g'(h) = \frac{\partial f}{\partial x} \frac{dx}{dh} + \frac{\partial f}{\partial y} \frac{dy}{dh} = f_x(x, y)a + f_y(x, y)b$$

Substituting $h = 0$, $x = x_0$, and $y = y_0$, we see that:

$$g'(0) = f_x(x_0, y_0)a + f_y(x_0, y_0)b$$

Which means that:

$$D_{\mathbf{u}}f(x_0, y_0) = f_x(x_0, y_0)a + f_y(x_0, y_0)b$$

So a directional derivative is:

The Directional Derivative

Let f be a differentiable function and $\mathbf{u}$ be a unit vector, $\mathbf{u} = [a, b]$. Then the directional derivative in the direction of $\mathbf{u}$ is:

$$D_{\mathbf{u}}f(x, y) = f_x(x, y)a + f_y(x, y)b = \mathbf{u}_x \left[\frac{\partial}{\partial x} f(x, y) \right] + \mathbf{u}_y \left[\frac{\partial}{\partial y} f(x, y) \right] \quad (6.1)$$

Where $\mathbf{u}_x$ and $\mathbf{u}_y$ are the x - and y -components of $\mathbf{u}$, respectively.

Example: Find the directional derivative $D_{\mathbf{u}}f(x, y)$ if $f(x, y) = y^3 - 3xy + 4x^2$ and $\mathbf{u}$ is the unit vector given by the angle $\theta = \pi/3$. What is the rate of change in the direction of $\mathbf{u}$ at $(1, 2)$?

Solution: We can describe $\mathbf{u}$ thusly:

$$\mathbf{u} = \left[\cos \frac{\pi}{3}, \sin \frac{\pi}{3} \right] = \left[\frac{1}{2}, \frac{\sqrt{3}}{2} \right]$$

And therefore:

$$\begin{aligned}
 D_{\mathbf{u}}f(x, y) &= f_x(x, y) \left(\frac{1}{2} \right) + f_y(x, y) \left(\frac{\sqrt{3}}{2} \right) \\
 &= \frac{\partial}{\partial x} (y^3 - 3xy + 4x^2) \left(\frac{1}{2} \right) + \frac{\partial}{\partial y} (y^3 - 3xy + 4x^2) \left(\frac{\sqrt{3}}{2} \right) \\
 &= \frac{1}{2} (-3y + 8x) + \frac{\sqrt{3}}{2} (3y^2 - 3x) \\
 &= \frac{-3}{2}y + 4x + \frac{3\sqrt{3}}{2}y^2 - \frac{3\sqrt{3}}{2}x = \frac{3\sqrt{3}}{2}y^2 + \frac{8 - 3\sqrt{3}}{2}x - \frac{3}{2}y
 \end{aligned}$$

And therefore $D_{\mathbf{u}}f(1, 2)$ is:

$$\begin{aligned}
 &= \frac{3\sqrt{3}}{2} (2)^2 + \frac{8 - 3\sqrt{3}}{2} (1) - \frac{3}{2} (2) = 6\sqrt{3} + 4 - \frac{3\sqrt{3}}{2} - 3 \\
 &= 1 + \frac{9\sqrt{3}}{2}
 \end{aligned}$$

6.2 Unit Vectors in Two Dimensions

What if the given vector is not a unit vector? We can scale the given vector to find a unit vector in the same direction:

Example: Find the directional derivative of $f(x, y) = 3x\sqrt{y}$ at $(1, 4)$ in the direction of $\mathbf{v} = [2, 1]$.

Solution: First, we need to find a unit vector in the same direction as $\mathbf{v}$. There are several ways to do this. In two dimensions, a unit vector in the same direction as $\mathbf{v}$ can be found using trigonometry (see figure 6.6 for an illustration).

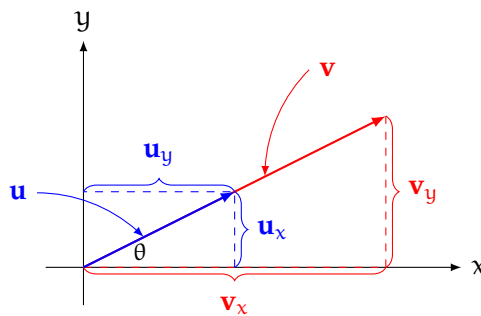


Figure 6.6: $\mathbf{u}$ is a unit vector in the same direction as $\mathbf{v}$

We know that $\theta = \arctan(v_y/v_x)$. Therefore, the x -component of the unit vector, $\mathbf{u}$, is given by:

$$\mathbf{u}_x = |\mathbf{u}| \cos \theta = \cos \left(\arctan \frac{v_y}{v_x} \right)$$

Similarly, we know that:

$$\mathbf{u}_y = |\mathbf{u}| \sin \theta = \sin \left(\arctan \frac{v_y}{v_x} \right)$$

(Recall that since $\mathbf{u}$ is a unit vector, $|\mathbf{u}| = 1$).

Let us use this method to find a unit vector, $\mathbf{u}$, in the same direction as $\mathbf{v} = [2, 1]$:

$$\mathbf{u}_x = \cos \left(\arctan \frac{1}{2} \right) \approx \cos(0.464) = \frac{2}{\sqrt{5}}$$

$$\mathbf{u}_y = \sin \left(\arctan \frac{1}{2} \right) \approx \sin(0.464) = \frac{1}{\sqrt{5}}$$

Therefore, a unit vector in the same direction as $\mathbf{v}$ is $\mathbf{u} = [2/\sqrt{5}, 1/\sqrt{5}]$.

And we can find the directional derivative:

$$\begin{aligned} D_{\mathbf{u}}(x, y) &= \mathbf{u}_x \left[\frac{\partial}{\partial x} f(x, y) \right] + \mathbf{u}_y \left[\frac{\partial}{\partial y} f(x, y) \right] \\ D_{\mathbf{u}}(x, y) &= \left(\frac{2}{\sqrt{5}} \right) \left[\frac{\partial}{\partial x} (3x\sqrt{y}) \right] + \left(\frac{1}{\sqrt{5}} \right) \left[\frac{\partial}{\partial y} (3x\sqrt{y}) \right] \\ D_{\mathbf{u}}(x, y) &= \left(\frac{2}{\sqrt{5}} \right) (3\sqrt{y}) + \left(\frac{1}{\sqrt{5}} \right) \left(\frac{3x}{2\sqrt{y}} \right) \\ D_{\mathbf{u}}(x, y) &= \frac{12y + 3x}{2\sqrt{5}y} \end{aligned}$$

To find the value of the directional derivative at $(1, 4)$, we substitute for x and y :

$$D_{\mathbf{u}}(1, 4) = \frac{12(4) + 3(1)}{2\sqrt{5}(4)} = \frac{51}{4\sqrt{5}} \approx 5.702$$

6.3 Unit Vectors in Higher Dimensions

The trigonometric explanation for finding unit vectors is more difficult to visualize in higher dimensions. However, there is another method that works well in 2, 3, and higher

dimensions. Recall that the magnitude of a vector, $\mathbf{v} = [\mathbf{v}_x, \mathbf{v}_y]$ is given by $|\mathbf{v}| = \sqrt{(\mathbf{v}_x)^2 + (\mathbf{v}_y)^2}$. For a vector with n dimensions, $\mathbf{v} = [\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n]$, the magnitude is given by $|\mathbf{v}| = \sqrt{(\mathbf{v}_1)^2 + (\mathbf{v}_2)^2 + \dots + (\mathbf{v}_n)^2}$.

To find a unit vector, $\mathbf{u}$, in the same direction as $\mathbf{v}$, we can scale $\mathbf{v}$ up or down so that its magnitude is 1. We can do this by dividing by $\mathbf{v}$'s magnitude. Consider the two-dimensional vector used in the last example, $\mathbf{v} = [2, 1]$. Its magnitude is:

$$|\mathbf{v}| = \sqrt{(2)^2 + (1)^2} = \sqrt{5}$$

Let us check if $\mathbf{v}/|\mathbf{v}|$ is a unit vector:

$$\frac{\mathbf{v}}{|\mathbf{v}|} = \left(\frac{1}{\sqrt{5}}\right) [2, 1] = \left[\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right]$$

And the magnitude of this scaled vector is:

$$\left|\frac{\mathbf{v}}{|\mathbf{v}|}\right| = \sqrt{\left(\frac{2}{\sqrt{5}}\right)^2 + \left(\frac{1}{\sqrt{5}}\right)^2} = \sqrt{\frac{4}{5} + \frac{1}{5}} = \sqrt{1} = 1$$

Notice our unit vector is the same as we found using the trigonometric method above.

Another way to think of the question is: what factor, k , can we multiply $\mathbf{v}$ by to yield a vector with a magnitude of 1? Let us see this method for the 3-dimensional vector $\mathbf{v} = [3, 2, 1]$. We are looking for a k such that:

$$|k\mathbf{v}| = 1$$

$$\begin{aligned} |k\mathbf{v}| &= |[3k, 2k, 1k]| = \sqrt{(3k)^2 + (2k)^2 + (1k)^2} \\ &= \sqrt{9k^2 + 4k^2 + k^2} = k\sqrt{14} = 1 \end{aligned}$$

Which implies that $k = 1/\sqrt{14}$, which is $1/|\mathbf{v}|$. And therefore a unit vector in the same direction as $\mathbf{v} = [3, 2, 1]$ is:

$$\mathbf{u} = \frac{1}{\sqrt{14}} [3, 2, 1] = \left[\frac{3}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{1}{\sqrt{14}}\right]$$

Exercise 24 Finding Directional Derivatives

Find the directional derivative of the

Working Space

function at the given point in the direction of the given vector.

1. $f(x, y) = e^{3x} \sin 2y$, $(0, \pi/6)$, $\mathbf{v} = [-3, 4]$
2. $f(x, y) = x^2y + xy^3$, $(2, 4)$, $\mathbf{v} = 2\mathbf{i} - \mathbf{j}$
3. $f(x, y, z) = \ln(x^2 + 3y - z)$, $(2, 2, 1)$, $\mathbf{v} = [1, 1, 1]$

Answer on Page 73

6.4 The Direction of Steepest Ascent

The directional derivative can be written as the dot product of two vectors:

$$D_{\mathbf{u}}f(x, y) = af_x(x, y) + bf_y(x, y) = [f_x(x, y), f_y(x, y)] \cdot \mathbf{u}$$

The first vector, $[f_x(x, y), f_y(x, y)]$, is called *the gradient of f* , and is noted as ∇f .

The Gradient

For a two-variable function, $f(x, y)$, the gradient of f is the vector:

$$\nabla f = [f_x(x, y), f_y(x, y)] = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j}$$

Where $\mathbf{i}$ and $\mathbf{j}$ are the unit vectors in the x - and y -directions, respectively.

Think back to the elevation example we opened the chapter with. What if we wanted to complete our ascent as quickly as possible? We would want to know the direction in which the elevation is changing the fastest. This occurs when the direction we are going is the same direction as the gradient vector, ∇f .

Recall that the dot product is defined as:

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}| |\mathbf{v}| \cos \theta$$

Where θ is the angle between the vectors $\mathbf{u}$ and $\mathbf{v}$. Applying this to the directional deriva-

tive, we see that:

$$D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} = |\nabla f| |\mathbf{u}| \cos \theta = |\nabla f| \cos \theta$$

Which is at its maximum when ∇f and $\mathbf{u}$ point in the same direction (because $\cos(0) = 1$). Therefore, the gradient vector points in the direction of maximum change and the magnitude of that vector is the rate of maximum change.

Example: Find the maximum rate of change of $f(x, y) = 4y\sqrt{x}$ at $(4, 1)$. In what direction does the maximum change occur?

Solution: We begin by finding ∇f :

$$\nabla f = \left[\frac{\partial}{\partial x} (4y\sqrt{x}), \frac{\partial}{\partial y} (4y\sqrt{x}) \right]$$

$$\nabla f = \left[\frac{2y}{\sqrt{x}}, 4\sqrt{x} \right]$$

And thus,

$$\nabla f(4, 1) = \left[\frac{2(1)}{\sqrt{4}}, 4\sqrt{4} \right] = [1, 8]$$

Therefore, the maximum rate of change at $(4, 1)$ is:

$$|\nabla f| = \sqrt{1^2 + 8^2} = \sqrt{65}$$

in the direction of the vector $[1, 8]$.

Exercise 25 Using the Gradient to find Maximum Change

Suppose you are climbing a mountain whose elevation is described by $z = 3000 - 0.01x^2 - 0.02y^2$. Take the positive x -direction to be east and the positive y -direction to be north.

Working Space

1. If you are at $(x, y) = (50, 50)$, what is your elevation?
2. If you walk south, will you ascend or descend?
3. If you walk northwest, will you ascend or descend? Will the rate of ele-

vation change be greater or less than if you walked south?

4. In what direction should you walk for the steepest ascent? What will your ascension rate be?

Answer on Page 75

6.5 Conclusion

In summary, the gradient of a multivariable function points in the direction of steepest increase:

$$\nabla f = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right] = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$$

The directional derivative is the dot product of the gradient with a unit vector:

$$D_{\mathbf{u}}f(\mathbf{a}) = \nabla f(\mathbf{a}) \cdot \mathbf{u}$$

It is best to think of the gradient as an arrow pointing in the steepest uphill direction (vector), and the directional derivative as the slope of any specific uphill direction (scalar).

The gradient points uphill, so it is the natural tool for finding the highest and lowest points of a function. That is the subject of the next chapter, Optimization with Gradients.

Optimization with Gradients

Where is the highest point on a mountain? Where does a company make the most profit, or a machine-learning model make the fewest mistakes? Each question asks for the maximum or minimum of a function of several variables. In the Gradients chapter, you learned that the gradient points uphill. This chapter uses that fact to find peaks and valleys, both exactly with algebra and step by step with gradient descent, and then meets some of the famous equations of physics and economics that are written with partial derivatives.

7.1 Optimization with Gradients

At the top of a hill, there is no uphill direction. The gradient is the zero vector. The same is true at the bottom of a valley. So to find the highest and lowest points of $f(x, y)$, we look for *critical points*, where

$$\nabla f = [0, 0], \quad \text{that is, } f_x = 0 \text{ and } f_y = 0$$

In one variable, a critical point could be a max, a min, or neither. In two variables there is a new possibility: a *saddle point*. The surface $f(x, y) = x^2 - y^2$ at the origin is shaped like a horse's saddle (or a Pringles chip): walking east-west, the origin is a low point; walking north-south, it is a high point. Its gradient is zero, but it is neither a max nor a min.

The Second Derivative Test for Two Variables

Suppose $\nabla f(a, b) = [0, 0]$. Let

$$D = f_{xx}(a, b) f_{yy}(a, b) - [f_{xy}(a, b)]^2$$

- If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.
- If $D > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum.
- If $D < 0$, then (a, b) is a saddle point.
- If $D = 0$, the test tells you nothing.

For $x^2 - y^2$: $f_{xx} = 2$, $f_{yy} = -2$, $f_{xy} = 0$, so $D = -4 < 0$: a saddle, as promised.

Example: Find and classify the critical points of $f(x, y) = x^2 + xy + y^2 - 3x$.

Solution: Set both partial derivatives to zero:

$$f_x = 2x + y - 3 = 0 \quad f_y = x + 2y = 0$$

From the second equation, $x = -2y$. Substituting into the first, $-4y + y - 3 = 0$, so $y = -1$ and $x = 2$. The second partials are $f_{xx} = 2$, $f_{yy} = 2$, and $f_{xy} = 1$, so $D = (2)(2) - 1^2 = 3 > 0$. Since $f_{xx} > 0$, the point $(2, -1)$ is a local minimum, with $f(2, -1) = -3$.

7.1.1 Gradient Descent

Solving $\nabla f = [0, 0]$ by algebra only works for simple functions. A computer can instead do what a lost hiker in the fog would do: feel which way is downhill and take a small step that way. Over and over. Because ∇f points uphill, each step goes in the direction of $-\nabla f$:

$$[x, y] \leftarrow [x, y] - r \nabla f(x, y)$$

where r is a small step size. This algorithm, *gradient descent*, is how most machine learning models are trained: the “function” is the model’s error, with millions of variables.

```

1  x, y = 0.0, 0.0      # starting guess
2  r = 0.1              # step size
3
4  for step in range(100):
5      fx = 2*x + y - 3  # partial derivatives of x^2 + xy + y^2 - 3x
6      fy = x + 2*y
7      x = x - r * fx
8      y = y - r * fy
9
10 print(x, y)
```

After 100 steps, this prints values within 0.0001 of $(2, -1)$, the minimum we found by hand.

Exercise 26 Finding a Maximum

A company sells two products. If it charges x dollars more than a base price for the first and y dollars more for the second, its profit (in thousands of dollars) is

$$P(x, y) = 6x + 4y - x^2 - y^2 - xy$$

Find the critical point, show that it is a maximum, and find the maximum

Working Space

profit.

Answer on Page 76

Exercise 27 Saddle or Not?

Find and classify the critical point of $f(x, y) = xy$.

Working Space

Answer on Page 77

7.2 Famous Equations with Partial Derivatives

7.2.1 Laplace's Equation

A partial differential equation that has applications in fluid dynamics and electronics is Laplace's Equation. Solutions to Laplace's Equation are called *harmonic functions*.

Laplace's Equation

Consider a twice-differentiable function, f . In two dimensions, Laplace's Equation is given by:

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

And in three dimensions,

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$$

Another way to represent Laplace's Equation is:

$$\Delta f = \nabla^2 f = \nabla \cdot \nabla f = 0$$

Where $\nabla^2 = \Delta$ is called the *Laplace operator*.

Example: Determine whether or not $f = x^2 + y^2$ is a solution to Laplace's Equation.

Solution: We are checking to see if $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$ for $f(x, y) = x^2 + y^2$. Finding $\partial^2 f / \partial x^2$:

$$\begin{aligned}\frac{\partial^2 f}{\partial x^2} &= \frac{\partial}{\partial x} \left[\frac{\partial}{\partial x} (x^2 + y^2) \right] \\ &= \frac{\partial}{\partial x} (2x) = 2\end{aligned}$$

And finding $\partial^2 f / \partial y^2$:

$$\begin{aligned}\frac{\partial^2 f}{\partial y^2} &= \frac{\partial}{\partial y} \left[\frac{\partial}{\partial y} (x^2 + y^2) \right] \\ &= \frac{\partial}{\partial y} (2y) = 2\end{aligned}$$

Then $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 2 + 2 = 4 \neq 0$. Therefore, $f(x, y) = x^2 + y^2$ is not a solution to Laplace's Equation.

Exercise 28 Solutions to Laplace's Equation

Determine whether the function is a solution to Laplace's Equation.

Working Space

1. $f(x, y) = x^2 - y^2$
2. $f(x, y) = \sin x \cosh y + \cos x \sinh y$
3. $f(x, y) = e^{-x} \cos y - e^{-y} \cos x$

Answer on Page 77

7.2.2 The Wave Equation

Another useful equation with partial derivatives is the Wave Equation:

$$\frac{\partial^2 f}{\partial t^2} = a^2 \frac{\partial^2 f}{\partial x^2}$$

Where f is a function of x and t and a is a constant. This equation describes waves, such as a vibrating string, light waves, or sound waves.

Example: Show that $f(x, t) = \sin(x - at)$ satisfies the Wave Equation.

Solution: First, we find the second partial derivatives:

$$\frac{\partial^2 f}{\partial t^2} = \frac{\partial}{\partial t} \left[\frac{\partial}{\partial t} (\sin(x - at)) \right] = \frac{\partial}{\partial t} [-a \cos(x - at)] = -a^2 \sin(x - at)$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x} \left[\frac{\partial}{\partial x} (\sin(x - at)) \right] = \frac{\partial}{\partial x} [\cos(x - at)] = -\sin(x - at)$$

And we see that:

$$a^2 \frac{\partial^2 f}{\partial x^2} = -a^2 \sin(x - at) = \frac{\partial^2 f}{\partial t^2}$$

Therefore, this function satisfies the Wave Equation.

Exercise 29 The Wave Equation

Show that the following functions satisfy the Wave Equation:

1. $f(x, t) = \cos(kx) \cos(akt)$
2. $f(x, t) = \sin(x - at) + \ln(x + at)$
3. $f(x, t) = \frac{t}{a^2 t^2 - x^2}$

Working Space

Answer on Page 77

7.2.3 Cobb-Douglas Production Function

The Cobb-Douglas function describes how production depends on capital and labor, as theorized by the economists Charles Cobb and Paul Douglas. Capital investments are things like new machinery, expanded factories, or raw materials. Labor investments involve hiring more workers or improving working conditions to improve work rates. We can describe total production, P , as a function of labor, L , and capital, K . Cobb and Dou-

glas posit three conditions:

1. Without either labor or capital, production will cease.
2. The marginal productivity of labor is proportional to the amount of production per unit of labor.
3. The marginal productivity of capital is proportional to the amount of production per unit of capital.

The marginal productivity of labor is given by the partial derivative, $\partial P / \partial L$ and the production per unit of labor is given by P/L . Therefore, statement 2 says that:

$$\frac{\partial P}{\partial L} = \alpha \frac{P}{L}$$

where α is some constant. Keeping K constant at $K = K_0$, we have the differential equation:

$$\frac{dP}{dL} = \alpha \frac{P}{L}$$

Solving, we find that:

$$P(L, K_0) = C_1 (K_0) L^\alpha$$

We make C_1 a function of K_0 because it could depend on K_0 . In a similar manner to above, we can write statement 3 as a mathematical statement:

$$\frac{\partial P}{\partial K} = \beta \frac{P}{K}$$

where β is also a constant. Keeping $L = L_0$ and solving, we see that:

$$P(L_0, K) = C_2(L_0) K^\beta$$

again, we assume C_2 is a function of the fixed labor, L_0 . Combining these equations, we get:

$$P(L, K) = b L^\alpha K^\beta$$

where b is a constant independent of capital and labor. Additionally, from statement 1, we know that $\alpha > 0$ and $\beta > 0$. What happens if both labor and capital are increased by a factor of n ? Let us examine the effect on P :

$$P(nL, nK) = b (nL)^\alpha (nK)^\beta$$

$$P(nL, nK) = n^{\alpha+\beta} b L^\alpha K^\beta = n^{\alpha+\beta} P(L, K)$$

Cobb and Douglas noted that if $\alpha + \beta = 1$, then $P(nL, nK) = nP(L, K)$, and therefore increasing labor and capital by a factor of n increases production by a factor of n as well. Therefore, the Cobb-Douglas equation assumes $\alpha + \beta = 1$ and can be written as:

$$P(L, K) = bL^\alpha K^{1-\alpha}$$

Exercise 30 Cobb-Douglas Production Model

Cobb and Douglas modeled production in the US from 1900 to 1922 with the equation $P(L, K) = 1.01L^{0.75}K^{0.25}$.

Working Space

1. Express the marginal productivity of labor as a function of L and K .
2. Express the marginal productivity of capital as a function of L and K .
3. In 1916, $L = 382$ and $K = 126$ (compared to initial values of 100 in 1900). What is the marginal productivity of labor in 1916? Of capital?
4. Based on your answer to the previous question, would you invest in capital or labor if you owned a factory in 1916? Why?

Answer on Page 79

Answers to Exercises

Answer to Exercise 1 (on page 6)

First, take the derivative of the function:

$$f'(x) = 2x - 6.$$

Set the derivative equal to zero to find the critical point:

$$2x - 6 = 0.$$

Solving this equation gives

$$x = 3.$$

Next, take the second derivative:

$$f''(x) = 2.$$

Since the second derivative is positive, the function is concave up, and the critical point corresponds to a minimum.

Finally, substitute $x = 3$ back into the original function:

$$f(3) = 3^2 - 6(3) + 5 = -4.$$

Therefore, the function has a minimum value of -4 at $x = 3$.

Answer to Exercise 2 (on page 8)

Because only three sides require fencing, the total amount of fencing is given by

$$x + 2y = 60.$$

Solving this equation for x gives

$$x = 60 - 2y.$$

The area of the enclosure is

$$A = xy.$$

Substituting for x yields

$$A(y) = y(60 - 2y) = 60y - 2y^2.$$

Take the derivative:

$$A'(y) = 60 - 4y.$$

Set the derivative equal to zero:

$$60 - 4y = 0.$$

Solving for y gives

$$y = 15.$$

The second derivative is

$$A''(y) = -4.$$

Since the second derivative is negative, this critical point corresponds to a maximum area.

Substituting $y = 15$ into the constraint gives

$$x = 60 - 2(15) = 30.$$

Therefore, the enclosure has dimensions 30 units by 15 units, and its maximum area is $30 \times 15 = 450$ square units.

Answer to Exercise 3 (on page 9)

The base is $(30 - 2x)$ by $(30 - 2x)$ and the height is x , so $V(x) = x(30 - 2x)^2$ with $0 < x < 15$. By the product and chain rules,

$$V'(x) = (30 - 2x)^2 + x \cdot 2(30 - 2x)(-2) = (30 - 2x)(30 - 6x)$$

$V' = 0$ at $x = 15$ (an endpoint, where the volume is 0) and $x = 5$. So cut 5 cm squares:
 $V(5) = 5 \cdot 20^2 = 2,000 \text{ cm}^3$.

Answer to Exercise 4 (on page 10)

Let p be the price increase in dollars. Revenue is price times attendance:

$$R(p) = (10 + p)(300 - 20p) = 3000 + 100p - 20p^2$$

$R'(p) = 100 - 40p = 0$ gives $p = 2.5$, and $R'' = -40 < 0$, so it is a maximum. Charge \$12.50; 250 students come, for \$3,125 in revenue (up from \$3,000).

Answer to Exercise 5 (on page 12)

1. For $f(x) = 20x - x^2$,

$$f'(x) = 20 - 2x.$$

Setting $f'(x) = 0$ gives $20 - 2x = 0$, so $x = 10$.

$$f''(x) = -2.$$

Since $f''(10) < 0$, the function is concave down at the critical point, so the critical point is a maximum.

2. For $f(x) = x^2 - 6x + 5$,

$$f'(x) = 2x - 6.$$

Setting $f'(x) = 0$ gives $2x - 6 = 0$, so $x = 3$.

$$f''(x) = 2.$$

Since $f''(3) > 0$, the function is concave up at the critical point, so the critical point is a minimum.

3. For $f(x) = x^3$,

$$f'(x) = 3x^2.$$

Setting $f'(x) = 0$ gives $x = 0$.

$$f''(x) = 6x.$$

Since $f''(0) = 0$, the second derivative test gives no classification, and the script reports the point as inconclusive. The graph shows that f increases on both sides of $x = 0$, so this critical point is neither a maximum nor a minimum (Figure 1.2).

4. For a quadratic function of the form $f(x) = ax^2 + bx + c$ with $a \neq 0$, there is exactly

one critical point at

$$x = -\frac{b}{2a}.$$

Also,

$$f''(x) = 2a.$$

If $a > 0$, then $f''(x) > 0$ and the critical point is a minimum. If $a < 0$, then $f''(x) < 0$ and the critical point is a maximum.

Answer to Exercise 6 (on page 17)

Using implicit differentiation, we see that:

$$\begin{aligned}\frac{d}{dx} \arcsin x &= \frac{d}{dx} \ln y \\ \frac{1}{\sqrt{1-x^2}} &= \frac{1}{y} \frac{dy}{dx}\end{aligned}$$

Multiplying both sides by y to isolate $\frac{dy}{dx}$, we find that:

$$\frac{dy}{dx} = \frac{y}{\sqrt{1-x^2}}$$

Answer to Exercise 7 (on page 17)

First, we need to find $\frac{d^2y}{dx^2}$:

$$\begin{aligned}\frac{d}{dx} \frac{dy}{dx} &= \frac{d}{dx} x^2 + \frac{d}{dx} y \\ &= 2x + \frac{dy}{dx} = 2x + x^2 + y\end{aligned}$$

At $(-1, -1)$, $\frac{dy}{dx} = (-1)^2 + (-1) = 0$ and $\frac{d^2y}{dx^2} = 2(-1) + (-1)^2 + (-1) = -2 < 0$. Since the slope of y is zero and the graph of y is concave down, $(-1, -1)$ is a local maximum. At $(1, -5)$, $\frac{dy}{dx} = 1^2 + -5 = -4 \neq 0$ and $\frac{d^2y}{dx^2} = 2(1) + 1^2 + (-5) = -2 \neq 0$. Since neither the first nor second derivative of y are zero, $(1, -5)$ is neither a local extremum nor an inflection point.

Answer to Exercise 8 (on page 17)

$$\frac{dy}{dx} = -\frac{x}{y} = -\frac{-4}{3} = \frac{4}{3}.$$

Answer to Exercise 9 (on page 18)

$2x + 8y \frac{dy}{dx} = 0$, so $\frac{dy}{dx} = -\frac{x}{4y}$. At $(2, 1)$ the slope is $-\frac{1}{2}$, and the tangent line is $y - 1 = -\frac{1}{2}(x - 2)$, or $y = 2 - \frac{x}{2}$.

Answer to Exercise 10 (on page 19)

First compute the derivatives:

$$\frac{dx}{dt} = 3t^2 - 3, \quad \frac{dy}{dt} = 2t.$$

Horizontal tangents occur when $\frac{dy}{dt} = 0$ and $\frac{dx}{dt} \neq 0$.

$$2t = 0 \Rightarrow t = 0.$$

At $t = 0$, the point on the curve is

$$x = 0, \quad y = 0.$$

Vertical tangents occur when $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} \neq 0$.

$$3t^2 - 3 = 0 \Rightarrow t = \pm 1.$$

At $t = 1$, the point is

$$(-2, 1),$$

and at $t = -1$, the point is

$$(2, 1).$$

Answer to Exercise 11 (on page 20)

First compute the derivatives with respect to t :

$$\frac{dx}{dt} = 2, \quad \frac{dy}{dt} = 2t.$$

The derivative of the curve is

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t}{2} = t.$$

At $t = 1$,

$$\frac{dy}{dx} = 1.$$

Thus, the slope of the tangent line at the corresponding point on the curve is 1.

Answer to Exercise 12 (on page 23)

Let x be the distance of the base from the wall and y the height of the top. Then $x^2 + y^2 = 100$, so $2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0$. When $x = 6$, $y = 8$, and $\frac{dx}{dt} = 1$, so $\frac{dy}{dt} = -\frac{6(1)}{8} = -0.75$ m/s. The top slides down at 0.75 m/s.

Answer to Exercise 13 (on page 23)

Let x be your distance from the light and s the length of your shadow. The light, the tip of the shadow, and the top of your head form similar triangles:

$$\frac{6}{x+s} = \frac{1.8}{s} \Rightarrow 6s = 1.8x + 1.8s \Rightarrow s = \frac{3}{7}x$$

1. $\frac{ds}{dt} = \frac{3}{7} \frac{dx}{dt} = \frac{3}{7}(1.5) \approx 0.64$ m/s.
2. The tip is at $x + s$, so it moves at $1.5 + 0.64 \approx 2.14$ m/s, faster than you walk. Neither answer depends on how far you are from the light.

Answer to Exercise 14 (on page 23)

$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$, so $-10 = 4\pi(25) \frac{dr}{dt}$ and $\frac{dr}{dt} = -\frac{10}{100\pi} \approx -0.032$ cm/min. The radius shrinks by about 0.32 mm per minute.

Answer to Exercise 15 (on page 28)

1. $f(2, 1) = 6 - 1 = 5$ and $f(0, -3) = 0 - 9 = -9$.

2. $V(3, 10) = 90\pi \approx 283$: a cylinder with radius 3 and height 10 holds about 283 cubic units.
3. Doubling h doubles V . Doubling r multiplies V by 4, because r is squared.

Answer to Exercise 16 (on page 28)

1. $x + y = c$ is the line $y = c - x$. The level curves are parallel lines with slope -1 , evenly spaced: the graph is a tilted plane.
2. $y - x^2 = c$ is the parabola $y = x^2 + c$. The level curves are copies of $y = x^2$ shifted up and down.

Answer to Exercise 17 (on page 28)

The west side. Closely spaced contour lines mean the elevation changes quickly over a short distance: the east side is four times as steep.

Answer to Exercise 18 (on page 33)

1. $f_x(x, y) = \frac{\partial}{\partial x} [3x^4 + 4x^2y^3] = 12x^3 + 8xy^3$ and $f_y(x, y) = \frac{\partial}{\partial y} [3x^4 + 4x^2y^3] = 12x^2y^2$
2. $f_x(x, y) = \frac{\partial}{\partial x} (xe^{-y}) = e^{-y}$ and $f_y(x, y) = \frac{\partial}{\partial y} (xe^{-y}) = -xe^{-y}$
3. $f_x(x, y) = \frac{\partial}{\partial x} \sqrt{3x + 4y^2} = \left(\frac{1}{2\sqrt{3x + 4y^2}} \right) \left(\frac{\partial}{\partial x} (3x + 4y^2) \right) = \frac{3}{2\sqrt{3x + 4y^2}}$ and $f_y(x, y) = \frac{\partial}{\partial y} \sqrt{3x + 4y^2} = \frac{1}{2\sqrt{3x + 4y^2}} \left(\frac{\partial}{\partial y} (3x + 4y^2) \right) = \frac{8y}{2\sqrt{3x + 4y^2}} = \frac{4y}{\sqrt{3x + 4y^2}}$
4. $f_x(x, y) = \frac{\partial}{\partial x} \sin(x^2y) = \cos(x^2y) \left(\frac{\partial}{\partial x} (x^2y) \right) = 2xy \cos(x^2y)$ and $f_y(x, y) = \frac{\partial}{\partial y} \sin(x^2y) = \cos(x^2y) \left(\frac{\partial}{\partial y} (x^2y) \right) = x^2 \cos(x^2y)$
5. $f_x(x, y) = \frac{\partial}{\partial x} \ln(x^y) = \frac{\partial}{\partial x} (y \ln x) = \frac{y}{x}$ and $f_y(x, y) = \frac{\partial}{\partial y} (y \ln x) = \ln x$

Answer to Exercise 19 (on page 34)

1. Finding f_x :

$$f_x = \frac{\partial}{\partial x} \left[\sin(x^2 - y^2) \cos(\sqrt{z}) \right] = \cos(x^2 - y^2) \cos(\sqrt{z}) \left[\frac{\partial}{\partial x} (x^2 - y^2) \right]$$

$$f_x = 2x \cos(x^2 - y^2) \cos(\sqrt{z})$$

Finding f_y :

$$f_y = \frac{\partial}{\partial y} \left[\sin(x^2 - y^2) \cos(\sqrt{z}) \right] = \cos(x^2 - y^2) \cos(\sqrt{z}) \left[\frac{\partial}{\partial y} (x^2 - y^2) \right]$$

$$f_y = -2y \cos(x^2 - y^2) \cos(\sqrt{z})$$

Finding f_z :

$$f_z = \frac{\partial}{\partial z} \left[\sin(x^2 - y^2) \cos(\sqrt{z}) \right] = \sin(x^2 - y^2) (-\sin \sqrt{z}) \cdot \left(\frac{\partial}{\partial z} \sqrt{z} \right)$$

$$f_z = \frac{-\sin(x^2 - y^2) \sin(\sqrt{z})}{2\sqrt{z}}$$

2. Finding q_t :

$$q_t = \frac{\partial}{\partial t} \sqrt[3]{t^3 + u^3 \sin(5v)} = \frac{1}{3(t^3 + u^3 \sin(5v))^{2/3}} \left(\frac{\partial}{\partial t} (t^3 + u^3 \sin(5v)) \right)$$

$$q_t = \frac{t^2}{(t^3 + u^3 \sin(5v))^{2/3}}$$

Finding q_u :

$$q_u = \frac{\partial}{\partial u} \sqrt[3]{t^3 + u^3 \sin(5v)} = \frac{1}{3(t^3 + u^3 \sin(5v))^{2/3}} \left(\frac{\partial}{\partial u} (t^3 + u^3 \sin(5v)) \right)$$

$$q_u = \frac{u^2 \sin(5v)}{(t^3 + u^3 \sin(5v))^{2/3}}$$

Finding q_v :

$$q_v = \frac{\partial}{\partial v} \sqrt[3]{t^3 + u^3 \sin(5v)} = \frac{1}{3(t^3 + u^3 \sin(5v))^{2/3}} \left(\frac{\partial}{\partial v} (t^3 + u^3 \sin(5v)) \right)$$

$$q_v = \frac{u^3 \cos(5v)}{3(t^3 + u^3 \sin(5v))^{2/3}} \left(\frac{\partial}{\partial v} (5v) \right) = \frac{5u^3 \cos(5v)}{3(t^3 + u^3 \sin(5v))^{2/3}}$$

3. Finding w_x :

$$w_x = \frac{\partial}{\partial x} (x^z y^x) = (x^z) \cdot \left(\frac{\partial}{\partial x} y^x \right) + (y^x) \cdot \left(\frac{\partial}{\partial x} x^z \right)$$

$$w_x = (x^z) (\ln(y) y^x) + (y^x) (zx^{z-1}) = (x^{z-1} y^x) (x \ln(y) + z)$$

Finding w_y :

$$w_y = \frac{\partial}{\partial y} (x^z y^x) = (x^z) \left(\frac{\partial}{\partial y} y^x \right) = x^z (x y^{x-1})$$

$$w_y = x^{z+1} y^{x-1}$$

Finding w_z :

$$w_z = \frac{\partial}{\partial z} (x^z y^x) = (y^x) \left(\frac{\partial}{\partial z} x^z \right) = (y^x) (\ln(x) x^z)$$

$$w_z = \ln(x) y^x x^z$$

Answer to Exercise 20 (on page 36)

$$\begin{aligned} 1. f_{xy} &= \frac{\partial}{\partial y} \left(\frac{\partial}{\partial x} f(x, y) \right) = \frac{\partial}{\partial y} \left[\frac{\partial}{\partial x} (e^{2xy} \sin x) \right] = \frac{\partial}{\partial y} \left[(e^{2xy}) \left(\frac{\partial}{\partial x} \sin x \right) + (\sin x) \left(\frac{\partial}{\partial x} e^{2xy} \right) \right] = \\ &= \frac{\partial}{\partial y} [e^{2xy} \cos x + 2ye^{2xy} \sin x] = \frac{\partial}{\partial y} (e^{2xy} \cos x) + \frac{\partial}{\partial y} (2ye^{2xy} \sin x) = 2xe^{2xy} \cos x + (2y) \left(\frac{\partial}{\partial y} e^{2xy} \sin x \right) + \\ &= (e^{2xy} \sin x) \left(\frac{\partial}{\partial y} 2y \right) = 2xe^{2xy} \cos x + 4xye^{2xy} \sin x + 2e^{2xy} \sin x \end{aligned}$$

$$\begin{aligned} f_{yx} &= \frac{\partial}{\partial x} \left(\frac{\partial}{\partial y} f(x, y) \right) = \frac{\partial}{\partial x} \left[\frac{\partial}{\partial y} (e^{2xy} \sin x) \right] = \frac{\partial}{\partial x} (2xe^{2xy} \sin x) = (2x) \left[\frac{\partial}{\partial x} (e^{2xy} \sin x) \right] + \\ &= (e^{2xy} \sin x) \left(\frac{\partial}{\partial x} 2x \right) = (2x) \left[(e^{2xy}) \left(\frac{\partial}{\partial x} \sin x \right) + (\sin x) \left(\frac{\partial}{\partial x} e^{2xy} \right) \right] + 2e^{2xy} \sin x = 2xe^{2xy} \cos x + \\ &= 4xye^{2xy} \sin x + 2e^{2xy} \sin x = f_{xy} \end{aligned}$$

$$\begin{aligned} 2. f_{xy} &= \frac{\partial}{\partial y} \left(\frac{\partial}{\partial x} f(x, y) \right) = \frac{\partial}{\partial y} \left[\frac{\partial}{\partial x} \left(\frac{x^2}{x+y} \right) \right] = \frac{\partial}{\partial y} \left[\frac{(x+y)(2x) - x^2(1)}{(x+y)^2} \right] = \frac{\partial}{\partial y} \left[\frac{x^2 + 2xy}{(x+y)^2} \right] = \frac{(x+y)^2(2x) - (x^2 + 2xy)(2(x+y))}{(x+y)^4} = \\ &= \frac{(x^2 + 2xy + y^2)(2x) - (x^2 + 2xy)(2x + 2y)}{(x+y)^4} = \frac{2x^3 + 4x^2y + 2xy^2 - 2x^3 - 2x^2y - 4x^2y - 4xy^2}{(x+y)^4} = \frac{-2x^2y - 2xy^2}{(x+y)^4} \end{aligned}$$

$$\begin{aligned} f_{yx} &= \frac{\partial}{\partial x} \left(\frac{\partial}{\partial y} f(x, y) \right) = \frac{\partial}{\partial x} \left[\frac{\partial}{\partial y} \left(\frac{x^2}{x+y} \right) \right] = \frac{\partial}{\partial x} \left[\frac{-x^2}{(x+y)^2} \right] = \frac{(x+y)^2(-2x) - (-x^2)(2(x+y))}{(x+y)^4} = \frac{(x^2 + 2xy + y^2)(-2x) + x^2(2x + 2y)}{(x+y)^4} = \\ &= \frac{-2x^3 - 4x^2y - 2xy^2 + 2x^3 + 2x^2y}{(x+y)^4} = \frac{-2x^2y - 2xy^2}{(x+y)^4} = f_{xy} \end{aligned}$$

$$3. f_{xy} = \frac{\partial}{\partial y} \left(\frac{\partial}{\partial x} f(x, y) \right) = \frac{\partial}{\partial y} \left[\frac{\partial}{\partial x} (\ln(2x + 3y)) \right] = \frac{\partial}{\partial y} \left[\frac{2}{2x + 3y} \right] = \frac{-2(3)}{(2x + 3y)^2} = \frac{-6}{(2x + 3y)^2}$$

$$f_{yx} = \frac{\partial}{\partial x} \left(\frac{\partial}{\partial y} f(x, y) \right) = \frac{\partial}{\partial x} \left[\frac{\partial}{\partial y} (\ln(2x + 3y)) \right] = \frac{\partial}{\partial x} \left(\frac{3}{2x + 3y} \right) = \frac{-3(2)}{(2x + 3y)^2} = \frac{-6}{(2x + 3y)^2} = f_{xy}$$

Answer to Exercise 21 (on page 36)

- $$f_{xx} = \frac{\partial}{\partial x} \left(\frac{\partial}{\partial x} f(x, y) \right) = \frac{\partial}{\partial x} \left[\frac{\partial}{\partial x} (x^5 y^2 - 3x^3 y^2) \right] = \frac{\partial}{\partial x} (5x^4 y^2 - 9x^2 y^2) = 20x^3 y^2 - 18x y^2.$$

$$f_{xy} = f_{yx} = \frac{\partial}{\partial y} \left(\frac{\partial}{\partial x} f(x, y) \right) = \frac{\partial}{\partial y} \left[\frac{\partial}{\partial x} (x^5 y^2 - 3x^3 y^2) \right] = \frac{\partial}{\partial y} (5x^4 y^2 - 9x^2 y^2) = 10x^4 y - 18x^2 y.$$

$$f_{yy} = \frac{\partial}{\partial y} \left(\frac{\partial}{\partial y} f(x, y) \right) = \frac{\partial}{\partial y} \left[\frac{\partial}{\partial y} (x^5 y^2 - 3x^3 y^2) \right] = \frac{\partial}{\partial y} (2x^5 y - 6x^3 y) = 2x^5 - 6x^3.$$
- $$v_{pp} = \frac{\partial}{\partial p} \left(\frac{\partial}{\partial p} v(p, q) \right) = \frac{\partial}{\partial p} \left[\frac{\partial}{\partial p} (\sin(p^3 + q^2)) \right] = \frac{\partial}{\partial p} (\cos(p^3 + q^2) (3p^2)) = \cos(p^3 + q^2) \cdot \frac{\partial}{\partial p} (3p^2) + 3p^2 \cdot \frac{\partial}{\partial p} (\cos(p^3 + q^2)) = 6p \cos(p^3 + q^2) - 9p^4 \sin(p^3 + q^2)$$

$$v_{pq} = v_{qp} = \frac{\partial}{\partial q} \left(\frac{\partial}{\partial p} v(p, q) \right) = \frac{\partial}{\partial q} \left[\frac{\partial}{\partial p} (\sin(p^3 + q^2)) \right] = \frac{\partial}{\partial q} (\cos(p^3 + q^2) (3p^2)) = \cos(p^3 + q^2) \frac{\partial}{\partial q} (3p^2) + 3p^2 \frac{\partial}{\partial q} \cos(p^3 + q^2) = 0 + 3p^2 (-\sin(p^3 + q^2)) \left(\frac{\partial}{\partial q} (p^3 + q^2) \right) = -6p^2 q \sin(p^3 + q^2)$$

$$v_{qq} = \frac{\partial}{\partial q} \left(\frac{\partial}{\partial q} v(p, q) \right) = \frac{\partial}{\partial q} \left[\frac{\partial}{\partial q} (\sin(p^3 + q^2)) \right] = \frac{\partial}{\partial q} [2q \cos(p^3 + q^2)] = 2q \left[\frac{\partial}{\partial q} \cos(p^3 + q^2) \right] + \cos(p^3 + q^2) \left[\frac{\partial}{\partial q} (2q) \right] = (2q) \cdot [-2q \sin(p^3 + q^2)] + 2 \cos(p^3 + q^2) = 2 \cos(p^3 + q^2) - 4q^2 \sin(p^3 + q^2)$$
- $$T_{rr} = \frac{\partial}{\partial r} \left(\frac{\partial}{\partial r} T(r, \theta) \right) = \frac{\partial}{\partial r} \left[\frac{\partial}{\partial r} (e^{-3r} \cos \theta^2) \right] = \frac{\partial}{\partial r} (-3e^{-3r} \cos \theta^2) = 9e^{-3r} \cos \theta^2$$

$$T_{r\theta} = T_{\theta r} = \frac{\partial}{\partial \theta} \left(\frac{\partial}{\partial r} T(r, \theta) \right) = \frac{\partial}{\partial \theta} [-3e^{-3r} \cos \theta^2] = 3e^{-3r} \sin \theta^2 \left(\frac{\partial}{\partial \theta} \theta^2 \right) = 6\theta e^{-3r} \sin \theta^2$$

$$T_{\theta\theta} = \frac{\partial}{\partial \theta} \left(\frac{\partial}{\partial \theta} T(r, \theta) \right) = \frac{\partial}{\partial \theta} \left[\frac{\partial}{\partial \theta} (e^{-3r} \cos \theta^2) \right] = \frac{\partial}{\partial \theta} [-e^{-3r} \sin \theta^2 \left(\frac{\partial}{\partial \theta} \theta^2 \right)] = \frac{\partial}{\partial \theta} (-2\theta e^{-3r} \sin \theta^2) = (-2\theta e^{-3r}) \left(\frac{\partial}{\partial \theta} \sin \theta^2 \right) + (\sin \theta^2) \left[\frac{\partial}{\partial \theta} (-2\theta e^{-3r}) \right] = (-2\theta e^{-3r}) (\cos \theta^2) \left(\frac{\partial}{\partial \theta} \theta^2 \right) + (\sin \theta^2) (-2e^{-3r}) = -4\theta^2 e^{-3r} \cos \theta^2 - 2e^{-3r} \sin \theta^2$$

Answer to Exercise 22 (on page 38)

- We are looking for dz/dt only:

$$\begin{aligned} \frac{dz}{dt} &= \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt} \\ &= \frac{\partial}{\partial x} [\sin x \cos y] \cdot \frac{d}{dt} [3\sqrt{t}] + \frac{\partial}{\partial y} [\sin x \cos y] \cdot \frac{d}{dt} [2/t] \\ &= (\cos x \cos y) \cdot \left(\frac{3}{2\sqrt{t}} \right) + (-\sin x \sin y) \cdot \left(-\frac{2}{t^2} \right) \\ &= \frac{3 \cos x \cos y}{2\sqrt{t}} + \frac{2 \sin x \sin y}{t^2} \end{aligned}$$

Substituting for x and y :

$$\frac{dz}{dt} = \frac{3 \cos(3\sqrt{t}) \cos(2/t)}{2\sqrt{t}} + \frac{2 \sin(3\sqrt{t}) \sin(2/t)}{t^2}$$

2. We are looking for dz/dt only:

$$\begin{aligned}\frac{dz}{dt} &= \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt} \\ &= \frac{\partial}{\partial x} [\sqrt{1+xy}] \cdot \frac{d}{dt} [\tan t] + \frac{\partial}{\partial y} [\sqrt{1+xy}] \cdot \frac{d}{dt} [\arctan t] \\ &= \left(\frac{y}{2\sqrt{1+xy}} \right) \cdot (\sec^2 t) + \left(\frac{x}{2\sqrt{1+xy}} \right) \cdot \left(\frac{1}{t^2+1} \right)\end{aligned}$$

Substituting for x and y :

$$\begin{aligned}\frac{dz}{dt} &= \frac{\tan t \sec^2 t}{2\sqrt{1+\tan t \arctan t}} + \frac{\tan t}{2\sqrt{1+\tan t \arctan t} (t^2+1)} \\ &= \frac{\tan t}{2\sqrt{1+\tan t \arctan t}} \left(\sec^2 t + \frac{1}{t^2+1} \right)\end{aligned}$$

3. Finding $\partial z/\partial s$:

$$\begin{aligned}\frac{\partial z}{\partial s} &= \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s} \\ &= \frac{\partial}{\partial x} [\arctan(x^2+y^2)] \cdot \frac{\partial}{\partial s} [t \ln s] + \frac{\partial}{\partial y} [\arctan(x^2+y^2)] \cdot \frac{\partial}{\partial s} [se^t] \\ &= \left(\frac{2x}{(x^2+y^2)^2+1} \right) \cdot \left(\frac{t}{s} \right) + \left(\frac{2y}{(x^2+y^2)^2+1} \right) \cdot (e^t)\end{aligned}$$

Substituting for x and y :

$$\begin{aligned}\frac{\partial z}{\partial s} &= \left(\frac{2(t \ln s)}{[(t \ln s)^2 + (se^t)^2]^2 + 1} \right) \cdot \left(\frac{t}{s} \right) + \left(\frac{2(se^t)}{[(t \ln s)^2 + (se^t)^2]^2 + 1} \right) \cdot (e^t) \\ &= \left(\frac{2}{[t^2 (\ln s)^2 + s^2 e^{2t}]^2 + 1} \right) \cdot \left(\frac{t^2 \ln s}{s} + se^{2t} \right)\end{aligned}$$

Finding $\partial z/\partial t$:

$$\begin{aligned}\frac{\partial z}{\partial t} &= \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t} \\ &= \frac{\partial}{\partial x} (\arctan(x^2+y^2)) \cdot \frac{\partial}{\partial t} (t \ln s) + \frac{\partial}{\partial y} (\arctan(x^2+y^2)) \cdot \frac{\partial}{\partial t} (se^t) \\ &= \left(\frac{2x}{(x^2+y^2)^2+1} \right) \cdot (\ln s) + \left(\frac{2y}{(x^2+y^2)^2+1} \right) \cdot (se^t)\end{aligned}$$

Substituting for x and y :

$$\frac{\partial z}{\partial t} = \left(\frac{2}{[(t \ln s)^2 + (se^t)^2] + 1} \right) \cdot [t (\ln s)^2 + s^2 e^{2t}]$$

4. Finding $\partial z / \partial s$:

$$\begin{aligned} \frac{\partial z}{\partial s} &= \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s} \\ &= \frac{\partial}{\partial x} (\sqrt{x} e^{xy}) \cdot \frac{\partial}{\partial s} (1 + st) + \frac{\partial}{\partial y} (\sqrt{x} e^{xy}) \cdot \frac{\partial}{\partial s} (s^2 - t^2) \\ &= \left[\frac{e^{xy} (2xy + 1)}{2\sqrt{x}} \right] \cdot (t) + [x^{3/2} e^{xy}] \cdot (2s) \\ &= \left[\frac{e^{xy}}{\sqrt{x}} \right] \cdot \left(\frac{(2xy + 1)t}{2} + x^2 (2s) \right) \end{aligned}$$

Substituting for x and y :

$$\frac{\partial z}{\partial s} = \left[\frac{e^{(1+st)(s^2-t^2)}}{\sqrt{1+st}} \right] \cdot \left(\frac{(2(1+st)(s^2-t^2) + 1)t}{2} + (1+st)^2 (2s) \right)$$

Finding $\partial z / \partial t$:

$$\begin{aligned} \frac{\partial z}{\partial t} &= \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t} \\ &= \frac{\partial}{\partial x} [\sqrt{x} e^{xy}] \cdot \frac{\partial}{\partial t} [1 + st] + \frac{\partial}{\partial y} [\sqrt{x} e^{xy}] \cdot \frac{\partial}{\partial t} [s^2 - t^2] \\ &= \left[\frac{e^{xy} (2xy + 1)}{2\sqrt{x}} \right] \cdot (s) + [x^{3/2} e^{xy}] \cdot (-2t) \\ &= \left[\frac{e^{xy}}{\sqrt{x}} \right] \cdot \left[\frac{(2xy + 1)s}{2} - 2tx^2 \right] \end{aligned}$$

Substituting for x and y :

$$\frac{\partial z}{\partial t} = \left[\frac{e^{(1+st)(s^2-t^2)}}{\sqrt{(1+st)}} \right] \cdot \left[\frac{(2(1+st)(s^2-t^2) + 1)s}{2} - 2t(1+st)^2 \right]$$

Answer to Exercise 23 (on page 42)

1. $z(1, -1) = (1)^2 e^{-1/1} = 1/e$. Therefore, we are looking for tangent lines through the point $(1, -1, 1/e)$. Finding a tangent line parallel to the x -axis: $\frac{\partial z}{\partial x} = \frac{\partial}{\partial x} (x^2 e^{y/x}) = x^2 \left(\frac{\partial}{\partial x} e^{y/x} \right) + e^{y/x} \left(\frac{\partial}{\partial x} x^2 \right) = x^2 e^{y/x} \left(\frac{\partial}{\partial x} \frac{y}{x} \right) + 2x e^{y/x} = x^2 e^{y/x} \left(\frac{-y}{x^2} \right) + 2x e^{y/x} = (2x - y) e^{y/x}$

and $z_x(1, -1) = (2(1) - (-1))e^{-1/1} = (3)e^{-1} = 3/e$. So, the slope of a line tangent to the surface at $(1, -1, 1/e)$ parallel to the x -axis is $3/e$ and an equation for that line is $z = 3/e(x - 1) + 1/e$.

Finding a tangent line parallel to the y -axis: $\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(x^2 e^{y/x}) = x e^{y/x}$ and $z_y(1, -1) = (1)e^{-1/1} = 1/e$. So, the slope of a line tangent to the surface at $(1, -1, 1/e)$ parallel to the y -axis is $1/e$ and an equation for that line is $z = 1/e(y + 1) + 1/e$.

The function is changing faster in the x -direction.

2. $z(\pi, \pi/2) = \cos(\pi) + \frac{\pi}{2} \sin(\pi/2) = \frac{\pi}{2} - 1$. Therefore, we are looking for tangent lines through the point $(\pi, \pi/2, \pi/2 - 1)$. Finding a tangent line parallel to the x -axis: $\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(\cos x + y \sin y) = -\sin x$ and $z_x(\pi, \pi/2) = -\sin \pi = 0$. So, the slope of a line tangent to the surface at $(\pi, \pi/2, \pi/2 - 1)$ parallel to the x -axis is 0 and an equation for that line is $z = \pi/2 - 1$.

Finding a tangent line parallel to the y -axis: $\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(\cos x + y \sin y) = y \left(\frac{\partial}{\partial y} \sin y \right) + \sin y \left(\frac{\partial}{\partial y} y \right) = y \cos y + \sin y$ and $z_y(\pi, \pi/2) = \left(\frac{\pi}{2} \right) \cos \left(\frac{\pi}{2} \right) + \sin \left(\frac{\pi}{2} \right) = 1$. So, the slope of a line tangent to the surface at $(\pi, \pi/2, \pi/2 - 1)$ parallel to the y -axis is 1 and an equation for that line is $z = (y - \pi/2) - (\pi/2 - 1) = y - \pi + 1$.

The function is changing faster in the y -direction.

3. $z(3, 2) = 3^2(2) - 3(3)(2^2) = 18 - 36 = -18$. Therefore, we are looking for tangent lines through the point $(3, 2, -18)$. Finding a tangent line parallel to the x -axis: $\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(x^2 y - 3xy^2) = 2xy - 3y^2$ and $z_x(3, 2) = 2(3)(2) - 3(2)^2 = 0$. So, the slope of a line tangent to the surface at $(3, 2, -18)$ is 0 and an equation for that line is $z = -18$.

Finding a tangent line parallel to the y -axis: $\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(x^2 y - 3xy^2) = x^2 - 6xy$ and $z_y(3, 2) = 3^2 - 6(3)(2) = 9 - 36 = -27$. So, the slope of a line tangent to the surface at $(3, 2, -18)$ is -27 and an equation for that line is $z = -27(y - 2) - 18 = -27y + 54 - 18 = 36 - 27y$.

The function is changing faster in the y -direction.

Answer to Exercise 24 (on page 49)

1. First, we define $\mathbf{u}$ such that $|\mathbf{u}| = 1$ and $\mathbf{u}$ is in the same direction as $\mathbf{v}$:

$$\mathbf{u} = k\mathbf{v} = [-3k, 4k]$$

$$\sqrt{(-3k)^2 + (4k)^2} = 1$$

$$\sqrt{9k^2 + 16k^2} = \sqrt{25k^2} = 5k = 1$$

$$k = \frac{1}{5}$$

Therefore, we define $\mathbf{u} = [-3/5, 4/5]$ and the directional derivative is given by:

$$\begin{aligned} D_{\mathbf{u}}(x, y) &= \left(\frac{-3}{5}\right) \frac{\partial}{\partial x} f(x, y) + \left(\frac{4}{5}\right) \frac{\partial}{\partial y} f(x, y) \\ &= \left(\frac{-3}{5}\right) \frac{\partial}{\partial x} [e^{3x} \sin 2y] + \left(\frac{4}{5}\right) \frac{\partial}{\partial y} [e^{3x} \sin 2y] \\ &= \left(\frac{-3}{5}\right) (3e^{3x} \sin 2y) + \left(\frac{4}{5}\right) (2e^{3x} \cos 2y) \end{aligned}$$

And substituting for $(x, y) = (0, \pi/6)$:

$$\begin{aligned} D_{\mathbf{u}}(0, \pi/6) &= \left(\frac{-3}{5}\right) \cdot [3e^{3 \cdot 0} \sin(\frac{\pi}{3})] + \left(\frac{4}{5}\right) \cdot [2e^{3 \cdot 0} \cos(\frac{\pi}{3})] \\ D_{\mathbf{u}}(0, \pi/6) &= \left(\frac{-3}{5}\right) \cdot \left[3 \cdot \frac{\sqrt{3}}{2}\right] + \left(\frac{4}{5}\right) \cdot \left[2 \cdot \frac{1}{2}\right] \\ D_{\mathbf{u}}(0, \pi/6) &= \left(\frac{-3}{5}\right) \cdot \left(\frac{3\sqrt{3}}{2}\right) + \left(\frac{4}{5}\right) \cdot (1) \\ D_{\mathbf{u}}(0, \pi/6) &= \frac{-9\sqrt{3}}{10} + \frac{8}{10} = \frac{8 - 9\sqrt{3}}{10} \approx -0.759 \end{aligned}$$

2. We can express $\mathbf{v}$ as $\mathbf{v} = [2, -1]$. And we define $\mathbf{u}$ such that $|\mathbf{u}| = 1$ and $\mathbf{u}$ is in the same direction as $\mathbf{v}$:

$$\begin{aligned} \mathbf{u} &= k\mathbf{v} = [2k, -k] \\ \sqrt{(2k)^2 + (-k)^2} &= 1 \\ \sqrt{4k^2 + k^2} &= \sqrt{5}k = 1 \\ k &= \frac{1}{\sqrt{5}} = \frac{\sqrt{5}}{5} \end{aligned}$$

Therefore, we define $\mathbf{u} = [2\sqrt{5}/5, -\sqrt{5}/5]$ and the directional derivative is given by:

$$\begin{aligned} D_{\mathbf{u}}(x, y) &= \left(\frac{2\sqrt{5}}{5}\right) \frac{\partial}{\partial x} f(x, y) + \left(\frac{-\sqrt{5}}{5}\right) \frac{\partial}{\partial y} f(x, y) \\ &= \left(\frac{2\sqrt{5}}{5}\right) \frac{\partial}{\partial x} [x^2y + xy^3] + \left(\frac{-\sqrt{5}}{5}\right) \frac{\partial}{\partial y} [x^2y + xy^3] \\ &= \left(\frac{2\sqrt{5}}{5}\right) [2xy + y^3] + \left(\frac{-\sqrt{5}}{5}\right) [x^2 + 3xy^2] \end{aligned}$$

And substituting $(x, y) = (2, 4)$:

$$D_u(2, 4) = \left(\frac{2\sqrt{5}}{5}\right) [2(2)(4) + 4^3] + \left(\frac{-\sqrt{5}}{5}\right) [2^2 + 3(2)(4^2)]$$

$$D_u(2, 4) = \left(\frac{2\sqrt{5}}{5}\right) [80] + \left(\frac{-\sqrt{5}}{5}\right) [100]$$

$$D_u(2, 4) = 32\sqrt{5} - 20\sqrt{5} = 12\sqrt{5} \approx 26.833$$

3. We define $\mathbf{u}$ such that $|\mathbf{u}| = 1$ and $\mathbf{u}$ is in the same direction as $\mathbf{v}$:

$$\mathbf{u} = k\mathbf{v} = [k, k, k]$$

$$\sqrt{k^2 + k^2 + k^2} = 1$$

$$\sqrt{3}k = 1$$

$$k = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

Therefore, we let $\mathbf{u} = \left[\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\right]$ and the directional derivative is given by:

$$\begin{aligned} D_u(x, y, z) &= \left(\frac{\sqrt{3}}{3}\right) \frac{\partial}{\partial x} f(x, y, z) + \left(\frac{\sqrt{3}}{3}\right) \frac{\partial}{\partial y} f(x, y, z) + \left(\frac{\sqrt{3}}{3}\right) \frac{\partial}{\partial z} f(x, y, z) \\ &= \left(\frac{\sqrt{3}}{3}\right) \left[\frac{\partial}{\partial x} \ln(x^2 + 3y - z) + \frac{\partial}{\partial y} \ln(x^2 + 3y - z) + \frac{\partial}{\partial z} \ln(x^2 + 3y - z) \right] \\ &= \left(\frac{\sqrt{3}}{3}\right) \left[\frac{2x}{x^2 + 3y - z} + \frac{3}{x^2 + 3y - z} + \frac{-1}{x^2 + 3y - z} \right] \\ &= \left(\frac{\sqrt{3}}{3}\right) \left[\frac{2x + 2}{x^2 + 3y - z} \right] = \frac{\sqrt{3}(2x + 2)}{3(x^2 + 3y - z)} \end{aligned}$$

And substituting $(x, y, z) = (2, 2, 1)$:

$$D_u(2, 2, 1) = \frac{\sqrt{3}(2(2) + 2)}{3(2^2 + 3(2) - 1)} = \frac{\sqrt{3}(6)}{3(9)} = \frac{2\sqrt{3}}{9} \approx 0.385$$

Answer to Exercise 25 (on page 51)

$$1. z = f(50, 50) = 3000 - 0.01(50)^2 - 0.02(50)^2 = 2925$$

2. A south-pointing unit vector is $\mathbf{u} = [0, -1]$. To find the rate of change, we find the

directional derivative in the direction of $\mathbf{u}$ at $(50, 50)$:

$$D_{\mathbf{u}}f(x, y) = (-1) \left[\frac{\partial}{\partial y} (3000 - 0.01x^2 - 0.02y^2) \right]$$

$$D_{\mathbf{u}}f(x, y) = (-1)(-0.04y) = 0.04y$$

And at $(50, 50)$, $D_{\mathbf{u}}f(50, 50) = 0.04(50) = 2 > 0$. Therefore, if you walk south, you will ascend.

3. A northwest-pointing unit vector is $\mathbf{u} = \left[-\sqrt{2}/2, \sqrt{2}/2 \right]$. To find the rate of change, we find the directional derivative at $(50, 50)$ in the direction of $\mathbf{u}$:

$$D_{\mathbf{u}}f(x, y) = \left(\frac{-\sqrt{2}}{2} \right) \left[\frac{\partial}{\partial x} f(x, y) \right] + \left(\frac{\sqrt{2}}{2} \right) \left[\frac{\partial}{\partial y} f(x, y) \right]$$

$$D_{\mathbf{u}}f(x, y) = \left(\frac{-\sqrt{2}}{2} \right) [-0.02x] + \left(\frac{\sqrt{2}}{2} \right) [-0.04y]$$

$$D_{\mathbf{u}}f(x, y) = 0.01\sqrt{2}x - 0.02\sqrt{2}y$$

$$D_{\mathbf{u}}f(50, 50) = 0.01\sqrt{2}(50) - 0.02\sqrt{2}(50) = \frac{-\sqrt{2}}{2} \approx -0.707$$

The rate of elevation change walking northwest is approximately -0.707 , so you will descend and your rate of elevation change would be less than if you walked south.

4. To find the direction of maximum elevation gain, we find the direction the gradient vector points in:

$$\nabla f = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right]$$

$$\nabla f = [-0.02x, -0.04y]$$

And at $(50, 50)$,

$$\nabla f(50, 50) = [-0.02(50), -0.04(50)] = [-1, -2]$$

Therefore, the rate of greatest elevation change is in a south-by-southwest direction indicated by the vector $[-1, -2]$ and the rate of elevation change is $|\nabla f(50, 50)| = \sqrt{(-1)^2 + (-2)^2} = \sqrt{5}$. Notice this is greater than the other two rates of change we have found.

Answer to Exercise 26 (on page 54)

$P_x = 6 - 2x - y = 0$ and $P_y = 4 - 2y - x = 0$. From the second, $x = 4 - 2y$; substituting, $6 - 8 + 4y - y = 0$, so $y = \frac{2}{3}$ and $x = \frac{8}{3}$. $P_{xx} = -2$, $P_{yy} = -2$, $P_{xy} = -1$, so $D = 4 - 1 = 3 > 0$ with $P_{xx} < 0$: a maximum. $P\left(\frac{8}{3}, \frac{2}{3}\right) = \frac{28}{3} \approx 9.33$, or about \$9,330.

Answer to Exercise 27 (on page 55)

$f_x = y$ and $f_y = x$, so the only critical point is $(0,0)$. $f_{xx} = 0$, $f_{yy} = 0$, $f_{xy} = 1$, so $D = 0 - 1 = -1 < 0$: a saddle point. (Along $y = x$, $f = x^2$ rises; along $y = -x$, $f = -x^2$ falls.)

Answer to Exercise 28 (on page 56)

1.

$$\begin{aligned}\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} &= \frac{\partial}{\partial x} \left[\frac{\partial}{\partial x} (x^2 - y^2) \right] + \frac{\partial}{\partial y} \left[\frac{\partial}{\partial y} (x^2 - y^2) \right] \\ &= \frac{\partial}{\partial x} [2x] + \frac{\partial}{\partial y} [-2y] = 2 - 2 = 0\end{aligned}$$

Therefore, $f(x, y) = x^2 - y^2$ is a solution to Laplace's Equation.

2.

$$\begin{aligned}\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} &= \frac{\partial}{\partial x} \left[\frac{\partial}{\partial x} (\sin x \cosh y + \cos x \sinh y) \right] + \frac{\partial}{\partial y} \left[\frac{\partial}{\partial y} (\sin x \cosh y + \cos x \sinh y) \right] \\ &= \frac{\partial}{\partial x} [\cos x \cosh y - \sin x \sinh y] + \frac{\partial}{\partial y} [\sin x \sinh y + \cos x \cosh y] \\ &= -\sin x \cosh y - \cos x \sinh y + \sin x \cosh y + \cos x \sinh y = 0\end{aligned}$$

Therefore, $f(x, y) = \sin x \cosh y + \cos x \sinh y$ is a solution to Laplace's Equation.

3.

$$\begin{aligned}\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} &= \frac{\partial}{\partial x} \left[\frac{\partial}{\partial x} (e^{-x} \cos y - e^{-y} \cos x) \right] + \frac{\partial}{\partial y} \left[\frac{\partial}{\partial y} (e^{-x} \cos y - e^{-y} \cos x) \right] \\ &= \frac{\partial}{\partial x} [-e^{-x} \cos y + e^{-y} \sin x] + \frac{\partial}{\partial y} [-e^{-x} \sin y + e^{-y} \cos x] \\ &= e^{-x} \cos y + e^{-y} \cos x - e^{-x} \cos y - e^{-y} \cos x = 0\end{aligned}$$

Therefore, $f(x, y) = e^{-x} \cos y - e^{-y} \cos x$ is a solution to Laplace's Equation.

Answer to Exercise 29 (on page 57)

1. Finding the partial derivatives:

$$\frac{\partial^2 f}{\partial t^2} = \frac{\partial}{\partial t} \left[\frac{\partial}{\partial t} (\cos(kx) \cos(akt)) \right] = \frac{\partial}{\partial t} [-ak \cos(kx) \sin(akt)]$$

$$\frac{\partial^2 f}{\partial t^2} = -a^2 k^2 \cos(kx) \cos(akt)$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x} \left[\frac{\partial}{\partial x} (\cos(kx) \cos(akt)) \right] = \frac{\partial}{\partial x} [-k \sin(kx) \cos(akt)]$$

$$\frac{\partial^2 f}{\partial x^2} = -k^2 \cos(kx) \cos(akt)$$

And we see that:

$$a^2 \frac{\partial^2 f}{\partial x^2} = -a^2 k^2 \cos(kx) \cos(akt) = \frac{\partial^2 f}{\partial t^2}$$

Therefore, $f(x, t) = \cos(kx) \cos(akt)$ satisfies the Wave Equation.

2. Finding the partial derivatives:

$$\frac{\partial^2 f}{\partial t^2} = \frac{\partial}{\partial t} \left[\frac{\partial}{\partial t} (\sin(x - at) + \ln(x + at)) \right]$$

$$\frac{\partial^2 f}{\partial t^2} = \frac{\partial}{\partial t} \left[-a \cos(x - at) + \frac{a}{x + at} \right] = -a^2 \sin(x - at) + \frac{-a^2}{(x + at)^2}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x} \left[\frac{\partial}{\partial x} (\sin(x - at) + \ln(x + at)) \right]$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x} \left[\cos(x - at) + \frac{1}{x + at} \right] = -\sin(x - at) + \frac{-1}{(x + at)^2}$$

And we see that:

$$a^2 \frac{\partial^2 f}{\partial x^2} = -a^2 \sin(x - at) + \frac{-a^2}{(x + at)^2} = \frac{\partial^2 f}{\partial t^2}$$

Therefore, $f(x, t) = \sin(x - at) + \ln(x + at)$ satisfies the Wave Equation.

3. Finding the partial derivatives:

$$\frac{\partial^2 f}{\partial t^2} = \frac{\partial}{\partial t} \left[\frac{\partial}{\partial t} \left(\frac{t}{a^2 t^2 - x^2} \right) \right] = \frac{\partial}{\partial t} \left[\frac{(a^2 t^2 - x^2) - t(2a^2 t)}{(a^2 t^2 - x^2)^2} \right]$$

$$\frac{\partial^2 f}{\partial t^2} = \frac{\partial}{\partial t} \left[\frac{a^2 t^2 - x^2 - 2a^2 t^2}{(a^2 t^2 - x^2)^2} \right] = \frac{\partial}{\partial t} \left[\frac{-a^2 t^2 - x^2}{(a^2 t^2 - x^2)^2} \right]$$

$$\frac{\partial^2 f}{\partial t^2} = \frac{(a^2 t^2 - x^2)^2 (-2a^2 t) - (-a^2 t^2 - x^2) (2(a^2 t^2 - x^2) (2a^2 t))}{(a^2 t^2 - x^2)^4}$$

$$\frac{\partial^2 f}{\partial t^2} = \frac{(a^2 t^2 - x^2) (-2a^2 t) - (-a^2 t^2 - x^2) (4a^2 t)}{(a^2 t^2 - x^2)^3}$$

$$\frac{\partial^2 f}{\partial t^2} = \frac{-2a^4 t^3 + 2a^2 t x^2 + 4a^4 t^3 + 4a^2 t x^2}{(a^2 t^2 - x^2)^3}$$

$$\frac{\partial^2 f}{\partial t^2} = \frac{2a^4 t^3 + 6a^2 t x^2}{(a^2 t^2 - x^2)^3} = 2a^2 t \left(\frac{a^2 t^2 + 3x^2}{(a^2 t^2 - x^2)^3} \right)$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x} \left[\frac{\partial}{\partial x} \left(\frac{t}{a^2 t^2 - x^2} \right) \right] = \frac{\partial}{\partial x} \left[\frac{(a^2 t^2 - x^2) (0) - t (-2x)}{(a^2 t^2 - x^2)^2} \right]$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x} \left[\frac{2tx}{(a^2 t^2 - x^2)^2} \right] = \frac{(a^2 t^2 - x^2)^2 (2t) - (2tx) (2(a^2 t^2 - x^2) (-2x))}{(a^2 t^2 - x^2)^4}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{(a^2 t^2 - x^2) (2t) - (2tx) (2) (-2x)}{(a^2 t^2 - x^2)^3}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{2a^2 t^3 - 2tx^2 + 8tx^2}{(a^2 t^2 - x^2)^3} = \frac{2a^2 t^3 + 6tx^2}{(a^2 t^2 - x^2)^3} = 2t \left(\frac{a^2 t^2 + 3x^2}{(a^2 t^2 - x^2)^3} \right)$$

And we see that:

$$a^2 \frac{\partial^2 f}{\partial x^2} = 2a^2 t \left(\frac{a^2 t^2 + 3x^2}{(a^2 t^2 - x^2)^3} \right) = \frac{\partial^2 f}{\partial t^2}$$

Therefore, $f(x, t) = \frac{t}{a^2 t^2 - x^2}$ satisfies the Wave Equation.

Answer to Exercise 30 (on page 59)

1. The marginal productivity of labor is given by $\partial P / \partial L$:

$$\frac{\partial P}{\partial L} = \frac{\partial}{\partial L} [1.01 L^{0.75} K^{0.25}] = 0.7575 L^{-0.25} K^{0.25}$$

2. The marginal productivity of capital is given by $\partial P / \partial K$:

$$\frac{\partial P}{\partial K} = \frac{\partial}{\partial K} [1.01 L^{0.75} K^{0.25}] = 0.2525 L^{0.75} K^{-0.75}$$

3. Finding the marginal productivity of labor in 1916:

$$\frac{\partial P}{\partial L} = 0.7575 (382)^{-.25} (126)^{0.25} \approx 0.574$$

And finding the marginal productivity of capital in 1916:

$$\frac{\partial P}{\partial K} = 0.2525 (382)^{0.75} (126)^{-0.75} \approx 0.580$$

4. Since the marginal productivity of capital is greater, I would invest in capital. This would yield a greater increase in production than the same investment in labor.



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