



CONTENTS

1	Graphing Polynomials	3
1.1	Leading term in graphing	4
2	Interpolating with Polynomials	7
2.1	Interpolating polynomials in python	10
3	Rational Functions	13
4	Partial Fractions	21
4.0.1	Improper fractions	22
4.0.2	Proper fractions	23
5	Limits	29
5.1	Continuity	38
5.1.1	Continuity Practice	40
5.2	Limits Rules	42
5.3	Squeeze Theorem	46
5.3.1	Squeeze Theorem Practice	47
5.4	Intermediate Value Theorem	49
A	Answers to Exercises	53
	Index	61

Graphing Polynomials

In using polynomials to solve real-world problems, it is often handy to know what the graph of the polynomial looks like. You have many of the tools you need to start to sketch out the the graphs:

- To find where the graph crosses the y-axis, you can evaluate the polynomial at $x = 0$.
- To find where the graph crosses the x-axis, you can find the roots of the polynomial.
- To find the level spots on the graph (often the top of a hump or the bottom of a dip), you can take the derivative of the polynomial (which is a polynomial), and find the roots of that.

FIXME: Diagram of those things

For example, if you wanted to graph the polynomial $f(x) = -x^3 - x^2 + 6x$, you might plug in a few values that are easy to compute:

- $f(-2) = -8$
- $f(-1) = -6$
- $f(0) = 0$
- $f(1) = 4$
- $f(2) = 0$

So, right away, we know two roots: $x = 0$ and $x = 2$. Are there others? We won't know until we factor the polynomial:

$$\begin{aligned} -x^3 - x^2 + 6x &= (-1x)(x^2 + x - 6) \\ &= (-1x)(x + 3)(x - 2) \end{aligned}$$

So, yes, there is a third root: $x = -3$

What about the level spots? $f'(x) = -3x^2 - 2x + 6$. Where is that zero?

$$-3x^2 - 2x + 6 = 0$$

$$x^2 + \frac{2}{3}x - 2 = 0$$

We have a formula for quadratics like this:

$$\begin{aligned}x &= -\frac{b}{2} \pm \frac{\sqrt{b^2 - 4c}}{2} \\&= -\frac{\frac{2}{3}}{2} \pm \frac{\sqrt{\left(\frac{2}{3}\right)^2 - 4(-2)}}{2} \\&= -\frac{1}{3} \pm \frac{\sqrt{\frac{4}{9} + 8}}{2} \\&= -\frac{1}{3} \pm \frac{\sqrt{\frac{85}{9}}}{2} \\&= -\frac{1}{3} \pm \frac{\sqrt{85}}{6} \\&\approx 1.20 \text{ and } -1.87\end{aligned}$$

Now, you might plug those numbers in:

- $f(1.2) \approx 4.0$
- $f(-1.87) \approx -8.2$

1.1 Leading term in graphing

There is one more trick you need before you can draw a good graph of a polynomial. As you go farther and farther to the left and right, where does the function go? In other words, does the graph go up on both ends (like a smile)? Or does it go down on both ends (like a frown)? Or does the negative end go down (frowny) while the positive end goes up (smiley)? Or does the negative end go up (smiley) and the positive end go down (frowny)?

Assuming the polynomial is not constant, there are only those four possibilities. It is determined entirely by the leading term of the polynomial. If the degree of the leading term is even, both ends go in the same direction (both are smiley or both are frowny). If the coefficient of the leading term is positive, the positive end is smiley.

The graph we are working on has a leading term of $-1x^3$. The degree is odd, thus the ends go in different directions. The coefficient is negative, so the positive end points down. Now, you can draw the graph, which should look something like this:

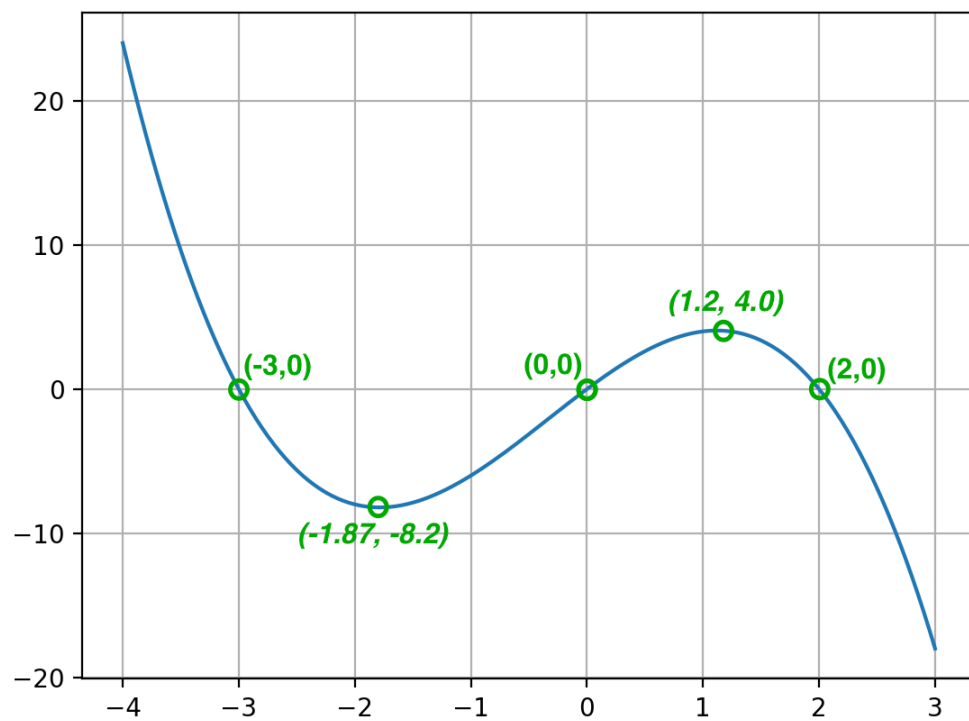


Figure 1.1: A graph with roots and significant points.

Interpolating with Polynomials

Let's say someone on a distant planet records video of a hammer being throw up into the air. They send you three random frames of the hammer in flight. Each frame has a timestamp, and you can clearly see how high the hammer is in each one. Can you create a 2nd degree polynomial that explains the entire flight of the hammer?

In other words, you have three points $(t_0, h_0), (t_1, h_1), (t_2, h_2)$. Can you find a, b, c such that the graph of $at^2 + bt + c = t$ passes through all three points?

The answer is yes. In fact, given any n points¹, there is exactly one $n-1$ degree polynomial that passes through all the points.

There are a lot of variables floating around. Let's make it concrete: The photos are taken at $t = 2$ seconds, $t = 3$ seconds, and $t = 4$ seconds. In those photos, the height of the hammer is 5m, 7m, and 6m. So, we want our polynomial to pass through these points: $(2, 5), (3, 7), (4, 6)$.

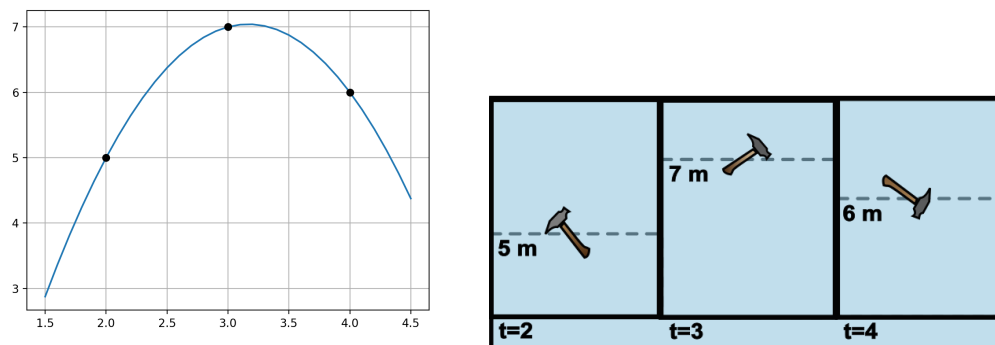


Figure 2.1: Left: the graph of the hammer going through the given points. Right: the three hammer frames.

How can you find that polynomial? Let's do it in small steps. Can you create a 2nd degree polynomial that is not zero at $t = 2$, but is zero at $t = 3$ and $t = 4$? Yes, you can: $(x-3)(x-4)$ has exactly two roots at $t = 3$ and $t = 4$. The value of this polynomial at $t = 2$ is $(2-3)(2-4) = 2$. We really want it to be 5m, so we can divide the whole polynomial by 2 and multiply it by 5.

¹ $n = 2$ will result in a line in slope intercept point, and $n = 1$ is a single point

Now we have the polynomial:

$$f_0(x) = \frac{5}{(2-3)(2-4)}(x-3)(x-4) = \frac{5}{2}x^2 - \frac{35}{2}x + 30$$

This is a second degree polynomial that is 5 at $t = 2$ and 0 at $t = 3$ and $t = 4$.

Now, we create a polynomial that is 7 at $t = 3$ and 0 at $t = 2$ and $t = 4$:

$$f_1(x) = \frac{7}{(3-2)(3-4)}(x-2)(x-4) = -7x^2 + 42x - 56$$

Finally, we create a polynomial that is 6 at $t = 4$ and zero at $t = 2$ and $t = 3$:

$$f_2(x) = \frac{6}{(4-2)(4-3)}(x-2)(x-3) = 3x^2 - 15x + 18$$

Adding these three polynomials together gives you a new polynomial that touches all three points:

$$f(x) = \frac{5}{2}x^2 - \frac{35}{2}x + 30 - 7x^2 + 42x - 56 + 3x^2 - 15x + 18 = -\frac{3}{2}x^2 + \frac{19}{2}x - 8$$

You can test this with your Polynomial class. Create a file called `test_interpolation.py`. Add this code:

```
from Polynomial import Polynomial
import matplotlib.pyplot as plt

in_x = [2,3,4]
in_y = [5,7,6]

pn = Polynomial([-8, 19/2, -3/2])
print(pn)

# These lists will hold our x and y values
x_list = []
y_list = []

# Starting x
current_x = 1.5

while current_x <= 4.5:
    # Evaluate pn at current_x
    current_y = pn(current_x)
```

```

# Add x and y to respective lists
x_list.append(current_x)
y_list.append(current_y)

# Move x forward
current_x += 0.05

# Plot the curve
plt.plot(x_list, y_list)

# Plot black circles on the given points
plt.plot(in_x, in_y, "ko")
plt.grid(True)
plt.show()

```

You should get a nice plot that shows the graph of the polynomial passing through those three points.

In general, then, if you provide any three points $(t_0, h_0), (t_1, h_1), (t_2, h_2)$, there is a second degree polynomial that pass through all three:

$$\frac{h_0}{(t_0 - t_1)(t_0 - t_2)}(x - t_1)(x - t_2) + \frac{h_1}{(t_1 - t_0)(t_1 - t_2)}(x - t_0)(x - t_2) + \frac{h_2}{(t_2 - t_0)(t_2 - t_1)}(x - t_0)(x - t_1)$$

What if you are given 9 points $((t_0, h_0), (t_1, h_1), \dots, (t_8, h_8))$ and want to find a 8th degree polynomial that passes through all of them? Just what you would expect:

$$\frac{h_0}{(t_0 - t_1)(t_0 - t_2) \dots (t_0 - t_8)}(x - t_1)(x - t_2) \dots (x - t_8) + \dots + \frac{h_8}{(t_8 - t_0) \dots (t_8 - t_7)}(x - t_0) \dots (x - t_7)$$

FIXME: Do I need to define summation and prod here?

The general solution is, given n points, the $n - 1$ degree polynomial that goes through them is

$$y = \sum_{i=0}^n \left(\prod_{\substack{0 \leq j \leq n \\ j \neq i}} \frac{x - t_j}{t_i - t_j} \right) h_i$$

That would be tedious for a person to compute, but computers are perfect for this stuff. Let's create a method that creates instances of Polynomial using interpolation.²

²Note that other textbooks or resources may refer to this as *Lagrange Polynomial Interpolation*.

2.1 Interpolating polynomials in python

Your method will take two lists of numbers, one contains x-values and the other contains y-values. So comment out the line that creates the polynomial in `test_interpolation.py` and create it from two lists:

```
in_x = [2,3,4]
in_y = [5,7,6]
# pn = Polynomial([-8, 19/2, -3/2])
pn = Polynomial.from_points(in_x, in_y)
print(pn)
```

Add the following method to your Polynomial class in `Polynomial.py`

```
@classmethod
def from_points(cls, x_values, y_values):
    coef_count = len(x_values)

    # Sums start with a zero polynomial
    sum_pn = Polynomial([0.0] * coef_count)
    for i in range(coef_count):

        # Products start with the constant 1 polynomial
        product_pn = Polynomial([1.0])
        for j in range(coef_count):

            # Must skip j=i
            if j != i:
                # (1x - x_values[j]) has a root at x_values[j]
                factor_pn = Polynomial([-1 * x_values[j], 1])
                product_pn = product_pn * factor_pn

        # Scale so product_pn(x_values[i]) = y_values[i]
        scale_factor = y_values[i] / product_pn(x_values[i])
        scaled_pn = scale_factor * product_pn

        # Add it to the sum
        sum_pn = sum_pn + scaled_pn

    return sum_pn
```

It should work exactly the same as before. You should get the same polynomial printed out as before as well as the same plot of the curve passing through the three points.

How about five points? Change `in_x` and `in_y` at the start of `test_interpolation.py`:

```
in_x = [1.7, 2, 2.7, 3.5, 4, 4.4]  
in_y = [8, 12, 1, 4, -1, 6]
```

You should get a polynomial that passes through all five points:

$$11.21x^5 - 171.05x^4 + 1019.44x^3 - 2957.53x^2 + 4161.78x - 2258.75$$

It should look like this:

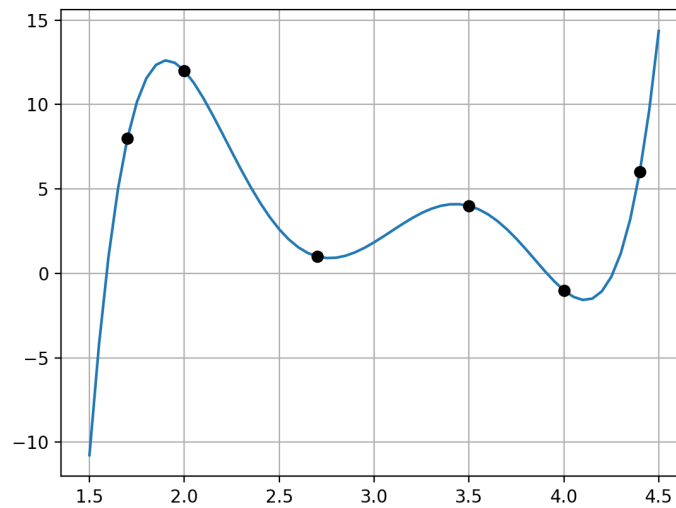


Figure 2.2: 5 point interpolation of points.

Rational Functions

We have discussed addition, subtraction, and multiplication of polynomials. What about division?

A quotient of polynomials is called a rational expression. When the polynomials are factored and the stars align, we can simplify the rational expression to a single polynomial, just like we might reduce a fraction to lowest terms.

Example

$$\begin{aligned}\frac{(x+1)(x+5)}{x+5} &= (x+1) \cdot \frac{x+5}{x+5} \\ &= x+1 \quad (x \neq 5)\end{aligned}\tag{3.1}$$

1

What if the polynomials are not factored? Factor them first.

Example

$$\frac{x^2 + 6x + 5}{x + 5} = \frac{(x+1)(x+5)}{x+5}$$

and simplify as in the previous example.

Now, let us consider a rational expression which can be simplified to a single polynomial — but in the denominator.

Example

$$\begin{aligned}\frac{x+5}{x^2 + 6x + 5} &= \frac{x+5}{(x+1)(x+5)} \\ &= \frac{1}{x+1} \cdot \frac{x+5}{x+5} \\ &= \frac{1}{x+1}\end{aligned}\tag{3.2}$$

Consider this expression as a function: $f(x) = \frac{1}{x+1}$. As you might have guessed, this is called a rational function. We did not bother looking at the result of the previous example

¹It is important to note that we need to restrict $x = 5$ since this would be division by zero. FIXME expand on this?

as a function, because we already know that function type: it is a line with slope 1 and y-intercept 1. However, this rational function is another animal entirely. Let us examine our first rational function with a familiar concept: the y-intercept.

y-intercept: $f(0) = \frac{1}{0+1} = \frac{1}{1} = 1$. The graph contains the point $(0, 1)$.

Does f have an x-intercept? That would be an x -value where $f(x) = 0$. But a fraction equals 0 only when its numerator equals 0; since the numerator of this expression is always 1, f has no x-intercept.

Knowing the y-intercept, and that there is no x-intercept, is a comforting start. But things get weird when we consider a concept that has previously seemed quite simple: domain. Recall that the domain of a function is the set of all values which can be used as inputs. In this case, the domain includes all real numbers, with one exception. The number -1 is not a valid input because $f(-1) = \frac{1}{-1+1} = \frac{1}{0}$, which is undefined. So, we say that the domain is all real numbers except -1 . This means the graph contains a point corresponding to every x -value except -1 .

There is no point at $x = -1$, but there is a point at every other x -value, such as, for example, -1.1 , or -0.99999 . So, what is happening near $x = -1$?

x	-1.1	-1.01	-1.001	-0.999	-0.99	-0.9
$f(x)$	-10	-100	-1000	1000	100	10

The function is going haywire. As we choose x -values closer and closer to -1 , the resulting function values are larger and larger in magnitude. Also, they are negative on one side, but positive on the other. So, how does a graph go from y -values of -10 , to -100 , to -1000 , all in a space of less than 0.1 on the x -axis? Then, from there, suddenly to big positive numbers on the other side of $x = -1$? All without ever crossing the x -axis (since there is no x-intercept)? Let's look at the graph.

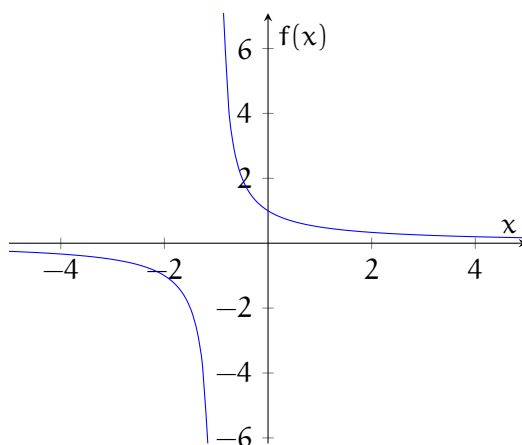


Figure 3.1: Graph of $f(x) = \frac{1}{x+1}$

We can see the y-intercept we found above. We can also see that the graph has no x-intercept, as expected. The phenomenon occurring at $x = -1$ is called a vertical asymptote. One other interesting feature of this graph is how it hugs the x-axis toward the left and right edges of the window. This makes the line $y = 0$ (the x-axis) a horizontal asymptote for this function. We can see why this is happening numerically by considering what happens for x-values far from 0. In this function, the result is a fraction with a numerator of 1 and a denominator that is large in size: a fraction that is close to 0.

x	-1000	-100	-10	10	100	1000
f(x)	-0.001	-0.01	-0.1	0.1	0.01	0.001

Let's examine another rational function. Begin by factoring to see if the function can be simplified.

$$g(x) = \frac{x^2 - 3x + 2}{x^2 - 4x + 3} = \frac{(x-1)(x-2)}{(x-1)(x-3)}$$

Consider the domain of g before continuing. Which values of x are valid inputs? Since substituting $x = 1$ or $x = 3$ would result in division by 0, these are not valid inputs. The domain of g is all real numbers except 1 and 3.

Now, for any x -value except 1, $\frac{x-1}{x-1} = 1$. This means that, for all x -values but 1, we can cancel those factors, leaving $g(x) = \frac{x-2}{x-3}$. (We will talk more about what is happening at $x = 1$ in a moment.)

This function has both x- and y-intercepts: y-intercept: $g(0) = \frac{0-2}{0-3} = \frac{2}{3}$. The graph contains the point $(0, \frac{2}{3})$. x-intercept: $g(x) = 0$ where the numerator equals 0 and the denominator does not equal 0. Since $x - 2 = 0$ when $x = 2$, the x-intercept is 2 and the graph contains the point $(2, 0)$.

The graph of g has a vertical asymptote at any x -value where substitution would result in dividing a nonzero number by zero. Thus, g has a vertical asymptote at $x = 3$.

Does g have a horizontal asymptote? Let us see what happens when we substitute x -values far from 0.

x	-1000	-100	-10	10	100	1000
g(x)	0.999	0.990	0.923	1.143	1.010	1.001

As we move further away from the y-axis, the y-values become closer to 1. The horizontal asymptote describes the end behavior of the function, or what the graph looks like far from the y-axis. In this case, if we ignore the portion close to the y-axis, the graph begins to look like the line $y = 1$, making this the horizontal asymptote of g .

So, what is happening at $x = 1$? The value is not in the domain of the function, but there is no vertical asymptote there. That is because substituting any other value for x , even

values very close to 1, into $\frac{(x-1)(x-2)}{(x-1)(x-3)}$ gives the exact same number as substituting into $\frac{x-2}{x-3}$. So, there is a hole in the graph at $x = 1$, but nothing strange is happening on either side of 1. (Depending on the graphing software, the hole may not be visible.)

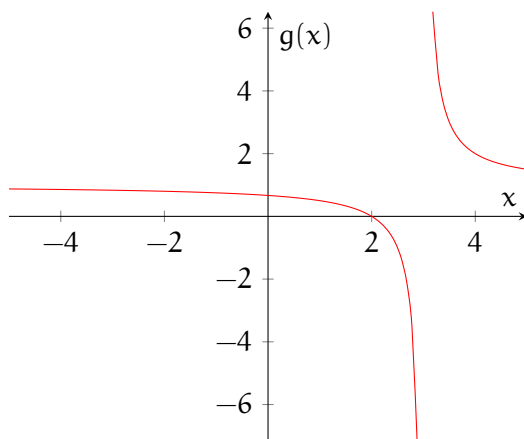


Figure 3.2: Graph of $g(x) = \frac{x^2 - 3x + 2}{x^2 - 4x + 3}$

Exercise 1 Rational Functions Practice 1

Determine the x - and y -intercepts and horizontal and vertical asymptotes of the rational function:

1. $\frac{2x+5}{x+4}$

Working Space

Answer on Page 53

Exercise 2

[This question was originally presented in the no-calculator section of the 2012 AP Calculus BC exam.] The line $y = 5$ is a horizontal asymptote of which of the following functions? Explain.

A. $y = \frac{\sin 5x}{x}$

B. $y = 5x$

C. $y = \frac{1}{x-5}$

D. $y = \frac{5x}{1-x}$

E. $y = \frac{20x^2 - x}{1 + 4x^2}$

*Working Space**Answer on Page 53*

In those examples, common factors cancel, leaving one polynomial. Of course, there is no guarantee that any two polynomials will have common factors, or even be factorable at all. Now, we consider an example that cannot be simplified. We will focus on just the asymptotes here.

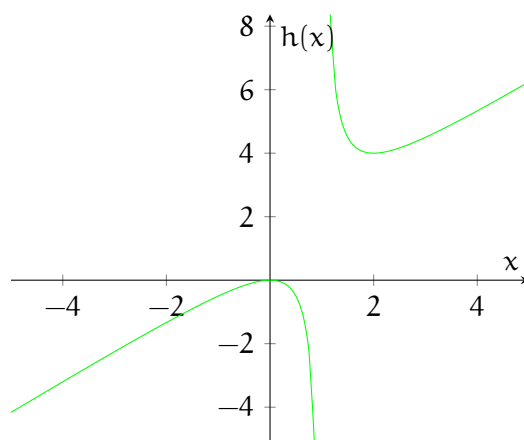
$$h(x) = \frac{x^2}{x-1}$$

We see that the x -value 1 gives division of a non-zero number by zero, giving a vertical asymptote at $x = 1$. How about a horizontal asymptote? We examine values of h for values of x far from 0.

x	-1000	-100	-10	10	100	1000
$h(x)$	-999	-99	-9	11	101	1001

Rather than seeing function values leveling off as in the previous examples, we see function values that grow in size along with x . The function h has no horizontal asymptote. Let's examine the graph:

This function exhibits a different type of end behavior: that of a line with slope 1. To see that, cover up the portion of the graph near the y -axis and focus on the left and right. The rather dull and time-consuming technique of polynomial long division can be used to rewrite the function as a quotient and a remainder. We encourage you to watch the Khan Academy video on the topic, but for now, let us instead use our knowledge of factoring techniques and a clever little trick.

Figure 3.3: Graph of $h(x) = \frac{x^2}{x-1}$

$$\begin{aligned}
 h(x) &= \frac{x^2}{x-1} \\
 &= \frac{x^2 - 1 + 1}{x-1} \\
 &= \frac{x^2 - 1}{x-1} + \frac{1}{x-1} \\
 &= \frac{(x-1)(x+1)}{x-1} + \frac{1}{x-1} \\
 &= x + 1 + \frac{1}{x-1}
 \end{aligned} \tag{3.3}$$

We obtain a quotient of $x + 1$ and a remainder of 1. It is the quotient that determines the end behavior of the graph. Why? Substituting x -values far from zero makes the remainder term very small, since it becomes a fraction with a large denominator but a numerator of only 1. So for x -values far from zero, the y -value is x plus 1 plus a very small number (so small that we can justifiably ignore it). This means that far from the y -axis, the function acts like the quotient: the line $y = x + 1$. We call this line an oblique asymptote. See below how the graph of $h(x)$ hugs that line.

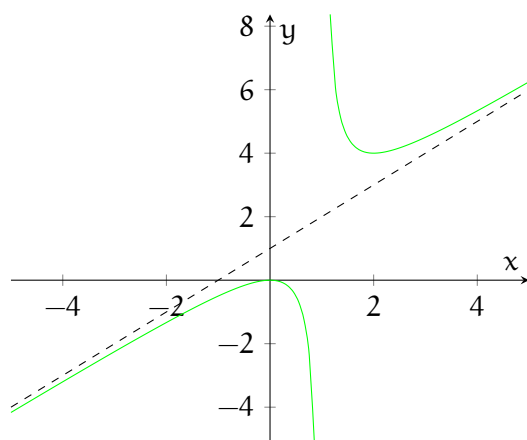


Figure 3.4: Graph of $h(x) = \frac{x^2}{x-1}$ and its oblique asymptote $y = x + 1$

Exercise 3 Rational Functions Practice 2

Factor and simplify the rational function, then determine any holes and vertical and oblique asymptotes of the rational function.

1. $\frac{x^3+2x^2}{x^2+x}$

Working Space

Answer on Page 53

We have seen lines act as end behaviors. Are there other possibilities? Sure! Here is an example with parabolic end behavior.

$$k(x) = \frac{x^3}{x-2}$$

We use our add-subtract trick to reveal the quotient, which describes the end behavior.

$$\begin{aligned}
 h(x) &= \frac{x^3}{x-2} \\
 &= \frac{x^3 - 8 + 8}{x-2} \\
 &= \frac{x^3 - 8}{x-2} + \frac{8}{x-2} \\
 &= \frac{(x-2)(x^2 + 2x + 4)}{x-2} + \frac{8}{x-2} \\
 &= x^2 + 2x + 4 + \frac{8}{x-2}
 \end{aligned} \tag{3.4}$$

The quotient, $x^2 + 2x + 4$, should describe the end behavior. We confirm by graphing both k and the quotient - the parabolic asymptote.

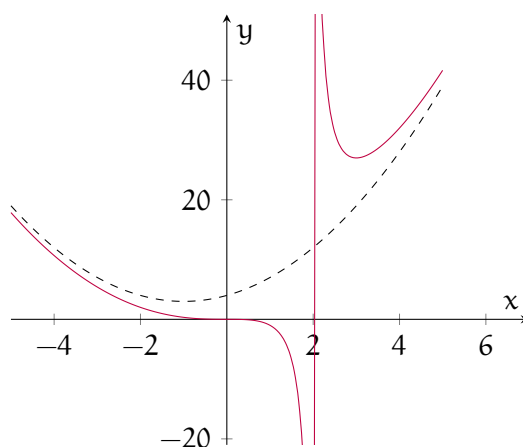


Figure 3.5: Graph of $k(x) = \frac{x^3}{x-2}$ and its parabolic asymptote $y = x^2 + 2x + 4$

Partial Fractions

How can you add fractions with different denominators, like $\frac{1}{x} + \frac{2}{x+3}$? You would need to make the denominators the same; after that, you could just add the numerators. You achieve this by multiplying the numerator and denominator of each fraction by the denominator of the other fraction:

$$\frac{1}{x} + \frac{2}{x+3} = \frac{1}{x} \left(\frac{x+3}{x+3} \right) + \frac{2}{x+3} \left(\frac{x}{x} \right)$$

Recall that when the numerator and denominator of a fraction are the same, the fraction is equal to one. So, we are not changing the *value* of each fraction, since we are just multiplying by one. Continuing, we can perform the multiplication and see that:

$$\begin{aligned} \frac{1}{x} + \frac{2}{x+3} &= \frac{x+3}{x(x+3)} + \frac{2x}{x(x+3)} \\ &= \frac{(x+3) + 2x}{x(x+3)} = \frac{3x+3}{x^2+3x} = \frac{3(x+1)}{x^2+3x} \end{aligned}$$

The inverse of this process is called **partial fraction decomposition** (or partial fraction expansion). This method has applications in many fields, but we will find it most useful as a tool to evaluate integrals in a later chapter.

Let $g(x)$ be a rational function such that

$$g(x) = \frac{P(x)}{Q(x)}$$

Where $P(x)$ and $Q(x)$ are polynomials. If $g(x)$ is proper (that is, the degree of P is less than the degree of $Q(x)$) then we can express $g(x)$ as the sum of simpler rational fractions. If $g(x)$ is improper (that is, the degree of P is greater than or equal to the degree of Q), then we must first perform long division to obtain a remainder, $R(x)$, where the degree of R is less than the degree of Q :

$$g(x) = \frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)}$$

4.0.1 Improper fractions

What is $\int \frac{x^3+x}{x-1} dx$. Using long division, we see that:

$$\frac{x^3+x}{x-1} = x^2 + x + 2 + \frac{2}{x-1}$$

(see figure 4.1 for an explanation). Then we can also say that:

$$\frac{x^3+x}{x-1} = x^2 + x + 2 + \frac{2}{x-1}$$

$$\begin{array}{r}
 x^2 + x + 2 \\
 x-1 \overline{) x^3 + 0x^2 + x} \\
 \underline{-(x^3 - x^2)} \quad \downarrow \\
 x^2 + x \\
 \underline{-(x^2 - x)} \\
 2x \\
 \underline{-(2x - 2)} \\
 2
 \end{array}$$

Figure 4.1: Evaluating $(x^3 + x) \div (x - 1)$ with the long division method

When you start with an improper fraction, use long division to reduce it to a term plus a proper fraction, then use the methods outlined below to further manipulate the proper fraction.

Exercise 4

Use long division to reduce the following improper rational functions to a term plus a proper rational fraction.

1. $\frac{x^4+x^3+2x^2+2x-3}{x^2-3x+2}$

2. $\frac{2x^3+5}{x^3-3x^2+2x-4}$

3. $\frac{3x^4-2x^3-x^2+1}{x^3-3x}$

Working Space

Answer on Page 53

4.0.2 Proper fractions

When the order of the numerator is less than or equal to the denominator, there are three further possibilities.

No repeated linear factors

In the first case, the denominator, $Q(x)$ is composed of distinct linear factors. In this case, we can say that $Q(x) = (a_1x + b_1)(a_2x + b_2) \cdots (a_nx + b_n)$, where no factor is repeated (including constant multiples). Then, there exists $A, B, C, \dots$, such that:

$$\frac{P(x)}{Q(x)} = \frac{A}{a_1x + b_1} + \frac{B}{a_2x + b_2} + \cdots$$

Let's see an example of this by decomposing $\frac{4x^2-7x-12}{x(x+2)(x-3)}$. We start by defining A, B , and C , such that:

$$\frac{4x^2 - 7x - 12}{x(x+2)(x-3)} = \frac{A}{x} + \frac{B}{x+2} + \frac{C}{x-3}$$

Multiplying both sides by $x(x+2)(x-3)$ we get:

$$4x^2 - 7x - 12 = A(x+2)(x-3) + B(x)(x-3) + C(x)(x+2)$$

We have 3 unknowns and only one equation! Don't worry — remember this equation is true for all x , so we can choose a convenient value of x to isolate each unknown in turn. Starting, let $x = 0$. Then:

$$\begin{aligned} 4(0)^2 - 7(0) - 12 &= A(0+2)(0-3) + B(0)(0-3) + C(0)(0+2) \\ -12 &= A(2)(-3) + 0 + 0 \end{aligned}$$

Notice that the B and C disappear, and we can solve for A :

$$A = \frac{-12}{-6} = 2$$

We can solve for B by setting $x = -2$ and for C by setting $x = 3$ (notice, we've used all three zeroes of the denominator polynomial):

$$\begin{aligned} 4(-2)^2 - 7(-2) - 12 &= A(-2+2)(-2-3) + B(-2)(-2-3) + C(-2)(-2+2) \\ 4(4) + 14 - 12 &= 0 + B(-2)(-5) + 0 \\ 16 + 2 &= 10B \\ B &= \frac{9}{5} \end{aligned}$$

and

$$4(3)^2 - 7(3) - 12 = A(3 + 2)(3 - 3) + B(3)(3 - 3) + C(3)(3 + 2)$$

$$4(9) - 21 - 12 = 0 + 0 + C(3)(5)$$

$$36 - 33 = 15C$$

$$C = \frac{1}{5}$$

We can then decompose our original fraction:

$$\frac{4x^2 - 7x - 12}{x(x + 2)(x - 3)} = \frac{2}{x} + \frac{9}{5(x + 2)} + \frac{1}{5(x - 3)}$$

You can check your answer by cross-multiplying and adding. You should get the same rational function back.

Repeated linear factors

The second case is if $Q(x)$ has repeated factors (such as $x^2 + 8x + 16 = (x + 4)^2$). Suppose the first linear factor, $(a_1x + b_1)$ is repeated r times (that is, $Q(x)$ contains the factor $(a_1x + b_1)^r$). Then, instead of $\frac{A}{a_1x + b_1}$, we should write:

$$\frac{A_1}{a_1x + b_1} + \frac{A_2}{(a_1x + b_1)^2} + \cdots + \frac{A_r}{(a_1x + b_1)^r}$$

Let's look at a concrete example to see how this works:

Example: Decompose $\frac{x^2 + x + 1}{(x + 1)^2(x + 2)}$

Solution: We start by defining:

$$\frac{x^2 + x + 1}{(x + 1)^2(x + 2)} = \frac{A}{x + 1} + \frac{B}{(x + 1)^2} + \frac{C}{x + 2}$$

Multiplying both sides by $(x + 1)^2(x + 2)$:

$$x^2 + x + 1 = A(x + 1)(x + 2) + B(x + 2) + C(x + 1)^2$$

Since there are only 2 roots to $(x + 1)^2(x + 2)$, we will use another method called "equating the coefficients" to find A , B , and C . We start by expanding the right side of the equation:

$$x^2 + x + 1 = A(x^2 + 3x + 2) + B(x + 2) + C(x^2 + 2x + 1)$$

Distributing and combining, we find that:

$$x^2 + x + 1 = Ax^2 + 3Ax + 2A + Bx + 2B + Cx^2 + 2Cx + C$$

$$x^2 + x + 1 = (A + C)x^2 + (3A + B + 2C)x + (2A + 2B + C)$$

For this equation to be true, we know that:

$$A + C = 1$$

$$3A + B + 2C = 1$$

$$2A + 2B + C = 1$$

(That is, the coefficient for x^2 on the left, 1, must be equal to the coefficient for x^2 on the right, $(A + C)$, and so on.) We now have a system of 3 equations and 3 unknowns. When you solve for each, you should find that:

$$A = -2$$

$$B = 1$$

$$C = 3$$

Therefore,

$$\frac{x^2 + x + 1}{(x + 1)^2(x + 2)} = \frac{-2}{x + 1} + \frac{1}{(x + 1)^2} + \frac{3}{x + 2}$$

Irreducible quadratic factors

Sometimes, we cannot express a polynomial as the product of two linear statements (that is, terms in the form $ax + b$). Take $x^2 + 1$, which cannot be expressed as the product of real, linear terms. What do you do if something like $x^2 + 1$ is in the denominator? In this case, when we write an expression for $\frac{P(x)}{Q(x)}$, we include a term in the form:

$$\frac{Ax + B}{ax^2 + bx + c}$$

For example, we can write:

$$\frac{x}{(x - 2)(x^2 + 1)(x^2 + 4)} = \frac{A}{x - 2} + \frac{Bx + C}{x^2 + 1} + \frac{Dx + E}{x^2 + 4}$$

Example: Decompose $\frac{2x^2 - x + 4}{x^3 + 4x}$

Solution: We begin by factoring the denominator:

$$x^3 + 4x = x(x^2 + 4)$$

Which cannot be factored further. Therefore, we define:

$$\frac{2x^2 - x + 4}{x(x^2 + 4)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 4}$$

$$2x^2 - x + 4 = A(x^2 + 4) + (Bx + C)x$$

$$2x^2 - x + 4 = Ax^2 + 4A + Bx^2 + Cx$$

Which implies that:

$$2 = A + B$$

$$C = -1$$

$$4A = 4$$

Therefore, $A = 1$, $B = 1$, and $C = -1$ and we can say that:

$$\frac{2x^2 - x + 4}{x^3 + 4x} = \frac{1}{x} + \frac{x - 1}{x^2 + 4}$$

Repeated irreducible quadratic factors

Lastly, the denominator might contain repeated irreducible quadratic factors. Similar to repeated linear factors, when setting up your partial fractions, instead of only writing

$$\frac{A}{ax^2 + bx + c}$$

For a quadratic factor that is repeated r times, your equation should include:

$$\frac{A_1}{ax^2 + bx + c} + \frac{A_2}{(ax^2 + bx + c)^2} + \cdots + \frac{A_r}{(ax^2 + bx + c)^r}$$

Exercise 5

Decompose the following proper fractions

1. $\frac{x-4}{x^2+5x-6}$

2. $\frac{x^2+x+1}{(x^2+1)^2}$

3. $\frac{x^2+x+1}{(x+1)^2(x+2)}$

Working Space

Answer on Page 53

Limits

FIXME define asymptotic and end behavior, not discussed before here

The asymptotic behavior we see in rational functions suggests that we need to expand our vocabulary of function characteristics. We examined vertical asymptotes and end behavior through graphs and tables, and discussed them in English. The language of limits enables us to discuss these attributes mathematically and with greater efficiency.

Let's revisit an example from the previous chapter. This function has a hole at $x = 1$, a vertical asymptote at $x = 3$, and a horizontal asymptote of $y = 1$.

$$f(x) = \frac{x^2 - 3x + 2}{x^2 - 4x + 3} = \frac{(x-1)(x-2)}{(x-1)(x-3)}$$

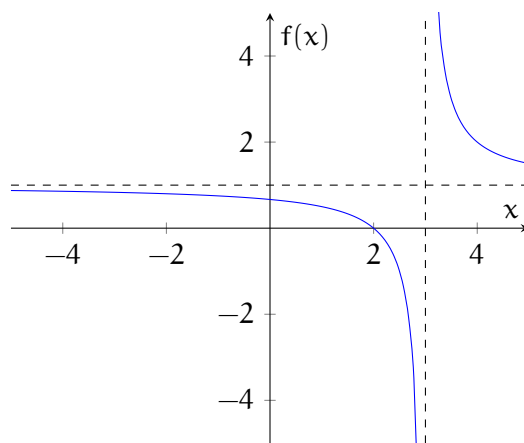


Figure 5.1: Graph of $f(x) = \frac{x^2 - 3x + 2}{x^2 - 4x + 3}$ with asymptotes

First, consider the vertical asymptote. We see that the graph goes down as it hugs the left side of the vertical asymptote, and goes up as it hugs the right side. We can describe these behaviors as the left- and right-hand limits, respectively. We say that the left-hand limit of f at $x = 3$ is negative infinity. Another way of communicating this is to say that as x approaches 3 from the left, the function approaches negative infinity. Symbolically, we summarize this as

$$\lim_{x \rightarrow 3^-} f(x) = -\infty$$

The little negative sign in $x \rightarrow 3^-$ indicates we are approaching $x = 3$ from the left (the negative side of the axis).

Similarly, the right-hand limit of f at $x = 3$ is positive infinity. In other words, as x approaches 3 from the right, the function approaches positive infinity. Symbolically, we write

$$\lim_{x \rightarrow 3^+} f(x) = \infty$$

This time, the little $^+$ indicates we are approaching the x -value from the right (positive) side of the axis.

The limit of a function at a particular x -value is the y -value that the function approaches as it approaches the given x -value. In the previous example, we could only specify the left- and right-hand limits, because they were different. In cases where the left- and right-hand limits are equal, we can say that the function has a limit there. The hole in our function f is one such value. We see that as we approach the hole from both the left and right, the function takes on values near $\frac{1}{2}$. This is more apparent numerically:

x	0.9	0.99	0.999	1	1.001	1.01	1.1
$f(x)$	0.5238	0.5025	0.5003	undefined	0.4998	0.4975	0.4737

We can also see this by zooming in on the graph (see Figure 5.2):

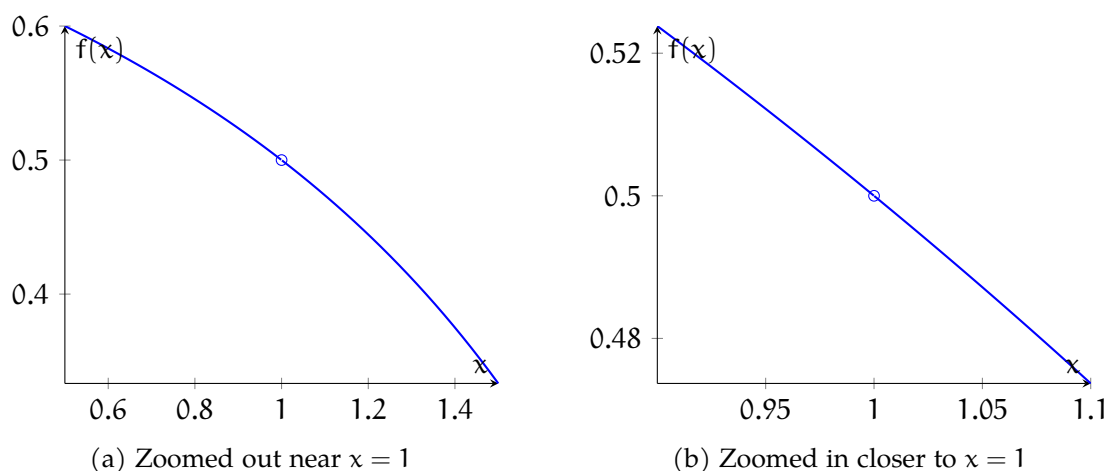


Figure 5.2: Two graphs of $f(x) = \frac{x^2 - 3x + 2}{x^2 - 4x + 3}$ zoomed in about $x = 1$

The left-hand and right-hand limits of f at 1 are both $\frac{1}{2}$. Since they are equal, we can also say that the limit of f at 1 is $\frac{1}{2}$. This allows us to efficiently discuss the behavior of f at 1, even though the function is not defined there, as substituting 1 into the function gives division by zero.

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1} f(x) = \frac{1}{2}$$

We can also talk about limits at x -values where nothing weird is happening (that is, no hole or vertical asymptote). For example, as x approaches 4 from the left and right, y approaches 2.

x	3.9	3.99	3.999	4	4.001	4.01	4.1
$f(x)$	2.1111	2.0101	2.0010	2	1.9990	1.9901	1.9091

In this case, since nothing weird is happening, the limit is equal to the function value. This is an example of continuity, which we will discuss in more detail in the next chapter. By contrast, at the vertical asymptote $x = 1$, since the left- and right-hand limits are not equal, we say the function does not have a limit, or the limit does not exist.

Finally, let's consider the horizontal asymptote of f . The graph hugs the line $y = 1$ as x goes far to the left and far to the right. We say that as x approaches negative infinity, f approaches 1; likewise, that as x approaches positive infinity, f approaches 1. We write these symbolically as $\lim_{x \rightarrow -\infty} f(x) = 1$ and $\lim_{x \rightarrow \infty} f(x) = 1$.

Exercise 6 Limits Practice 1

Determine the left- and right-hand limits of the function as x approaches the given values. At x -values where the limit exists, determine it.

1. $p(x) = \frac{x+3}{x^2+9x+18}$, $x = -6, -5, -3, \infty$

Working Space

Answer on Page 54

We have seen two weird behaviors of rational functions at certain x -values: holes and vertical asymptotes. Now, we will examine another type of weird behavior: jumps. This is a characteristic of some piecewise-defined functions. In piecewise-defined functions, the domain is divided into two or more pieces, and a different expression is used to give the

y-value depending on which piece contains the x -value. One common piecewise-defined function is the floor function (shown in figure 5.3), sometimes denoted $\lfloor x \rfloor$. The standard floor function rounds any real number down to the nearest integer. So, for a price quoted in dollars and cents, the floor would just be the number of dollars.

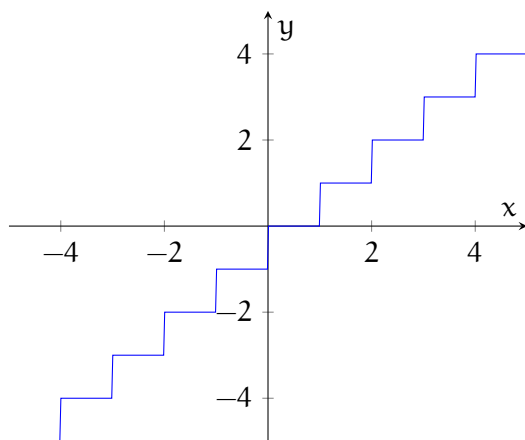


Figure 5.3: Graph of $y = \lfloor x \rfloor$

When x is exactly 1, the function value is 1: the number of dollars in a price of \$1.00. When x is any number greater than 1 but less than 2, the function value is still 1. Also, $\lfloor 1.01 \rfloor$, $\lfloor 1.5 \rfloor$, and $\lfloor 1.99999 \rfloor$ are all 1. As we continue to look to the right, once x equals exactly 2, it jumps up to the value 2. So, $\lim_{x \rightarrow 2^-} \lfloor x \rfloor = 1$, while $\lim_{x \rightarrow 2^+} \lfloor x \rfloor = 2$.

Besides rational and piecewise defined functions, there are other functions with interesting limits. Consider the standard exponential function, $y = e^x$ (shown in figure 5.4).

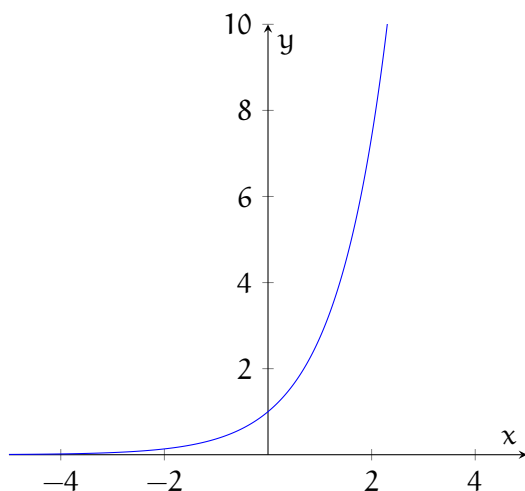


Figure 5.4: Graph of $y = e^x$

As x increases, y increases without bound; that is, $\lim_{x \rightarrow \infty} e^x = \infty$. However, looking far to the left, we see that y hugs the x -axis. This is because raising e to a large negative exponent is the same as 1 divided by e raised to a large positive exponent; that is, 1

divided by a very large number, which yields a very small positive number. In limit notation, $\lim_{x \rightarrow -\infty} e^x = 0$. This example illustrates that horizontal asymptotes need not model end behavior in both directions. Note that this reasoning holds for $y = b^x$ for any $b > 1$, so all such functions have the same horizontal asymptote, $y = 0$.

We know that the natural logarithm function, $y = \ln x$, is the inverse of $y = e^x$. Since inverse functions swap the role of x and y , it stands to reason that a horizontal asymptote in one function corresponds with a vertical asymptote in the other function, and that is indeed the case (see figure 5.5).

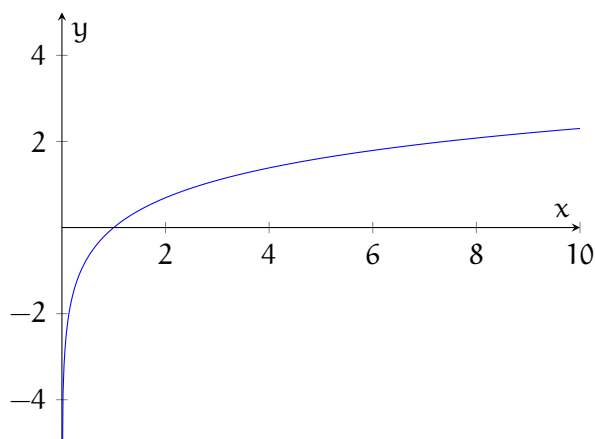


Figure 5.5: Graph of $y = \ln x$

An untransformed logarithm function is defined only for positive inputs. That is because it is not possible to find an exponent of a positive number that will yield a negative or zero result. What type of exponent on a positive number yields a number near zero? That would be a large-magnitude negative number. So, on the logarithm graph, large negative y -values correspond with x -values only slightly greater than zero. So, $\ln x$ (and $\log_2 x$, and indeed $\log_b x$ for any $b > 1$) approaches negative infinity as x approaches 0 from the right. There is no left-hand limit at 0, however. In limit notation, $\lim_{x \rightarrow 0^+} \ln x = -\infty$.

Exercise 7 **Limits Practice 2**

State the asymptotes of the following transformed exponential and logarithmic functions. Give the limit statement which describes the behavior of the function along the asymptote.

1. $y = 3^x + 1$
2. $y = \log_2(x - 4)$
3. $y = 2^{1-x}$
4. $y = \log_{10}(-2x)$

*Working Space**Answer on Page 54*

We next consider two functions that each have two horizontal asymptotes. These two seemingly obscure functions are quite important in data science.

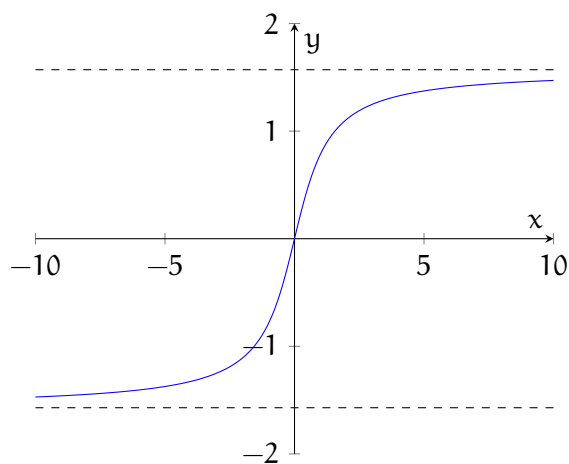


Figure 5.6: Graph of $y = \arctan x$

We know that the arctangent, or inverse tangent, function is the inverse of the piece of the tangent function which passes through the origin. The vertical asymptotes bounding this piece become horizontal asymptotes when the function is inverted.

Here are the equation and graph of the logistic function:

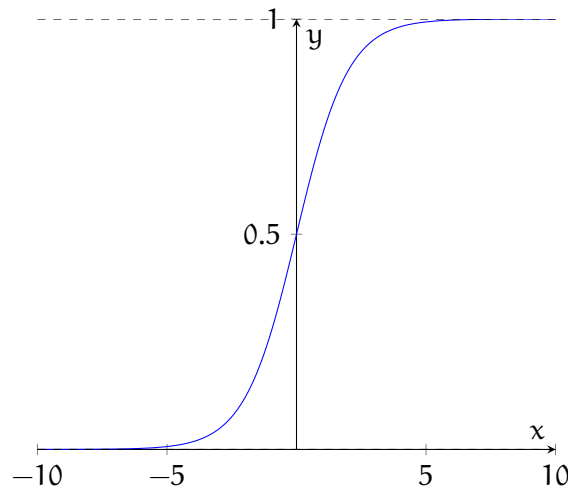


Figure 5.7: Graph of the logistic function, $y = \frac{1}{1+e^{-x}}$

For large magnitude negative values of x , the exponential term in the denominator becomes a very large positive value. The fraction thus becomes a positive number very close to zero. For large magnitude positive values of x , that exponential term becomes a very small positive number. Adding it to 1 yields a denominator just barely greater than 1. Dividing 1 by this number therefore yields a function value just barely less than 1. So, the logistic function yields values between 0 and 1, though never equaling either of these values exactly. It is precisely this characteristic which makes the logistic function so useful.

Exercise 8 Limits Practice 3

Using limit notation, state the limits as x approaches negative and positive infinity for the inverse tangent and logistic functions given above.

Working Space

Answer on Page 54

As seen above, the limit of a function from the left may be different from the limit of the function from the right. Additionally, the actual *value* of the function may be different from the limit. Consider the piecewise function $h(x)$:

$$h(x) = \begin{cases} -x^2 + 3, & \text{if } x < 0 \\ 2, & x = 0 \\ -x + 3, & \text{if } x > 0 \end{cases}$$

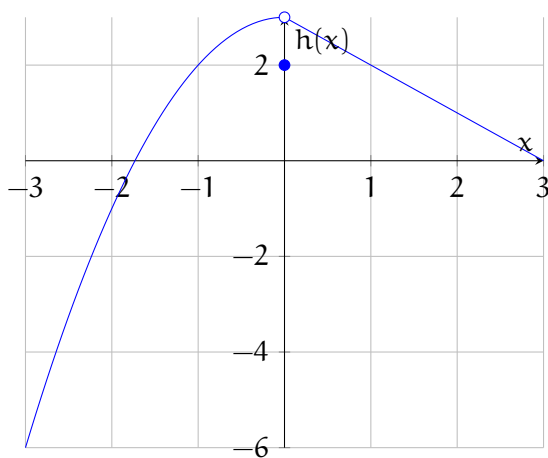


Figure 5.8: Graph of the piecewise function, $h(x)$

From examining the graph, we see that

$$\lim_{x \rightarrow 0^-} h(x) = \lim_{x \rightarrow 0^+} h(x) = 3$$

However, $h(0) = 2 \neq 3$. So, does this limit exist? It does! The limit of a function describes the *behavior* of the function around a particular value, not the value of the function itself. In order for a limit to exist, the limits from the left and right must be equal to each other, but not necessarily the actual value of the function.

Use the graph of $h(x)$ above and the graphs of $f(x)$ and $g(x)$ below to complete the following exercise.

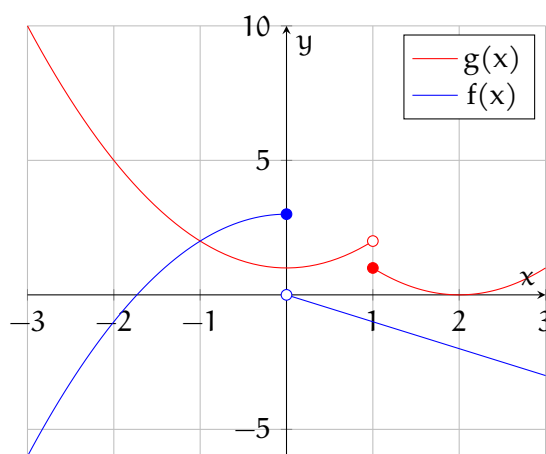


Figure 5.9: Piecewise functions $f(x)$ and $g(x)$

Exercise 9 Limits Practice 4

Determine the limit from the left and the right for each function at the given value(s). State the limit at that value, if it exists.

1. $h(x), x = -1, 0, 1$
2. $f(x), x = -1, 0, 2$
3. $g(x), x = -2, 0, 1, 2$

Working Space

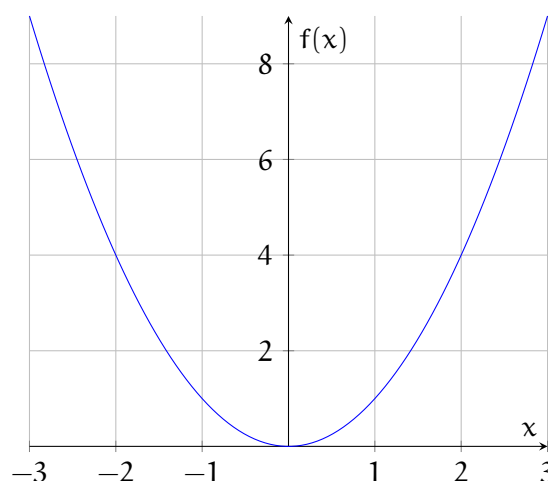
Answer on Page 54

5.1 Continuity

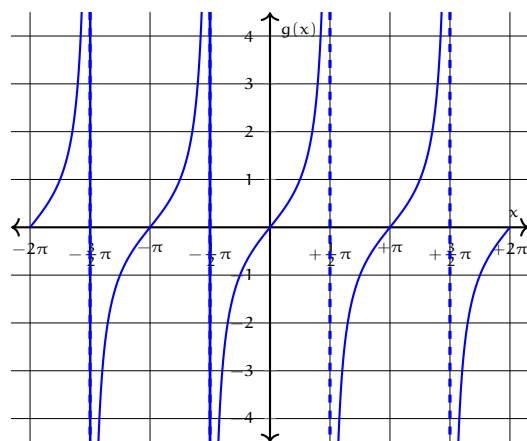
A note about continuity:

In order to be able to talk more about limits and know when we can apply certain rules and theorems, we first must discuss continuity. A function is continuous if there are no "jumps" or "gaps" in the graph of the function. For example, the function $f(x) = x^2$ is continuous for all real values of x . On the other hand, the function $g(x) = \tan(x)$ has many discontinuities, including at $x = \frac{\pi}{2}$. Let's examine the graph of each of these functions:

If you wanted, you could trace your finger along the graph of $f(x)$ from $x = -3$ to $x = 3$ without ever picking up your finger. This means the function is continuous in the domain from $-3 \leq x \leq 3$. In this case, the domain of continuity *includes* the end points ($x = 3$ and


 Figure 5.10: Graph of $f(x) = x^2$

$x = -3$). This is called a closed interval. In other cases, the function will be continuous right up to, but not including, the endpoints, as with the domains of continuity for our other example, $g(x) = \tan x$. This is called an open interval. Let's learn more about intervals of continuity by examining $g(x) = \tan x$.


 Figure 5.11: Graph of $g(x) = \tan x$

As you can see, if you trace your finger along the graph of the function starting at $x = 0$, you can continue without lifting your finger to $x = \frac{\pi}{2}$. As you approach $x = \frac{\pi}{2}$ from the left, the value of $g(x)$ approaches ∞ . In order to continue tracing the function PAST $x = \frac{\pi}{2}$, you have to lift your finger and bring it down to $-\infty$. The function then continues continuously again until $x = \frac{3\pi}{2}$.

In the case of $g(x) = \tan x$, the function is continuous on *open intervals*, including the open interval $\frac{\pi}{2} < x < \frac{3\pi}{2}$.

There is a shorter way to represent open and closed domain intervals. We can represent

that $f(x) = x^2$ is continuous on the closed interval $-3 \leq x \leq 3$ in the following way:

$$x \in [-3, 3]$$

This reads as “ x contained in the domain -3 to 3 , inclusive”. That is, all the values from -3 to 3 , including the endpoints. The inclusion of the endpoints is implied by the use of *brackets*. For open intervals, we use parentheses to communicate that the interval goes up to, but does not include, the endpoints. For $g(x) = \tan x$, we can use parentheses:

$$x \in \left(-\frac{3\pi}{2}, \frac{\pi}{2} \right)$$

because the $g(x) = \tan x$ is not continuous at $x = \frac{-3\pi}{2}$ or at $x = \frac{\pi}{2}$.

Formally, a function $f(x)$ is continuous at $x = a$ if $\lim_{x \rightarrow a} f(x)$ exists *and* $\lim_{x \rightarrow a} f(x) = f(a)$. In other words, the limit is equal to the actual value of the function. Re-examine the graph of $h(x)$. We have already seen that $\lim_{x \rightarrow 0} h(x)$ exists and is equal to 3 . However, $h(0) = 2 \neq \lim_{x \rightarrow 0} h(x)$. So $h(x)$ is not continuous at $x = 0$. Because $-x^2 + 3$ is evaluable all the way to $-\infty$ and $-x + 3$ is evaluable all the way to ∞ , the function $h(x)$ is continuous everywhere *except* $x = 0$. We can represent this mathematically by saying $h(x)$ is continuous on the domain $x \in (-\infty, 0) \cup (0, \infty)$. We use parentheses for $\pm\infty$ because we can never actually reach ∞ . Additionally, the function is continuous up to, but not including 0 , and the use of parentheses excludes $x = 0$ from the domain of continuity.

5.1.1 Continuity Practice

Exercise 10

[This problem was originally presented as a calculator-allowed, multiple-choice question on the 2012 AP Calculus BC exam.] Suppose a function f is continuous at $x = 3$. Classify the following statements as always true, sometimes true, or never true. Explain your answers.

1. $f(3) < \lim_{x \rightarrow 3} f(x)$
2. $\lim_{x \rightarrow 3^+} f(x) \neq \lim_{x \rightarrow 3^-} f(x)$
3. $f(3) = \lim_{x \rightarrow 3^+} f(x) = \lim_{x \rightarrow 3^-} f(x)$
4. The derivative of f at $x = 3$ exists.
5. The derivative of f is positive for $x < 3$ and negative for $x > 3$.

*Working Space**Answer on Page 55***Exercise 11 Limits Practice 5**

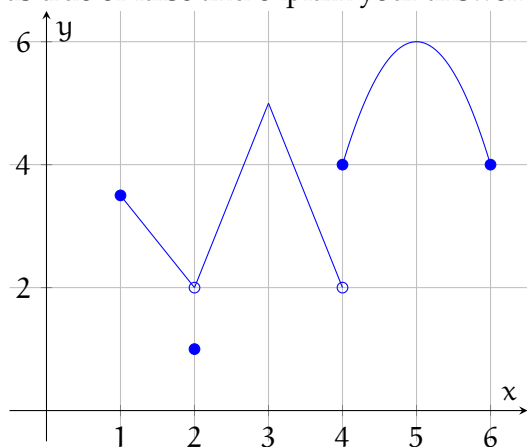
State the location of discontinuities (if any) and explain why the function is discontinuous at that location:

1. $f(x) = \frac{3x^2 - 8x - 3}{x - 3}$
2. $f(x) = \begin{cases} \frac{2}{x^4}, & \text{if } x \neq 0 \\ 2, & \text{if } x = 0 \end{cases}$
3. $f(x) = \begin{cases} \frac{3x^2 - 8x - 3}{x - 3}, & \text{if } x \neq 3 \\ 1, & \text{if } x = 3 \end{cases}$

*Working Space**Answer on Page 55*

Exercise 12

The graph of a function, $h(x)$, is shown. Classify each of the following statements as true or false and explain your answer.



1. $\lim_{x \rightarrow 2} h(x)$ exists
2. $\lim_{x \rightarrow 3} h(x)$ does not exist
3. $\lim_{x \rightarrow 4} h(x)$ exists
4. $h(x)$ is continuous at $x = 5$
5. $h(x)$ is not continuous at $x = 4$
6. $\lim_{x \rightarrow 2} h(x) = h(2)$

Working Space

Answer on Page 56

5.2 Limits Rules

There are some mathematical properties of limits which allow us to determine the limit of complex functions without seeing a graph or using a calculator to generate a table.

The following laws are true given that c is a constant, $\lim_{x \rightarrow a} f(x)$ exists, and $\lim_{x \rightarrow a} g(x)$ exists.

1. Sum Law $\lim_{x \rightarrow a} [f(x) + g(x)] = \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x)$

2. Difference Law $\lim_{x \rightarrow a} [f(x) - g(x)] = \lim_{x \rightarrow a} f(x) - \lim_{x \rightarrow a} g(x)$
3. Constant Multiple Law $\lim_{x \rightarrow a} [cf(x)] = c \cdot \lim_{x \rightarrow a} f(x)$
4. Product Law $\lim_{x \rightarrow a} [f(x)g(x)] = \lim_{x \rightarrow a} f(x) \cdot \lim_{x \rightarrow a} g(x)$
5. Quotient Law $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)}$ given that $\lim_{x \rightarrow a} g(x) \neq 0$

These laws are fairly obvious — the limit of the sum of two functions is equal to the sum of the limits of each function individually. The only tricky one is the last: The limit of the quotient of two functions is equal to the quotient of the limits if and only if the limit of the function in the denominator does not equal zero. This makes sense, as we know dividing by zero yields an undefined result.

Let's practice applying these laws to evaluate the limits of the functions $f(x)$, shown in blue below, and $g(x)$, shown in red below:

$$f(x) = \begin{cases} -x^2 + 3, & \text{if } x \leq 0 \\ -x, & \text{if } x > 0 \end{cases}$$

$$g(x) = \begin{cases} x^2 + 1, & \text{if } x < 1 \\ (x - 2)^2, & \text{if } x \geq 1 \end{cases}$$

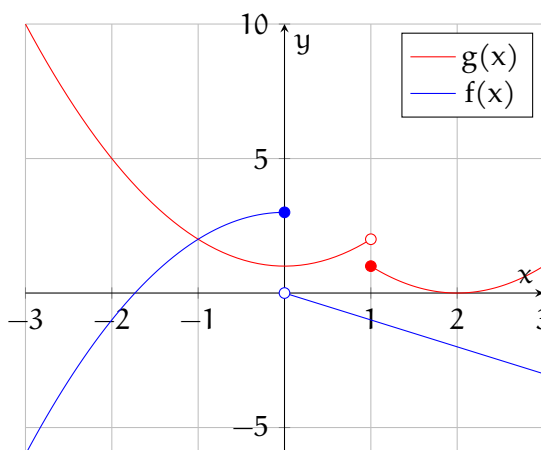


Figure 5.12: Graphs of the piecewise functions $f(x)$ and $g(x)$

We can use these laws to evaluate limits involving $f(x)$ and $g(x)$ (shown on the graph above). Here are some examples: Use the graphs of $f(x)$ and $g(x)$ given above to evaluate each limit, if it exists. If the limit does not exist, explain why. Two examples are given first:

Example 1: Evaluate $\lim_{x \rightarrow 0} f(x) \cdot g(x)$

Solution 1: From the Product Law, we know that:

$$\lim_{x \rightarrow 0} f(x) \cdot g(x) = \lim_{x \rightarrow 0} f(x) \cdot \lim_{x \rightarrow 0} g(x)$$

Looking at the graph, we can see that

$$\lim_{x \rightarrow 0} g(x) = 1$$

and there is a discontinuity in $f(x)$ at $x = 0$. Therefore,

$$\lim_{x \rightarrow 0} f(x) = \text{DNE}$$

Substituting this, we get:

$$\lim_{x \rightarrow 0} f(x) \cdot \lim_{x \rightarrow 0} g(x) = \text{DNE} \cdot 1 = \text{DNE}$$

Therefore, *the limit does not exist*.

Example 2: Evaluate $\lim_{x \rightarrow 2} f(x) - g(x)$

Solution 2: Applying the Difference Law, we see that:

$$\lim_{x \rightarrow 2} [f(x) - g(x)] = \lim_{x \rightarrow 2} f(x) - \lim_{x \rightarrow 2} g(x)$$

Examining the graph, we see that

$$\lim_{x \rightarrow 2} f(x) = -2$$

and

$$\lim_{x \rightarrow 2} g(x) = 0$$

Substituting these values, we get:

$$\lim_{x \rightarrow 2} f(x) - g(x) = -2 - 0 = -2$$

Exercise 13 **Limits Practice 6***Working Space*

1. $\lim_{x \rightarrow -3} \frac{f(x)}{g(x)}$
2. $\lim_{x \rightarrow 2} [f(x) + 5g(x)]$
3. $\lim_{x \rightarrow -1} \frac{3g(x)}{f(x)}$
4. $\lim_{x \rightarrow 0} f(x) \cdot 5g(x)$
5. $\lim_{x \rightarrow -1} f(x) - 3g(x)$

Answer on Page 56

Recall that exponents represent repeated multiplication. Therefore, if we apply the Product Law multiple times, we obtain the Power Law for limits:

6. Power Law $\lim_{x \rightarrow \infty} [f(x)]^n = [\lim_{x \rightarrow \infty} f(x)]^n$ where n is a positive integer

There are two special limits that will be useful to us and are intuitively obvious, but we won't formally prove here.

7. $\lim_{x \rightarrow a} c = c$
8. $\lim_{x \rightarrow a} x = a$

Combining Law 8 with the Power Law, we find that:

9. $\lim_{x \rightarrow a} x^n = a^n$

And similarly, for square roots:

10. $\lim_{x \rightarrow a} \sqrt[n]{x} = \sqrt[n]{a}$ (if n is even, we assume $a > 0$)

Direct substitution property: If f is a polynomial or rational function and a is in the domain for f , then

$$\lim_{x \rightarrow a} f(x) = f(a)$$

Often, rational functions can be simplified. In an above example, we computed the limit by simplifying $f(x) = \frac{3x^2 - 8x - 3}{x - 3}$ to the simpler $g(x) = 3x + 1$. This is a valid strategy because $\frac{3x^2 - 8x - 3}{x - 3} = 3x + 1$ when $x \neq 3$. Remember: a limit describes how a function behaves *as it approaches* a , not its value/behavior when x *actually equals* a . This reveals the following useful rule:

If $f(x) = g(x)$ when $x \neq a$, then $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} g(x)$, provided the limit exists.

5.3 Squeeze Theorem

The Squeeze Theorem states that if $f(x) \leq g(x) \leq h(x)$ when x is near a (except at a) and

$$\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} h(x) = L$$

, then

$$\lim_{x \rightarrow a} g(x) = L$$

In other words, if $g(x)$ is between $f(x)$ and $h(x)$ near a , and f and h have the same limit, L , then the limit of g must also be L .

Example: Let's examine the graph of $g(x) = x^2 \sin \frac{1}{x}$ and determine $\lim_{x \rightarrow 0} g(x)$:

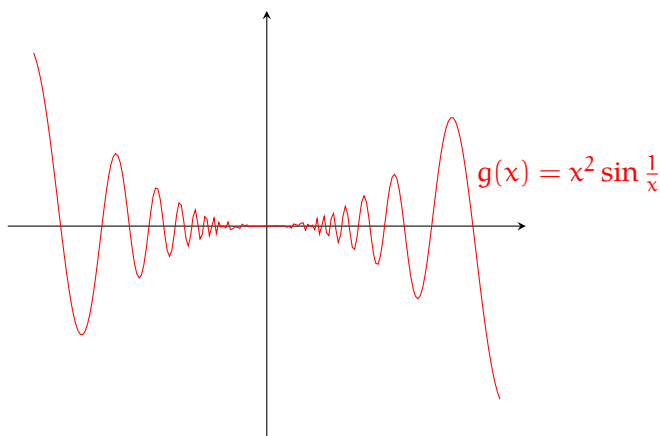


Figure 5.13: Graph of $g(x) = x^2 \sin \frac{1}{x}$

Solution: Because $\sin \frac{1}{x}$ is undefined at $x = 0$, we cannot compute the limit directly. However, from examining the graph, we can guess that $\lim_{x \rightarrow 0} g(x) = 0$. Feel free to confirm this with your calculator. We need to choose two functions: one that is larger than $g(x)$ near $x = 0$ and one that is smaller. Since $|\sin \frac{1}{x}| \leq 1$ (when $x \neq 0$), then

$$|x^2 \sin \frac{1}{x}| \leq x^2$$

and

$$-x^2 \leq x^2 \sin \frac{1}{x} \leq x^2$$

Let's confirm this by plotting $f(x) = -x^2$, $g(x)$, and $h(x) = x^2$ on the same graph:

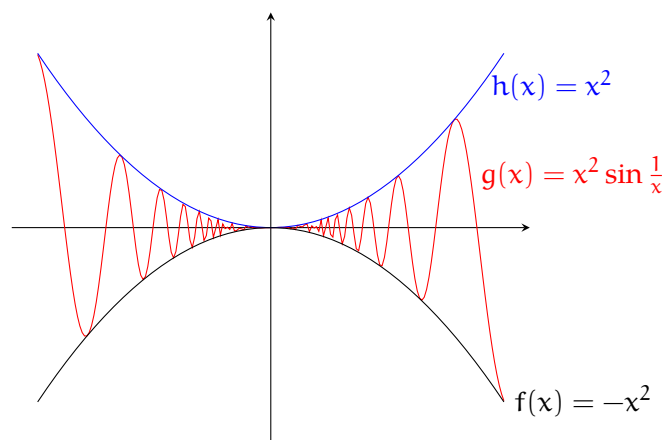


Figure 5.14: Squeeze Theorem example

As you can see, when x is near 0, $f(x) \leq g(x) \leq h(x)$. Because $f(x)$ and $h(x)$ are both polynomials, their limits are straightforward:

$$\lim_{x \rightarrow 0} -x^2 = 0 \text{ and } \lim_{x \rightarrow 0} x^2 = 0$$

Then, by the Squeeze Theorem, we can say that:

$$\lim_{x \rightarrow 0} x^2 \sin \frac{1}{x} = 0$$

5.3.1 Squeeze Theorem Practice

Exercise 14 Squeeze Theorem 1

Use the Squeeze Theorem to show that $\lim_{x \rightarrow 0} \sqrt{x^3 + x^2} \cos \frac{1}{x} = 0$. Illustrate by graphing the functions you define as f , g , and h on the same plot.

Working Space

Answer on Page 58

Exercise 15 Squeeze Theorem 2

If $2x + 3 \leq f(x) \leq x^2 - 2x + 7$ for $x \geq 0$,
find $\lim_{x \rightarrow 2} f(x)$

Working Space

Answer on Page 58

Exercise 16 Squeeze Theorem 3

Prove that $\lim_{x \rightarrow 0^+} \sqrt{x} e^{\sin \frac{\pi}{x}} = 0$

Working Space

Answer on Page 59

5.4 Intermediate Value Theorem

When considering functions that are continuous on a closed interval, the Intermediate Value Theorem can help us: Given a function, $f(x)$, that is continuous on the closed interval $[a, b]$ and $f(a) \neq f(b)$, there is at least one number c such that $f(c) = N$, where N is any number between $f(a)$ and $f(b)$. The theorem is illustrated in figures [5.15](#) and [5.16](#):

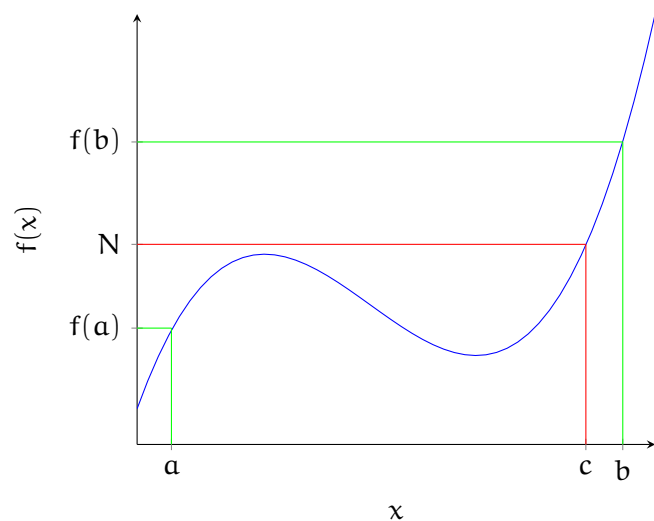


Figure 5.15: An example where one solution satisfies the Intermediate Value Theorem

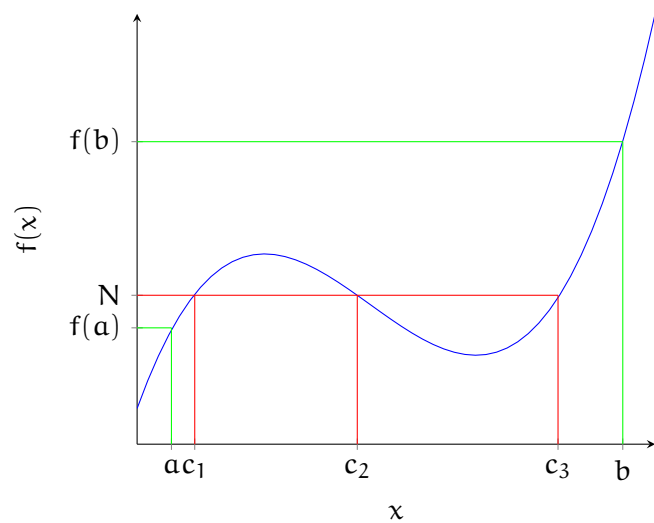


Figure 5.16: An example where more than one solution satisfies the Intermediate Value Theorem

Logically, we can think of the IVT this way: If a function is continuous on a closed interval, there are no gaps or breaks. If there are no gaps or breaks, then the function must pass through the line $y = N$, since it cannot jump over the line. Your graphing calculator uses IVT to find roots of functions:

Example: Show that there is at least one root of the equation $2x^3 - 6x^2 + 3x - 1 = 0$ between $x = 2$ and $x = 3$.

Solution: For IVT to apply, we must first check that the function is continuous on the closed interval $x \in [2, 3]$. We define $f(x) = 2x^3 - 6x^2 + 3x - 1$, which is continuous everywhere, because it is a polynomial function. *For more complex functions, always be sure to check the endpoints of an interval, since IVT only applies on closed intervals of continuity.* We will take $a = 2$, $b = 3$, and $N = 0$. We find the values of $f(x)$ at the endpoints:

$$f(2) = 2(2)^3 - 6(2)^2 + 3(2) - 1 = 16 - 24 + 6 - 1 = -3$$

$$f(3) = 2(3)^3 - 6(3)^2 + 3(3) - 1 = 54 - 54 + 9 - 1 = 8$$

Therefore, $f(2) < 0 < f(3)$ and according to IVT, there must exist some c such that $f(c) = 0$ and there is a root to the equation in the interval $x \in (2, 3)$.

Exercise 17 **Intermediate Value Theorem Practice**

Use the IVT to show there is a solution
the given equation on the stated interval:

Working Space

1. $2x^4 + x - 12 = 0$, $(1, 2)$

2. $\ln(x) = 3x - 4\sqrt{x}$, $(2, 3)$

3. $2\sin x = 3x^2 - 2x$, $(1, 2)$

Answer on Page 59

Answers to Exercises

Answer to Exercise 1 (on page 16)

x-intercept: $(-5/2, 0)$; y-intercept: $(0, 5/4)$; horizontal asymptote: $y = 2$; vertical asymptote: $x = -4$

Answer to Exercise 3 (on page 19)

The correct answer is E. To explain, we examine the behavior of each function as $x \rightarrow \infty$.

A. $\lim_{x \rightarrow \infty} \frac{\sin 5x}{x} = \pm\infty \neq 5$

B. $\lim_{x \rightarrow \infty} 5x = \infty \neq 5$

C. $\lim_{x \rightarrow \infty} \frac{1}{x-5} = 0 \neq 5$ (this function does have a *vertical* asymptote at $x = 5$).

D. $\lim_{x \rightarrow \infty} \frac{5x}{1-x} = \frac{5}{-1} = -5 \neq 5$ (this function has a horizontal asymptote at $x = -5$).

E. $\lim_{x \rightarrow \infty} \frac{20x^2 - x}{1 + 4x^2} = \frac{20}{4} = 5$.

Answer to Exercise 3 (on page 19)

Factored form: $\frac{x^2(x+2)}{x(x+1)}$; hole: $(0, 0)$; vertical asymptote: $x = -1$; oblique asymptote: $y = x + 1$

Answer to Exercise 4 (on page 22)

1. $x^2 + 4x + 12 \frac{30x-27}{x^2-3x+2}$

2. $2 + \frac{6x^2-4x+13}{x^3-3x^2+2x-4}$

3. $3x - 2 + \frac{8x^2-6x+1}{x^3-3x}$

Answer to Exercise 5 (on page 27)

1. $\frac{10}{7(x+6)} + \frac{-3}{7(x-1)}$
2. $\frac{1}{x^2+1} + \frac{x}{(x^2+1)^2}$
3. $\frac{-2}{x+1} + \frac{1}{(x+1)^2} + \frac{3}{x+2}$

Answer to Exercise 6 (on page 31)

$$\begin{aligned}\lim_{x \rightarrow -6^-} p(x) &= -\infty, \lim_{x \rightarrow -6^+} p(x) = \infty \\ \lim_{x \rightarrow -5^-} p(x) &= \lim_{x \rightarrow -5^+} p(x) = \lim_{x \rightarrow -5} p(x) = 1 \\ \lim_{x \rightarrow -3^-} p(x) &= \lim_{x \rightarrow -3^+} p(x) = \lim_{x \rightarrow -3} p(x) = \frac{1}{3} \\ \lim_{x \rightarrow \infty} p(x) &= 0 \text{ called simply a limit, although it is a left-hand limit}\end{aligned}$$

Answer to Exercise 7 (on page 34)

$$\lim_{x \rightarrow -\infty} 3^x + 1 = 1; \lim_{x \rightarrow 4^+} \log_2(x-4) = -\infty; \lim_{x \rightarrow \infty} 2^{1-x} = 0; \lim_{x \rightarrow 0^-} \log_{10}(-2x) = -\infty$$

Answer to Exercise 8 (on page 36)

$$\lim_{x \rightarrow -\infty} \tan^{-1}x = -\frac{\pi}{2}, \lim_{x \rightarrow \infty} \tan^{-1}x = \frac{\pi}{2}; \lim_{x \rightarrow -\infty} \frac{1}{1+e^{-x}} = 0, \lim_{x \rightarrow \infty} \frac{1}{1+e^{-x}} = 1$$

Answer to Exercise 9 (on page 38)

1. $\lim_{x \rightarrow -1^-} h(x) = 2$ and $\lim_{x \rightarrow -1^+} h(x) = 2$, therefore the limit exists and $\lim_{x \rightarrow -1} h(x) = 2$
 $\lim_{x \rightarrow 0^-} h(x) = 3$ and $\lim_{x \rightarrow 0^+} h(x) = 3$, therefore the limit exists and $\lim_{x \rightarrow 0} h(x) = 3$
 $\lim_{x \rightarrow 1^-} h(x) = 2$ and $\lim_{x \rightarrow 1^+} h(x) = 2$, therefore the limit exists and $\lim_{x \rightarrow 1} h(x) = 2$
2. $\lim_{x \rightarrow -1^-} f(x) = 2$ and $\lim_{x \rightarrow -1^+} f(x) = 2$, therefore the limit exists and $\lim_{x \rightarrow -1} f(x) = 2$.
 $\lim_{x \rightarrow 0^-} f(x) = 3$ and $\lim_{x \rightarrow 0^+} f(x) = 0$, and because $\lim_{x \rightarrow 0^-} f(x) \neq \lim_{x \rightarrow 0^+} f(x)$, the

limit does not exist.

$\lim_{x \rightarrow 2-} f(x) = -2$ and $\lim_{x \rightarrow 2+} f(x) = -2$, therefore the limit exists and $\lim_{x \rightarrow 2} f(x) = -2$.

3. $\lim_{x \rightarrow -2-} g(x) = -1$ and $\lim_{x \rightarrow -2+} g(x) = -1$, therefore the limit exists and $\lim_{x \rightarrow -2} g(x) = -1$.

$\lim_{x \rightarrow 0-} g(x) = 1$ and $\lim_{x \rightarrow 0+} g(x) = 1$, therefore the limit exists and $\lim_{x \rightarrow 0} g(x) = 1$

$\lim_{x \rightarrow 1-} g(x) = 2$ and $\lim_{x \rightarrow 0+} g(x) = 1$, and because $\lim_{x \rightarrow 1-} g(x) = 2 \neq \lim_{x \rightarrow 0+} g(x)$, the limit does not exist.

$\lim_{x \rightarrow 2-} g(x) = 0$ and $\lim_{x \rightarrow 2+} g(x) = 0$, therefore the limit exists and $\lim_{x \rightarrow 2} g(x) = 0$

Answer to Exercise 10 (on page 41)

1. Never true. If a function is continuous at a , then $f(a) = \lim_{x \rightarrow a} f(x)$.
2. Never true. If a function is continuous at a , then $\lim_{x \rightarrow a^+} f(x) = \lim_{x \rightarrow a^-} f(x)$.
3. Always true. This is the definition of continuity.
4. Sometimes true. The derivative of f at $x = 3$ exists for $f(x) = x^2$ but not for $f(x) = |x - 3|$.
5. Sometimes true. This statement is true for $f(x) = -(x - 3)^2$ but not for $f(x) = 4x$.

Answer to Exercise 11 (on page 41)

1. $f(x)$ is not defined at $x = 3$. Therefore, it is also discontinuous at $x = 3$. As we learn about the continuity of polynomials, we will see why $f(x)$ is continuous everywhere else.
2. Here, $f(0)$ is defined, so we need to check if $\lim_{x \rightarrow 0} f(x) = f(0)$. The left and right limits as x approaches 0 are the same (∞), so the limit exists. However, $f(0) = 1 \neq \lim_{x \rightarrow 0} f(x)$. Therefore, the function is discontinuous at $x = 0$.
3. In this function, $f(3)$ is defined, so we need to check if the limit equals the function value. The limit of $f(x)$ as x approaches 3 is:

$$\lim_{x \rightarrow 3} \frac{3x^2 - 8x - 3}{x - 3} = \lim_{x \rightarrow 3} \frac{(3x + 1)(x - 3)}{x - 3} = \lim_{x \rightarrow 3} 3x + 1 = 10$$

So the limit exists, but $\lim_{x \rightarrow 3} f(x) \neq f(3)$, and we see that the function is discontinuous at $x = 3$.

Answer to Exercise 13 (on page 45)

1. True, $h(x)$ approaches 2 from the left and right, therefore the limit exists
2. False, $h(x)$ approaches 5 from the left and right, therefore the limit exists
3. False, $h(x)$ approaches 2 from the left and 4 from the right, therefore the limit does not exist
4. True, $\lim_{x \rightarrow 5} h(x) = h(5)$, therefore $h(x)$ is continuous at $x = 5$
5. True, $\lim_{x \rightarrow 4} h(x)$ does not exist, therefore $h(x)$ is discontinuous at $x = 4$
6. False, $h(2) = 1 \neq 2 = \lim_{x \rightarrow 2} h(x)$

Answer to Exercise 13 (on page 45)

1. From the quotient law, we know that:

$$\lim_{x \rightarrow 3} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow 3} f(x)}{\lim_{x \rightarrow 3} g(x)}$$

From the graph, we see that:

$$\lim_{x \rightarrow 3} f(x) = -3$$

and that:

$$\lim_{x \rightarrow 3} g(x) = 1$$

Substituting these values, we get:

$$\lim_{x \rightarrow 3} \frac{f(x)}{g(x)} = \frac{-3}{1} = -3$$

2. From the Sum Law, we know that:

$$\lim_{x \rightarrow 2} [f(x) + 5g(x)] = \lim_{x \rightarrow 2} f(x) + \lim_{x \rightarrow 2} 5g(x)$$

and applying the Constant Multiple Law, we see that:

$$\lim_{x \rightarrow 2} [f(x) + 5g(x)] = \lim_{x \rightarrow 2} f(x) + 5 \lim_{x \rightarrow 2} g(x)$$

Examining the graph of $f(x)$ and $g(x)$, we can determine that

$$\lim_{x \rightarrow 2} f(x) = -2$$

and

$$\lim_{x \rightarrow 2} g(x) = 0$$

Substituting these values, we get:

$$\lim_{x \rightarrow 2} [f(x) + 5g(x)] = -2 + 5 \cdot 0 = -2$$

3. From the quotient law, we see that:

$$\lim_{x \rightarrow -1} \frac{3g(x)}{f(x)} = \frac{\lim_{x \rightarrow -1} 3g(x)}{\lim_{x \rightarrow -1} f(x)}$$

Applying the Constant Multiple Law, we get:

$$\lim_{x \rightarrow -1} \left[\frac{3g(x)}{f(x)} \right] = \frac{3 \lim_{x \rightarrow -1} g(x)}{\lim_{x \rightarrow -1} f(x)}$$

From the graph, we see that:

$$\lim_{x \rightarrow -1} f(x) = 2$$

and

$$\lim_{x \rightarrow -1} g(x) = 2$$

Substituting, we get:

$$\lim_{x \rightarrow -1} \left[\frac{3g(x)}{f(x)} \right] = \frac{3 \cdot 2}{2} = 3$$

4. Applying the Product and Constant Multiple Laws, we get:

$$\lim_{x \rightarrow 0} [f(x) \cdot 5g(x)] = \lim_{x \rightarrow 0} f(x) \cdot 5 \cdot \lim_{x \rightarrow 0} g(x)$$

Examining the graphs, we see that $\lim_{x \rightarrow 0} f(x)$ does not exist and $\lim_{x \rightarrow 0} g(x) = 1$. Because $\lim_{x \rightarrow 0} f(x)$ does not exist, $\lim_{x \rightarrow 0} f(x) \cdot 5 \cdot \lim_{x \rightarrow 0} g(x)$ also does not exist.

5. Applying the Difference and Constant Multiple Laws, we see that:

$$\lim_{x \rightarrow -1} [f(x) - 3g(x)] = \lim_{x \rightarrow -1} f(x) - 3 \cdot \lim_{x \rightarrow -1} g(x)$$

Examining the graphs, we see that:

$$\lim_{x \rightarrow -1} f(x) = 2$$

and

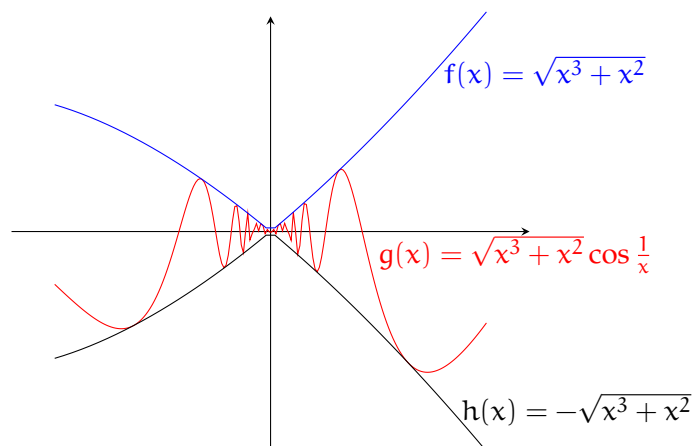
$$\lim_{x \rightarrow -1} g(x) = 2$$

Substituting, we get that:

$$\lim_{x \rightarrow -1} [f(x) - 3g(x)] = 2 - 3 \cdot 2 = 2 - 6 = -4$$

Answer to Exercise 14 (on page 48)

Let $f(x) = -\sqrt{x^3 + x^2}$ and $h(x) = \sqrt{x^3 + x^2}$. Near 0, $f(x) \leq \sqrt{x^3 + x^2} \cos \frac{1}{x} \leq h(x)$. Additionally, $\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} h(x) = 0$. Therefore, by the Squeeze Theorem, we can state that $\lim_{x \rightarrow 0} \sqrt{x^3 + x^2} \cos \frac{1}{x} = 0$. Plotting all three functions, we can confirm our answer:



Answer to Exercise 15 (on page 49)

The question tells us that the function in question, $f(x)$, is between two other functions.

$\lim_{x \rightarrow 2} 2x + 3 = 2(2) + 3 = 7$ and $\lim_{x \rightarrow 2} x^2 - 2x + 7 = 2^2 - 2(2) + 7 = 7$. Since the limits are equal, by Squeeze Theorem we can also know that $\lim_{x \rightarrow 2} f(x) = 7$.

Answer to Exercise 16 (on page 49)

Note that we can only evaluate the limit from the right, as the domain for this function is $x \geq 0$. Since the range of the sine function is $[-1, 1]$, we can state that

$$-1 \leq \sin \frac{\pi}{x} \leq 1$$

and therefore

$$\frac{1}{e} \leq e^{\sin \frac{\pi}{x}} \leq e$$

. Because we assume the positive root, it is also true that

$$\frac{\sqrt{x}}{e} \leq \sqrt{x} e^{\sin \frac{\pi}{x}} \leq \sqrt{x} e$$

. Taking the limits of the border functions, we see that

$$\lim_{x \rightarrow 0^+} \frac{\sqrt{x}}{e} = \lim_{x \rightarrow 0^+} \sqrt{x} e = 0$$

Therefore,

$$\lim_{x \rightarrow 0^+} \sqrt{x} e^{\sin \frac{\pi}{x}} = 0$$

Answer to Exercise 17 (on page 52)

1. define $f(x) = 2x^4 + x - 12$, $a = 1$, $b = 2$, and $N = 0$. Calculate $f(a)$ and $f(b)$:

$$f(1) = -9 \text{ and } f(2) = 22$$

Since $f(x)$ is a polynomial, it is continuous on the interval $x \in [1, 2]$ and we see that $f(a) < 0 < f(b)$. Therefore, there exists some $c \in [1, 2]$ such that $f(c) = 0$.

2. First, we can rearrange the equation we are considering and define $f(x) = \ln x - 3x + 4\sqrt{x}$, and realize we are looking for values where $f(x) = 0$. Both $\ln x$ and $\sqrt{x}$ are only continuous for $x > 0$. The interval we are interested in, $x \in [2, 3]$, is in the domain of continuity for both $\ln x$ and $\sqrt{x}$. Defining $a = 2$, $b = 3$, and $N = 0$, we find that $f(a) = 0.35 > N > -0.973 = f(b)$. Since $N \in [f(b), f(a)]$, there must exist some c such that $f(c) = N = 0$ and there is a solution to the equation $\ln x = 3x - 4\sqrt{x}$ on the interval $x \in (2, 3)$.

3. Similar to above, define $f(x) = 2 \sin x - 3x^2 + 2x$, $a = 1$, $b = 2$, and $N = 0$. Calculate that $f(a) = 0.683$ and $f(b) = -6.181$. Since $f(x)$ is continuous on the interval $x \in [1, 2]$ and $f(b) < N < f(a)$, there exists some $c \in (1, 2)$ such that $f(c) = 0$. Therefore, there is a solution to $\sin x = 3x^2 - 2x$ on the interval $x \in (1, 2)$.



INDEX

- end
 - behavior, [15](#)
- hole, [16](#)
- horizontal
 - asymptote, [15](#)
- oblique
 - asymptote, [18](#)
- partial fraction decomposition, [21](#)
- polynomial
 - graphing, [3](#)
- rational
 - expression, [13](#)
 - function, [13](#)
- vertical
 - asymptote, [15](#)