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Differentiating Polynomials

In the last chapter, every derivative took a page of algebra with limits. Good news: for polynomials, you can skip all of it. If you know a function that gives the height of an object, three rules turn it into a function that gives its velocity. Finding that new function is called *differentiation* or *finding the derivative*. The rules:

- The derivative of a sum is equal to the sum of the derivatives.
- The derivative of a constant is zero.
- The derivative of a nonconstant monomial at^b (a and b are constant numbers, t is time) is abt^{b-1} . This is the **power rule**.

Power Rule

For $f(x) = ax^n$, the derivative is $f'(x) = anx^{n-1}$.

The power rule works for any exponent n , not just positive integers: the derivative of x^{-2} is $-2x^{-3}$, and the derivative of $x^{1/2}$ is $\frac{1}{2}x^{-1/2}$.

Say, for example, that the height in meters of a quadcopter at second t is $2t^3 - 5t^2 + 9t + 200$. Term by term, its vertical velocity is $6t^2 - 10t + 9$. The constant 200 (the height of the hill it took off from) disappears: it does not change, so it contributes nothing to the velocity.

Remember the notation: the derivative of $h(t)$ is $h'(t)$, read “ h prime of t .”

Exercise 1 Differentiation of polynomials

Differentiate the following functions.

Working Space

1. $f(t) = 2t^3 - 3t^2 - 4t$

2. $g(t) = 2t^{-3/4}$

3. $F(r) = \frac{5}{r^3}$

4. $H(u) = (3u - 1)(u + 2)$

Answer on Page 67

Notice that the degree of the derivative is one less than the degree of the original polynomial. (Unless, of course, the degree of the original is already zero.)

Now, if you know that a position is given by a polynomial, you can differentiate it to find the object's velocity at any time (Figure 1.1).

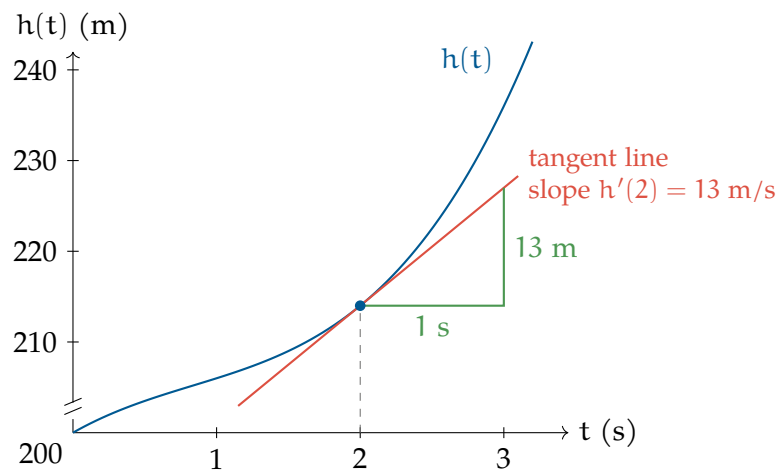


Figure 1.1: The tangent line to $h(t) = 2t^3 - 5t^2 + 9t + 200$ at $t = 2$ has slope $h'(2) = 13$, the vertical velocity of the quadcopter in meters per second at that moment.

The same trick works for acceleration: Let us say you know a function that gives an object's velocity. To find its acceleration at any time, you take the derivative of the velocity function (the second derivative).

Exercise 2 Differentiation of polynomials in Python

Write a function that returns the derivative of a polynomial in `poly.py`. It should look like this:

Working Space

```
1 def derivative_of_polynomial(pn):
2     ...Your code here...
```

When you test it in `test.py`, it should look like this:

```
1 import poly
2
3 # 3x**3 + 2x + 5
4 p1 = [5.0, 2.0, 0.0, 3.0]
5 d1 = poly.derivative_of_polynomial(p1)
6 # d1 should be 9x**2 + 2
7 print("Derivative of",
8     ↪ poly.polynomial_to_string(p1), "is",
9     ↪ poly.polynomial_to_string(d1))
10
11 # Check constant polynomials
12 p2 = [-9.0]
13 d2 = poly.derivative_of_polynomial(p2)
14 # d2 should be 0.0
15 print("Derivative of",
16     ↪ poly.polynomial_to_string(p2), "is",
17     ↪ poly.polynomial_to_string(d2))
```

[Answer on Page 67](#)

1.1 Second order and higher derivatives

As seen from the example, with height, velocity, and acceleration, you can take the derivative of a derivative, which is called the second derivative and is indicated with two marks, like so:

$$\frac{d}{dx}f'(x) = f''(x)$$

When you have the height function (or position function, in the case of horizontal motion) of an object, the first derivative describes the velocity of the object, and the second derivative describes the acceleration. Suppose the motion of a particle is given by $s(t) = t^3 - 5t$, where s is in meters and t is in seconds. What is the acceleration when the velocity is 0? First, we find the velocity function, $s'(t)$, and the acceleration function, $s''(t)$:

$$s'(t) = 3t^2 - 5$$

$$s''(t) = 6t$$

To find where the velocity is 0, set $s'(t) = 0$:

$$3t^2 - 5 = 0$$

$$3t^2 = 5$$

$$t^2 = \frac{5}{3}$$

$$t = \sqrt{\frac{5}{3}} \approx 1.29\text{s}$$

(We ignore the other solution, $t = -\sqrt{\frac{5}{3}}$, because the clock starts at zero.)

Next, we use $t \approx 1.29\text{s}$ in the acceleration function, $s''(t)$:

$$s''\left(\sqrt{\frac{5}{3}}\right) = 6\sqrt{\frac{5}{3}} \approx 7.75 \frac{\text{m}}{\text{s}^2}$$

For higher order derivatives, you just keep taking the derivative! So a third derivative is found by taking the derivative of the second derivative, and so on.

Exercise 3 Using Derivatives to Describe Motion

The position of a particle is described by the equation $s(t) = t^4 - 2t^3 + t^2 - t$, where s is in meters and t is in seconds.

Working Space

(a) Find the velocity and acceleration as functions of t .

(b) Find the velocity after 1.5 s.

(c) Find the acceleration after 1.5 s.

(d) Is the object speeding up or slowing down at $t = 1.5$? How do you know?

Answer on Page 68

Exercise 4 Penalty Kick

A goalkeeper boots a ball straight up. Its height in meters after t seconds is $h(t) = 1 + 15t - 4.9t^2$.

Working Space

1. Find its velocity at $t = 1$ and at $t = 2$.
Which way is it moving each time?
2. Find its acceleration. Why does it not depend on t ?

Answer on Page 68

Rules for Finding Derivatives

You can now differentiate any polynomial in your head. But what about $\sqrt{x^2 + 1}$, or $x \sin x$, or $\frac{e^x}{x}$? Grinding through the limit definition every time would be miserable. Instead, a handful of rules lets you take apart any function built by multiplying, dividing, and nesting simpler ones. Each rule gives the slope of the tangent line (Figure 2.1).

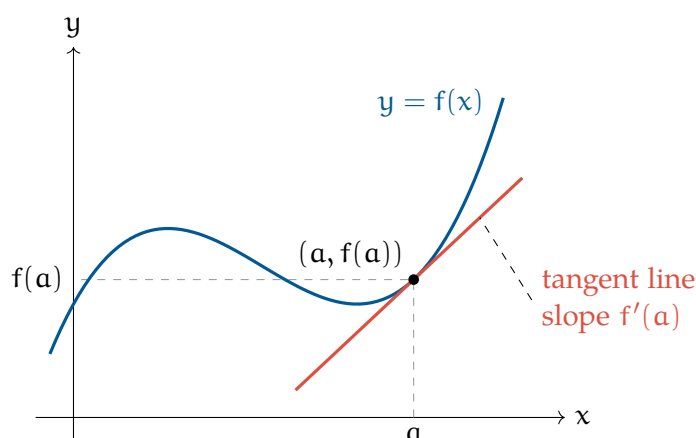


Figure 2.1: The tangent line touches the curve $y = f(x)$ at the point $(a, f(a))$ and has slope $f'(a)$.

2.1 Constant, Sum, and Power Rules

You met these in the last chapter. If c is a constant:

$$\frac{d}{dx}c = 0 \quad (2.1)$$

$$\frac{d}{dx}(cf) = cf' \quad (2.2)$$

$$\frac{d}{dx}(f + g) = f' + g' \quad (2.3)$$

$$\frac{d}{dx}x^n = nx^{n-1} \quad \text{for any real number } n \quad (2.4)$$

2.2 Product Rule

The derivative of the product of two functions is:

$$\frac{d}{dx}(fg) = f'g + fg' \quad (2.5)$$

Notice that it is *not* $f'g'$. To see why, picture a rectangle f wide and g tall, with both sides growing. In a tiny moment, the width grows by a sliver and the height grows by a sliver. The area gains a strip along the top (f times the change in g) and a strip down the side (g times the change in f). The little corner where the two strips overlap is a sliver times a sliver, so small it vanishes in the limit.

Example: Find $\frac{d}{dx} [x^3(2x + 1)]$.

Solution: With $f = x^3$ and $g = 2x + 1$:

$$3x^2(2x + 1) + x^3(2) = 8x^3 + 3x^2$$

Check by expanding first: $x^3(2x + 1) = 2x^4 + x^3$, whose derivative is $8x^3 + 3x^2$. It matches.

2.3 Quotient Rule

The derivative of the quotient of two functions is:

$$\frac{d}{dx} \left(\frac{f}{g} \right) = \frac{f'g - fg'}{g^2} \quad (2.6)$$

Because of the minus sign, order matters. Students remember it with a chant: “low d-high minus high d-low, over the square of what is below.”

Example: Find $\frac{d}{dx} \frac{x}{x^2 + 1}$.

Solution:

$$\frac{(1)(x^2 + 1) - x(2x)}{(x^2 + 1)^2} = \frac{1 - x^2}{(x^2 + 1)^2}$$

2.4 Chain Rule

The derivative of a composition of functions is:

$$\frac{d}{dx}(f(g(x))) = f'(g(x)) \cdot g'(x) \quad (2.7)$$

Think of gears. If gear A turns 3 times for every turn of gear B, and gear B turns 2 times for every turn of gear C, then gear A turns $3 \times 2 = 6$ times for every turn of gear C. Rates of change multiply. If we write $u = g(x)$ and $y = f(u)$, the chain rule looks like fractions canceling:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

In practice: take the derivative of the outside function (leaving the inside alone), then multiply by the derivative of the inside.

Example: Find $\frac{d}{dx}(x^2 + 1)^5$.

Solution: The outside is u^5 ; the inside is $u = x^2 + 1$.

$$5(x^2 + 1)^4 \cdot 2x = 10x(x^2 + 1)^4$$

(You could expand $(x^2 + 1)^5$ first, but that is a lot of work.)

Example: Find $\frac{d}{dx}\sqrt{3x+1}$.

Solution: Write it as $(3x+1)^{1/2}$:

$$\frac{1}{2}(3x+1)^{-1/2} \cdot 3 = \frac{3}{2\sqrt{3x+1}}$$

2.5 Exponentials and Logarithms

Here is what makes e special: e^x is its own derivative. At every point, the slope of $y = e^x$ equals its height.

$$\frac{d}{dx}e^x = e^x \quad (2.8)$$

$$\frac{d}{dx}\ln x = \frac{1}{x} \quad (x > 0) \quad (2.9)$$

With the chain rule, $\frac{d}{dx}e^{kx} = ke^{kx}$. For any other base, rewrite $a^x = e^{x \ln a}$, which gives $\frac{d}{dx}a^x = a^x \ln a$.

Example: A colony of bacteria has a population of $P(t) = 100e^{0.3t}$ after t hours. How fast is it growing at $t = 5$?

Solution: $P'(t) = 30e^{0.3t}$, so $P'(5) = 30e^{1.5} \approx 134$ bacteria per hour. Notice that $P' = 0.3P$: the colony always grows at 30% of its current size per hour. That is what exponential growth means.

Exercise 5 Exponential Derivatives

Find the derivative of each function.

Working Space

1. $f(x) = 5e^{2x}$
2. $g(x) = \ln(x^2 + 1)$
3. $h(x) = 2^x$
4. $k(x) = xe^x$

Answer on Page 68

2.6 Derivatives of Trigonometric Functions

In the trigonometric functions chapter, you saw that the slope of the sine curve at any point equals the cosine there, and that the slope of the cosine curve is $-\sin$. Now you have the tools to prove it and to use it.

Exercise 6 Derivative of cosine

Use the limit definition of a derivative to show that $\frac{d}{dx} \cos x = -\sin x$

Working Space

Answer on Page 68

The derivatives of all the trigonometric functions are presented below:

$\frac{d}{dx} \sin x = \cos x$	$\frac{d}{dx} \csc x = -\csc x \cdot \cot x$
$\frac{d}{dx} \cos x = -\sin x$	$\frac{d}{dx} \sec x = \sec x \cdot \tan x$
$\frac{d}{dx} \tan x = \sec^2 x$	$\frac{d}{dx} \cot x = -\csc^2 x$

Example: Find the derivative of $f(x)$ if $f(x) = x^2 \sin x$ **Solution:** Using the product rule, we find that:

$$\frac{d}{dx} f(x) = (x^2) \frac{d}{dx} (\sin x) + (\sin x) \frac{d}{dx} (x^2)$$

Taking the derivatives:

$$= x^2 (\cos x) + 2x (\sin x)$$

Exercise 7 Derivatives of trig functions

Find the derivative of the following functions:

Working Space

1. $f(x) = \frac{\sec x}{1 + \tan x}$
2. $y = \sec t \tan t$
3. $f(\theta) = \frac{\theta}{4 - \tan \theta}$
4. $f(t) = 2 \sec t - \csc t$
5. $f(\theta) = \frac{\sin \theta}{1 + \cos \theta}$
6. $f(x) = \sin x \cos x$

Answer on Page 69

2.7 Derivatives of Inverse Trigonometric Functions

The inverse trigonometric functions from volume 10 have derivatives that contain no trig functions at all:

f	f'
$\arcsin x$	$\frac{1}{\sqrt{1-x^2}}$
$\arccos x$	$-\frac{1}{\sqrt{1-x^2}}$
$\arctan x$	$\frac{1}{1+x^2}$
$\operatorname{arccsc} x$	$-\frac{1}{x\sqrt{x^2-1}}$
$\operatorname{arcsec} x$	$\frac{1}{x\sqrt{x^2-1}}$
$\operatorname{arccot} x$	$-\frac{1}{1+x^2}$

Worked example. What is the slope of $y = \arcsin x$ at $x = \frac{1}{2}$?

Solution. $\frac{1}{\sqrt{1-\frac{1}{4}}} = \frac{1}{\sqrt{3/4}} = \frac{2}{\sqrt{3}} \approx 1.155$.

Exercise 8

Find f' . Simplify your answer.

Working Space

1. $f(x) = (\arctan x)^2$

2. $f(x) = x \operatorname{arcsec}(x^3)$

3. $f(x) = \arcsin \frac{1}{x}$

Answer on Page 69

Exercise 9 Derivative of Arctangent

Find the derivative of $\arctan x$ at $x = 1$.

Working Space

Answer on Page 70

2.8 Practice

Exercise 10

If f is the function given, find f' .

Working Space

1. $f(x) = x \sin x$

2. $f(x) = (x^3 - \cos x)^5$

3. $f(x) = \sin^3 x$

*Answer on Page 70***Exercise 11**

Let $f(x) = 7x - 3 + \ln x$. Find $f'(x)$ and $f'(1)$

Working Space

*Answer on Page 70***Exercise 12**

[This question was originally presented as a multiple-choice, no-calculator question on the 2012 AP Calculus BC exam.]
The position of a particle in the xy -plane is given by the parametric equations $x(t) = t^3 - 3t^2$ and $y(t) = 12t - 3t^2$. State a coordinate point (x, y) at which the particle is at rest.

Working Space

*Answer on Page 70***Exercise 13**

Let $f(x) = \sqrt{x^2 - 4}$ and $g(x) = 3x - 2$. Find the derivative of $f(g(x))$ at $x = 3$.

Working Space

Answer on Page 71

Exercise 14

The particle's position on the x -axis is given by $x(t) = (t - a)(t - b)$, where a and b are constants and $a \neq b$. At what time(s) is the particle at rest?

Working Space

Answer on Page 71

Exercise 15

[This question was originally presented as a multiple-choice, no-calculator question on the 2012 AP Calculus BC exam.] Let $f(x) = \frac{x}{x+2}$. At what values of x does f have the property that the line tangent to f has a slope of $\frac{1}{2}$?

Working Space

Answer on Page 71

Exercise 16

For $t \geq 0$, the position of a particle moving along the x -axis is given by $x(t) = \sin t - \cos t$. (a) When does the velocity first equal 0? (b) What is the acceleration at the time when the velocity first equals 0?

Working Space

Answer on Page 71

Exercise 17

The graph of $y = e^{\tan x} - 2$ crosses the x -axis at one point on the interval $[0, 1]$. What is the slope of the graph at this point?

*Working Space**Answer on Page 72***Exercise 18**

The function f is defined by $f(x) = \sqrt{25 - x^2}$ for $-5 \leq x \leq 5$.

- (a) Find $f'(x)$.
(b) Write an equation for the line tangent to the graph at $x = -3$.

*Working Space**Answer on Page 72***Exercise 19**

For $0 \leq t \leq 12$, a particle moves along the x -axis. The velocity of the particle at a time t is given by $v(t) = \cos \frac{\pi}{6}t$. What is the acceleration of the particle at time $t = 4$?

*Working Space**Answer on Page 73***Exercise 20**

[This question was originally presented

Working Space

as a multiple-choice, calculator-allowed question on the 2012 AP Calculus BC exam.] Let f and g be the functions given by $f(x) = e^x$ and $g(x) = x^4$. On what intervals is the rate of change of $f(x)$ greater than the rate of change of $g(x)$?

Answer on Page 73

2.9 Conclusion

With the constant, sum, power, product, quotient, and chain rules, plus the derivatives of $\sin$, $\cos$, e^x , and $\ln x$, you can differentiate nearly any function you can write down. Everything else in differential calculus is built on these.

The next chapter, Applications of Derivatives, puts these rules to work.

Applications of Derivatives

What is $\frac{\ln x}{x-1}$ when x is 1? Plugging in gives $\frac{0}{0}$, which tells you nothing. If you drive 120 miles in 2 hours, must your speedometer have read exactly 60 mph at some moment? In the Rules for Finding Derivatives chapter, you learned to differentiate almost any function. This chapter puts those rules to work: to settle limits that look like $\frac{0}{0}$, to prove facts about average and instantaneous rates, and to connect position, velocity, acceleration, and force.

3.1 Applications in Mathematics

3.1.1 l'Hôpital's Rule

Consider the function $h(x) = \frac{\ln x}{x-1}$ and suppose we are interested in the behavior of $h(x)$ around $x = 1$. As x approaches 1, the numerator $\ln x$ approaches 0, and so does the denominator $x - 1$. Plugging in gives the indeterminate result $\frac{0}{0}$, which could turn out to be anything. This is exactly the situation where l'Hôpital's rule applies.

Now, looking at the graph of $h(x)$ (see figure 3.1), we can guess that $\lim_{x \rightarrow 1} \frac{\ln x}{x-1} = 1$.

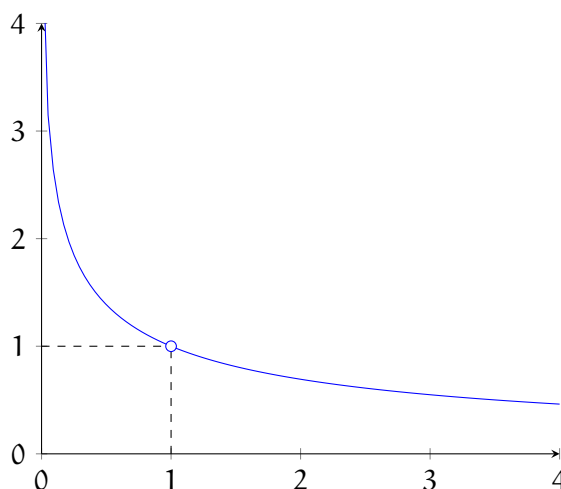
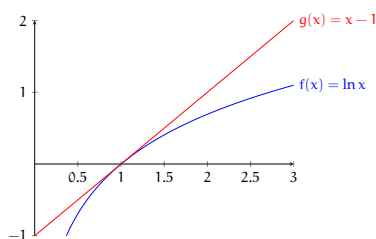
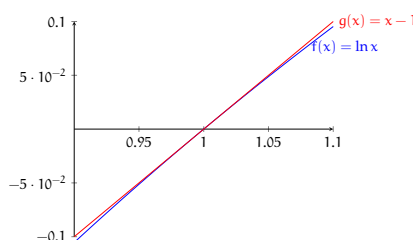


Figure 3.1: $h(x) = \frac{\ln x}{x-1}$

Let us examine the numerator and denominator separately: we will define $f(x) = \ln x$ and $g(x) = x - 1$ (see figure 3.2).

Figure 3.2: Examining each part of $\frac{\ln x}{x-1}$ separately

If we zoom in very far around $x = 1$, the graphs begin to look linear (see figure 3.3):

Figure 3.3: As we zoom in, the graph of $\ln x$ appears linear

We can approximate these graphs as linear functions with slopes m_1 and m_2 , so that the blue curve is approximated as $y = m_1(x - 1)$ and the red curve is approximated as $y = m_2(x - 1)$. The ratio of the functions would then be

$$\frac{m_1(x - 1)}{m_2(x - 1)} = \frac{m_1}{m_2}$$

which is the same as the ratio of the derivatives of our linear approximations. This suggests l'Hôpital's rule, that the limit of a ratio is the same as the limit of the ratio of the derivatives for certain indeterminate forms:

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$

Careful: this is the derivative of the top over the derivative of the bottom, not the quotient rule.

Let us apply l'Hôpital's rule to our limit of $h(x)$:

$$\lim_{x \rightarrow 1} \frac{\ln x}{x - 1} = \lim_{x \rightarrow 1} \frac{\frac{d}{dx} \ln x}{\frac{d}{dx} (x - 1)} = \lim_{x \rightarrow 1} \frac{\frac{1}{x}}{1} = 1$$

Notice our result with l'Hôpital's rule matches our guess based on the graph of $h(x) = \frac{\ln x}{x-1}$.

L'Hôpital's rule also applies to the indeterminate result $\frac{\pm\infty}{\pm\infty}$. For a limit of the form $\lim_{x \rightarrow a} \frac{f(x)}{g(x)}$, l'Hôpital's rule applies if:

1. the original limit is of the indeterminate form $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$
2. f and g are differentiable on an interval containing a (but possibly not differentiable at a)
3. $g'(x) \neq 0$ on said interval

Example: Determine $\lim_{x \rightarrow \infty} \frac{e^x}{x^2}$.

Solution: We begin by evaluating the limit:

$$\lim_{x \rightarrow \infty} \frac{e^x}{x^2} = \frac{e^\infty}{\infty^2} = \frac{\infty}{\infty}$$

This is an indeterminate form that we can apply l'Hôpital's rule to:

$$= \lim_{x \rightarrow \infty} \frac{\frac{d}{dx} e^x}{\frac{d}{dx} x^2} = \lim_{x \rightarrow \infty} \frac{e^x}{2x}$$

Evaluating this limit, we get another indeterminate form:

$$= \frac{e^\infty}{2 \cdot \infty} = \frac{\infty}{\infty}$$

Do not panic! We can apply l'Hôpital's rule again (in fact, we can apply l'Hôpital's rule as many times as needed to evaluate a limit, as long as we keep getting $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$):

$$= \lim_{x \rightarrow \infty} \frac{\frac{d}{dx} e^x}{\frac{d}{dx} 2x} = \lim_{x \rightarrow \infty} \frac{e^x}{2} = \frac{\infty}{2} = \infty$$

and therefore, $\lim_{x \rightarrow \infty} \frac{e^x}{x^2} = \infty$.

Exercise 21

What is $\lim_{x \rightarrow 0} \frac{\tan x - x}{x^3}$?

Working Space

Answer on Page 73

Note that this rule only applies if it is in *indeterminate form*. The most common indeterminate forms are

$$\frac{0}{0} \quad \text{and} \quad \frac{\infty}{\infty},$$

which you may end up with when evaluating quotients!

However, indeterminate forms can also appear in other ways. Some common examples include:

$$0 \cdot \infty, \quad \infty - \infty, \quad 0^0, \quad 1^\infty, \quad \infty^0.$$

These expressions are indeterminate because their values cannot be determined without evaluating further. In many cases, such limits can be rewritten into a quotient that results in either

$$\frac{0}{0} \quad \text{or} \quad \frac{\infty}{\infty}.$$

Then l'Hôpital's rule applies.

Example: Evaluate

$$\lim_{x \rightarrow 0^+} x \ln x.$$

Solution: As $x \rightarrow 0^+$, we have $x \rightarrow 0$ and $\ln x \rightarrow -\infty$, producing the indeterminate form

$$0 \cdot (-\infty).$$

Rewrite the expression as a quotient:

$$x \ln x = \frac{\ln x}{1/x}.$$

Now the limit has the form

$$\frac{-\infty}{\infty},$$

so l'Hôpital's rule applies. Differentiate the numerator and denominator:

$$\lim_{x \rightarrow 0^+} \frac{\ln x}{1/x} = \lim_{x \rightarrow 0^+} \frac{1/x}{-1/x^2}.$$

Simplifying,

$$\lim_{x \rightarrow 0^+} (-x) = 0.$$

Therefore,

$$\lim_{x \rightarrow 0^+} x \ln x = 0.$$

Exercise 22

Evaluate each of the following limits, using l'Hôpital's rule where needed.

Working Space

1. $\lim_{x \rightarrow 3} \frac{x-3}{x^2-9}$

2. $\lim_{x \rightarrow 1/2} \frac{6x^2+5x-4}{4x^2+16x-9}$

3. $\lim_{x \rightarrow 0^+} \frac{\ln x}{\sqrt{x}}$

4. $\lim_{x \rightarrow \pi} \frac{1+\cos x}{1-\cos x}$

5. $\lim_{x \rightarrow 1} \frac{x \sin(x-1)}{2x^2-x-1}$

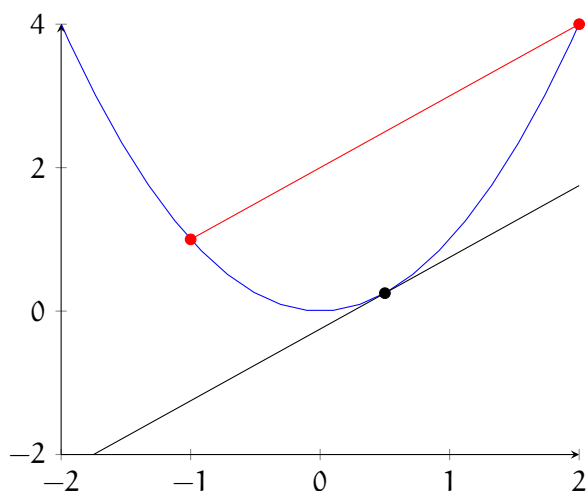
Answer on Page 74

3.1.2 Mean Value Theorem

The Mean Value Theorem (MVT) states that on an interval $[a, b]$ where a continuous function f is differentiable on an open interval (a, b) , there is at least one point where the tangent line to f has the same slope as a line connecting the points $(a, f(a))$ and $(b, f(b))$. Consider the graph of $f(x) = x^2$ (see figure 3.4). The line connecting the points $(-1, 1)$ and $(2, 4)$ has a slope of $\frac{1}{2}$. MVT tells us there must be *at least one* point, c , on the interval $x \in (-1, 2)$ where $f'(c) = \frac{1}{2}$. We can find this point by setting $f'(x)$ equal to $\frac{1}{2}$:

$$2x = \frac{1}{2} \rightarrow x = \frac{1}{4}$$

Examining the figure 3.4, you can see that the tangent at $f(\frac{1}{4})$ (the black line) is parallel to the red line connecting $(-1, f(-1))$ and $(2, f(2))$.

Figure 3.4: $f(x) = x^2$

Note that MVT does not tell us *where* $f'(x)$ is parallel to the line connecting $(a, f(a))$ and $(b, f(b))$, just that some value c exists that satisfies the condition.

Example: A hammer is tossed straight up at $5 \frac{\text{m}}{\text{s}}$ on Earth (where the acceleration due to gravity is approximately $-9.8 \frac{\text{m}}{\text{s}^2}$). Show that at some moment on the way up, its velocity equals its average velocity for the climb.

Solution: The hammer's rise can be described with the function $y(t) = 5t - 4.9t^2$. The hammer reaches its peak at approximately $t = 0.51$. So, we are looking for some value, c , such that

$$y'(c) = \frac{y(0.51) - y(0)}{0.51 - 0} = \frac{5(0.51) - 4.9(0.51^2)}{0.51} = \frac{1.2755}{0.51} = 2.5$$

Solving $y'(t) = 5 - 9.8t = 2.5$, we find that the c that satisfies the MVT is approximately 0.255. This result is illustrated in figure 3.5:

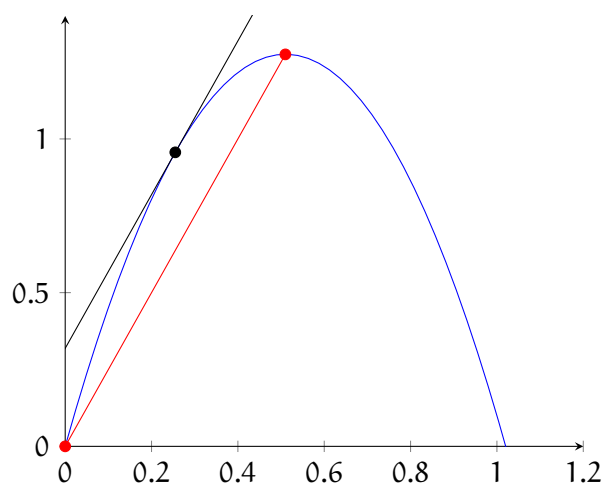


Figure 3.5: The height of a hammer tossed upwards at $5 \frac{\text{m}}{\text{s}}$

3.1.3 MVT Practice

Exercise 23

At 3:30 PM, a car's speedometer reads $30 \frac{\text{mi}}{\text{hr}}$. At 3:40 PM, it reads $50 \frac{\text{mi}}{\text{hr}}$. Show that at some time between 3:30 and 3:40 PM, the car's acceleration is exactly $120 \frac{\text{mi}}{\text{hr}^2}$.

Working Space

Answer on Page 75

Exercise 24

Find the number c that satisfies the MVT on the given interval.

(a) $f(x) = \sqrt{x}$, $[0, 4]$

(b) $f(x) = e^{-x}$, $[0, 2]$

Working Space

(c) $f(x) = \ln x$, $[1, 4]$

Answer on Page 75

3.2 Applications in Physics

In physics, derivatives play a vital role in describing how quantities change with respect to one another.

3.2.1 Velocity and Acceleration

In kinematics, the derivative of the position function with respect to time gives the velocity function, and further taking the derivative of the velocity function gives the acceleration function. For example, if $s(t)$ represents the position of an object at time t , then the velocity $v(t)$ and acceleration $a(t)$ are given by:

$$v(t) = \frac{ds}{dt} \quad \text{and} \quad a(t) = \frac{dv}{dt} = \frac{d^2s}{dt^2} \quad (3.1)$$

3.2.2 Practice

A particle's motion is described by $s(t) = t^3 - 6t^2 + 6t$, where t is measured in seconds and s is measured in meters. Answer the following questions about the particle's motion:

Exercise 25

Find the velocity at time t .

Working Space

Answer on Page 76

Exercise 26

What is the velocity after 2s? After 4s?

Working Space

Answer on Page 77

Exercise 27

When is the particle at rest?

Working Space

Answer on Page 77

3.2.3 Force and Momentum

In mechanics, the derivative of the momentum of an object with respect to time gives the net force acting on the object, as stated by Newton's second law of motion:

$$F = \frac{dp}{dt} \quad (3.2)$$

where F is the force, p is the momentum, and t is the time.

First and Second Derivatives and the Shape of a Function

A roller coaster designer cares about three things: where the track is high, where it is climbing or falling, and where it curves up (pressing riders into their seats) or curves down (lifting them out). The first and second derivatives answer all three questions without drawing a single point.

4.1 Using first derivatives to describe a function

4.1.1 Critical Values

Let us re-examine our graph showing the height of a hammer tossed in the air:

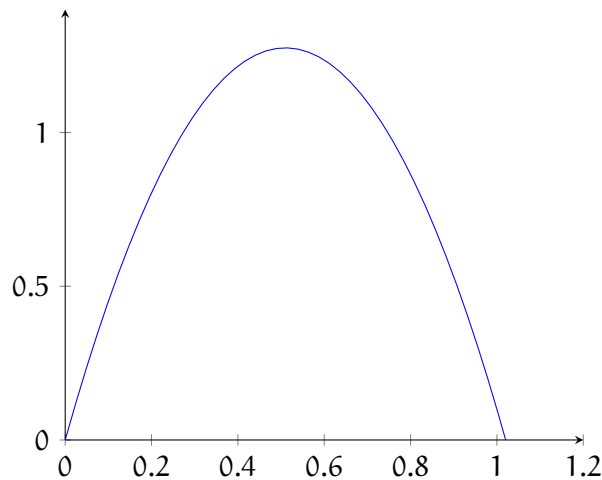


Figure 4.1: Height of a hammer over time

As you can see, the hammer reaches its peak around $t \approx 0.5$ s (see figure 4.1). Let us add tangent lines just before and after the peak of the hammer's path, so we can more easily examine how the slope of the graph changes:

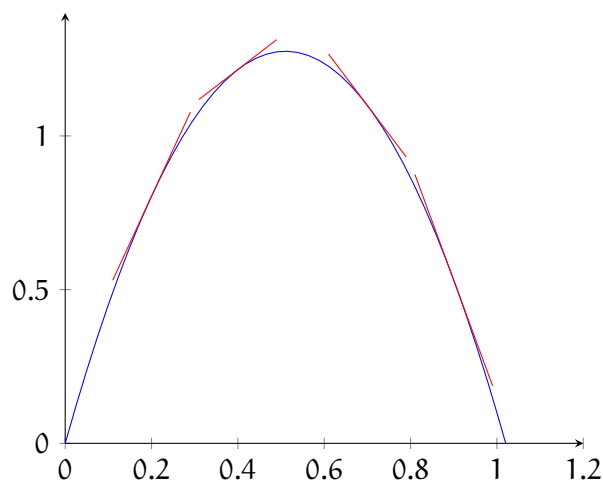


Figure 4.2: height of a hammer over time

In figure 4.2, we see that the slope changes from positive to negative as t increases. That implies that $f'(t)$ also changes from positive to negative. In fact, at the highest point of the hammer's flight, the slope (and therefore $f'(t)$) is exactly zero! In general,

1. If $f'(x) > 0$ (positive slope) on an interval, then $f(x)$ is increasing on that interval.
2. If $f'(x) < 0$ (negative slope) on an interval, then $f(x)$ is decreasing on that interval.

Example 1: Find where the function $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ is increasing.

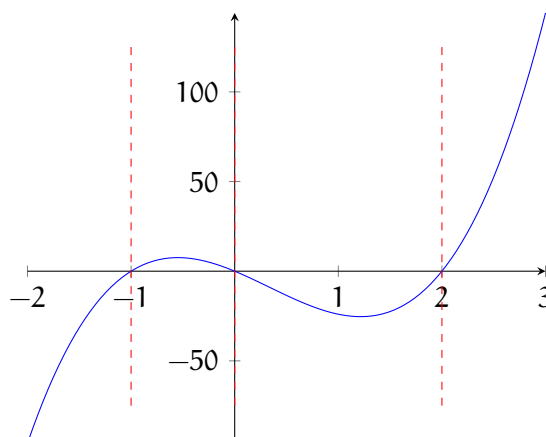
Solution: We want to find the intervals where $f'(x) > 0$. First, we take the derivative to find $f'(x)$:

$$f'(x) = 12x^3 - 12x^2 - 24x$$

It will be easier to analyze the value of $f'(x)$ if we factor it so:

$$f'(x) = 12x(x - 2)(x + 1)$$

To determine where $f'(x) > 0$, we start by finding where $f'(x) = 0$ (in this case, this is true when $x = -1, 0, 2$). These values of x are called *critical values*, and we will use them to divide $f'(x)$ into intervals. (Critical values are also called critical numbers, and we will use both in this text.) On each of these intervals, $f'(x)$ must be always positive or always negative. This is shown in the graph below:

Figure 4.3: $f'(x)$ with critical values

As you can see in figure 4.3, $f'(x) > 0$ on two intervals: $x \in (-1, 0)$ and $x \in (2, \infty)$. These are open intervals because $f'(x) = 0$ at $x = -1$, $x = 0$, and $x = 2$. But what if we had a more complex function, or did not have the resources to graph it?

We can use a table to help us analyze the value of $f'(x)$ (and therefore the behavior of $f(x)$). For each interval around the critical values, we can determine if $f'(x)$ is positive or negative by noting the value of the factors of $f'(x)$, which are $12x$, $x - 2$, and $x + 1$ in this case. For example, for $x < -1$, $12x < 0$, $(x - 2) < 0$, and $(x + 1) < 0$. Three negatives multiplied together is also negative. Therefore, for $x < -1$, $f'(x)$ is negative and $f(x)$ is decreasing. We can analyze all of the intervals similarly and log the results in a table:

x	$12x$	$x - 2$	$x + 1$	$f'(x)$	$f(x)$
$x < -1$	negative	negative	negative	negative	decreasing
$-1 < x < 0$	negative	negative	positive	positive	increasing
$0 < x < 2$	positive	negative	positive	negative	decreasing
$2 < x$	positive	positive	positive	positive	increasing

Notice the table method yields the same result as examining the graph: $f(x)$ is increasing for $x \in (-1, 0)$ and $x \in (2, \infty)$, which can also be written as $x \in (-1, 0) \cup (2, \infty)$.

Exercise 28

Let g be the function given by $g(x) = x^2 e^{kx}$, where k is a constant. For what value(s) of k does g have a critical value at $x = \frac{2}{3}$?

Working Space

Answer on Page 77

4.1.2 Local Extrema

Examine the graphs of x^2 , $\sin x$, and $y = \sqrt{4 - x^2}$ below. Each has a dot at a local extreme (either a local minimum or local maximum). Sketch what you think the tangent line to the graph would be at each local extreme. Use this to estimate the value of the derivative at that point.

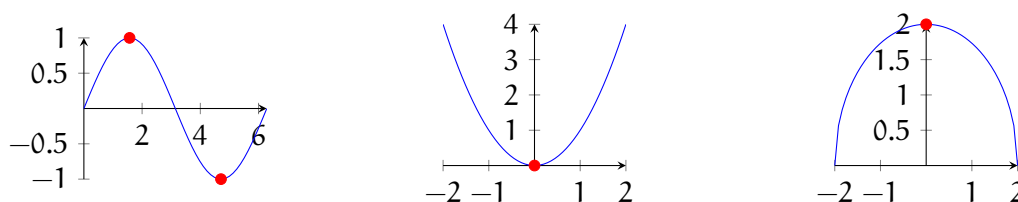
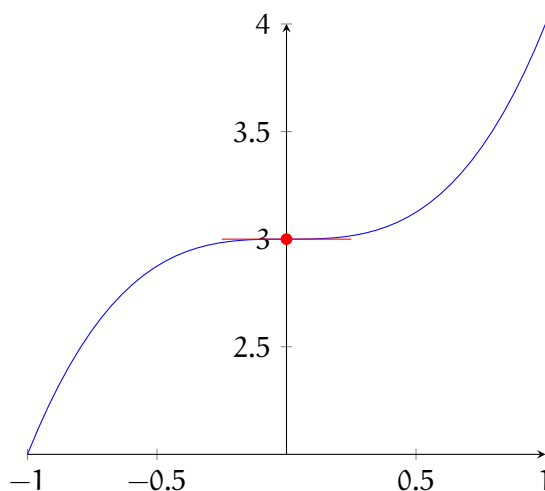
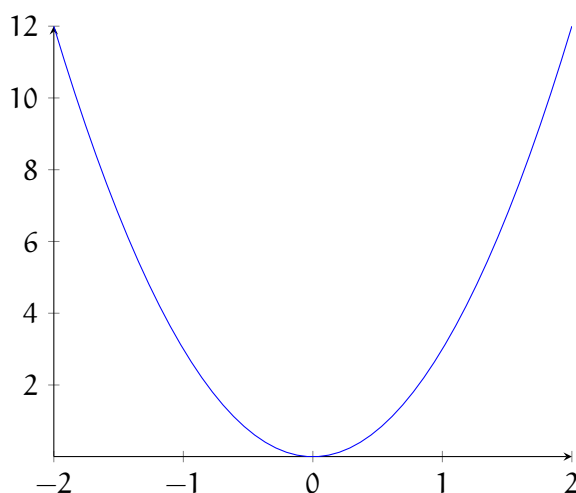


Figure 4.4: Three functions with highlighted points.

You should notice that all of the tangent lines are horizontal. Since the tangent lines at these local extrema have a slope of 0, that tells us $f'(x) = 0$ at these points as well. In fact, at *every* local minimum or maximum of a smooth function, the derivative is zero. (At a sharp corner, like the bottom of $|x|$, the derivative does not exist instead.) However, the converse statement is not necessarily true; just because the derivative is zero at some $x = c$, it does not mean there is a local extremum at $x = c$. Consider $f(x) = x^3 + 3$, shown in figure 4.5:

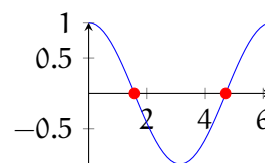
Figure 4.5: $f(x) = x^3 + 3$

At $x = 0$, $f'(x) = 0$, but there is not a local extreme. For a local extreme to exist, the graph of $f(x)$ must change from increasing to decreasing, or vice versa. Look closely at figure 4.5: the function is increasing for $x < 0$ and $x > 0$. Another way of saying this is to note that the graph of $f'(x)$ touches but does not cross the x -axis in this case:

Figure 4.6: $f'(x) = 3x^2$

If $f(x)$ changes from increasing to decreasing, then $f'(x)$ is changing from positive to negative (i.e. crossing the x -axis). Look at the derivative of $f(x) = \sin x$, $f'(x) = \cos x$, presented in figure 4.7. The x -values where local extrema exist on $f(x)$ are marked in red (recall $\sin x = \pm 1$ when $x = \frac{\pi}{2} + n\pi$):

As you can see, local extrema are indicated when $f'(x)$ crosses the x -axis. If $f'(x)$ is negative to the left of $x = c$ and positive to the right, then $f(x)$ has a local minimum at $x = c$. On the other hand, if $f'(x)$ is positive to the left of $x = c$ and negative to the right, then $f(x)$ has a local maximum at $x = c$. Any value of $x = c$ where $f'(c) = 0$ is called a **critical number** or a **critical value**. Values in the domain where $f'(c)$ does not exist are also critical numbers.

Figure 4.7: $f'(x) = \cos x$

4.1.3 Practice: Interval of Increasing and Decreasing, Local Extrema

Exercise 29

Let f be the function given by $f(x) = 300x - x^3$. On what interval is f increasing?

Working Space

Answer on Page 78

Exercise 30

Find the intervals on which $f(x) = x^3 - 3x^2 - 9x + 4$ is increasing or decreasing. Then, find all local minimum and/or maximum values of $f(x)$.

Working Space

Answer on Page 78

4.1.4 Global Extrema

Now that we have learned how to identify local minima and maxima, let us expand the discussion to include global extrema. A global extreme is an absolute minimum or maximum value of a function over a particular interval or the entire domain of the function. Let us examine the graph of $f(x) = x^4 - 5x^3 + 6x^2$ over the domain $x \in [-1, 4]$.

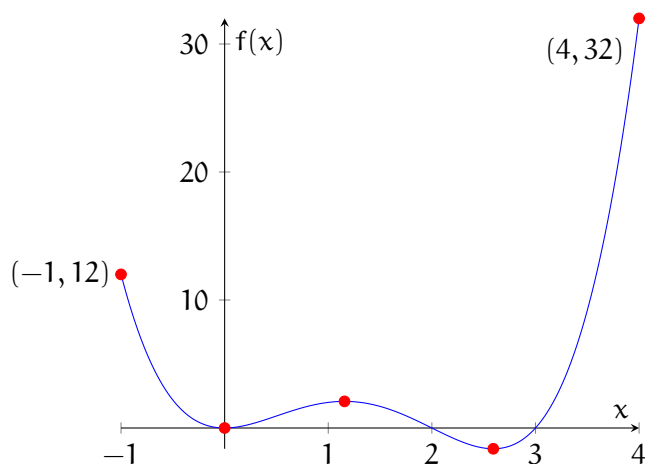


Figure 4.8: Graph of $f(x) = x^4 - 5x^3 + 6x^2$

As you can see in figure 4.8, $f(x)$ has two local minima and one local maximum. Additionally, the endpoints are labeled. To determine the *global* extrema, we need to examine the any local extrema (identified here graphically, but you can also identify them mathematically using that you learned in the “Local Extrema” subsection) **and** the endpoints of the domain (or the function’s behavior at $\pm\infty$, if you are not restricted to a specific domain).

In the case of $f(x) = x^4 - 5x^3 + 6x^2$, for $x \in [-1, 4]$, the global maximum value is 32 at $x = 4$ and the global minimum is -1.623 at $x = 2.593$.

If a function is continuous on an interval, then there must exist a global maximum and global minimum on that interval. These global extrema may also be local extrema (as is the case for $f(2.593)$ in the example above) or not (as is the case for $f(4)$). Applying the Closed Interval Method is a straightforward way to identify global (absolute) extrema.

To find the global extrema of a continuous function, f , on a closed interval $[a, b]$:

1. Find the values of f at the critical numbers of f in (a, b) .
2. Find the values of f at the endpoints of the interval.
3. The largest of the values from steps 1 and 2 is the absolute maximum; the smallest of the values is the absolute minimum.

Let us use the Closed Interval Method to determine the global extrema for the function $g(x) = x - 3 \sin x$ on the interval $x \in [0, 2\pi]$.

To find the value of g at any critical numbers, we must first identify the critical numbers. Recall that critical numbers are values where the first derivative of the function is 0 or does not exist. To find critical numbers, we set g' equal to 0:

$$g'(x) = 1 - 3 \cos x = 0$$

$$3 \cos x = 1$$

$$\cos x = \frac{1}{3}$$

$$x = 1.23, 5.052$$

Now, we substitute these critical numbers back into $g(x)$:

$$g(1.23) \approx -1.60$$

$$g(5.052) = 7.881$$

Now we need to check the endpoints:

$$g(0) = 0 - 3 * 0 = 0$$

$$g(2\pi) = 2\pi - 3 * 0 = 2\pi \approx 6.28$$

The results are presented in the table below:

x	$g(x)$
0	0
1.23	-1.60
5.052	7.881
6.28	6.28

Therefore, for $g(x) = x - 3 \sin x$ on the interval $x \in [0, 2\pi]$, the global maximum is $g(5.052) = 7.881$ and the global minimum is $g(1.23) = -1.60$.

4.1.5 Practice: Global Extrema

Exercise 31

Let f be the function defined by $f(x) = \frac{\ln x}{x}$. What is the absolute maximum value of f ?

Working Space

Answer on Page 79

Exercise 32

Find the global minimum and maximum values on the stated interval.

Working Space

1. $f(x) = 12 + 4x - x^2$, $[0, 5]$
2. $f(t) = \frac{\sqrt{t}}{1+t^2}$, $[0, 2]$
3. $f(t) = 2 \cos t + \sin 2t$, $[0, \frac{\pi}{2}]$
4. $f(x) = \ln x^2 + x + 1$, $[-1, 1]$

Answer on Page 79

4.2 Sketching f from f'

Now that we know how the shape of f is related to the value of f' , we can predict the shape of f if we are given f' . Take the example $f'(x) = -(x-1)(x-5)$, shown in figure 4.9:

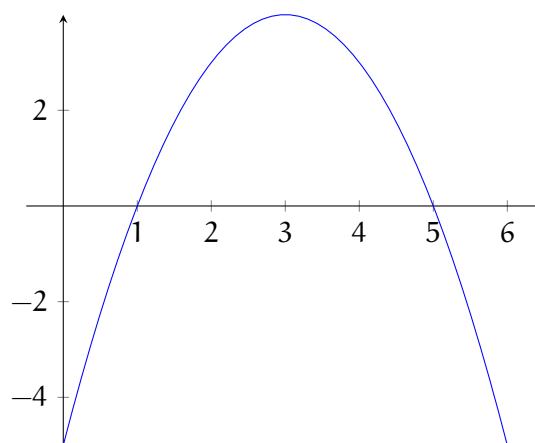


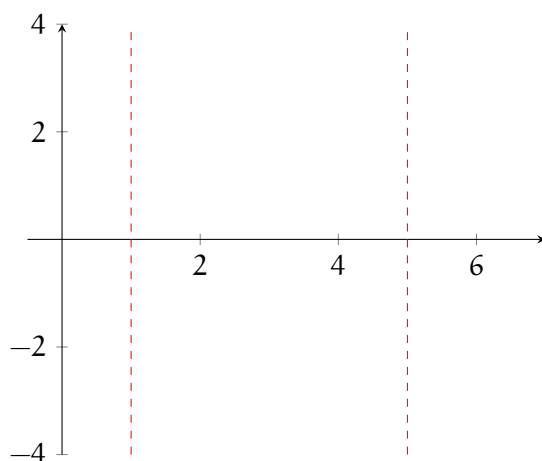
Figure 4.9: Graph of $f' = -(x-1)(x-5)$

Using the graph of f' , we can construct an approximate sketch of f . First, let us identify the critical numbers. Where does $f' = 0$? Take a second to examine the graph of f' above and jot down what you think the critical numbers are.

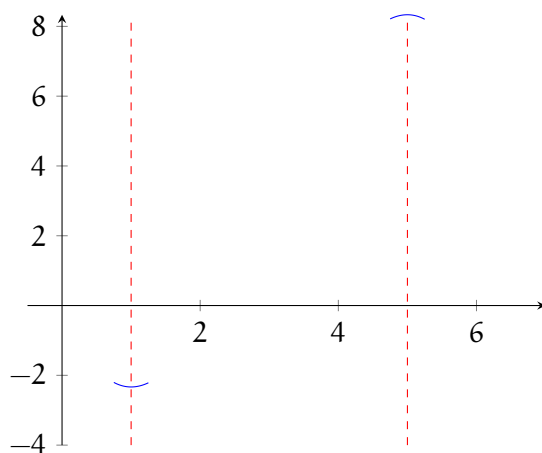
You should recall that critical numbers are x -values where $f' = 0$. Examining the graph of f' , we see that $f' = 0$ at $x = 1$ and $x = 5$. We can now use a table to describe the behavior of f (remember the minus sign in front of f'):

x	$x-1$	$x-5$	f'	behavior of f
$x < 1$	negative	negative	negative	decreasing
$x = 1$	zero	negative	zero	local minimum
$1 < x < 5$	positive	negative	positive	increasing
$x = 5$	positive	zero	zero	local maximum
$x > 5$	positive	positive	negative	decreasing

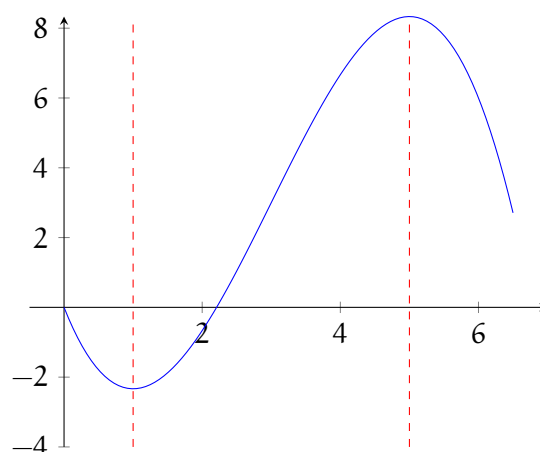
We can use this information to sketch a possible graph of f . We start by noting the location of local extrema:

Figure 4.10: Possible graph of f

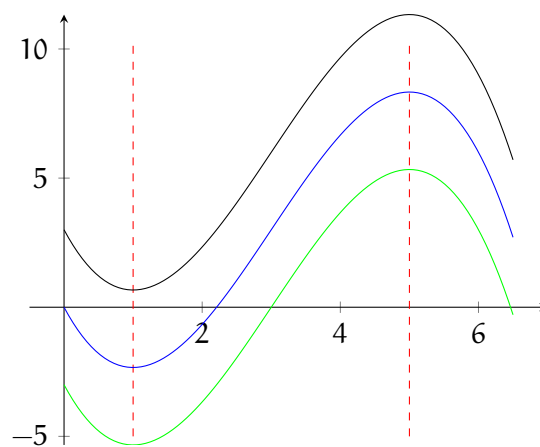
We know there is a local minimum at $x = 1$ and a local maximum at $x = 5$. We can add sketches around these values to indicate what we know about f :

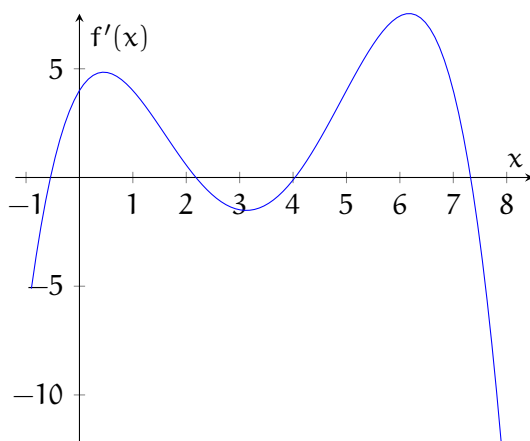
Figure 4.11: Possible graph of f

Lastly, we know f is increasing on $1 < x < 5$ and decreasing everywhere else, so we fill in the space between our local extrema:

Figure 4.12: Possible graph of f

However, figure 4.12 is only a *possible* graph of f . Analyzing f' reveals the shape of f , but not how high or low it is on the y -axis. Recall that the derivative of a constant is zero. Therefore, any $+c$ (where c is a constant) is lost when taking the derivative. So, there are many sketches of f that fulfill the behavior of f indicated by f' . You can see several of the possible sketches for f in figure 4.13.

Figure 4.13: Possible graphs of f

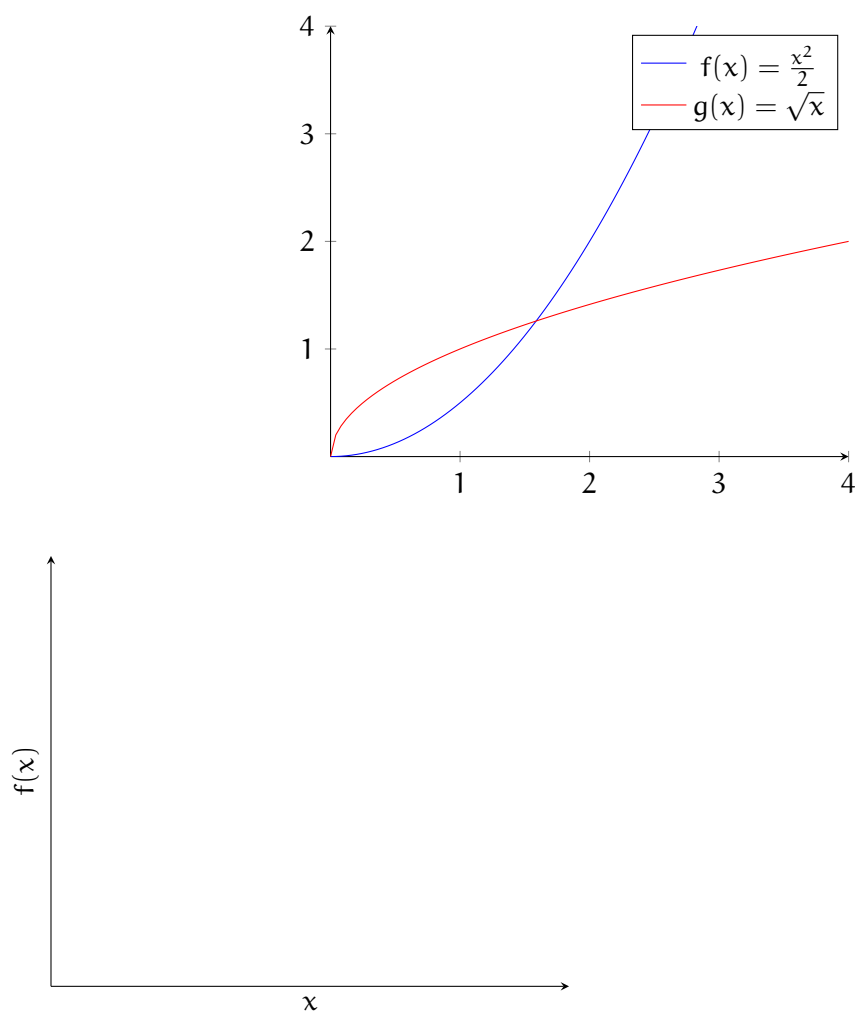
4.2.1 Practice Sketching f from f' Figure 4.14: Graph of $f'(x)$ **Exercise 33**

Use figure 4.14 to answer the following questions:

Working Space

1. On what approximate intervals is f increasing or decreasing?
2. At what approximate values of x does f have a local maximum or minimum?
3. Sketch a possible graph of f in the space below:

Answer on Page 81



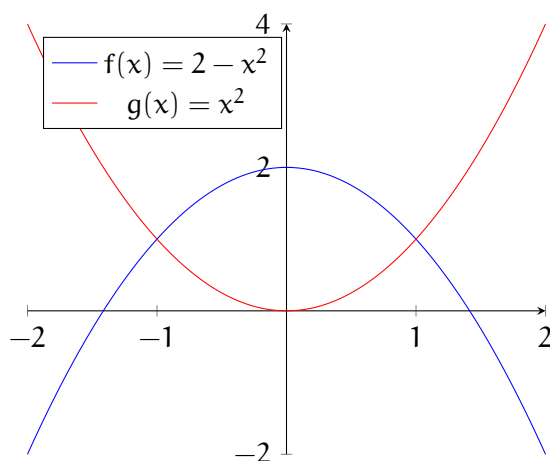
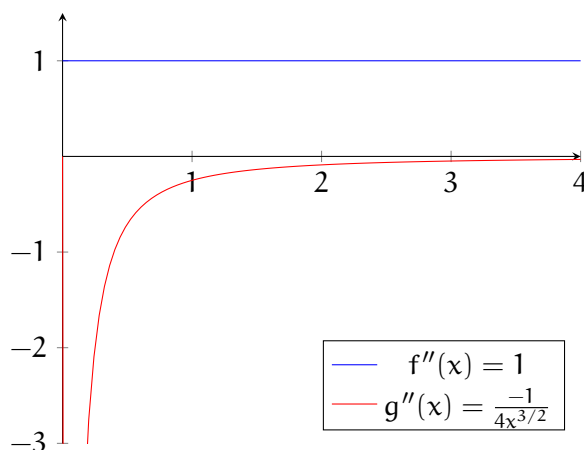
4.3 Using second derivatives to describe a function

4.3.1 Concavity

Let us examine two increasing functions, $f(x) = \frac{x^2}{2}$ and $g(x) = \sqrt{x}$:

Even though both of these functions are increasing, they have different shapes. $f(x)$ looks like a bowl. On the other hand, $g(x)$ looks like an upside-down bowl. These shapes are called *concave up* (in the case of $f(x)$) and *concave down* (in the case of $g(x)$). Both functions are increasing on the interval $x \in [0, 4]$, and therefore both $f'(x)$ and $g'(x)$ are positive on the stated interval. Let us look at their second derivatives, $f''(x)$ and $g''(x)$:

As you can see, $f''(x) > 0$ and $g''(x) < 0$. The second derivative tells us if a function is concave up or concave down. In general:



1. If $f''(x) > 0$ for all x in a given interval, then the graph of f is concave up on the interval.
2. If $f''(x) < 0$ for all x in a given interval, then the graph of f is concave down on the interval.

Additionally, the second derivative can help us determine if there is a local minimum or maximum at critical numbers. Look at the graphs of $f(x) = 2 - x^2$ and $g(x) = x^2$, which both have first derivatives equal to 0 at $x = 0$:

When the graph is concave up, there is a local minimum where the first derivative equals 0. When the graph is concave down, there is a local maximum where the first derivative equals 0. This is summarized with the Second Derivative Test:

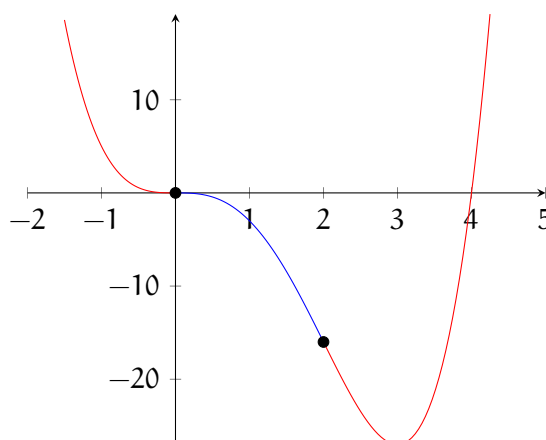
Suppose f'' is continuous near c . Then,

1. If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c .
2. If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c .

4.3.2 Inflection Points

If f is concave up when $f'' > 0$ and concave down when $f'' < 0$, what about when $f'' = 0$? This is the value at which f changes from concave up to concave down (or vice versa), which is called an *inflection point*. Similar to local extrema with f' , if there is an inflection point at $x = c$, then $f''(c) = 0$, but the converse is not necessarily true. To check if $x = c$ is an inflection point, then f'' should change signs on either side of $x = c$ (either from positive to negative or from negative to positive).

Look at the graph of $f(x) = x^4 - 4x^3$. The concave up areas are shown in red, and the concave down in blue:



Let us examine f'' to confirm the inflection points are at $(0, 0)$ and $(2, -16)$. First, we note that $f''(x) = 12x^2 - 24x$. Factoring, we see that $f''(x) = 12x(x - 2)$, which has zeroes at $x = 0$ and $x = 2$. For $x < 0$, $f'' > 0$, and for $0 < x < 2$, $f'' < 0$; therefore, there is an inflection point in f at $(0, 0)$.

Exercise 34

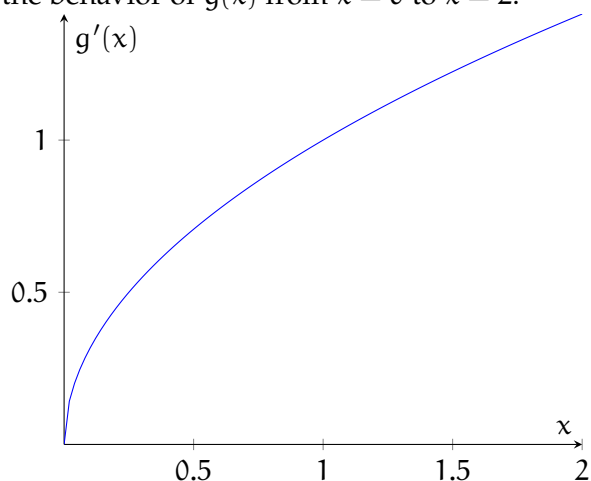
Prove that the other inflection point for $f(x) = x^4 - 4x^3$ is $(2, -16)$.

Working Space

Answer on Page 82

Exercise 35

The graph below shows $g'(x)$. Describe the behavior of $g(x)$ from $x = 0$ to $x = 2$.

*Working Space**Answer on Page 82***Exercise 36**

[This question was originally presented as a calculator-allowed, multiple-choice problem on the 2012 AP Calculus BC exam.] For $-1.5 < x < 1.5$, let f be a function with first derivative given by $f'(x) = e^{(x^4 - 2x^2 + 1)} - 2$. State the interval(s) (to three decimal places) for which f is concave down.

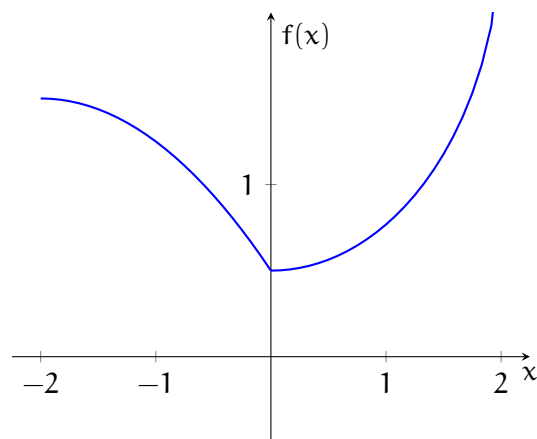
*Working Space**Answer on Page 82***Exercise 37**

[The following problem was origi-

Working Space

nally presented as a calculator-allowed, multiple-choice question on the 2012 AP Calculus BC exam.] Consider the function, f , whose graph is shown below. Classify each of the following statements as true or false and explain.

1. $f' > 0$ for $x \in (-2, 0)$.
2. f is differentiable at $x = 0$.
3. $f'' > 0$ for $x \in (0, 2)$
4. f has a critical value at $x = 0$



Answer on Page 83

Exercise 38

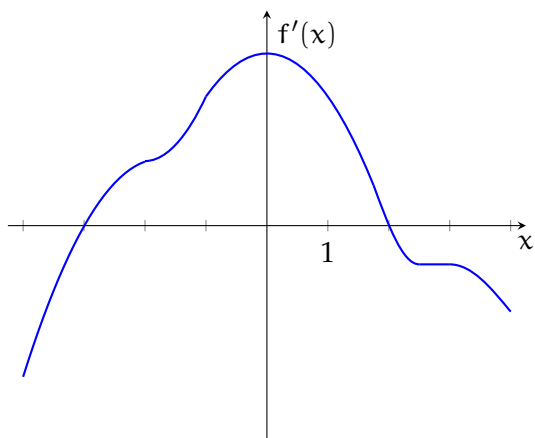
[The following problem was originally presented as a calculator-allowed, multiple-choice question on the 2012 AP Calculus BC exam.] The graph of f' , the derivative of f , is shown below. Classify each of the following statements as true or false and explain your answer.

1. f has a relative minimum at $x = -3$.
2. The graph of f has a point of inflec-

Working Space

tion at $x = -2$.

3. The graph of f is concave down for $0 < x < 2$.



Answer on Page 83

Exercise 39

[The following problem was originally presented as a calculator-allowed, multiple-choice question on the 2012 AP Calculus BC exam.] Let f be a function that is twice differentiable on $-2 < x < 2$ and satisfies the conditions in the table below. If $f(x) = f(-x)$, what are the x -coordinates of the points of inflection of the graph of f on $-2 < x < 2$?

	$0 < x < 1$	$1 < x < 2$
$f(x)$	Positive	Negative
$f'(x)$	Negative	Negative
$f''(x)$	Negative	Positive

Working Space

Answer on Page 84

Vector-valued Functions

In the drag chapter, you simulated the flight of an artillery shell. For any time t , you could find a vector [distance, height]. And in the Parametric Functions chapter, a parametric curve gave a point $(x(t), y(t))$ for every t . Both are functions that take a number and return a vector. We call this a *vector-valued* function, from $\mathbb{R} \rightarrow \mathbb{R}^2$ (or $\mathbb{R} \rightarrow \mathbb{R}^3$ in three dimensions)¹.

5.1 Vector-valued functions: position

We often make a vector-valued function by defining several real-valued functions. For example, if you threw a hammer with an initial upward speed of 12 m/s and a horizontal speed of 4 m/s along the x axis from the point $(1, 6, 2)$, its position at time t (during its flight) would be given by:

$$f(t) = [4t + 1, 6, -4.9t^2 + 12t + 2]$$

In other words, x is increasing with t , y is constant, and z is a parabola.

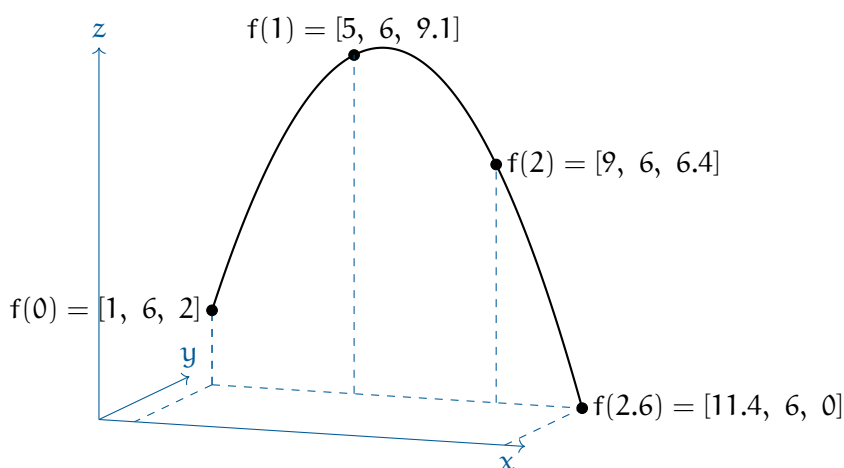


Figure 5.1: An example of a vector-valued function.

¹the $\mathbb{R}$ symbol represents the set of all real numbers; the $\mathbb{R}^2$ symbol represents the set of all 2-dimensional vectors, and $\mathbb{R}^3$ represents the set of all 3-dimensional vectors

5.2 Finding the velocity vector

Velocity is the rate of change of position, so each component of the velocity is the slope (the derivative) of the matching component of position. Using the power rule from the Differentiating Polynomials chapter (each ct^n becomes nct^{n-1}), we differentiate each component separately:

$$f'(t) = [4, 0, -9.8t + 12]$$

In other words, the velocity is constant along the x -axis, zero along the y -axis, and decreasing with time along the z axis.

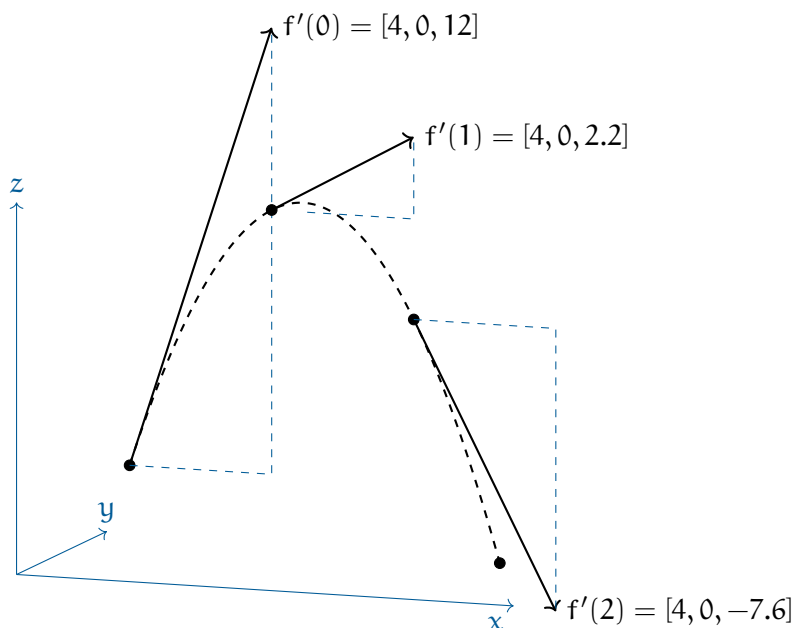


Figure 5.2: The derivatives of the position function (velocity) with respect to time.

5.3 Finding the acceleration vector

Now that we have its velocity, we can get its acceleration as a vector-valued function:

$$f''(t) = [0, 0, -9.8]$$

There is no acceleration along the x or y axes. It is accelerating down at a constant 9.8m/s^2 .

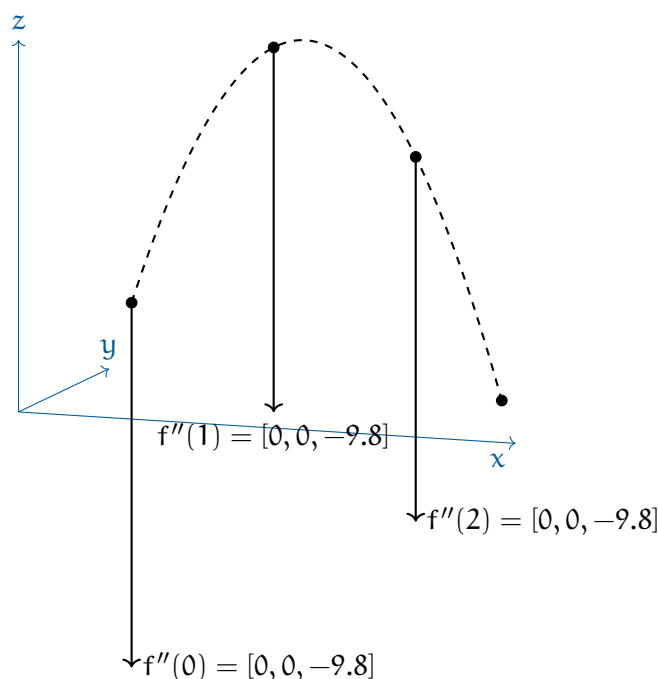


Figure 5.3: The acceleration vector is constant and points downward.

Worked example. A point moves along $\vec{r}(t) = \langle t, t^2 \rangle$. Find its velocity and acceleration.

Solution. Differentiate each component: $\vec{v}(t) = \langle 1, 2t \rangle$ and $\vec{a}(t) = \langle 0, 2 \rangle$. The point moves right at a steady speed while speeding up upward, tracing the parabola $y = x^2$.

Exercise 40 Circular Path

(Recall from the Rules for Finding Derivatives chapter that the derivative of $\sin t$ is $\cos t$ and the derivative of $\cos t$ is $-\sin t$.) Let $\vec{r}(t) = \langle \cos t, \sin t \rangle$. Find the speed $|\vec{v}(t)|$.

Working Space

Answer on Page 84

Experimental Design

Does listening to music while you study raise your test scores? You could ask your classmates, and you might find that the music listeners do better. But maybe the music listeners also study longer, or sleep more. Surveys tell you what people think or report. Well-designed experiments can show what causes what. The distinction determines what conclusions you are entitled to draw when the data comes in.

6.1 Experiments vs. Observational Studies

The defining question is whether the researcher assigns the conditions being studied or simply observes them as they exist.

In an *observational study*, the researcher collects data on *subjects*, the people or things being studied, without intervening. The variables of interest already exist in the subjects; the researcher records what is there. A sociologist who tracks the grades and sleep hours of university students over a semester, without changing anyone's sleep schedule, is conducting an observational study.

In an *experiment*, the researcher imposes conditions on subjects and then measures the effect or change. The researcher creates the differences rather than observing them.

Why does the distinction matter? Only a properly designed experiment can support a cause-and-effect conclusion. An observational study can show that two variables are associated, but a third variable you did not account for could explain the relationship entirely. This uninvited third variable is called a *confounding variable*: one whose effect on the response cannot be separated from the effect of the variable you are actually studying.

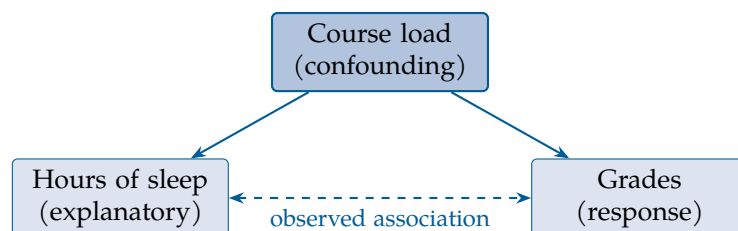


Figure 6.1: A confounding variable affects both the explanatory and response variables, so the observed association cannot be attributed to sleep alone.

In the observational sleep study, students who sleep more might also exercise more, eat better, or have lower course loads. Any of these could be responsible for better grades. The

data give no way to tell whether sleep or one of these other factors explains the difference.

There are situations where an experiment is unethical or impossible: you cannot randomly assign people to smoke for twenty years, or randomly select cities to experience floods. In these cases, observational data is all we have, and conclusions must be drawn with caution.

Exercise 41 Experiment or Observational Study?

For each study, identify whether it is an experiment or an observational study. Then state whether a cause-and-effect conclusion is justified and why.

Working Space

Study	Type	Causal conclusion?
Researchers randomly assign 180 seedlings to three groups and water each group with a different fertilizer solution. After eight weeks they measure plant height.		
A researcher records the screen time and attention span of 500 children aged 8 to 12, then looks for a relationship between the two.		
A company randomly selects half its customer service team to complete a new communication training program, then compares customer satisfaction scores between the two groups.		
A historian finds that cities with more libraries per capita had higher literacy rates in the early 20th century.		

Answer on Page 84

6.2 Vocabulary for Experiments

Experiments have their own vocabulary. These terms build on each other, so it helps to learn them in order.

Explanatory variable Also called the independent variable. The variable whose effect

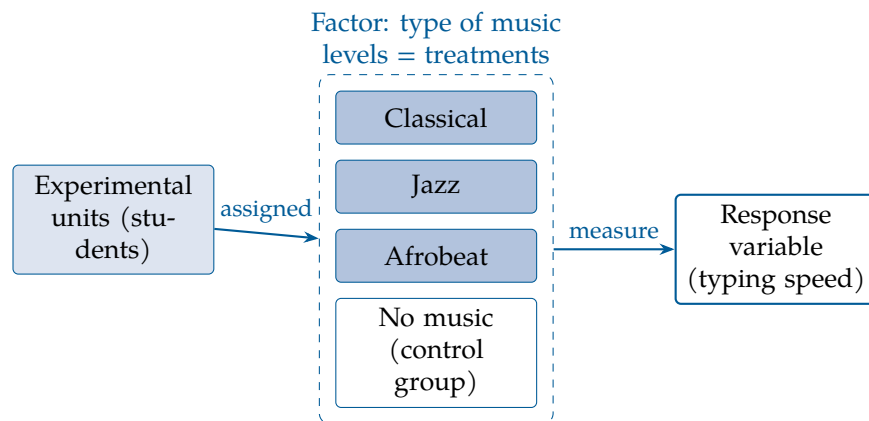


Figure 6.2: The parts of a one-factor experiment on background music and typing speed.

on the response is being studied. In a study of whether background music affects typing speed, the type of music is the explanatory variable.

Response variable Also called the dependent variable. The outcome being measured. In the same study, typing speed is the response variable.

Experimental unit The smallest unit to which a treatment is applied. In a study of plants, each plant is an experimental unit. In a study of people, each person is an experimental unit.

Factor A variable whose effect on the response is of interest. Factors can be qualitative (type of music: classical, jazz, Afrobeat) or quantitative (temperature in degrees Celsius).

Levels The specific values of a factor used in the experiment. If the factor is type of music with three options, there are three levels.

Treatment A specific combination of factor levels applied to experimental units. With one factor and three levels, there are three treatments. With two factors at two levels each, there are $2 \times 2 = 4$ treatments.

Control group A group that receives no treatment or a standard baseline condition. It provides a reference point for comparison with the treatment groups.

Placebo A dummy treatment that resembles the real treatment but contains no active component. Used to isolate the genuine effect of the treatment from the *placebo effect*: the tendency for people to show change simply because they believe they are receiving treatment.

6.3 The Three Principles of Experimental Design

A well-designed experiment rests on three principles.

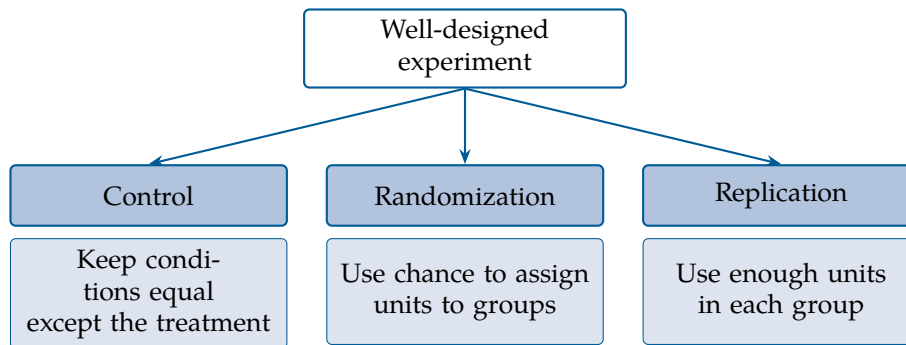


Figure 6.3: The three principles of experimental design.

Control means keeping everything the same for all groups except the treatment. It is not just having a control group. Control helps make sure the treatment is the only real difference.

Randomization means using chance to place experimental units into groups. It helps keep the groups fair and spreads unknown factors across all groups.

Replication means including enough experimental units in each group so that chance variation among individuals does not hide the treatment effect. If a new training method is tested on only two employees per group, a single unusually high or low performer could dominate the group's result. With 30 employees per group, individual extremes average out.

Randomization also protects against a *lurking variable*: a variable that affects the response but is not measured or included in the study. A lurking variable becomes a confounding variable when it is also linked to the explanatory variable, so that its effect cannot be separated from the effect of the treatment. Random assignment spreads lurking variables across all groups by chance, which keeps them from lining up with one treatment.

Exercise 42 Applying the Three Principles

A horticulture researcher wants to test whether exposing seedlings to classical music for two hours per day increases their growth rate compared to no music exposure. They have 60 seedlings of the same species, grown from the same seed batch, available for the study.

Working Space

1. Describe how randomization should be applied.
2. What does control mean in this con-

text? List at least three variables that should be held constant.

3. The researcher initially plans to use 5 seedlings per group. A colleague recommends 30 per group. Which principle does this change address, and why?
4. Should there be a control group? What would it look like?

Answer on Page 84

6.4 Blinding

Subjects who know their treatment may behave or report differently because of that knowledge. The person measuring outcomes faces the same risk, since knowing a subject's group can sway their judgment, often without their awareness. Blinding prevents both kinds of bias.

In a *single-blind experiment*, either the subjects do not know which treatment they received, or the person measuring outcomes does not know, but not both.

In a *double-blind experiment*, neither the subjects nor the people measuring outcomes know the treatment assignments. Double-blind experiments are preferred whenever possible because they remove bias from both sources at once.

Double-blinding is not always possible. If subjects can easily identify their treatment condition from its appearance or feel, blinding them is impossible. A study comparing two different workout splits cannot prevent participants from knowing which one they are following. In such cases, the best available design blinds the people measuring outcomes, even though the subjects know their treatment.

6.5 Random Selection vs. Random Assignment

Two design choices decide what you may conclude: (1) Were subjects randomly assigned to treatments? (2) Were they randomly selected from a larger population? These are separate decisions, and each one supports a different type of conclusion.

Random assignment of treatments is what allows you to conclude that a treatment caused an observed difference. When treatments are assigned by chance, any pre-existing differences among subjects are distributed randomly across groups. If one group then performs meaningfully better than another, the treatment is the only consistent explanation left. A difference large enough to be unlikely due to chance alone is called *statistically significant*. For example, suppose a noisy-room group scores 8 points lower than a quiet-room group, and random assignment alone would rarely produce a gap that large. The gap is then statistically significant. Statistically significant differences between treatment groups are evidence that the treatment caused the effect.

Random selection of subjects from a population is what allows you to generalize conclusions beyond the people in your study. If subjects were randomly drawn from a well-defined population, the results of the experiment can be extended to that population. If subjects were not randomly selected, the conclusions apply only to the subjects in the study, or to others who are very similar to them.

These two ideas operate independently. An experiment can have one, both, or neither. The table below summarizes what each combination allows.

	Random assignment: Yes	Random assignment: No
Random selection: Yes	Can conclude causation. Can generalize to the population.	Cannot conclude causation. Can generalize to the population.
Random selection: No	Can conclude causation. Cannot generalize beyond the study subjects.	Cannot conclude causation. Cannot generalize beyond the study subjects.

Remember: the two are separate ideas, and a causal claim about a broader population needs both.

Exercise 43 Random Assignment vs. Random Selection

For each study, answer three questions:

(1) Was there random assignment? (2) Was there random selection? (3) What conclusions are justified?

Working Space

1. A researcher posts a sign-up sheet in a university cafeteria asking for volunteers to participate in a study on the effect of background noise on reading comprehension. Sixty students volunteer. The researcher randomly assigns 30 to read in a noisy

room and 30 to read in a quiet room. Students in the noisy room score significantly lower on a comprehension test.

2. A public health agency randomly selects 400 adults from the voter registration rolls of a large city. Researchers observe whether each person exercises regularly and whether they report high stress levels. Those who exercise regularly report significantly lower stress.

Answer on Page 85

Exercise 44 Random Assignment vs. Random Selection, continued

For each study, answer three questions:

(1) Was there random assignment? (2) Was there random selection? (3) What conclusions are justified?

Working Space

3. A school district randomly selects 10 of its 60 elementary schools to pilot a new reading curriculum. All students in those 10 schools use the new curriculum; the remaining 50 schools continue with the standard curriculum. At year end, students in the pilot schools score significantly higher on a standardized reading test.
4. A fitness company recruits 80 volunteers through social media and randomly assigns 40 to a new training app and 40 to a standard printed workout plan. After 12 weeks, the app group shows significantly greater

improvement in endurance.

Answer on Page 85

Experimental Designs

Suppose your track coach wants to know whether a new warm-up makes people faster. The team has sprinters and distance runners, and the sprinters are faster no matter what. If chance happens to put most of the sprinters in the new-warm-up group, the warm-up will look great for the wrong reason. How you arrange an experiment decides whether a difference like that can fool you.

There are three experimental designs you should be able to recognize, describe, and distinguish from one another. Each one handles *variability*, the natural differences among experimental units, in its own way.

7.1 Completely Randomized Design

The simplest structure is a *completely randomized design*. All available experimental units are assigned to treatments purely by chance, with no prior sorting or grouping. Randomization alone is responsible for balancing any existing differences among units across the treatment groups.

Suppose an agricultural research station wants to compare the yield of three varieties of wheat: a standard variety currently in use, and two new varieties developed in the lab. The station has 90 available plots of farmland. In a completely randomized design, the experiment proceeds as follows:

1. Number the plots 1 to 90 and use a random number generator to assign 30 plots to variety 1, 30 to variety 2, and 30 to variety 3.
2. Plant the assigned wheat variety in each plot and manage all plots identically in terms of water, fertilizer, and pest control.
3. At harvest, measure the yield from each plot.
4. Compare average yields across the three groups.

This design works well when the experimental units are reasonably similar to begin with. If the 90 plots varied dramatically in soil quality, however, randomization might not perfectly balance soil quality across the three groups, particularly with a smaller experiment. This is the motivation for blocking.

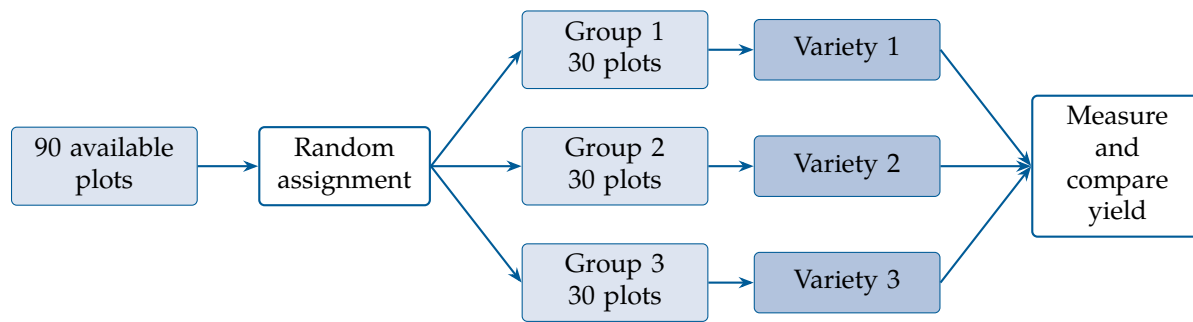


Figure 7.1: Schematic diagram of a completely randomized experiment.

Exercise 45 Designing a Completely Randomized Experiment

Working Space

A sports science department wants to test whether three different warm-up routines (static stretching, dynamic stretching, or no warm-up) affect sprint times in recreational runners. Sixty volunteers have been recruited.

1. Identify the experimental units, the factor, the levels, and the treatments.
2. Describe a complete procedure for randomly assigning the 60 volunteers to the three groups.
3. At the end of the study, the dynamic stretching group has the fastest average sprint times. Can the researchers conclude that dynamic stretching caused the improvement? Explain.

Answer on Page 86

7.2 Randomized Block Design

When experimental units differ in some important, identifiable way before the experiment begins, a completely randomized design may leave a large amount of unexplained variability in the results. A more efficient approach is to group similar units into blocks before

assigning treatments.

In a *randomized block design*, experimental units are first divided into groups of similar units called blocks. Within each block, all treatments are randomly assigned. Because units within a block are similar to each other, the comparison between treatments within a block is cleaner than a comparison that mixes unlike units together. The blocking factor is a variable known to affect the response that the researcher controls for by separating units before randomization.

Return to the wheat experiment. Suppose the research station has two soil zones: sandy soil in the northern half of the land and clay soil in the southern half. Soil type is known to affect yield. If the three varieties are distributed randomly across all 90 plots without regard to soil type, differences in soil quality may obscure or exaggerate differences between varieties. The solution is to block on soil type.

1. Divide the 90 plots into two blocks: 45 sandy-soil plots and 45 clay-soil plots.
2. Within the sandy-soil block, randomly assign 15 plots to variety 1, 15 to variety 2, and 15 to variety 3.
3. Within the clay-soil block, randomly assign 15 plots to variety 1, 15 to variety 2, and 15 to variety 3.
4. Plant, manage, and harvest as before.
5. Compare yields within each block, then examine how results differ across soil types.

Notice that results must be compared within blocks. Comparing a variety 1 plot on sandy soil to a variety 2 plot on clay soil would mix up the variety effect with the soil effect. The whole point of blocking is to make comparisons fair by ensuring the only difference within each comparison is the treatment.

A good blocking variable is one you can observe before the experiment begins and one you expect to affect the response. In the wheat experiment, soil type is visible and known to matter. In a study of a new study-skills program, prior academic performance would be a natural blocking variable. The rule: block on what you can see and control; randomize against what you cannot.

Exercise 46 Identifying the Block Design

For each study, identify the blocking factor and explain why blocking is preferable to a completely randomized design.

Working Space

1. A study tests two methods of teaching reading comprehension to pri-

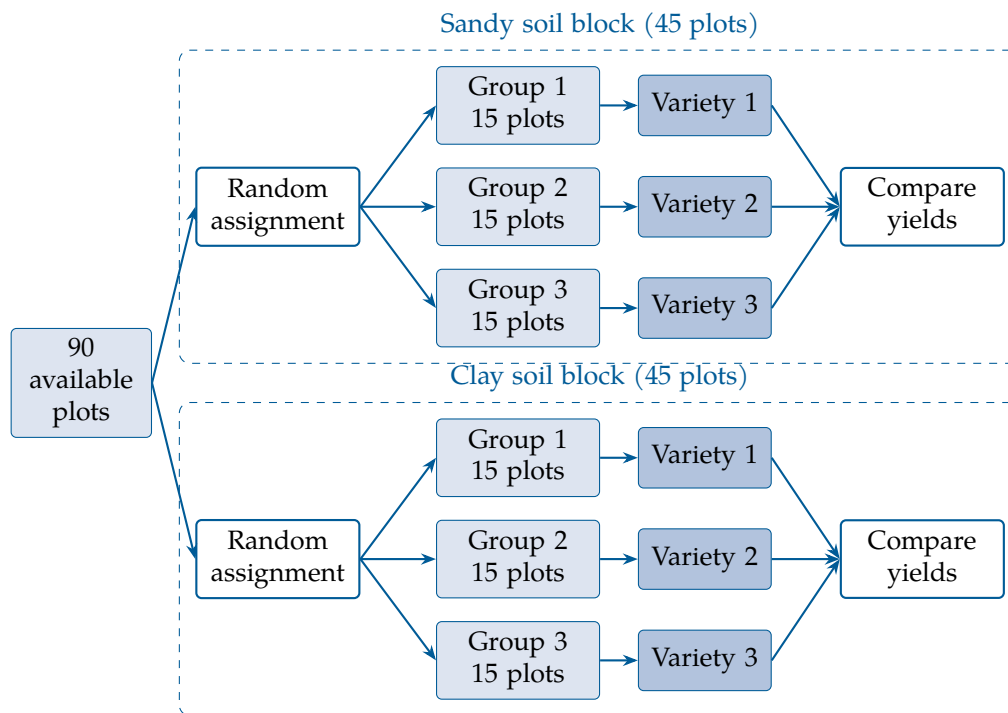


Figure 7.2: Schematic diagram of a randomized block design.

mary school students. Researchers believe that students who already read above grade level will respond differently from those reading at or below grade level.

2. An engineering team tests four surface coatings on metal panels exposed to salt water. The available panels come from two different steel mills and are known to differ in base composition.

7.3 Matched-Pairs Design

A *matched-pairs design* is a special case of blocking in which each block contains exactly two experimental units and there are exactly two treatments to compare.

Pairs of units are matched on some relevant characteristic, and within each pair, one unit is randomly assigned to each treatment. Because the two units in each pair are as similar as possible, any difference in their responses can be attributed to the treatment rather than to pre-existing differences.

There are two common versions of this design.

In the first, two different subjects are matched. A study comparing two physiotherapy protocols for knee rehabilitation might pair patients by age and injury severity. Within each pair, one patient is randomly assigned to protocol 1 and the other to protocol 2. The comparison is made within pairs, not across the whole sample.

In the second version, each subject serves as their own pair, receiving both treatments in a randomly determined order. This is sometimes called a *repeated measures design*. A washout period between treatments allows the first treatment's effect to clear before the second begins. Because each subject is compared only to themselves, all individual differences are perfectly controlled.

Consider a study testing whether caffeine affects short-term memory performance. Each participant completes a memory task twice: once after consuming a caffeinated drink, and once after consuming an identical-looking decaffeinated drink. The order is randomly determined for each participant. This is a matched-pairs design because each participant's caffeinated performance is paired with their own decaffeinated performance. Comparing one participant's caffeinated score to a different participant's decaffeinated score would not be effective.

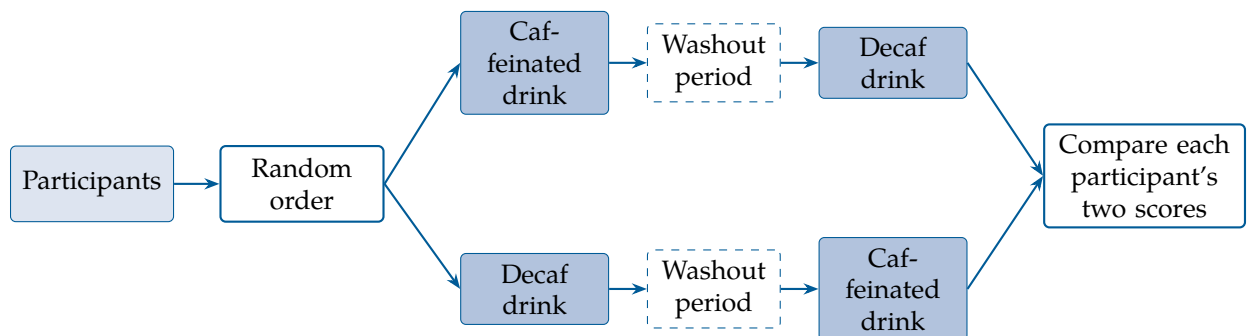


Figure 7.3: Schematic diagram of a matched-pairs design (repeated measures version).

Exercise 47 Matched Pairs or Randomized Block?

Classify each study as a matched-pairs

Working Space

design or a randomized block design with more than two units per block. Explain your reasoning.

1. A study tests three different revision strategies (practice testing, summarizing, and re-reading) on university students. Students are divided into blocks by their first-year GPA: high (3.5 and above), mid (2.5 to 3.49), and low (below 2.5). Within each GPA block, students are randomly assigned to one of the three strategies.
2. A study compares the effectiveness of two soil treatment methods on crop yield. Farmers are paired by farm size and climate zone. Within each pair, a coin flip decides which farm uses method 1; the other uses method 2.
3. A study tests whether a standing desk reduces reported lower back discomfort. Each participant uses a standard seated desk for four weeks, then a standing desk for four weeks, with the order randomly determined for each person.

Answer on Page 86

7.4 Choosing a Design

Knowing what each experimental design is called is not enough. The AP exam regularly asks you to justify why a particular design is appropriate for a given situation. That requires understanding what problem each design solves and when those problems arise.

The starting point is always the completely randomized design. It is the simplest structure, and when experimental units are reasonably similar to one another, it is the right choice. Random assignment spreads pre-existing differences among units across the treatment

groups by chance. With enough units, those differences tend to balance out. If you have no strong reason to expect a particular characteristic of the units to consistently affect the response, use a completely randomized design.

Blocking makes sense when, before the experiment begins, you can identify a characteristic of the units that you expect to affect the response. Ignoring such a characteristic adds extra variability to your results. Units in the same treatment group will differ in ways unrelated to the treatment, and those differences make a real treatment effect harder to detect. Blocking solves this by grouping similar units first. Treatments are then compared within groups of units that are alike in the relevant way, which reduces variability and makes the experiment more sensitive.

To decide whether to block, ask yourself: is there a variable I can observe right now, before any treatment is applied, that I expect to affect how units respond? If yes, and if it is practical to measure and group by that variable, block on it.

Use matched pairs when there are exactly two treatments. Either pair up units that match on a relevant characteristic, or give both treatments to each unit. Because the two units in a pair are as similar as possible, the within-pair comparison is very clean. When each unit receives both treatments (with a washout period between them), individual differences drop out of the comparison entirely.

A design justification on the AP exam names the design, the variable it accounts for, and why that variable matters. Here is a model for the wheat experiment:

Use a randomized block design, blocking on soil type. Soil type can be observed before planting, and it is expected to affect yield. Comparing the three varieties within each soil type removes soil-related variability from the comparison, so a difference in yield is easier to attribute to the variety.

The same three parts work for any design: name it, name the variable it handles, and explain how handling that variable makes the treatment comparison fairer.

Exercise 48 Justifying a Design Choice

For each situation, recommend a design (completely randomized, randomized block, or matched pairs) and write a clear justification that explains why that design is more appropriate than the alternatives.

Working Space

1. A software company wants to test whether a new user interface reduces

the time it takes to complete a standard task, compared to the existing interface. It has 80 volunteers available. None of them have used either interface, and all have similar levels of computing experience.

2. A sleep research lab wants to compare the effect of two room temperatures (cool vs. warm) on sleep quality. They have 40 adult volunteers. Each person's natural sleep patterns vary night to night, but researchers expect consistent individual differences in baseline sleep quality across people.
3. Bluewave School District wants to compare three different formats for a professional development workshop (in-person, hybrid, and fully online) on teachers' reported confidence in teaching a new curriculum. The district's teaching staff includes both primary school teachers and secondary school teachers, and prior research suggests that these two groups respond very differently to professional development formats.
4. A skincare company wants to test whether a new moisturizer reduces skin dryness compared to a standard moisturizer. They recruit 50 volunteers and note that participants' baseline skin dryness varies widely.

Answers to Exercises

Answer to Exercise 1 (on page 3)

1. $f'(t) = 6t^2 - 6t - 4$
2. $g'(t) = (\frac{-3}{4})2t^{-3/4-1} = \frac{-3}{2}t^{-7/4}$
3. Rewrite $F(r) = 5r^{-3}$, so $F'(r) = -15r^{-4} = \frac{-15}{r^4}$
4. First, we expand the function by multiplying out the two binomials: $(3u - 1)(u + 2) = 3u^2 + 6u - u - 2$. Therefore, $H(u) = 3u^2 + 5u - 2$, and we can differentiate using what we have learned about differentiating polynomials. $H'(u) = 6u + 5$. In a later chapter, you will learn the Product rule, which will allow you to differentiate this function without multiplying out the binomials.

Answer to Exercise 2 (on page 4)

```
1 def derivative_of_polynomial(pn):
2
3     # We can ignore the constant term (skip the first coefficient)
4     current_degree = 1
5     result = []
6
7     # Differentiate each monomial
8     while current_degree < len(pn):
9         coefficient = pn[current_degree]
10        result.append(coefficient * current_degree)
11        current_degree = current_degree + 1
12
13    # No terms? Make it the zero polynomial
14    if len(result) == 0:
15        result.append(0.0)
16
17    return result
```

Answer to Exercise 3 (on page 6)

(a) Velocity is the first derivative of the position function, $s'(t) = 4t^3 - 6t^2 + 2t - 1$. Acceleration is the derivative of the velocity function, $s''(t) = 12t^2 - 12t + 2$.

(b) $s'(1.5) = 4(1.5)^3 - 6(1.5)^2 + 2(1.5) - 1 = 2$ We should note that this is a measurement and needs units to make sense. Since s is in meters and t is in seconds, our velocity should have units of $\frac{\text{m}}{\text{s}}$, so our final answer is $s'(1.5\text{s}) = 2 \frac{\text{m}}{\text{s}}$.

(c) $s''(1.5) = 12(1.5)^2 - 12(1.5) + 2 = 11$. Similarly to part (b), our answer needs units. The units for acceleration are the units for velocity divided by the unit for time (because acceleration is a rate of change of velocity), and our final answer should be $s''(1.5\text{s}) = 11 \frac{\text{m}}{\text{s}^2}$.

(d) When velocity and acceleration are occurring in the same direction (i.e. have the same sign), the speed (the absolute value of velocity) is increasing. Since $s'(1.5\text{s})$ and $s''(1.5\text{s})$ are both > 0 , the speed of the object is increasing.

Answer to Exercise 4 (on page 6)

1. $h'(t) = 15 - 9.8t$. $h'(1) = 5.2 \text{ m/s}$ (still rising) and $h'(2) = -4.6 \text{ m/s}$ (falling).
2. $h''(t) = -9.8 \text{ m/s}^2$. The only force on the ball (ignoring air) is gravity, which pulls the same way all the time.

Answer to Exercise 5 (on page 12)

1. $f'(x) = 10e^{2x}$
2. By the chain rule, $g'(x) = \frac{1}{x^2+1} \cdot 2x = \frac{2x}{x^2+1}$
3. $h'(x) = 2^x \ln 2$
4. By the product rule, $k'(x) = e^x + xe^x = (1+x)e^x$

Answer to Exercise 6 (on page 12)

We start by writing out the limit:

$$\frac{d}{dx} \cos x = \lim_{h \rightarrow 0} \frac{\cos(x+h) - \cos x}{h}$$

Applying the sum formula for $\cos(x + h)$, we get:

$$= \lim_{h \rightarrow 0} \frac{\cos x \cos h - \sin x \sin h - \cos x}{h}$$

Rearranging to group the $\cos x$ and applying the Difference Rule:

$$= \lim_{h \rightarrow 0} \frac{\cos x \cos h - \cos x}{h} - \lim_{h \rightarrow 0} \frac{\sin x \sin h}{h}$$

Applying the Constant Multiple Rule:

$$= \cos x \lim_{h \rightarrow 0} \frac{\cos h - 1}{h} - \sin x \lim_{h \rightarrow 0} \frac{\sin h}{h}$$

Recalling that $\lim_{h \rightarrow 0} \frac{\sin h}{h} = 1$,

$$= \cos x \lim_{h \rightarrow 0} \frac{\cos h - 1}{h} - \sin x \cdot 1$$

Recalling that $\lim_{h \rightarrow 0} \frac{\cos h - 1}{h} = 0$:

$$= \cos x \cdot 0 - \sin x = -\sin x$$

Therefore, $\frac{d}{dx} \cos x = -\sin x$

Answer to Exercise 7 (on page 13)

1. $\frac{\sec x(\tan x - 1)}{(1 + \tan x)^2}$
2. $\sec t[\sec^2 t + \tan^2 t]$
3. $\frac{4 - \tan \theta + \theta \sec^2 \theta}{(4 - \tan \theta)^2}$
4. $2 \sec t \tan t + \csc t \cot t$
5. $\frac{1}{1 + \cos \theta}$
6. $\cos^2 x - \sin^2 x$

Answer to Exercise 8 (on page 14)

1. By the chain rule, $f'(x) = 2 \arctan x \times \frac{d}{dx} \arctan x = 2 \arctan x \frac{1}{1+x^2}$
2. By the Product rule, $f'(x) = x \frac{d}{dx} \operatorname{arcsec}(x^3) + \operatorname{arcsec}(x^3)$. Further, by the chain rule, $\frac{d}{dx} \operatorname{arcsec}(x^3) = \frac{1}{(x^3)\sqrt{(x^3)^2-1}} \times \frac{d}{dx}(x^3) = \frac{3x^2}{x^3\sqrt{x^6-1}}$. Therefore, $f'(x) = \frac{3}{\sqrt{x^6-1}} + \operatorname{arcsec}(x^3)$
3. By the chain rule, $f'(x) = \frac{1}{\sqrt{1-(1/x)^2}} \times \left(-\frac{1}{x^2}\right) = -\frac{1}{x^2\sqrt{1-\frac{1}{x^2}}} = -\frac{1}{|x|\sqrt{x^2-1}}$

Answer to Exercise 9 (on page 14)

$\frac{d}{dx} \arctan x = \frac{1}{1+x^2}$. At $x = 1$, this is $\frac{1}{2}$.

Answer to Exercise 10 (on page 14)

1. $\frac{dy}{dx} = \frac{d}{dx}[x \sin x] = x \frac{d}{dx} \sin x + \sin x \frac{d}{dx} x = x \cos x + \sin x(1) = x \cos x + \sin x$
2. By the chain rule, $f'(x) = 5(x^3 - \cos x)^4 \cdot \frac{d}{dx}(x^3 - \cos x) = 5(x^3 - \cos x)^4 \cdot (3x^2 + \sin x)$
3. By the chain rule, $f'(x) = \frac{d}{d(\sin x)}[\sin^3 x] \times \frac{d}{dx} \sin x = 3 \sin^2 x \cdot \cos x$

Answer to Exercise 11 (on page 15)

$$f'(x) = \frac{d}{dx}(7x) - \frac{d}{dx}(3) + \frac{d}{dx}(\ln x) = 7 - 0 + \frac{1}{x} = 7 + \frac{1}{x} \text{ and } f'(1) = 7 + \frac{1}{1} = 8$$

Answer to Exercise 12 (on page 15)

The particle is at rest when $x'(t) = y'(t) = 0$. First, we find each of the derivatives:

$$x'(t) = 3t^2 - 6t$$

$$y'(t) = 12 - 6t$$

We can solve $y' = 0$ for t and find that the y -velocity is 0 when $t = 2$. Substituting $t = 2$ into our expression for x' , we find $x'(2) = 3(2)^2 - 6(2) = 0$. Therefore, the particle is at rest when $t = 2$. To find the xy -coordinate, we substitute $t = 2$ into $x(t)$ and $y(t)$:

$$x(2) = (2)^3 - 3(2)^2 = 8 - 12 = -4$$

$$y(2) = 12(2) - 3(2)^2 = 24 - 12 = 12$$

Therefore, the particle is at rest when it is located at $(-4, 12)$.

Answer to Exercise 13 (on page 15)

$f(g(x)) = \sqrt{(3x-2)^2 - 4} = \sqrt{9x^2 - 12x}$ and $\frac{d}{dx}f(g(x)) = \frac{18x-12}{2\sqrt{9x^2-12x}}$. Substituting $x = 3$, we find $(f \circ g)'(3) = \frac{18(3)-12}{2\sqrt{9(3)^2-12(3)}} = \frac{42}{2\sqrt{45}} = \frac{21}{3\sqrt{5}} = \frac{7}{\sqrt{5}}$

Answer to Exercise 14 (on page 16)

First, recall that the velocity of a particle is the derivative of its position function. Therefore, $v(t) = x'(t) = \frac{d}{dt}[(t-a)(t-b)]$. Applying the Product Rule for derivatives, we see that $v(t) = (t-a)(1) + (t-b)(1) = 2t - a - b$. To find the time(s) when the particle is at rest, we set $v(t) = 0$ and solve for t .

$$0 = 2t - a - b$$

$$2t = a + b$$

$$t = \frac{a+b}{2}$$

Answer to Exercise 15 (on page 16)

The question is asking when the derivative of f is $\frac{1}{2}$. We will take the derivative and set it equal to $\frac{1}{2}$.

$$f'(x) = \frac{(x+2)(1) - x(1)}{(x+2)^2} = \frac{2}{(x+2)^2}$$

$$\frac{2}{(x+2)^2} = \frac{1}{2}$$

$$4 = (x+2)^2$$

$$\pm 2 = x + 2$$

$$x = 2 - 2 = 0 \text{ and } x = -2 - 2 = -4$$

Answer to Exercise 16 (on page 16)

(a) Let t_0 be the time at which the particle is first at rest. The velocity of the particle is

given by $v(t) = x'(t) = \cos t + \sin t$. Setting $v(t) = 0$, we find:

$$\cos t = -\sin t$$

which is true for $t = \frac{3\pi}{4} + n\pi$, where n is an integer. Therefore, the first time the velocity is 0 is $t_0 = \frac{3\pi}{4}$.

(b) To find the acceleration at $t = \frac{3\pi}{4}$, we take the derivative of the velocity function to yield the acceleration function.

$$a(t) = v'(t) = -\sin t + \cos t$$

Substituting $t = \frac{3\pi}{4}$, we find the acceleration is $-\sin \frac{3\pi}{4} + \cos \frac{3\pi}{4} = \frac{-\sqrt{2}}{2} - \frac{\sqrt{2}}{2} = -\sqrt{2}$

Answer to Exercise 17 (on page 17)

First, we find the x such that $y = 0$

$$0 = e^{\tan x} - 2$$

$$2 = e^{\tan x}$$

$$\ln 2 = \tan x$$

$$x = \arctan(\ln 2) = \arctan 0.693 \approx 0.606$$

Then, we find the slope of the function at $x = 0.606$ by finding $y'(0.606)$

$$y' = e^{\tan x}(\sec x)^2 = \frac{e^{\tan x}}{(\cos x)^2}$$

$$y'(0.606) = \frac{e^{\tan 0.606}}{(\cos 0.606)^2} = 2.961$$

Answer to Exercise 18 (on page 17)

(a) Apply the chain rule to find $f'(x)$

$$f'(x) = \frac{1}{2\sqrt{25-x^2}} \cdot (-2x) = \frac{-x}{\sqrt{25-x^2}}$$

(b) First, substitute $x = -3$ into $f'(x)$

$$f'(-3) = \frac{-(-3)}{\sqrt{25-(-3)^2}} = \frac{3}{\sqrt{16}} = \frac{3}{4}$$

This is the slope of the line. To complete an equation for the tangent line, we need a point. We know the tangent line touches $f(x)$ at $x = -3$, so the tangent line must pass through the point $(-3, f(-3))$.

$$f(-3) = \sqrt{25 - (-3)^2} = 4$$

We use $m = \frac{3}{4}$ and the coordinate point $(x_1, y_1) = (-3, 4)$ to complete the equation $y - y_1 = m(x - x_1)$

$$y - 4 = \frac{3}{4}(x + 3)$$

Answer to Exercise 19 (on page 17)

$$a(t) = v'(t) = -\frac{\pi}{6} \sin \frac{\pi}{6} t$$

$$a(4) = -\frac{\pi}{6} \sin \frac{2\pi}{3} = -\frac{\pi}{6} \cdot \frac{\sqrt{3}}{2} = -\frac{\pi\sqrt{3}}{12}$$

Answer to Exercise 20 (on page 17)

Recall that the rate of change of a function is given by the derivative of that function. Therefore, we are looking for the interval(s) where $f'(x) > g'(x)$. First, we find each derivative:

$$f'(x) = e^x$$

$$g'(x) = 4x^3$$

We are looking for x -values, such that $e^x > 4x^3$. This inequality can be restated as $e^x - 4x^3 > 0$. Using a calculator, you should find that $e^x - 4x^3 = 0$ when $x \approx 0.831$ and $x \approx 7.384$. We will check values on either side of and in the interval $x \in (0.831, 7.384)$ to determine the sign value of $e^x - 4x^3$. We know that when $x = 0$, $e^x - 4x^3 > 0$, when $x = 5$, $e^x - 4x^3 < 0$, and when $x = 10$, $e^x - 4x^3 > 0$. Therefore, $f'(x)$ is greater than $g'(x)$ on the open intervals $x \in (-\infty, 0.831) \cup (7.384, \infty)$.

Answer to Exercise 21 (on page 21)

First, let us confirm that l'Hôpital's rule applies here:

$$\lim_{x \rightarrow 0} \frac{\tan x - x}{x^3} = \frac{0 - 0}{0} = \frac{0}{0}$$

Therefore, we can apply l'Hôpital's rule:

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{\tan x - x}{x^3} &= \lim_{x \rightarrow 0} \frac{\frac{d}{dx}(\tan x - x)}{\frac{d}{dx}x^3} \\ &= \lim_{x \rightarrow 0} \frac{\sec^2 x - 1}{3x^2} = \frac{1 - 1}{0} = \frac{0}{0}\end{aligned}$$

which is an indeterminate form. We apply l'Hôpital's rule again:

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{\tan x - x}{x^3} &= \lim_{x \rightarrow 0} \frac{\frac{d}{dx}(\sec^2 x - 1)}{\frac{d}{dx}3x^2} \\ &= \lim_{x \rightarrow 0} \frac{2 \tan x \sec^2 x}{6x} = \frac{2(0)(1^2)}{6 \cdot 0} = \frac{0}{0}\end{aligned}$$

which is also an indeterminate form. We apply l'Hôpital's rule again:

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{\tan x - x}{x^3} &= \lim_{x \rightarrow 0} \frac{\frac{d}{dx}(2 \tan x \sec^2 x)}{\frac{d}{dx}6x} \\ &= \lim_{x \rightarrow 0} \frac{2 \sec^2 x [2 \tan^2 x + \sec^2 x]}{6} = \frac{2 \cdot 1 \cdot [2 \cdot 0 + 1]}{6} \\ &= \frac{2}{6} = \frac{1}{3}\end{aligned}$$

Answer to Exercise 22 (on page 23)

1. $\lim_{x \rightarrow 3} \frac{x-3}{x^2-9} = \frac{0}{0}$, so we apply l'Hôpital's rule. $\lim_{x \rightarrow 3} \frac{x-3}{x^2-9} = \lim_{x \rightarrow 3} \frac{1}{2x} = \frac{1}{6}$
2. $\lim_{x \rightarrow 1/2} \frac{6x^2+5x-4}{4x^2+16x-9} = \frac{0}{0}$, so we apply l'Hôpital's rule. $\lim_{x \rightarrow 1/2} \frac{6x^2+5x-4}{4x^2+16x-9} = \lim_{x \rightarrow 1/2} \frac{12x+5}{8x+16} = \frac{11}{20}$
3. $\lim_{x \rightarrow 0^+} \frac{\ln x}{\sqrt{x}} = \frac{-\infty}{0} = -\infty$. This limit is not indeterminate, so l'Hôpital's rule does not apply (and is not needed).
4. $\frac{1+\cos \pi}{1-\cos \pi} = \frac{1-1}{1+1} = \frac{0}{2} = 0$. Not indeterminate, so just plug in.
5. $\frac{1 \cdot \sin 0}{2-1-1} = \frac{0}{0}$, so we apply l'Hôpital's rule: $\lim_{x \rightarrow 1} \frac{x \cos(x-1) + \sin(x-1)}{4x-1} = \frac{1 \cdot 1 + 0}{4-1} = \frac{1}{3}$.

Answer to Exercise 23 (on page 25)

The speed of a car must be a continuous, differentiable function, since your car cannot "jump" from one speed to another: it must smoothly accelerate from one speed to another. Therefore, the Mean Value Theorem applies. The average acceleration from 3:30 PM to 3:40 PM is given by:

$$\frac{\text{change in speed}}{\text{change in time}} = \frac{50 \frac{\text{mi}}{\text{hr}} - 30 \frac{\text{mi}}{\text{hr}}}{3:40\text{PM} - 3:30\text{PM}}$$

Simplifying and converting minutes to hours, we see the average acceleration is:

$$\frac{20 \frac{\text{mi}}{\text{hr}}}{\frac{1}{6} \text{hr}} = 120 \frac{\text{mi}}{\text{hr}^2}$$

Therefore, by MVT, there must be some time between 3:30 and 3:40 PM where the car's acceleration is exactly $120 \frac{\text{mi}}{\text{hr}^2}$.

Answer to Exercise 24 (on page 25)

(a) For the domain given, $f(x)$ is defined and differentiable. Finding the slope of the secant line connecting the endpoints:

$$\frac{f(b) - f(a)}{b - a} = \frac{\sqrt{4} - \sqrt{0}}{4 - 0} = \frac{2}{4} = \frac{1}{2}$$

So we are looking for some number c such that $f'(c) = \frac{1}{2}$. Let us find $f'(x)$:

$$f'(x) = \frac{d}{dx} \sqrt{x} = \frac{1}{2\sqrt{x}}$$

Setting this equal to $\frac{1}{2}$ to find c :

$$f'(c) = \frac{1}{2\sqrt{c}} = \frac{1}{2}$$

$$\sqrt{c} = 1$$

$$c = 1$$

(b) For the domain given, $f(x)$ is defined and differentiable. Finding the slope of the secant

line connecting the endpoints:

$$\frac{f(2) - f(0)}{2 - 0} = \frac{e^{-2} - e^0}{2} = \frac{1 - e^2}{2e^2} \approx -0.432$$

And find $f'(x)$:

$$f'(x) = -e^{-x}$$

According to MVT, there must be some c such that $f'(c) \approx -0.432$:

$$-e^{-c} \approx -0.432$$

$$e^{-c} \approx 0.432$$

$$-c \approx \ln 0.432$$

$$c \approx -\ln 0.432 \approx 0.839$$

(c) For the domain given, $f(x)$ is defined and differentiable. Finding the secant line connecting the endpoints:

$$\frac{f(b) - f(a)}{b - a} = \frac{\ln 4 - \ln 1}{4 - 1} = \frac{\ln 4}{3} \approx 0.462$$

And find $f'(x)$:

$$f'(x) = \frac{1}{x}$$

According to MVT, there must be some c such that $f'(c) \approx 0.462$

$$f'(c) = \frac{1}{c} \approx 0.462$$

$$c \approx \frac{1}{0.462} = 2.164$$

Answer to Exercise 25 (on page 26)

Velocity is the derivative of position. Therefore, $v(t) = s'(t) = 3t^2 - 12t + 6$.

Answer to Exercise 26 (on page 27)

$$v(2) = 3(2)^2 - 12(2) + 6 = -6 \frac{\text{m}}{\text{s}}$$

$$v(4) = 3(4)^2 - 12(4) + 6 = 6 \frac{\text{m}}{\text{s}}$$

Answer to Exercise 27 (on page 27)

When the particle is at rest, $v(t) = 0$.

$$3t^2 - 12t + 6 = 0$$

$$3(t^2 - 4t + 2) = 0$$

$$t^2 - 4t + 2 = 0$$

This is not easily factorable, so we will use the quadratic formula:

$$t = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(2)}}{2(1)}$$

$$t = \frac{4 \pm \sqrt{16 - 8}}{2} = 2 \pm \sqrt{2} \approx 0.586, 3.414$$

Therefore, the particle is at rest at 0.586s and 3.414s.

Answer to Exercise 28 (on page 31)

Recall that critical values are values of x where $g'(x) = 0$ or is undefined. We need to find an expression for $g'(x)$, set it equal to zero when $x = \frac{2}{3}$, and solve for k . By the product and chain rules,

$$g'(x) = x^2 \cdot ke^{kx} + 2xe^{kx}$$

$$g'\left(\frac{2}{3}\right) = \left(\frac{2}{3}\right)^2 ke^{\frac{2k}{3}} + \frac{4}{3} \exp \frac{2k}{3} = 0$$

$$\frac{4k}{9} e^{\frac{2k}{3}} + \frac{4}{3} e^{\frac{2k}{3}} = 0$$

$$\left(\frac{4k}{9} + \frac{4}{3}\right) e^{\frac{2k}{3}} = 0$$

There are no real values of k such that $e^{\frac{2k}{3}} = 0$, therefore, we will examine the other factor:

$$\frac{4k}{9} + \frac{4}{3} = 0$$

$$\frac{4k}{9} = \frac{-4}{3}$$

$$\frac{k}{3} = -1$$

$$k = -3$$

Therefore, $g(x)$ has a critical value at $x = \frac{2}{3}$ when $k = -3$.

Answer to Exercise 29 (on page 33)

First, we will find f' and set it equal to zero:

$$f'(x) = 300 - 3x^2 = 0$$

$$300 = 3x^2 \rightarrow x = \pm\sqrt{100} = \pm 10$$

(Note: $f'(x) = 3(10 - x)(10 + x)$, which implies roots at $x = \pm 10$. Now, we will evaluate the value of $f'(x)$ for $x < -10$, $-10 < x < 10$, and $x > 10$.

Value of x	$(10-x)$	$(10+x)$	$f'(x)$	$f(x)$ behavior
$x < -10$	positive	negative	negative	decreasing
$-10 < x < 10$	positive	positive	positive	increasing
$x > 10$	negative	positive	negative	decreasing

Therefore, the function is increasing on the interval $x \in [-10, 10]$ because $f'(x) > 0$ for $x \in [-10, 10]$.

Answer to Exercise 30 (on page 34)

Given $f(x) = x^3 - 3x^2 - 9x + 4$, it follows that $f'(x) = 3x^2 - 6x - 9$. Factoring, we find that $f'(x) = 3(x - 3)(x + 1)$ and $f'(x) = 0$ when $x = 3$ and $x = -1$. We construct our table to help us analyze the value of $f'(x)$ and behavior of $f(x)$ on the whole domain of the function:

Value of x	$(x - 3)$	$(x + 1)$	$f'(x)$	$f(x)$ behavior
$x < -1$	negative	negative	positive	increasing
$-1 < x < 3$	negative	positive	negative	decreasing
$x > 3$	positive	positive	positive	increasing

So, $f(x)$ is increasing for $x \in (-\infty, -1) \cup (3, \infty)$ and decreasing for $x \in (-1, 3)$. Since $f'(-1) = 0$ and changes from positive to negative, $f(x)$ has a local maximum at $x = -1$. And since $f'(3) = 0$ and changes from negative to positive, $f(x)$ has a local minimum at $x = 3$.

Answer to Exercise 31 (on page 36)

First, we identify any critical numbers:

$$f'(x) = \frac{x * (\frac{1}{x}) - \ln x * 1}{x^2} = \frac{1 - \ln x}{x^2}$$

Recall that critical numbers are values where $f'(x) = 0$ or does not exist. We might identify $x = 0$ as a critical number, but the presence of $\ln x$ limits the domain of the function to $x \in (0, \infty)$, excluding $x = 0$. For all $x \in (0, \infty)$, $f'(x)$ exists. So, we look for values where $f'(x) = 0$.

$$\frac{1 - \ln x}{x^2} = 0$$

$$1 - \ln x = 0$$

$$1 = \ln x$$

$$x = e$$

Finding the value of $f(x)$ at $x = e$:

$$f(e) = \frac{\ln e}{e} = \frac{1}{e}$$

Because the domain of $f(x)$ is on an *open interval*, instead of checking the endpoints directly, we will take the limits as x approaches 0 and ∞ .

$$\lim_{x \rightarrow 0} \frac{\ln x}{x} = -\infty < \frac{1}{e}$$

$$\lim_{x \rightarrow \infty} \frac{\ln x}{x} = 0 < \frac{1}{e}$$

Therefore, the absolute maximum values of $f(x) = \frac{\ln x}{x}$ is $\frac{1}{e}$ at $x = e$.

Answer to Exercise 32 (on page 36)

1. $f'(x) = 4 - 2x$ and to find the critical numbers, we set $f'(x) = 0$:

$$4 - 2x = 0$$

$$x = 2$$

We evaluate $f(x)$ at $x = 0, 2, 5$:

$$f(0) = 12 + 4(0) - 0^2 = 12$$

$$f(2) = 12 + 4(2) - 2^2 = 12 + 8 - 4 = 16$$

$$f(5) = 12 + 4(5) - 5^2 = 12 + 20 - 25 = 7$$

Therefore, the global maximum is $f(2) = 16$ and the global minimum is $f(5) = 7$.

2. $f(t) = \frac{\sqrt{t}}{1+t^2}, [0, 2]$.

We write $f(t) = t^{1/2}(1+t^2)^{-1}$ and differentiate:

$$f'(t) = \frac{1-3t^2}{2\sqrt{t}(1+t^2)^2}.$$

Critical points occur when the numerator is zero (and t is in the interval):

$$1-3t^2=0 \Rightarrow t = \frac{1}{\sqrt{3}}.$$

We evaluate $f(t)$ at $t = 0, \frac{1}{\sqrt{3}}, 2$:

$$f(0) = 0, \quad f\left(\frac{1}{\sqrt{3}}\right) = \frac{3^{3/4}}{4}, \quad f(2) = \frac{\sqrt{2}}{5}.$$

Since $\frac{3^{3/4}}{4}$ is the largest of these and 0 is the smallest, the global maximum is

$$f\left(\frac{1}{\sqrt{3}}\right) = \frac{3^{3/4}}{4},$$

and the global minimum is

$$f(0) = 0.$$

3. $f(t) = 2\cos t + \sin(2t), [0, \frac{\pi}{2}]$.

We differentiate:

$$f'(t) = -2\sin t + 2\cos(2t).$$

Set $f'(t) = 0$:

$$-2\sin t + 2\cos(2t) = 0 \Rightarrow \cos(2t) = \sin t.$$

Use $\cos(2t) = 1 - 2\sin^2 t$:

$$1 - 2\sin^2 t = \sin t \Rightarrow 2\sin^2 t + \sin t - 1 = 0.$$

Let $u = \sin t$:

$$2u^2 + u - 1 = 0 \Rightarrow u = \frac{1}{2} \text{ or } u = -1.$$

On $[0, \frac{\pi}{2}]$, $\sin t = \frac{1}{2}$ gives $t = \frac{\pi}{6}$ (and $\sin t = -1$ is not in this interval).

Evaluate $f(t)$ at $t = 0, \frac{\pi}{6}, \frac{\pi}{2}$:

$$f(0) = 2\cos 0 + \sin 0 = 2,$$

$$f\left(\frac{\pi}{6}\right) = 2 \cos\left(\frac{\pi}{6}\right) + \sin\left(\frac{\pi}{3}\right) = 2 \cdot \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{2},$$

$$f\left(\frac{\pi}{2}\right) = 2 \cos\left(\frac{\pi}{2}\right) + \sin(\pi) = 0 + 0 = 0.$$

Hence the global maximum is

$$f\left(\frac{\pi}{6}\right) = \frac{3\sqrt{3}}{2},$$

and the global minimum is

$$f\left(\frac{\pi}{2}\right) = 0.$$

4. $f(x) = \ln(x^2 + x + 1)$, $[-1, 1]$.

Differentiate:

$$f'(x) = \frac{2x + 1}{x^2 + x + 1}.$$

To find critical points, set the numerator equal to 0:

$$2x + 1 = 0 \Rightarrow x = -\frac{1}{2}.$$

Evaluate $f(x)$ at $x = -1, -\frac{1}{2}, 1$:

$$f(-1) = \ln(1 - 1 + 1) = \ln 1 = 0,$$

$$f\left(-\frac{1}{2}\right) = \ln\left(\left(-\frac{1}{2}\right)^2 - \frac{1}{2} + 1\right) = \ln\left(\frac{1}{4} - \frac{1}{2} + 1\right) = \ln\left(\frac{3}{4}\right),$$

$$f(1) = \ln(1 + 1 + 1) = \ln 3.$$

Since $\ln\left(\frac{3}{4}\right) < 0 < \ln 3$, the global minimum is

$$f\left(-\frac{1}{2}\right) = \ln\left(\frac{3}{4}\right),$$

and the global maximum is

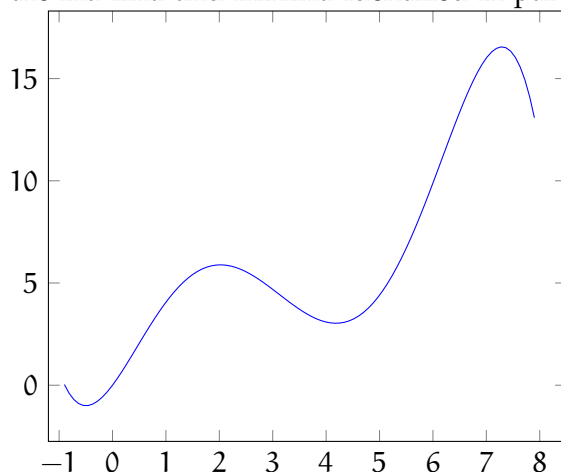
$$f(1) = \ln 3.$$

Answer to Exercise 33 (on page 40)

[Your answers are meant to be estimates; anything within ± 0.1 of the given answers are reasonable estimates.]

1. $f(x)$ is increasing on the intervals $x \in (-0.5, 2.2) \cup (4, 7.3)$. $f(x)$ is decreasing on the intervals $x \in (-\infty, -0.5) \cup (2.2, 4) \cup (7.3, \infty)$.
2. $f(x)$ has local maxima at $x = 2.2, 7.3$ and local minima at $x = -0.5, 4$.

3. Your sketch should show the maxima and minima identified in part 2. One possible



solution is shown below.

Answer to Exercise 34 (on page 43)

Noting that $f''(2) = 0$, we examine the value of f'' around $x = 2$. For $0 < x < 2$, $f'' < 0$, which indicates f is concave down in the domain $x \in (0, 2)$. For $x > 2$, $f'' > 0$, which indicates f is concave up. Therefore, there is an inflection point at $x = 2$ for f . Recalling that $f(x) = x^4 - 4x^3$, we find the coordinate of the inflection point by substituting $x = 2$:

$$f(2) = 2^4 - 4 \cdot 2^3 = 16 - 4 \cdot 8 = 16 - 32 = -16$$

Therefore, $f(x)$ has an inflection point at $(2, -16)$.

Answer to Exercise 35 (on page 44)

According to the graph, g' is positive and increasing. Therefore, g is increasing (because g' is positive) and concave up (because g' is increasing, and therefore g'' is positive).

Answer to Exercise 36 (on page 44)

Since the question asks about concavity, we need to examine the second derivative:

$$f''(x) = \frac{d}{dx} f'(x) = \frac{d}{dx} [e^{(x^4-2x^2+1)} - 2]$$

By the chain rule,

$$f''(x) = e^{(x^4-2x^2+1)} (4x^3 - 4x)$$

The exponential is never zero, so the second derivative equals zero only when $4x^3 - 4x = 4x(x^2 - 1) = 0$, which gives roots $x = 0$, $x = 1$, and $x = -1$. So the intervals we need to test are $(-1.5, -1)$, $(-1, 0)$, $(0, 1)$, and $(1, 1.5)$. To test $x \in (-1.5, -1)$, we will substitute $x = -1.25$ into $f''(x)$:

$$f''(-1.25) = -3.85928 < 0$$

Therefore, $f(x)$ is concave down on the interval $x \in (-1.5, -1)$. Next, we test $x \in (-1, 0)$:

$$f''(-0.5) = 2.63258 > 0$$

So, we eliminate $x \in (-1, 0)$. Next, we test $x \in (0, 1)$:

$$f''(0.5) = -2.63258 < 0$$

And $f(x)$ is concave down on the interval $x \in (0, 1)$. Finally, we test the interval $x \in (1, 1.5)$:

$$f''(1.25) = 3.85928 > 0$$

Which eliminates that interval. Therefore, $f(x)$ is concave down on the intervals $x \in (-1.5, -1)$ and $x \in (0, 1)$.

Answer to Exercise 37 (on page 44)

1. False. For $x \in (-2, 0)$, the slope of $f(x)$ is negative, which implies that $f'(x) < 0$ for $x \in (-2, 0)$.
2. False. The graph comes to a point at $x = 0$, therefore $\lim_{x \rightarrow 0^+} f'(x) \neq \lim_{x \rightarrow 0^-} f'(x)$, which means the limit does not exist and f is not differentiable at $x = 0$.
3. True. The graph of $f(x)$ is concave up for $x \in (0, 2)$, which means the second derivative is positive.
4. True. Recall that critical values are where derivatives equal 0 or do not exist. Since we have established that $f'(x)$ does not exist at $x = 0$, then there is a critical value at $x = 0$.

Answer to Exercise 38 (on page 45)

1. True. $f'(-3) = 0$, and f' changes from negative to positive there, so f switches from decreasing to increasing: a local minimum at $x = -3$.
2. False. Though it appears that $f'' = 0$ at $x = -2$, the slope of f' is positive before and after. Therefore, f'' does not cross the x -axis and there is not an inflection point at $x = -2$.
3. True. For $0 < x < 2$, the slope of f' is negative, which means f'' is negative, which

means f is concave down.

Answer to Exercise 39 (on page 46)

The graph of f has inflection points at $x = -1$ and $x = 1$. Since $f(x) = f(-x)$, we can expand

the table to include the entire window we are investigating:

	$-2 < x < -1$	$-1 < x < 0$	$0 < x < 1$	$1 < x < 2$
$f(x)$	Negative	Positive	Positive	Negative
$f'(x)$	Negative	Negative	Negative	Positive
$f''(x)$	Positive	Negative	Negative	Positive

Recall that inflection points occur when f'' changes from positive to negative or from negative to positive. Examining the table, we see that the sign of f'' changes at $x = -1$ and $x = 1$.

Answer to Exercise 40 (on page 49)

$\vec{v}(t) = \langle -\sin t, \cos t \rangle$, so $|\vec{v}(t)| = \sqrt{\sin^2 t + \cos^2 t} = 1$. This matches the Uniform Circular Motion and Centripetal Force chapter, where you found the same velocity with geometry: the speed is $r\omega = 1 \cdot 1 = 1$.

Answer to Exercise 41 (on page 52)

- **Seedlings:** Experiment. Fertilizer was randomly assigned. Causal conclusion is justified.
- **Screen time and attention:** Observational study. No assignment was made. Causal conclusion is not justified; many confounding variables (parenting style, diet, sleep) could explain the relationship.
- **Customer service training:** Experiment. The training was randomly assigned. Causal conclusion is justified.
- **Libraries and literacy:** Observational study. No assignment was made. Causal conclusion is not justified; wealthier cities may have had both more libraries and higher literacy for unrelated reasons.

Answer to Exercise 42 (on page 54)

1. Number the 60 seedlings from 1 to 60. Use a random number generator to select 30 for

the music group. The remaining 30 form the no-music group.

2. Control means growing all seedlings under identical conditions except for the music. Variables to hold constant include: soil type and quantity, pot size, amount and frequency of watering, light exposure, temperature, and humidity.
3. Replication. With only 5 seedlings per group, a single unusually fast or slow grower could distort the group average. With 30 per group, individual variation averages out and a real effect of the music is more detectable.
4. Yes. The no-music group serves as the control group, establishing the baseline growth rate without the treatment. Without it, there is nothing to compare the music group's growth against.

Answer to Exercise 43 (on page 56)

1. (1) Yes, random assignment. (2) No, volunteers are self-selected. Conclusion: the noisy room likely caused lower comprehension scores, but this result cannot be generalized to all university students or beyond. Volunteers may differ from the broader student population in ways that affect how noise impacts their reading.
2. (1) No random assignment. The researchers observed existing behavior and imposed no treatment. (2) Yes, random selection from voter rolls. Conclusion: the association between exercise and lower stress can be generalized to the adult population of the city. We cannot conclude that exercise caused lower stress because no treatment was assigned. A confounding variable such as income or work conditions could explain both.

Answer to Exercise 44 (on page 57)

3. (1) Yes, random assignment at the school level. The district used chance to decide which 10 of its 60 schools received the new curriculum, so the schools are the experimental units. Students were not individually assigned. (2) No random selection from a larger population. Every elementary school in the district took part, so the study covers the district's schools and makes no attempt to sample from other districts. Conclusion: the new curriculum likely caused the higher scores in this district's elementary schools. The result cannot be extended to schools in other districts.
4. (1) Yes, random assignment to app or printed plan. (2) No, volunteers are recruited through social media. Conclusion: the app likely caused greater endurance improvement compared to the printed plan. However, this result cannot be generalized to all people interested in fitness. Social media recruits may be younger, more comfortable with mobile apps, or more motivated than the broader population.

Answer to Exercise 45 (on page 60)

1. Experimental units: the 60 volunteers. Factor: warm-up routine. Levels: three (static stretching, dynamic stretching, no warm-up). Treatments: the three levels, since there is only one factor.
2. Number the volunteers from 01 to 60. Use `randInt(1,60,20)` on a TI-84, ignoring repeats, to select 20 volunteers for group 1 (static stretching). From the remaining 40, use `randInt` again to select 20 for group 2 (dynamic stretching). The remaining 20 form group 3 (no warm-up).
3. Yes. Because treatments were randomly assigned, a causal conclusion is appropriate, provided the experiment was otherwise well-controlled (same track, same time of day, same measurement method, etc.). Random assignment is what licenses the claim of causation.

Answer to Exercise 46 (on page 61)

1. Blocking factor: prior reading level (above grade level vs. at or below). If above-level readers respond better to one method regardless of which method it is, mixing them randomly with below-level readers would make it harder to detect the true method effect. Blocking ensures each method is compared within similar groups.
2. Blocking factor: steel mill source (mill A vs. mill B). Differences in base composition could affect how well coatings adhere, confounding the coating comparison. Blocking by mill source ensures all four coatings are tested on panels from each mill.

Answer to Exercise 47 (on page 63)

1. Randomized block design. There are three treatments and each GPA block contains more than two units. It is not a matched-pairs design because the block size exceeds two and there are more than two treatments.
2. Matched-pairs design. There are exactly two treatments and each block (pair of farms) contains exactly two units, one receiving each treatment.
3. Matched-pairs design (repeated measures). Each participant receives both treatments in random order and serves as their own pair of size two.

Answer to Exercise 48 (on page 65)

1. Completely randomized design. The volunteers are similar in relevant ways (comparable computing experience, no prior exposure to either interface). There is no identified variable that would consistently affect how they respond to one interface versus the other. Randomly assign 40 volunteers to the new interface and 40 to the existing interface.
2. Matched pairs (repeated measures). Each participant has their own baseline sleep patterns that differ from others. Using each person as their own pair eliminates individual differences in baseline sleep quality from the comparison. Randomly determine which temperature each volunteer experiences first, include a washout night between conditions, then compare each person's cool-night and warm-night sleep quality directly.
3. Randomized block design, blocking on school level (primary vs. secondary). Prior research indicates that the two groups respond differently to professional development formats. If teachers are randomly assigned to workshop formats without blocking, differences in how primary and secondary teachers respond could mask or distort the true effect of format. Block by school level and randomly assign teachers to the three formats within each block.
4. Randomized block design or matched pairs, blocking on baseline skin dryness level. Because baseline dryness varies widely, a participant who starts very dry might show a large reduction regardless of which moisturizer they use. Grouping participants by baseline dryness (for example, low, medium, high) and assigning both moisturizers within each group removes this variability from the comparison. Even better: have each participant use one moisturizer on their left arm and the other on their right, with a coin flip deciding which arm gets which. This matched-pairs design compares each person's skin to their own. Both designs earn credit here because each one accounts for baseline dryness; the matched-pairs version is stronger when repeated use is practical.



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