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Springs and Simple Harmonic Motion

A guitar string, a car's suspension over a bump, a skyscraper swaying in the wind, the quartz crystal ticking inside a watch: all of them vibrate back and forth in the same characteristic way. In the last chapters, you studied objects moving around circles. Now we look at objects that move back and forth.

In this chapter, we will study oscillatory motion, which is a type of motion where an object moves back and forth around an equilibrium point. Oscillations are common in many physical systems, such as springs, pendulums, and even electrical circuits. Oscillations are important not only because they appear everywhere in physical systems, but also because they provide a connection between simple motion and more complex behaviors. Small oscillations often behave in a very predictable way and can be described using *simple harmonic motion* (SHM), which is governed by restoring forces proportional to displacement.

(Engineers model oscillations with *differential equations*, which you will meet in the calculus volumes. Here, we will get far with the slopes of sine and cosine that you already know.)

1.1 Sine and Cosine Graphs [Review]

Let us recall the two pillars of trigonometry and, consequently, simple harmonic motion: Sine and Cosine. Recall that these two functions are periodic, meaning they repeat their values in regular intervals or periods. The sine and cosine functions have a period of 2π radians or 360° , repeating their values every 2π radians.

As discussed in the trigonometric functions chapter, the sine function represents the y-coordinate of a point on the unit circle as it moves around the circle, while the cosine function represents the x-coordinate. This is the fundamental behavior of the unit circle, where a point moves around the circle at a constant angular speed ω . The position of the point at any time t can be described using the following parametric equations:

$$x(t) = \cos(\omega t), \quad y(t) = \sin(\omega t)$$

The general forms of the sine and cosine functions can be expressed as:

$$x(t) = A \cos(\omega t + \phi), \quad y(t) = A \sin(\omega t + \phi)$$

where:

- A is the amplitude, representing the maximum displacement from the equilibrium position.
- ω is the angular speed, which determines how quickly the oscillations occur.
- ϕ is the phase shift, which determines the initial position of the oscillation at $t = 0$.

Recall from the Uniform Circular Motion and Centripetal Force chapter what happens when a point moves around a circle of radius A at angular speed ω . Its velocity is $-A\omega \sin(\omega t)$ in the x direction, and its acceleration points toward the center with size $A\omega^2$, so its x component is $-A\omega^2 \cos(\omega t)$.

Now watch only the x -coordinate, the shadow of the point on the x -axis:

- position: $x(t) = A \cos(\omega t)$
- velocity: $v(t) = -A\omega \sin(\omega t)$
- acceleration: $a(t) = -A\omega^2 \cos(\omega t)$

Compare the first and last lines: the acceleration is the position times $-\omega^2$:

$$a(t) = -\omega^2 x(t)$$

That is the signature of every oscillation in this chapter: the farther the object is from the middle, the harder it is pulled back.

1.2 Simple Harmonic Motion

Simple Harmonic Motion will reference many definitions we have seen before and will continue to see in this chapter. Here are some key terms to know:

Cycle A single complete execution of a periodically repeated motion. Another way to think about it is a particle, block, or object making a *round-trip*.

Period The time it takes to complete one cycle of motion. An identity that comes up often is $T = \frac{2\pi}{\omega}$.

Frequency The number of cycles completed per unit time. It is the reciprocal of the

period, represented as **Hz**.¹

Equilibrium The position at which the “default” position of an oscillating object is located. In terms of mechanics, this is the position where all the forces acting on the object are balanced or zero.

Restoring Force The force that acts to bring an oscillating object back to its equilibrium position. It is typically proportional to the displacement from equilibrium and acts in the opposite direction.

Angular Speed The rate at which an object moves around a circle, measured in radians per second.

So far, we have discussed oscillatory motion in general terms. Now, let us focus on a specific type of oscillatory motion known as Simple Harmonic Motion (SHM). SHM is characterized by the following properties:

- The motion occurs around an equilibrium position.
- The *restoring force* is directly proportional to the displacement from the equilibrium position and acts in the opposite direction.
- The motion is sinusoidal in nature, meaning it can be described using sine or cosine functions.
- The period and frequency of the motion are constant, regardless of the amplitude.
- The acceleration always points toward equilibrium and increases in size as the displacement grows.

So the motion continuously repeats itself in a regular pattern. It speeds up near equilibrium, and slows down to a stop at maximum displacement. Examples of these systems are

- A mass attached to a spring, either vertically or horizontally
- A simple pendulum
- Vibrations of a tuning fork or guitar string

In the pendulums chapter, you will see a swinging mass with both centripetal acceleration and acceleration due to gravity, oscillating back and forth with changing speed. Like all mechanical systems, an SHM system has both kinetic and potential energy. For general simple harmonic motion systems, we can describe its energy as follows:

¹One **Hz** is equal to one cycle per second. For example, a frequency of 5 Hz means that 5 cycles are completed in one second. On a monitor, a refresh rate of 60 Hz means the screen updates 60 times per second. Higher refresh rates can lead to smoother motion perception.

- The total mechanical energy E of the system is the sum of its kinetic energy K and potential energy U :

$$E = K + U$$

- The kinetic energy K , system dependent, is highest when the object passes through the equilibrium position.
- The potential energy U depends on the specific system, such as spring potential energy or gravitational potential energy, but is highest at the maximum displacement from equilibrium.

1.3 Springs

Let us talk about springs! Springs are mechanical devices, usually made out of some rigid metal material, that store and release energy through deformation. When a spring is stretched or compressed from its equilibrium position, it exerts a restoring force that tries to bring it back to its original shape. The most common type of spring is a coil spring, which is a wire wound into a helix.

1.3.1 Hooke's Law

In 1676, physicist Robert Hooke formulated a principle that describes the behavior of springs, now known as Hooke's Law. Hooke's Law states that the force exerted by an ideal spring is directly proportional to the displacement from its equilibrium position.

Mathematically, Hooke's Law can be expressed as:

$$F_s = -kx$$

where:

- F_s is the restoring force exerted by the spring (in newtons, N)
- k is the spring constant (in newtons per meter, N/m), which measures the stiffness of the spring. It is spring-specific
- x is the displacement from the equilibrium position (in meters, m)
- The negative sign represents the fact that the force attempts to bring the spring back to its equilibrium position.

Since we know that force is related to mass and acceleration through Newton's Second Law, we can combine Hooke's Law with Newton's Second Law to analyze the motion of

a mass attached to a spring. See Figure 1.1 for a diagram of a mass-spring system.

$$F_s = ma = -kx \quad (1.1)$$

$$ma = -kx \quad (1.2)$$

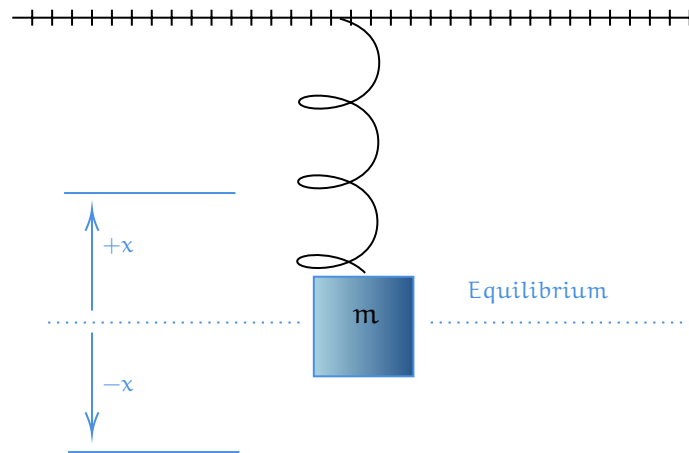


Figure 1.1: A diagram showing a block of mass m displaced $+x$ and $-x$ from equilibrium.

Dividing by m :

$$a = -\frac{k}{m}x \quad (1.3)$$

This is exactly the signature $a = -\omega^2 x$ from the start of the chapter, so the mass moves in simple harmonic motion with

$$\omega = \sqrt{\frac{k}{m}}$$

since the number in front of x in equation (1.3) plays the role of ω^2 . So the period is

$$T = 2\pi\sqrt{\frac{m}{k}}$$

and the frequency as

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

Notice that the amplitude does not appear. Pull the mass back a little or a lot, and it takes the same time per cycle.

Worked example. A 2 kg mass hangs on a spring with $k = 200$ N/m. What are its angular frequency and period?

Solution. $\omega = \sqrt{\frac{200}{2}} = 10$ rad/s, and $T = \frac{2\pi}{10} \approx 0.628$ s.

Exercise 1 **Mass-Spring Frequency Change**

A block is attached to a spring and set into oscillatory motion, and its frequency is measured. If this block were removed and replaced with a block of $1/2$ the mass, how would the frequency of oscillations compare to the original frequency?

Working Space

1. The frequency would be halved.
2. The frequency would remain the same.
3. The frequency would increase by a factor of $\sqrt{2}$.
4. The frequency would decrease by a factor of $\sqrt{2}$.

Answer on Page 87

1.3.2 Spring Potential Energy

Let us recall that Work from a physics standpoint is defined as the transfer of energy through a certain distance via a force. When we alter the length of a spring through either compression or extension, we are performing work on the spring. This work is stored as potential energy within the spring, which can be released when the spring returns to its equilibrium position.

How much energy is stored? Work is force times distance, but here the force changes as you stretch: it grows from 0 to kx_f . When force changes, work is the area under the force-versus-displacement graph. For a spring, that graph is a straight line, so the area is a triangle with base x_f and height kx_f (figure 1.2):

$$W = \frac{1}{2}(x_f)(kx_f) = \frac{1}{2}kx_f^2$$

This work is stored as potential energy in the spring:

$$\Delta U_s = \frac{1}{2}kx_f^2 \tag{1.4}$$

We can see this graphically in Figure 1.2, where the area under the force vs. displacement graph represents the work done on the spring, which is equal to the change in potential energy.

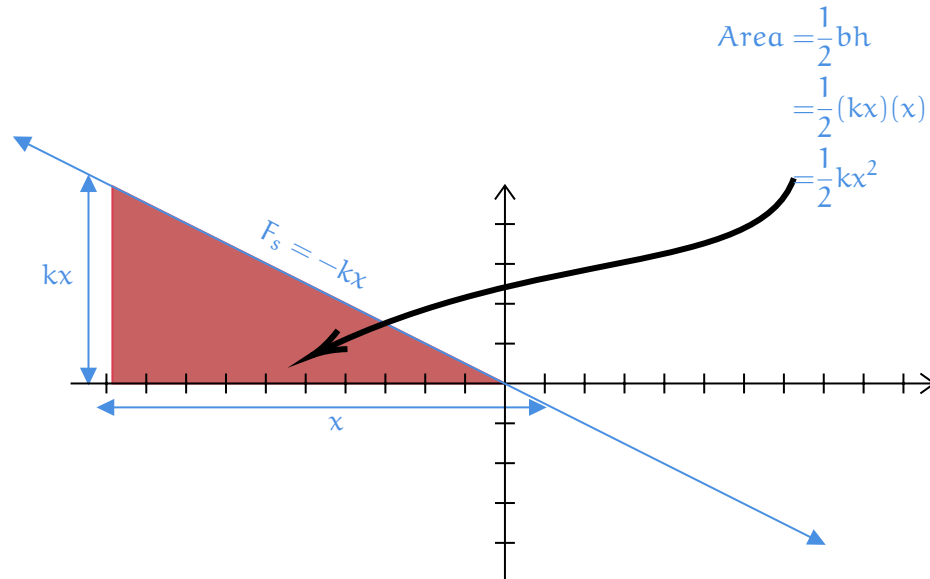


Figure 1.2: A graph of force versus displacement for a spring, equal to the work done and change in potential energy.

Notice that Equation (1.4) is similar in structure to the kinetic energy equation, $K = \frac{1}{2}mv^2$. This is not a coincidence, as both equations describe forms of energy in mechanical systems, and are derived from linear systems. When the effort (force) needed increases in direct proportion to what you are trying to change, the energy required ends up depending on the square of that change. For a spring

- the farther you stretch or compress it, the more force it exerts to return to equilibrium (Hooke's Law)
- the more energy is stored in the spring (spring potential energy)

For kinetic energy

- the faster you try to move an object, the more force is needed to accelerate it (Newton's Second Law)
- the more energy the object has due to its motion (kinetic energy)

Exercise 2 Finding the equation for a mass-spring system

This question is from an AP-style physics review. A block of mass 4 kg on a frictionless, horizontal table is attached to a spring with spring constant $k = 400 \text{ N/m}$ and undergoes simple harmonic oscillations about its equilibrium position and its amplitude $A = 6 \text{ cm}$. If the block is at $x = 6 \text{ cm}$ at time $t = 0$ and has a velocity of 0 cm/s at that time, which of the following equations (with x in cm and t in seconds) describes the block's position as a function of time?

Working Space

- (A) $x(t) = 6 \cos(10t)$
(B) $x(t) = 6 \sin(10t)$
(C) $x(t) = 6 \sin(10t + \frac{1}{2}\pi)$
(D) $x(t) = 6 \sin(10\pi t + \frac{1}{2}\pi)$
(E) $x(t) = 6 \sin(10\pi t - \frac{1}{2}\pi)$

*Answer on Page 87***Exercise 3** Energy Transfer in a Mass-Spring System

A pinball machine uses a spring to launch a ball of mass 0.2 kg . The spring has a spring constant of 500 N/m and is compressed by 0.24 m before release. Assuming energy is conserved and no frictional losses occur, what is the speed of the ball as it leaves the spring?

*Working Space**Answer on Page 88*

Exercise 4 A Lighter Mass

The same spring, with $k = 200$ N/m, now holds a 0.5 kg mass. Find the angular frequency and the period.

Working Space

Answer on Page 88

Pendulums and Alternating Current

2.1 Pendulums

A grandfather clock keeps time with a swinging weight, and a playground swing goes back and forth at its own steady rhythm no matter how high you pump it. Both are pendulums, and in this chapter you will find out what sets their rhythm, then see the same mathematics in the electricity in your walls.

You may be surprised to know that pendulums act the exact same way as roller coasters from the circular chapter, in terms of forces. But, similar to springs, they oscillate back and forth unless acted upon by a damping force. A pendulum has an equilibrium position $\theta = 0$, hanging straight down. A pendulum also has a maximum angle $\theta_{\max}$ from its equilibrium position. It will swing through a set arc in a repetitive motion, which is a type of motion we call simple harmonic motion, similar to the springs in the oscillations chapter. The tension force is always directed towards the pivot. This is diagramed below in Figure 2.1

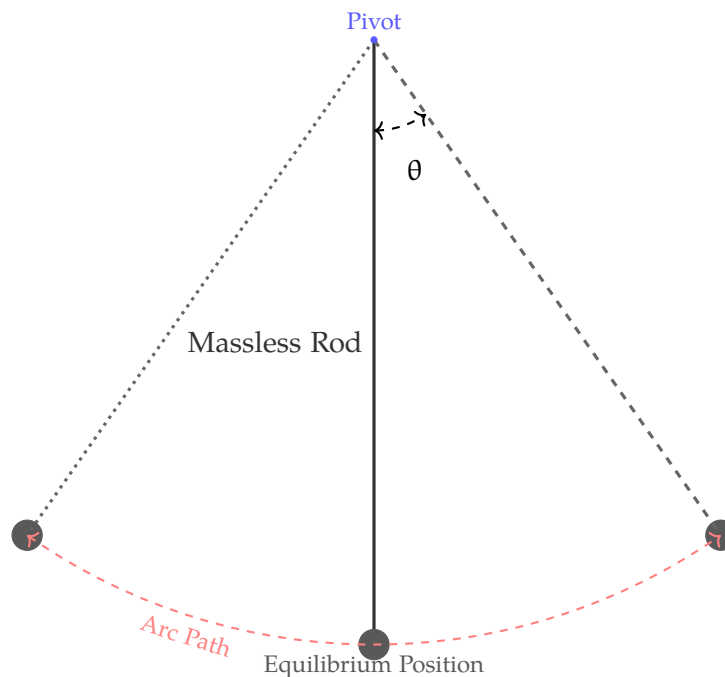


Figure 2.1: The anatomy of a pendulum.

When displaced by an angle θ from the vertical equilibrium position ($\theta = 0$), gravity provides a *restoring force* that acts along the arc of motion. This restoring force is given by $F = -mg \sin \theta$, where the negative sign indicates that the force always points back toward the equilibrium position. The tension in the string keeps the bob moving along its curved path, while gravity continually tries to return it to rest. The net force shifts such that the acceleration arrow is *not* always pointing to the center. The net force is a combination of the tension (centripetal) force and the gravitational (tangential), so rather it points at different points along the arc of the circle, as it is always changing due to the height of the pendulum. This is demonstrated in Figure 2.2. Note the following differences in energy and forces at the equilibrium position versus the maximum angle positions, summarized in Table 2.1. Note here that the magnitude of the restoring force $-mg \sin(\theta)$, which is

At $\theta = 0$	At $\theta = \theta_{\max}$
$F_{\text{restoring}} = 0$	$F_{\text{restoring}} = \max$
a_t (tangential acceleration) $= 0$	$a_t = \max$
$PE = 0$	$PE = mgh$
$KE = \max$	$KE = 0$
$v = \max$	$v = 0$

Table 2.1: Comparison of pendulum quantities at equilibrium and maximum displacement

not *directly* proportional to the angular displacement θ . However, for small angles (less than about 15° , or 0.26 radians), we can use the small-angle approximation $\sin(\theta) \approx \theta$ (in radians), and state the restoring force as $F \approx -mg\theta$. This allows us to treat the pendulum’s motion as simple harmonic motion for small oscillations, where the restoring force is approximately proportional to the angular displacement.

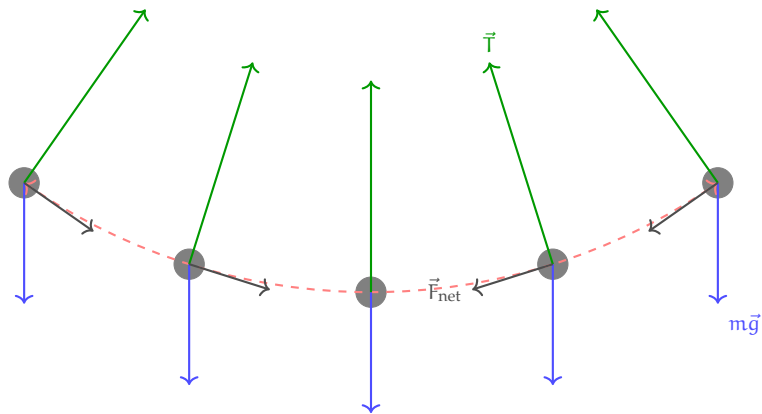


Figure 2.2: The net force at different locations of the arc.

Exercise 5 Pendulum Period Dependence

A simple pendulum consists of a mass

Working Space

m attached to a massless rod L , swinging under the influence of gravity. Derive the equation for the period of a simple pendulum. Which of the following changes would affect the period T of the pendulum's oscillation?

1. Increasing the mass m of the pendulum bob.
2. Increasing the length L of the string.
3. Increasing the amplitude (maximum angle) of the swing.
4. Decreasing the acceleration due to gravity g .

Answer on Page 88

The period of a simple pendulum depends on its length and on gravity, but not on the mass of the bob or (for small swings) the amplitude. Galileo noticed this around 1600, supposedly by timing a swinging lamp in a cathedral against his pulse, and it is why pendulums made the first accurate clocks. Like a mass on a spring, the pendulum's angle follows a sine wave: $\theta(t) = \theta_{\max} \sin\left(\sqrt{\frac{g}{L}} t + \phi_0\right)$.

Exercise 6 Pendulum on the Moon

A simple pendulum has a period of 2 seconds when measured on Earth. If the same pendulum were taken to the Moon, where the acceleration due to gravity is approximately $\frac{1}{6}$ of Earth's, what would be its new period?

Working Space

- a Approximately 0.8 seconds
- b Approximately 2 seconds
- c Approximately 4.9 seconds

d Approximately 6.8 seconds

[Answer on Page 89](#)

2.2 Alternating Current

We have talked about Alternating Current (AC) in previous chapters, but now we will explore its connection to oscillations. AC is an electric current that periodically reverses direction, in contrast to Direct Current (DC), which flows in a single direction and provides constant voltage. The voltage and current in an AC circuit vary sinusoidally with time, making them ideal for modeling using oscillatory functions.

In a DC circuit, the voltage is constant: $V = V_0$.

In an AC circuit, the voltage oscillates exactly like the position of a mass on a spring:

$$V(t) = V_{\text{peak}} \sin(\omega t + \phi) \quad (2.4)$$

where

- V_{peak} is the maximum voltage reached in either direction
- ω is the angular frequency, related to standard frequency by $\omega = \frac{2\pi}{T} = 2\pi f$
- ϕ which is the phase constant, or the starting position of the wave

Note that, because of the trigonometric nature of sine, the wave fluctuates between V_{peak} and $-V_{\text{peak}}$.

As you saw in the alternating current chapter, the average power from a sine-wave voltage is the same as from a steady voltage of $V_{\text{rms}} = \frac{V_{\text{peak}}}{\sqrt{2}}$.

2.2.1 Impedance

In DC, we use Resistance (R). In AC oscillations, we use *Impedance*. Impedance is the “total opposition” to the oscillation and is a combination of standard resistance and Reactance (X).

Capacitors oppose changes in voltage, which we call Capacitive Reactance:

$$X_C = \frac{1}{\omega C}$$

Inductive Reactance comes from inductors which oppose changes in current (acting like “electrical inertia”).

$$X_L = \omega L$$

Similarly, in AC circuits, the voltage and current waves do not always peak at the same time.

- In a purely inductive circuit, the current falls behind the voltage by 90° ($\pi/2$).
- In a purely capacitive circuit, the current leads the voltage by 90° ($\pi/2$).

2.2.2 Comparing Mass-Springs and AC

Mechanical System (Mass-Spring)	Electrical System (AC Circuit)
Displacement (x)	Charge (q)
Velocity (v)	Current (i)
Mass (m)	Inductance (L)
Spring Constant (k)	Reciprocal Capacitance (1/C)
Damping/Friction (b)	Resistance (R)

2.3 Summary

In this chapter, we have discussed oscillations via Simple Harmonic Motion, mass-springs, spring energy, pendulums, and AC currents.

- **Simple Harmonic Motion (SHM):** periodic motion in which a restoring force, proportional to displacement, pulls an object back toward equilibrium. One full cycle takes $T = 2\pi/\omega$.
- **Hooke’s Law:** For an ideal spring, $F_s = -kx$. This linear relationship leads to an angular frequency of $\omega = \sqrt{k/m}$.
- **Pendulums:** For small swings, $T = 2\pi\sqrt{L/g}$, independent of mass and amplitude.
- **Energy Dynamics:** In a closed system, energy oscillates between Kinetic (K) and Potential (U). Total mechanical energy remains constant ($E = \frac{1}{2}kA^2$), with K peaking at equilibrium and U peaking at maximum displacement.

- **AC circuits:** Voltage oscillates like a mass on a spring, with inductance playing the role of mass and $1/C$ the role of the spring constant.

This chapter leads us into our next chapter on orbits, the oscillations of planetary motion.

Worked example. What is the period of a pendulum 1 m long?

Solution. $T = 2\pi\sqrt{\frac{1}{9.8}} \approx 2.01$ s: about one second per swing in each direction.

Exercise 7 **Shorter Pendulum**

A pendulum has length 0.25 m. What is its period?

Working Space

Answer on Page 90

CHAPTER 3

Orbits

Right now, about 400 km over your head, the International Space Station is racing around Earth at 7.7 kilometers per second, circling the planet every hour and a half. It is falling the whole time, and it never lands. In this chapter, you will see how, and you will calculate the speeds that put satellites in orbit and rockets on their way to other planets.

Gravity is the force that attracts two bodies toward each other. Gravity is described by Newton's Law of Universal Gravitation, which states that every point mass attracts every other point mass with a force that is directly proportional to the product of their masses and inversely proportional to the square of the distance between them. The formula for gravitational force is:

$$F_g = \frac{Gm_1m_2}{r^2}$$

Where F_g is the gravitational force between two objects, G is the gravitational constant, m_1 and m_2 are the masses of the objects, and r is the distance between the centers of the two objects. G is a constant with a value of approximately $6.674 \times 10^{-11} \text{ N m}^2/\text{kg}^2$.

If we think about acceleration due to gravity where one mass is significantly larger than the other, we can rewrite the equation as:

$$g = \frac{F_g}{m_2} = \frac{Gm_1m_2}{r^2m_2} = \frac{Gm_1}{r^2}$$

Where m_1 is the mass of the more massive object and m_2 is the mass of the significantly less massive object. This cancelling of the m_2 mass is why the acceleration due to gravity is independent of the mass of the object in free fall.

A satellite stays in orbit around the planet because the pull of the planet's gravity causes it to accelerate toward the center of the planet.

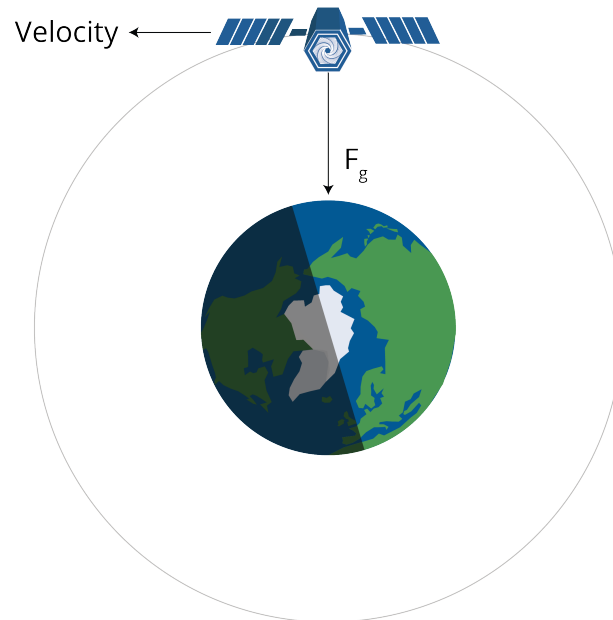


Figure 3.1: A satellite's centripetal force is gravity.

The satellite must be moving at a very particular speed to keep a constant distance from the planet — to travel in a circular orbit. If it is moving too slowly, it will get closer to the planet. If it is going too fast, it will get farther from the planet.

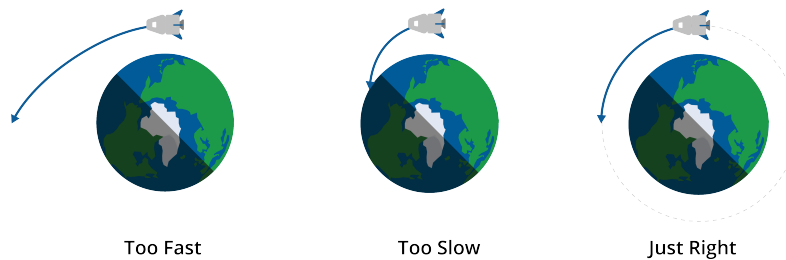


Figure 3.2: A diagram showing the required speeds for entering orbit.

The International Space Station orbits about 420 km above the ground. Earth's radius is 6,371 km, so the station's orbit has a radius of about 6.79 million meters. At that distance, $g = \frac{GM}{r^2}$ works out to about 8.6 m/s^2 : only about 12% less than the 9.8 m/s^2 on the surface.

How fast does the station need to move in a circle of radius 6.79 million meters to have a centripetal acceleration of 8.6 m/s^2 ? Real fast.

Recall that the acceleration vector is

$$a = \frac{v^2}{r}$$

Thus the velocity v needs to be:

$$v = \sqrt{ar} = \sqrt{8.64(6.79 \times 10^6)} \approx 7,660 \text{ m/s}$$

That is about 17,000 miles per hour. At that speed, one trip around the planet ($2\pi r \approx 42.7$ million meters) takes about 93 minutes, so astronauts on board see 16 sunrises a day.

When a satellite falls out of orbit, it hits the atmosphere at nearly that speed. The air in front of it is squeezed so violently that it heats to thousands of degrees, and the satellite usually burns up.

3.1 Astronauts are not weightless

Some people see astronauts floating inside an orbiting spacecraft and think there is no gravity: that the astronauts are so far away that the gravity of the planet does not affect them. This is incorrect. On the space station, gravity is still about 8.6 newtons per kg, almost 90% of what it is on the ground. The astronauts feel weightless because they and the spacecraft are falling together, so nothing pushes on them, just like you feel for a moment at the top of a roller coaster drop. They are moving so fast (in a direction perpendicular to gravity) that they do not collide with the planet.

Exercise 8 Mars Orbit

Working Space

The radius of Mars is 3.39 million meters. The atmosphere goes up another 11 km. Let us say you want to put a satellite in a circular orbit around Mars with a radius of 3.4 million meters.

The acceleration due to gravity on the surface of Mars is 3.721 m/s^2 . We can safely assume that it is approximately the same 11 km above the surface.

How fast does the satellite need to be traveling in its orbit? How long will each

orbit take?

Answer on Page 90

3.2 Geosynchronous Orbits

The planet Earth rotates once a day. Satellites in low orbits circle the Earth many times a day. Satellites in very high orbits circle less than once per day. There is a radius at which a satellite orbits exactly once per day. Satellites at this radius are known as “geosynchronous” or “geostationary”, because they are always directly over a place on the planet.

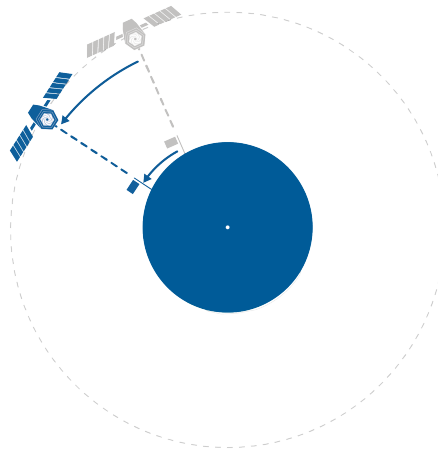


Figure 3.3: A satellite in geosynchronous orbit.

The radius of a circular geosynchronous orbit is 42.164 million meters, about 35,800 km above the surface of the earth.

A geosynchronous satellite travels at a speed of 3,070 m/s.

Geosynchronous satellites are used for weather monitoring and for communications such as satellite TV: point a dish at one spot in the sky, and it never needs to move. (GPS satellites, by contrast, fly lower, about 20,200 km up, and circle the Earth twice a day.)

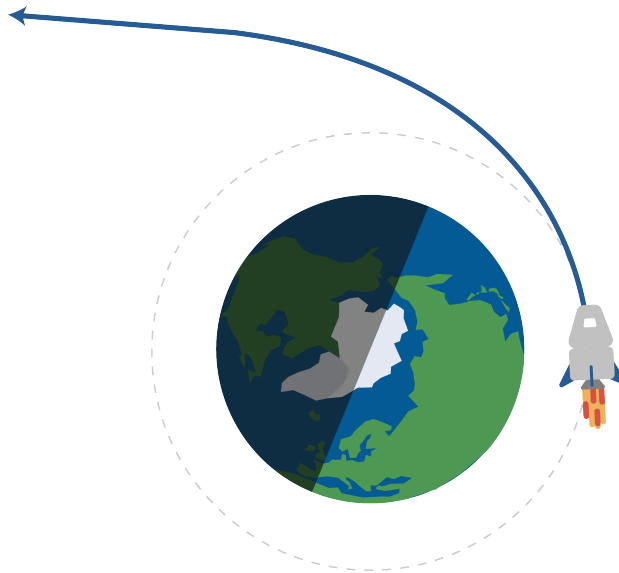


Figure 3.4: A satellite can reach a speed at which it "escapes" earth's orbit (centripetal force).

3.3 Kepler's Laws

There are three laws that describe the motion of planets and their elliptical properties.

1. Every planet moves along an ellipse with the sun at one focus.

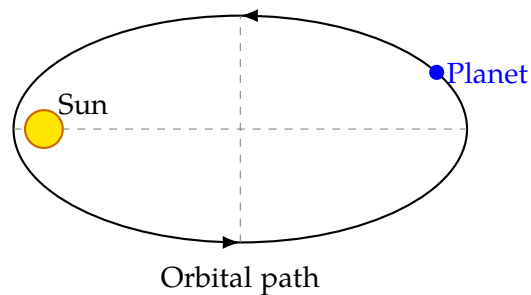


Figure 3.5: A diagram of Kepler's First Law.

2. As a planet moves around its orbit, it sweeps out equal areas in equal times. This means that a planet moves faster when it is closer to the sun and slower when it is farther from the sun. Note that this directly comes from the *Conservation of Angular Momentum*.

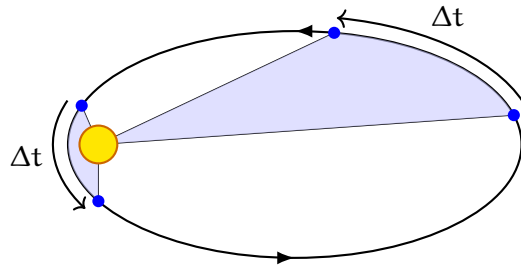


Figure 3.6: A diagram of Kepler's Second Law.

3. If T is the period of a planet's orbit, and a is the length of the semi-major axis of the ellipse, then $\frac{T^2}{a^3}$ is the same for all planets orbiting the same star.

3.4 Escape velocity

The escape velocity is the minimum speed needed for an object to “break free” from the gravitational attraction of a massive body or planet, without further propulsion. Gravity will never pull it back, because we assume no further forces act on the object. This is most commonly associated with a spaceship or satellite escaping from a planet's gravitational pull.

We need to understand that escape velocity relies on energy, rather than a force or acceleration. An object must have enough kinetic energy to overcome the gravitational potential energy pulling it back toward the planet. To escape from an orbit, the object's mechanical energy (the sum of its kinetic and potential energy) must be greater than or equal to zero.

We can derive its initial energy as: $E = K + U = \frac{1}{2}mv^2 - \frac{GMm}{R}$, where m is the mass of the object, M is the mass of the planet, R is the distance from the center of the planet to the object, and G is the gravitational constant.

After its escape, the object will be infinitely far away from the planet, and its potential energy will be zero. If we assume it just barely escapes, its kinetic energy will also be zero. Therefore, we set the initial energy equal to zero:

$$\frac{1}{2}mv^2 - \frac{GMm}{R} = 0 \quad (3.1)$$

$$\frac{1}{2}mv^2 = \frac{GMm}{R} \quad (3.2)$$

$$v^2 = \frac{2GM}{R} \quad (3.3)$$

$$v_{\text{esc}} = \sqrt{\frac{2GM}{R}} \quad (3.4)$$

Notice that the mass of the object m cancels out, meaning that the escape velocity is

independent of the mass of the escaping object. It only depends on the mass of the planet and the distance from its center.

Further, **notice that the escape velocity is $\sqrt{2}$ times the orbital velocity**, $v_{\text{orb}} = \sqrt{GM/R}$ at the same distance from the planet's center. This means that to escape from a circular orbit, an object must increase its speed by about 41.4%. **Worked example.** What is the escape velocity from Earth's surface? Earth has mass $M = 5.972 \times 10^{24}$ kg and radius $R = 6.371 \times 10^6$ m.

Solution. $v = \sqrt{\frac{2(6.674 \times 10^{-11})(5.972 \times 10^{24})}{6.371 \times 10^6}} \approx 11,200$ m/s, or about 11.2 km/s (25,000 mph). This means that any object, regardless of its mass, must reach this speed to escape Earth's gravitational pull without further propulsion.

Exercise 9 Escape Velocity from Mars

Working Space

The radius of Mars is 3.4×10^6 meters. The mass of Mars is 6.42×10^{23} kg. A spaceship is sitting on the surface of Mars.

1. Find the escape velocity from the surface of Mars.
2. Suppose the spaceship is in a circular orbit 11 km above the surface of Mars. What is the escape velocity from that orbit?

Answer on Page 90

Note that giving an object an extra height h causes the escape velocity to decrease (as escape velocity is inversely proportional to the square root of the radius plus height). This is due to gravitation potential energy being less negative at larger radii. However, the effect is small compared to the overall radius of the planet. Problems may give you a radius plus height, or just the radius of the planet, or a total distance from the center, depending on the context, so read each problem carefully.

Exercise 10 Escape from Mars

Mars has mass 6.42×10^{23} kg and radius 3.39×10^6 m. Find its escape velocity.

Working Space

Answer on Page 91

Simulation with Vectors

Astronomers cannot rewind the solar system to see how the Moon formed, and NASA cannot test-fly a probe to Jupiter before launch. They simulate instead. In earlier chapters, you simulated a skate on a spring and an artillery shell one small time step at a time. Now you will simulate a whole system of bodies pulling on each other, keeping track of positions, velocities, accelerations, and forces as vectors.

In this chapter, you are going to simulate two moons that, as they wandered through the vast universe, get caught in each other's gravity well. We will assume there are no other forces acting upon the moons.

4.1 Force, Acceleration, Velocity, and Position

Worked example: one step of a simulation. A particle has position $x = 0$ and velocity $v = 2$ m/s. With time step $\Delta t = 0.5$ s, the next position is $x + v \Delta t = 0 + 2(0.5) = 1$ m.

We talked about the magnitude of a gravitational attraction between two masses:

$$F = G \frac{m_1 m_2}{r^2}$$

where F is the magnitude of the force in newtons, m_1 and m_2 are the masses in kg, r is the distance between them in meters, and G is the universal gravitational constant: 6.67430×10^{-11} N·m²/kg².

What is the direction? For the two moons, the force on moon 1 will pull toward moon 2. Likewise, the force on moon 2 will pull toward moon 1.

Of course, if something is big (like the sun), you need to be more specific: The force points directly at the center of mass of the object that is generating the force.

Each of the moons will start off with a velocity vector. That velocity vector will change over time as the moon is accelerated by the force of gravity. If you have a mass m with an initial velocity vector of $\vec{v}_0$ that is being accelerated with a constant force vector $\vec{F}$, at time t , the new velocity vector will be:

$$\vec{v}_t = \vec{v}_0 + \frac{t}{m} \vec{F}$$

If an object is at an initial position vector of $\vec{p}_0$ and moves with a constant velocity vector $\vec{v}$ for time t , the new position will be given by

$$\vec{p}_t = \vec{p}_0 + t\vec{v}$$

4.2 Simulations and Step Size

As two moons orbit each other, the force, acceleration, velocity, and position are changing smoothly and continuously. It is difficult to simulate truly continuous things on a digital computer.

However, think about how a movie shows you many frames each second. Each frame is a still picture of the state of the system. The more frames per second, the smoother it looks.

We do a similar trick in simulations. We say “We are going to run our simulation in 2 hour steps. We will assume that the acceleration and velocity were constant for those two hours. We will update our position vectors accordingly, then we will recalculate our acceleration and velocity vectors.”

Generally, as you make the step size smaller, your simulation will get more accurate and take longer to execute.

4.3 Make a Text-based Simulation

To start, you are going to write a Python program that simulates the moons and prints out their position for every time step. Later, we will add graphs and even animation.

We are going to assume the two moons are traveling the same plane so we can do all the math and graphing in 2 dimensions.

Each moon will be represented by a dictionary containing the state of the moon:

- Its mass in kilograms
- Its position — A 2-dimensional vector represent x and y coordinates of the center of the moon.
- Its velocity — A 2-dimensional vector
- Its radius — Each moon has a radius so we know when the centers of the two moons are so close to each other that they must have collided.
- Its color — We will use that when do the plots and animations. One moon will be red,

the other blue.

There will then be a loop where we will update the positions of the moons and then recalculate the acceleration and velocities.

How much time will be simulated? 100 days or until the moons collide, whichever comes first.

We will use numpy arrays to represent our vectors.

Create a file called `moons.py`, and type in this code:

```
1 import numpy as np
2
3 # Constants
4 G = 6.67430e-11          # Gravitational constant (Nm^2/kg^2)
5 SEC_PER_DAY = 24 * 60 * 60 # How many seconds in a day?
6 MAX_TIME = 100 * SEC_PER_DAY # 100 days
7 TIME_STEP = 2 * 60 * 60   # Update every two hours
8
9 # Create the initial state of Moon 1
10 m1 = {
11     "mass": 6.0e22, # kg
12     "position": np.array([0.0, 200_000_000]), # m
13     "velocity": np.array([100.0, 25.0]), # m/s
14     "radius": 1_500_000.0, # m
15     "color": "red" # For plotting
16 }
17
18 # Create the initial state of Moon 2
19 m2 = {
20     "mass": 11.0e22, # kg
21     "position": np.array([0.0, -150_000_000]), # m
22     "velocity": np.array([-45.0, 2.0]), # m/s
23     "radius": 2_000_000.0, # m
24     "color": "blue" # For plotting
25 }
26
27 # Lists to hold positions and time
28 position1_log = []
29 position2_log = []
30 time_log = []
31
32 # Start at time zero seconds
33 current_time = 0.0
34
35 # Loop until current time exceed Max Time
36 while current_time <= MAX_TIME:
37
38     # Add time and positions to log
```

```

39 time_log.append(current_time)
40 position1_log.append(m1["position"])
41 position2_log.append(m2["position"])
42
43 # Print the current time and positions
44 print(f"Day {current_time/SEC_PER_DAY:.2f}:")
45 print(f"\tMoon 1:({m1['position'][0]:.1f},{m1['position'][1]:.1f})")
46 print(f"\tMoon 2:({m2['position'][0]:.1f},{m2['position'][1]:.1f})")
47
48 # Update the positions based on the current velocities
49 m1["position"] = m1["position"] + m1["velocity"] * TIME_STEP
50 m2["position"] = m2["position"] + m2["velocity"] * TIME_STEP
51
52 # Find the vector from moon1 to moon2
53 delta = m2["position"] - m1["position"]
54
55 # What is the distance between the moons?
56 distance = np.linalg.norm(delta)
57
58 # Have the moons collided?
59 if distance < m1["radius"] + m2["radius"]:
60     print(f"*** Collided {current_time:.1f} seconds in!")
61     break
62
63 # What is a unit vector that points from moon1 toward moon2?
64 direction = delta / distance
65
66 # Calculate the magnitude of the gravitational attraction
67 magnitude = G * m1["mass"] * m2["mass"] / (distance**2)
68
69 # Acceleration vector of moon1 (a = f/m)
70 acceleration1 = direction * magnitude / m1["mass"]
71
72 # Acceleration vector of moon2
73 acceleration2 = (-1 * direction) * magnitude / m2["mass"]
74
75 # Update the velocity vectors
76 m1["velocity"] = m1["velocity"] + acceleration1 * TIME_STEP
77 m2["velocity"] = m2["velocity"] + acceleration2 * TIME_STEP
78
79 # Update the clock
80 current_time += TIME_STEP
81
82 print(f"Generated {len(position1_log)} data points.")

```

When you run the simulation, you will see the positions of the moons for 100 days:

```

1 > python3 moons.py
2 Day 0.00:
3     Moon 1:(0.0,200,000,000.0)
4     Moon 2:(0.0,-150,000,000.0)

```

```

5 Day 0.08:
6     Moon 1: (720,000.0, 200,180,000.0)
7     Moon 2: (-324,000.0, -149,985,600.0)
8 Day 0.17:
9     Moon 1: (1,439,990.7, 200,356,896.1)
10    Moon 2: (-647,995.0, -149,969,507.0)
11    ...
12 Day 100.00:
13    Moon 1: (119,312,305.5, 283,265,313.5)
14    Moon 2: (17,393,287.9, -60,319,261.9)
15 Generated 1201 data points.

```

Look over the code. Make sure you understand what every line does.

4.4 Graph the Paths of the Moons

Now, you will use the matplotlib to graph the paths of the moons. Add this line to the beginning of `moons.py`.

```

1 import matplotlib.pyplot as plt

```

Add this code to the end of your `moons.py`:

```

1 # Convert lists to np.arrays
2 positions1 = np.array(position1_log)
3 positions2 = np.array(position2_log)
4
5 # Create a figure with a set of axes
6 fig, ax = plt.subplots(1, figsize=(7.2, 10))
7
8 # Label the axes
9 ax.set_xlabel("x (m)")
10 ax.set_ylabel("y (m)")
11 ax.set_aspect("equal", adjustable='box')
12
13 # Draw the path of the two moons
14 ax.plot(positions1[:, 0], positions1[:, 1], m1["color"], lw=0.7)
15 ax.plot(positions2[:, 0], positions2[:, 1], m2["color"], lw=0.7)
16
17 # Save out the figure
18 fig.savefig("plotmoons.png")

```

When you run it, your `plotmoons.png` should look like this:

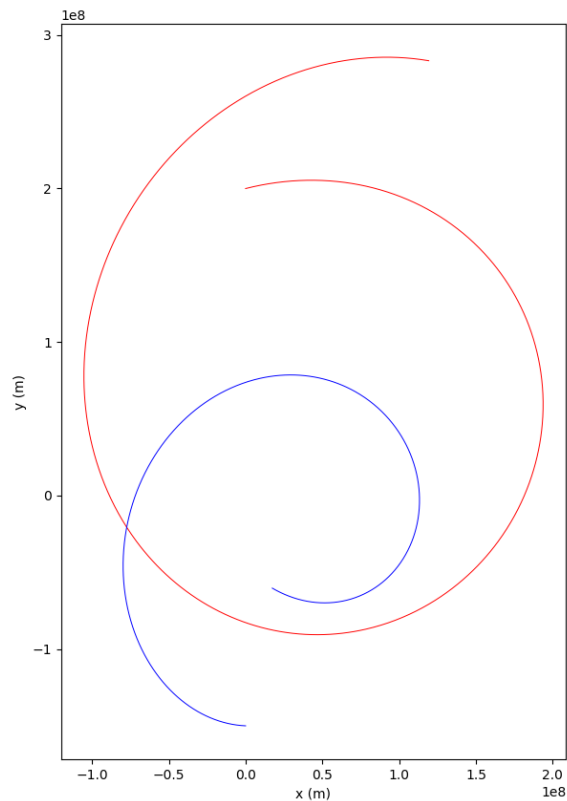


Figure 4.1: Output of moons_01.py.

It is nifty to see the paths, but we do not know where each moon was at a particular time. In fact, it is difficult to figure out which end of each curve was the beginning and which was the ending.

What if we added some lines and labels every 300 steps to put a sense of time into the plot? Add one more constant after the import statements:

```
1 PAIR_LINE_STEP = 300 # How many time steps between pair lines
```

Immediately before you save the figure to the file, add the following code:

```
1 # Draw some pair lines that help the
2 # viewer understand time in the graph
3 i = 0
4 while i < len(positions1):
5
6     # Where are the moons at the ith entry?
7     a = positions1[i, :]
8     b = positions2[i, :]
9     ax.plot([a[0], b[0]], [a[1], b[1]], "--", c="gray", lw=0.6, marker=".")
```

```

10
11     # What is the time at the ith entry?
12     t = time_log[i]
13
14     # Label the location of moon 1 with the day
15     ax.text(a[0], a[1], f"{t/SEC_PER_DAY:.0f} days")
16     i += PAIR_LINE_STEP

```

When you run it, your plot should look like this:

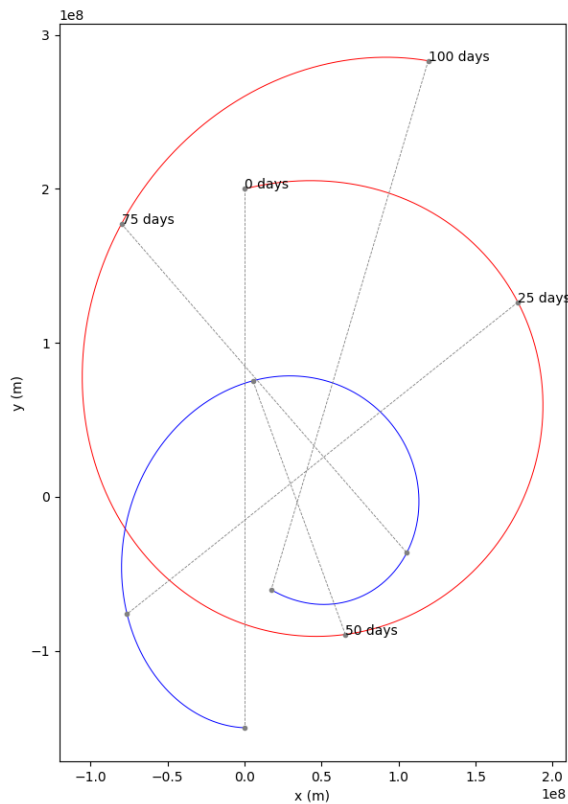


Figure 4.2: Plot moons_02.py.

Now you can get a feel for what happened. The moons were attracted to each other by gravity and started to circle each other. The heavier moon accelerates less quickly, so it makes a smaller loop.

Maybe we will get a better feel for what is happening if we look at more time. Let us increase it to 400 days. Change the relevant constant:

```

1 MAX_TIME = 400 * SEC_PER_DAY # 400 days

```

Now it should look like this:

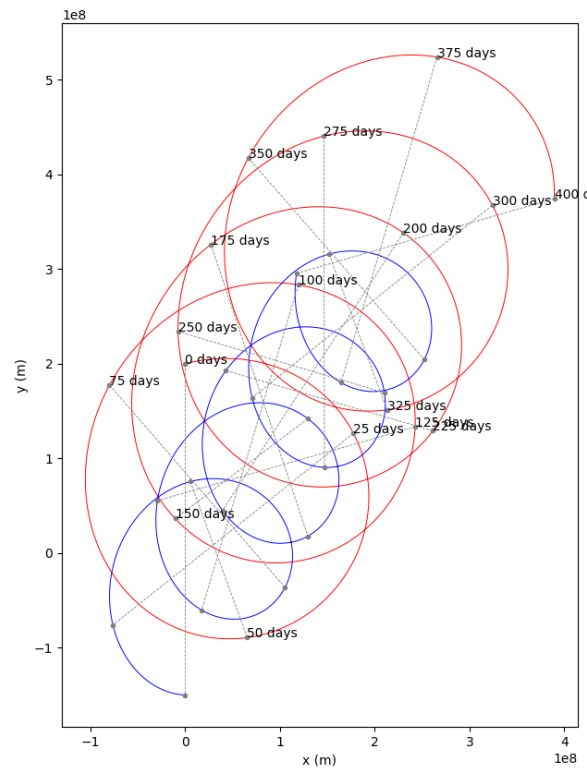


Figure 4.3: Output of plot moons_03.py.

Now you can see the pattern. They are rotating around each other and the pair is gradually migrating up and to the right.

4.5 Conservation of Momentum

Recall the idea of momentum being conserved in a system. In these programs, you are observing an extra important idea: *the momentum of a system will be conserved*. That is, absent forces from outside the system, the velocity of the center of mass will not change.

We can compute the initial center of mass and its velocity. In both cases, we just do a weighted average using the mass of the moon as the weight.

Immediately after you initialize the state of two moons, calculate the initial center of mass and its velocity:

```
1 # Calculate the initial position and velocity of the center of mass
2 tm = m1["mass"] + m2["mass"] # Total mass
3 cm_position = (m1["mass"] * m1["position"] + m2["mass"] * m2["position"]) / tm
```

```
4 cm_velocity = (m1["mass"] * m1["velocity"] + m2["mass"] * m2["velocity"]) / tm
```

Let us record the center of mass for each time. Before the loop starts, create a list to hold them:

```
1 cm_log = []
```

Inside the loop (before any calculations), append the current center of mass position to the log:

```
1 cm_log.append(cm_position)
```

Anywhere later in the loop (after you update the positions of the moon), update `cm_position`:

```
1 # Update the center of mass
2 cm_position = cm_position + cm_velocity * TIME_STEP
```

Now, let us look at the positions of the moons relative to the center of mass. Before you do any plotting, convert the list to a numpy array and subtract it from the positions:

```
1 cms = np.array(cm_log)
2
3 # Make positions relative to the center of mass
4 positions1 = positions1 - cms
5 positions2 = positions2 - cms
```

When you run it, you can really see what is happening:

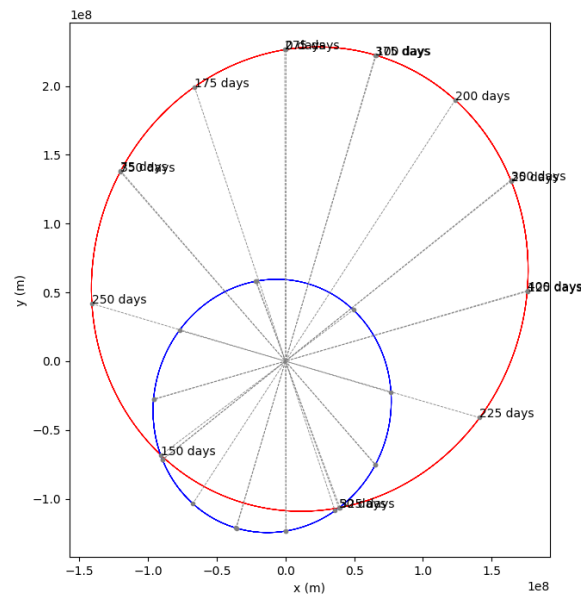


Figure 4.4: Output of plotmoons_04.py.

The moons are tracing elliptical paths. The center of mass is the focus point for both of them.

4.6 Animation

One of the features of matplotlib that not a lot of people understand is how to make animations with it. This seems like a really great opportunity to make an animation showing the position, velocity, acceleration of the moons. We will also show the center of mass.

The trick to animations is that you create a bunch “artist” objects. You create a function that updates the artists. matplotlib will call your functions, tell the artists to draw themselves, and make a movie out of that.

Make a copy of moons.py called animate_moons.py.

Edit it to look like this:

```

1 import numpy as np
2 import matplotlib.pyplot as plt
3
4 # Import animation support and artists
5 from matplotlib.animation import FuncAnimation
6 from matplotlib.patches import Circle, FancyArrow

```

```

7 from matplotlib.text import Text
8
9 # Constants
10 G = 6.67430e-11 # Gravitational constant (Nm^2/kg^2)
11 SEC_PER_DAY = 24 * 60 * 60 # How many seconds in a day?
12 MAX_TIME = 400 * SEC_PER_DAY # 400 days
13 TIME_STEP = 12 * 60 * 60 # Update every 12 hours
14 FRAMECOUNT = MAX_TIME / TIME_STEP # How many frames in animation
15 ANI_INTERVAL = 1000 / 50 # ms for each frame in animation
16
17 # The velocity and acceleration vectors are invisible
18 # unless we scale them up. A lot.
19 VSCALE = 140000.0
20 ASCALE = VSCALE * 800000.0
21
22 # Create the initial state of Moon 1
23 m1 = {
24     "mass": 6.0e22, # kg
25     "position": np.array([0.0, 200_000_000]), # m
26     "velocity": np.array([100.0, 25.0]), # m/s
27     "radius": 1_500_000.0, # m
28     "color": "red", # For plotting
29 }
30
31 # Create the initial state of Moon 2
32 m2 = {
33     "mass": 11.0e22, # kg
34     "position": np.array([0.0, -150_000_000]), # m
35     "velocity": np.array([-45.0, 2.0]), # m/s
36     "radius": 2_000_000.0, # m
37     "color": "blue", # For plotting
38 }
39
40 # Calculate the initial position and velocity of the center of mass
41 tm = m1["mass"] + m2["mass"] # Total mass
42 cm_position = (m1["mass"] * m1["position"] + m2["mass"] * m2["position"]) / tm
43 cm_velocity = (m1["mass"] * m1["velocity"] + m2["mass"] * m2["velocity"]) / tm
44
45 # Start at time zero seconds
46 current_time = 0.0
47
48 # Create the figure and axis
49 fig, ax = plt.subplots(1, figsize=(7.2, 10))
50
51 # Set up the axes
52 ax.set_xlabel("x (m)")
53 ax.set_xlim((-1.2e8, 4e8))
54 ax.set_ylabel("y (m)")
55 ax.set_ylim((-1.6e8, 5.5e8))
56 ax.set_aspect("equal", adjustable="box")
57 fig.tight_layout()
58
59 # Create artists that will be edited in animation

```

```

60 time_text = ax.add_artist(Text(0.03, 0.95, "", transform=ax.transAxes))
61 circle1 = ax.add_artist(Circle((0, 0), radius=m1["radius"], color=m1["color"]))
62 circle2 = ax.add_artist(Circle((0, 0), radius=m2["radius"], color=m2["color"]))
63 circle_cm = ax.add_artist(Circle((0, 0), radius=m2["radius"], color="purple"))
64 varrow1 = ax.add_artist(FancyArrow(0, 0, 0, 0, color="green",
↳ head_width=m1["radius"]))
65 varrow2 = ax.add_artist(FancyArrow(0, 0, 0, 0, color="green",
↳ head_width=m2["radius"]))
66 acc_arrow1 = ax.add_artist(
67     FancyArrow(0, 0, 0, 0, color="purple", head_width=m1["radius"])
68 )
69 acc_arrow2 = ax.add_artist(
70     FancyArrow(0, 0, 0, 0, color="purple", head_width=m2["radius"])
71 )
72
73
74 # This function will get called for every frame
75 def animate(frame):
76
77     # Global variables needed in scope from the model
78     global cm_position, cm_velocity, current_time, m1, m2
79
80     # Global variables needed in scope from the artists
81     global time_text, varrow1, varrow2, acc_arrow1, acc_arrow2, circle1, circle2,
↳ circle_cm
82
83     print(f"Updating artists for day {current_time/SEC_PER_DAY:.1f}.")
84
85     # Update the positions based on the current velocities
86     m1["position"] = m1["position"] + m1["velocity"] * TIME_STEP
87     m2["position"] = m2["position"] + m2["velocity"] * TIME_STEP
88
89     # Update day label
90     time_text.set_text(f"Day {current_time/SEC_PER_DAY:.0f}")
91
92     # Update positions of circles
93     circle1.set_center(m1["position"])
94     circle2.set_center(m2["position"])
95
96     # Update velocity arrows
97     varrow1.set_data(
98         x=m1["position"][0],
99         y=m1["position"][1],
100         dx=VSCALE * m1["velocity"][0],
101         dy=VSCALE * m1["velocity"][1],
102     )
103     varrow2.set_data(
104         x=m2["position"][0],
105         y=m2["position"][1],
106         dx=VSCALE * m2["velocity"][0],
107         dy=VSCALE * m2["velocity"][1],
108     )
109

```

```

110 # Update the center of mass
111 cm_position = cm_position + cm_velocity * TIME_STEP
112 circle_cm.set_center(cm_position)
113
114 # Find the vector from moon1 to moon2
115 delta = m2["position"] - m1["position"]
116
117 # What is the distance between the moons?
118 distance = np.linalg.norm(delta)
119
120 # Have the moons collided?
121 if distance < m1["radius"] + m2["radius"]:
122     print(f"*** Collided {current_time:.1f} seconds in!")
123
124 # What is a unit vector that points from moon1 toward moon2?
125 direction = delta / distance
126
127 # Calculate the magnitude of the gravitational attraction
128 magnitude = G * m1["mass"] * m2["mass"] / (distance**2)
129
130 # Acceleration vector of moons (a = f/m)
131 acceleration1 = direction * magnitude / m1["mass"]
132 acceleration2 = (-1 * direction) * magnitude / m2["mass"]
133
134 # Update the acceleration arrows
135 acc_arrow1.set_data(
136     x=m1["position"][0],
137     y=m1["position"][1],
138     dx=ASCALE * acceleration1[0],
139     dy=ASCALE * acceleration1[1],
140 )
141 acc_arrow2.set_data(
142     x=m2["position"][0],
143     y=m2["position"][1],
144     dx=ASCALE * acceleration2[0],
145     dy=ASCALE * acceleration2[1],
146 )
147
148 # Update the velocity vectors
149 m1["velocity"] = m1["velocity"] + acceleration1 * TIME_STEP
150 m2["velocity"] = m2["velocity"] + acceleration2 * TIME_STEP
151
152 # Update the clock
153 current_time += TIME_STEP
154
155 # Return the artists that need to be redrawn
156 return (
157     time_text,
158     varrow1,
159     varrow2,
160     acc_arrow1,
161     acc_arrow2,
162     circle1,

```

```

163         circle2,
164         circle_cm,
165     )
166
167
168 # Make the rendering happen
169 animation = FuncAnimation(
170     fig,
171     animate,
172     np.arange(FRAMECOUNT),
173     interval=ANI_INTERVAL
174 )
175
176 # Save the rendering to a video file
177 animation.save("moonmovie.mp4")

```

When you run this, it will take longer than the previous versions. You should have a video file that shows a simulation of the moons tracing their elliptical paths around their center of mass:

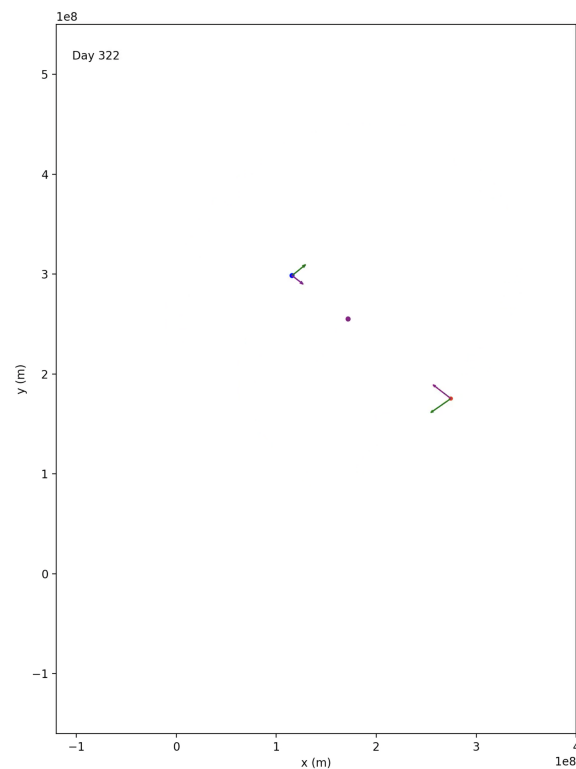


Figure 4.5: A still of the animation produced.

4.7 Challenge: The Three-Body Problem

It is time to stretch a little as a physicist and programmer: You are going to make a new version of `moons.py` that handles three moons instead of just two.

This is known as “the three-body problem.” Since Newton’s time, people have tried to find a formula that tells (from the starting positions and velocities) where three bodies will be at any time t , without doing a simulation. In general, no one has, and in the 1880s Henri Poincaré showed why: the motion can be chaotic.

For a lot of problems, the outcome is not very sensitive to the initial conditions. For example, consider the flight of a cannonball: If it leaves the muzzle of the cannon a little faster, it will go a little farther.

For the three-body problem, the outcome can be radically different even if the initial conditions are very similar.

(There is a whole field of mathematics studying systems that are very sensitive to initial conditions. It is known as *dynamical systems* or *chaos theory*.)

Copy `moons.py` to `3moons.py`. Here is a reasonable initial state for your third moon:

```
1 m3 = {  
2     "mass": 4.0e22, # kg  
3     "position": np.array([50_000_000, 80_000_000]), # m  
4     "velocity": np.array([-30.0, -35.0]), # m/s  
5     "radius": 1_700_000.0, # m  
6     "color": "green"  
7 }
```

If we run that simulation for 100 days, we get a plot like this:

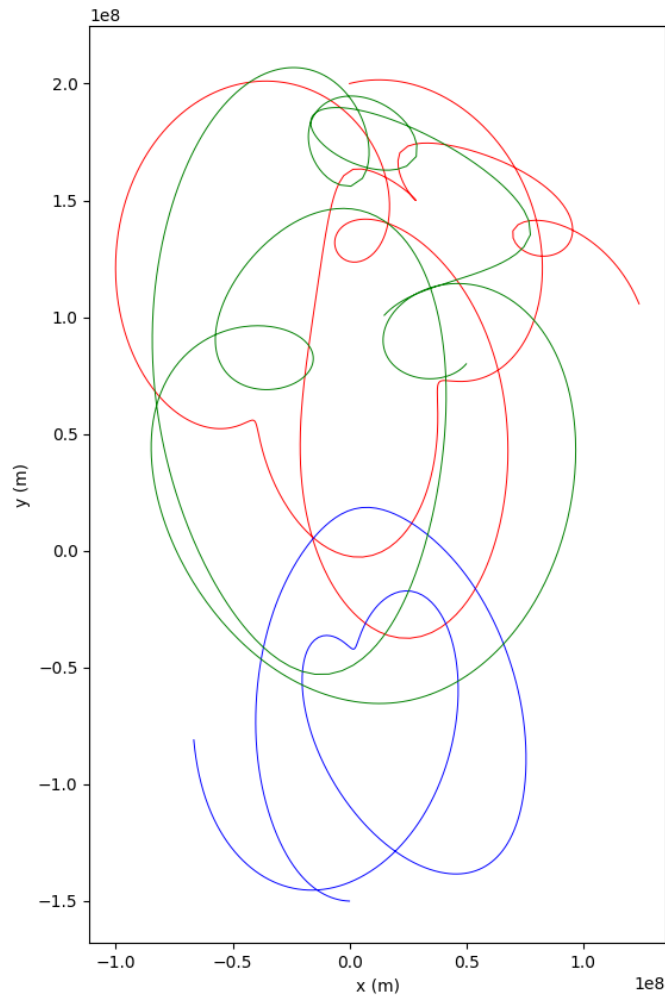


Figure 4.6: The Three Body Problem.

This is very different from the two-body problem, which just traced ellipses around the center of mass.

Exercise 11 Butterfly effect

In `3moons.py`, change the third moon's starting x position from 50,000,000 m to 50,001,000 m, just 1 km different, and run it again. Then try a similar 1 km change in the two-moon `moons.py` (move moon 1's starting x from 0 to 1,000 m). Which simulation's outcome changes more? What does that tell you

Working Space

about predicting three-body systems?

Answer on Page 91

Exercise 12 **Another Step**

A particle at $x = 1$ m has velocity $v = -3$ m/s. With $\Delta t = 0.1$ s, what is the next position?

Working Space

Answer on Page 91

Tides and Eclipses

Your life on earth involves many orbital paths:

- The Earth is spinning. If you are standing at the equator, you are traveling at 1,674 km per hour around the center of the planet. We are all spinning east, which is why the sun comes up in the east and sets in the west.
- The Earth is orbiting the sun. It takes 365.242 days for the Earth to go once around the sun. This is why different constellations appear at different times during the year — we only see the stars at night and the direction of night shifts as the Earth moves around the sun.
- The moon is orbiting the Earth. The moon travels once around the Earth once every 27.3 days.

You can see the effects of these orbits on our planet. Let us go over a few.

5.1 Leap Years

Note that it takes 365.242 days for the Earth to go around the sun. If we declared “The calendar will *always* be 365 days per year!” then the seasons would gradually shift by 0.242 days every year. After a century, they would have migrated 24 days.

So, we made a rule: “Every fourth year, we will add an extra day to the calendar!” The years 2021, 2022, and 2023 get no February 29th, but 2024 does.

That got us a calendar with an average 365.25 days per year, so the seasons would not have migrated as quickly, but they still would have migrated about three days every four hundred years.

So, we made another rule: ‘There will be no February 29th in the three century years (multiples of 100) that are not multiples of 400.’ So the year 1900 had no Feb 29, but the year 2000 had one. Now, the average number of days per year is 365.2425, very close to the true 365.242.

Worked example. Will 2100 be a leap year?

Solution. 2100 is divisible by 4 and by 100, but not by 400, so it is not a leap year.

5.2 Phases of the Moon

The Earth, the moon, and the sun form a triangle. If you were standing on the moon, you could measure the angle between the light coming from the sun and the light going to the Earth. That angle would fluctuate between 0 degrees and 180 degrees.

- When the angle was close to 0, the people on Earth would see a full moon (fully illuminated disc).
- When the angle was close to 90 degrees, the people on Earth would see a half moon.
- When the angle was nearing 180 degrees, the people on Earth would see a slim crescent.
- When the angle was very close to 180 degrees, the moon would be dark. This is called a new moon (fully darkened disc).

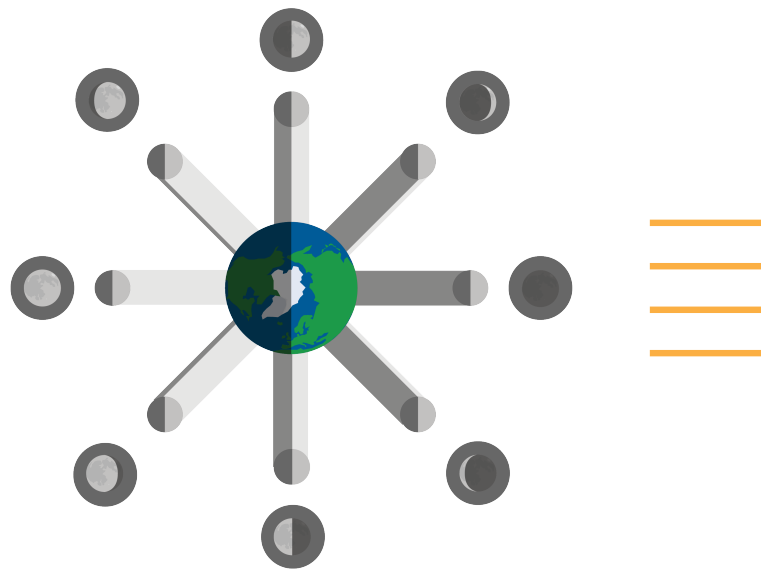


Figure 5.1: The moon revolves around the Earth in different phases every month.

Even though it takes 27.3 days for the moon to travel around the Earth once, it takes 29.5 days to get from one full moon to the next. Why? In the 27.3 days that it took the moon to travel around the Earth, the Earth has moved about 27 degrees around the sun. To get back into the same triangle configuration takes another 2.2 days.

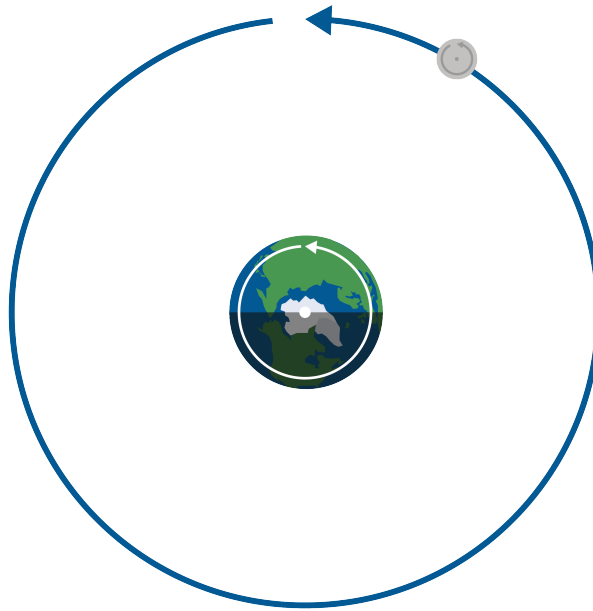


Figure 5.2: The Moon orbits the Earth while the Earth spins on its axis.



Figure 5.3: A half moon circumstance in space.

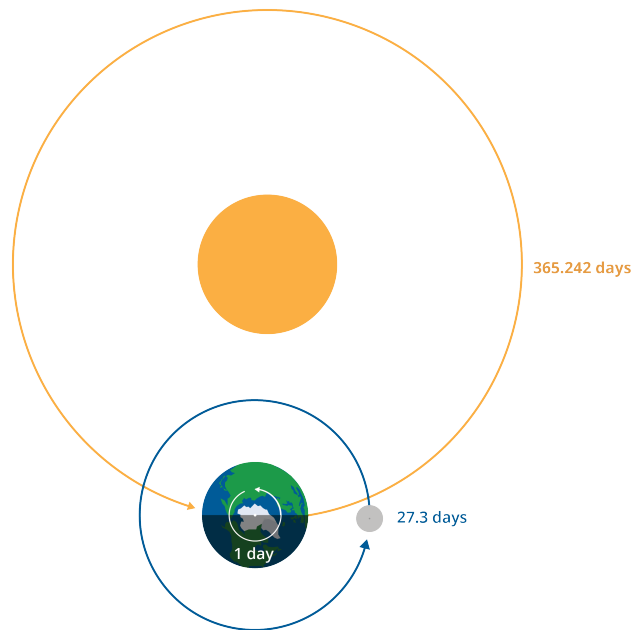


Figure 5.4: The Earth spins once a day, the Moon orbits the Earth every 27.3 days, and the Earth orbits the Sun every 365.242 days.

To explain why we often see a curve in the shadow of the moon, we can look at a ball that has one side painted yellow and the other red.



Figure 5.5: A rotating representation of the moon's phases.

As we rotate the ball, we can see that the straight color boundary between each hemisphere begins to look curved. The curve we see in the moon is due to this same basic principle of how the shading of spheres works.

The diagrams in this chapter are not to scale; if they were, you could not see anything. In reality, the Moon is about 384,000 km away, roughly 30 Earth diameters. Here is the true scale:



Figure 5.6: The Earth and the Moon, with their sizes and distance to scale.

5.3 Eclipses

While the Earth orbits the sun and the moon orbits the Earth, the two orbits are *not in the same plane*. We call the plane that the Earth orbits the sun in the *ecliptic plane*. The plane of the moon's orbit is about 5° tilted from the ecliptic plane.

Note that the moon passes through the ecliptic plane only twice every 27.3 days. Imagine that the instant it passed through the ecliptic plane was also the precise instant of a full moon. The sun, the Earth, and the moon would be in a straight line! The earth would cast a shadow upon the moon — it would go from a bright full moon to a dark moon until the moon moved back out of the shadow of the earth. This is known as a *lunar eclipse*.

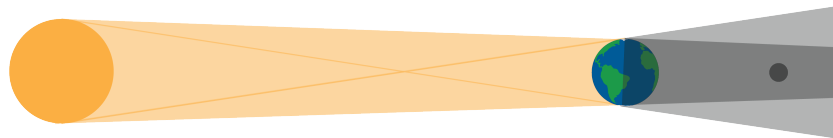


Figure 5.7: Lunar Eclipses happen when the Earth comes between the moon and the sun, causing the moon to be darkened by the Earth's shadow.

The diameter of the moon is a little more than a quarter the diameter of the Earth, so they do not have to be in perfect alignment for the moon to be darkened. Lunar eclipses actually happen once or twice per year.

Now, imagine that the instant the moon passed through the ecliptic plane was also the precise instant of a new moon. The sun, the moon, and the Earth would be in a straight line! The moon would cast a shadow upon some part of the Earth. To a person in that shadow, the sun disappears behind the moon. This is known as a *solar eclipse*.

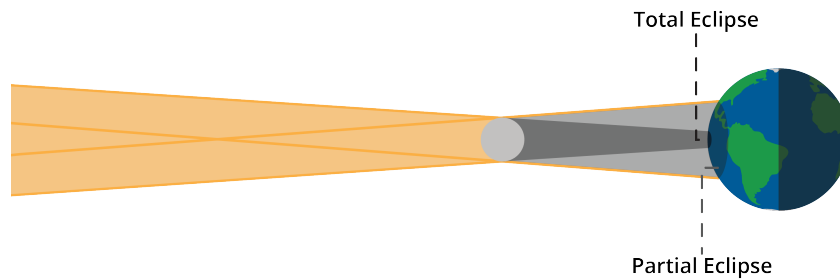


Figure 5.8: Partial and total solar eclipses.

The sun is pretty big, so if the moon blots out just part of it, we call it a *partial solar eclipse*, as seen in Figure 5.8. There are a few partial solar eclipses every year. Note that because the moon's shadow is too small to shade the whole Earth, only certain parts of the world will experience any solar eclipses.

Every 18 months or so, there is a total eclipse of the sun. Once again, only certain parts of the world experience it. You can expect to experience a total eclipse of the sun at your home about once every 375 years.

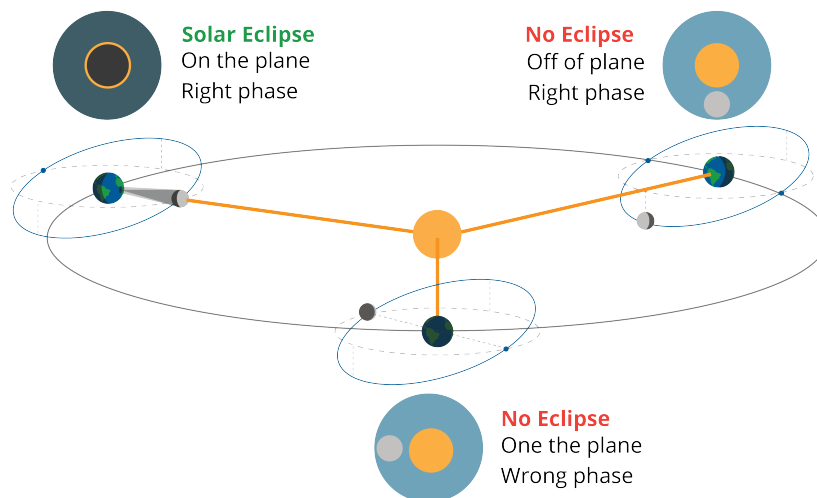


Figure 5.9: Different options for an eclipse.

5.4 The Far Side of the Moon

Like the Earth, the Moon spins on its axis. Long ago it spun faster, but Earth's gravity tugged on the Moon's own tiny tidal bulges and slowed its spin until it matched its orbit exactly: the Moon turns once on its axis for every trip around the Earth. Astronomers call this *tidal locking*. As a result, we always see the same side of the Moon.

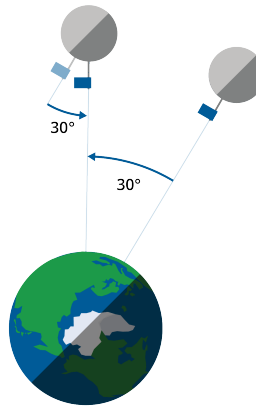


Figure 5.10: As the Moon moves 30° around its orbit, it also turns 30° on its axis, so the flag keeps pointing at Earth.

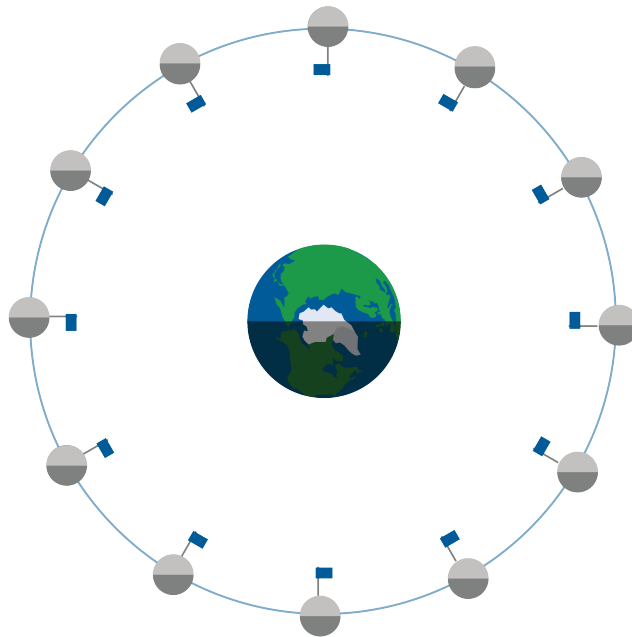


Figure 5.11: All the way around its orbit, the same face (with the flag) points toward Earth.

No one had seen the far side until the Soviet probe Luna 3 photographed it in 1959. Some people call it “the dark side of the Moon,” but it gets just as much sunshine as the side that faces us. The name stuck because we lose radio contact with spacecraft (like the Apollo missions) when they pass behind the Moon: they “go dark.”

5.5 Tides

Twice a day, the ocean rises and falls at the beach. The Moon causes it, but not in the way most people think. If the Moon simply pulled the ocean toward it, there would be one high tide a day, on the side facing the Moon. Instead there are two, one on each side of the planet. Why?

When we say “the Moon orbits the Earth,” that is a bit of an oversimplification. The force of gravity that pulls the Moon toward the Earth also pulls the Earth toward the Moon. The Earth is about 81 times heavier than the Moon, so the Moon moves more, but the Moon definitely moves the Earth. The two swing around their shared center of mass, the *barycenter*, which is inside the Earth, about 4,700 km from its center.

Here is the key idea. Forget for a moment that the Earth spins once a day. As the Earth swings around the barycenter each month, it does not turn; it just shifts around in a small circle, the way you might slide a coffee mug around in a circle on a table without turning it. Every part of the Earth, near side and far side alike, moves in an identical circle. So

every kilogram of the Earth needs exactly the same centripetal force, and that force is supplied by the Moon's gravity at the *center* of the Earth.

But the Moon's pull is not the same everywhere:

- On the side facing the Moon, the water is closer to the Moon, so the Moon pulls it a little harder than needed. The extra pull draws the water toward the Moon.
- On the far side, the water is farther from the Moon, so the Moon pulls it a little *less* than needed. That water falls behind, bulging away from the Moon.

The water in the oceans bulges a little both toward the Moon and away from it. The effect is tiny. You lose only about one nine-millionth of your weight when the Moon is overhead. But the ocean is huge, and a tiny force on every drop adds up.

The Earth is still rotating under these bulges. If you are at the beach as your longitude slides into one of them, you say "Hey, the tide is rising!" The peak of a bulge is *high tide*. Because there is a bulge on each side of the planet, high tide comes about twice a day.

This is a *lunar tide*, because it is caused by the Moon. The Sun makes tides the same way, but it is so far away that its pull hardly changes from one side of the Earth to the other, so solar tides are less than half as strong as lunar tides. When the Sun and the Moon line up and work together, the tides are stronger. This is called a *spring tide*. Spring tides do not happen in springtime; they happen near full moons and new moons.

When the Moon and the Sun are at right angles, working against each other, the tides are weaker. This is called a *neap tide*. Neap tides happen when you see a half moon in the sky.

5.5.1 Computing the Forces

We are enumerating several forces that shape the water on the planet. All these forces are pulling on your body too. In these exercises, you are going to calculate how each force would affect a 1 kg mass on the surface of the earth.

Here are some numbers you will need:

- The mass of the Earth: 5.97219×10^{24} kg
- The mass of the sun: 1.9891×10^{30} kg
- The mass of the moon: 7.347673×10^{22} kg
- Radius of the Earth at the equator: 6,371 km
- Average distance from the center of the Earth to the center of the sun: 149.6×10^6 km

- Average distance from the center of the moon to the center of the Earth: 384,467 km.

Exercise 13 Life Among the Orbits 1: Earth Gravity

Working Space

If the Earth were still and alone in the universe, there would still be the force of gravity. We have said that a kilogram on the surface of the Earth is pulled toward the center of the earth with a force of 9.8 N.

Confirm that the gravity of the Earth pulls a 1kg mass on the surface of the planet with a force of about 9.8 N.

You will need the formula for gravitation:

$$F_g = \frac{Gm_1m_2}{r^2}$$

If we measure distance in km and mass in kg, the gravitational constant G is 6.67430×10^{-17} . (Usually $G = 6.67430 \times 10^{-11}$ with distances in meters; a km is 10^3 m, and r^2 is in the denominator, so we divide by 10^6 .)

Answer on Page 91

Exercise 14 Life Among the Orbits 2: Earth Centripetal Force

Working Space

What if we add the spinning of the Earth? The spinning would try to throw the kg into space. The formula for centripetal force is

$$F_c = \frac{mv^2}{r}$$

Calculate the centripetal force on a 1 kg mass on the surface of the earth. It does not fly off into space, so the force due to gravity must be bigger. How many times bigger?

Assume that the mass is on the equator, thus rotating around the Earth at 465 m/s.

Does the centripetal force increase, decrease, or stay the same as you get closer to the north pole?

Answer on Page 92

Exercise 15 Life Among the Orbits 3: The Moon's Gravity

Working Space

Now we add the moon's gravitational force to our model.

When the moon is directly overhead, how strongly will it pull at the 1 kg mass on the equator?

When the moon is directly underfoot, how strongly will it pull at the 1 kg mass on the equator?

Is that a big difference?

Answer on Page 92

Exercise 16 **Life Among the Orbits 4: The Swing of the Moon**

Working Space

Now we add the moon's motion. The moon and the Earth swing each other around. This creates a centripetal force. They both travel in nearly a circle centered at their center of mass.

How far is the center of mass of the moon and the Earth from the center of the Earth? (You can imagine a see-saw with the center of the Earth on one end and the center of the moon on the other. Where would the balance point be?)

Is the barycenter inside or outside the Earth?

Answer on Page 93

Exercise 17 **Life Among the Orbits 5: The Needed Pull**

Working Space

The Earth's center swings around the barycenter once every 27.3 days, in a circle of radius 4,673 km. Every kilogram of the Earth moves in a circle of the same size. Using $F_c = m r \omega^2$, find the centripetal force needed for each kilogram. Compare it with the Moon's pull on 1 kg at the center of the Earth, $\frac{G m_{\text{moon}}}{d^2}$ with $d = 384,467$ km.

Answer on Page 93

Exercise 18 **Life Among the Orbits 6: The Tidal Force**

Using your answers to exercises 3 and 5, find the leftover force on 1 kg at the point closest to the Moon and at the point farthest from it: the difference between the Moon's actual pull there and the 3.318×10^{-5} N needed. Which way does each point?

*Working Space**Answer on Page 93***5.5.2 Solar Tidal Forces**

The Sun has a much larger gravitational effect on the Earth than the Moon does:

- The Sun pulls on a 1 kg mass with a force of about 0.00593 N.
- The Moon pulls on a 1 kg mass with a force of about 0.0000332 N.

So why are lunar tides about twice as strong as solar tides? Tides do not come from the size of the pull; they come from how much the pull *changes* from one side of the Earth to the other. The Earth is 1/30 of the way to the Moon, so the Moon's pull changes by several percent across the planet. The Earth is only 1/12,000 of the way to the Sun, so the Sun's pull changes far less. Working it out, the Sun's tidal force is about 0.46 of the Moon's.

Exercise 19 **The Year 1900**

Was 1900 a leap year?

*Working Space**Answer on Page 94*

Polar Coordinates

Picture a radar screen in an air traffic control tower. A sweeping line circles the screen, and each plane shows up as a blip. The controller does not say “that plane is 3 km east and 4 km north”; they say “that plane is 5 km out, at this heading.” Distance and direction: that is the idea behind polar coordinates.

So far we have located points with (x, y) coordinates: how far to go horizontally and how far vertically. This is the *Cartesian* coordinate system. It is not the only way to describe 2D space.

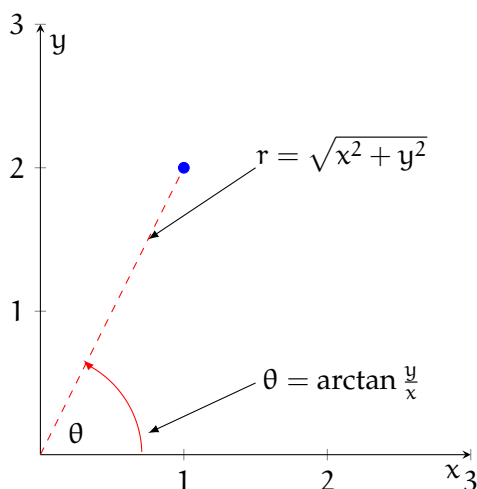


Figure 6.1: The point $(1, 2)$ is $\sqrt{5}$ units from the origin and approximately 1.107 radians counterclockwise from horizontal

Instead of thinking about the horizontal and vertical position, we could think about distance from the origin and rotation about the origin. Take the Cartesian coordinate point $(1, 2)$ (see figure 6.1). How far is $(1, 2)$ from the origin, $(0, 0)$? We can create a right triangle, where the legs are parallel to the x and y axes. This means the leg lengths are 1 and 2, and we can use the Pythagorean theorem to find the length of the hypotenuse (which is the distance from the origin to the point):

$$c^2 = a^2 + b^2$$

$$c^2 = 1^2 + 2^2 = 1 + 4 = 5$$

$$c = \sqrt{5}$$

Therefore, the Cartesian point $(1, 2)$ is $\sqrt{5}$ units from the origin. This is not enough to find our point: there are infinite points that are $\sqrt{5}$ from the origin (see 6.2). To identify a particular point that is a distance of $\sqrt{5}$ from the origin, we also need an *angle of rotation*. By convention, angles are measured from the positive x -axis. This means points on the positive x -axis have an angle of $\theta = 0$, points on the positive y -axis have an angle of $\theta = \frac{\pi}{2}$, and so on.

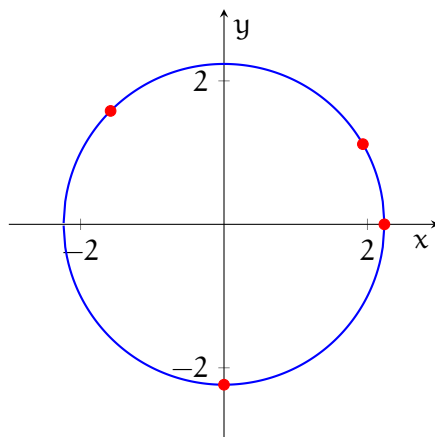


Figure 6.2: There are infinite points $\sqrt{5}$ from the origin, represented by the circle with a radius of $\sqrt{5}$ centered about the origin

We can use trigonometry to find the appropriate angle of rotation for our Cartesian point. There are many ways to do this, but using $\arctan$ is the most straightforward. Recall that:

$$\tan \theta = \frac{\text{opposite}}{\text{adjacent}}$$

That is, for a given angle in a right triangle, the tangent of that angle is given by the length of the opposite leg divided by the adjacent leg. In our case, the opposite leg is the vertical distance (y -value of the Cartesian point) and the adjacent leg is the horizontal distance (x -value of the Cartesian point), which means:

$$\tan \theta = \frac{2}{1}$$

$$\theta = \arctan 2 \approx 1.107 \text{ radians}$$

6.1 Plotting Polar Coordinate Points

How do we plot polar coordinate points? Begin by locating the angle given by the second coordinate (remember, the angle is measured counterclockwise from the horizontal). Your point will lie somewhere on this line. Next, move outwards along the angle by the radius given by the first coordinate.

Example: Plot the polar coordinate point $(2, \frac{\pi}{3})$.

Solution: Begin by locating $\theta = \frac{\pi}{3}$ (see figure 6.3)

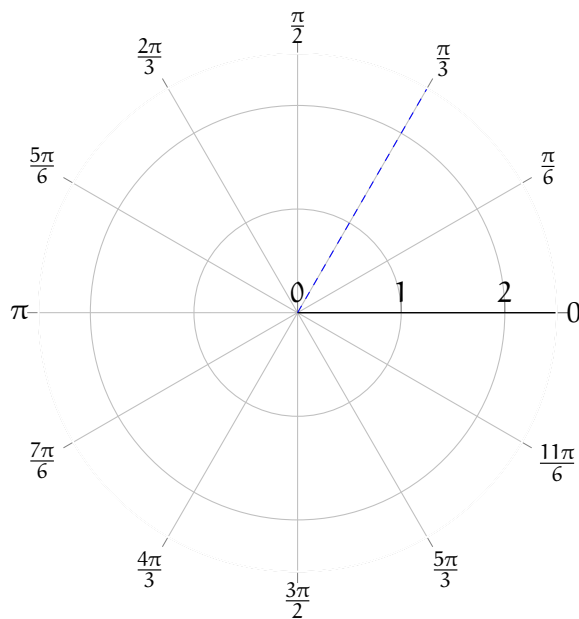
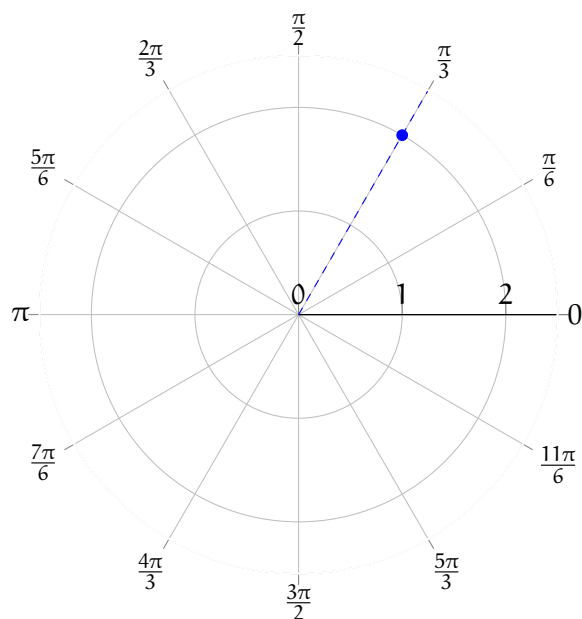


Figure 6.3: $\theta = \frac{\pi}{3}$

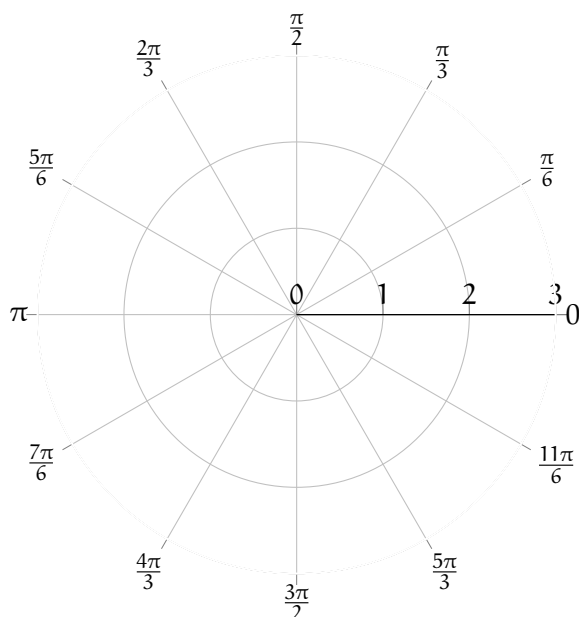
Then, move your finger or pencil along the line $\theta = \frac{\pi}{3}$ until you reach $r = 2$ (see figure 6.4).

Figure 6.4: $(2, \frac{\pi}{3})$ **Exercise 20**

Plot the following polar coordinate points on the provided polar axis (hint: negative angles are measured clockwise):

1. $(1, \pi)$
2. $(1.5, \frac{\pi}{2})$
3. $(1.5, -\frac{\pi}{6})$
4. $(2, \frac{3\pi}{4})$

Working Space



Answer on Page 94

6.2 Equivalent Points

Unlike the Cartesian coordinate system, two different coordinates may lie at the same location. Consider the points $(1, \frac{\pi}{4})$ and $(-1, \frac{5\pi}{4})$ (see figure 6.5). When a radius is negative, you move *backwards* through the origin, like a mirror image.

Going all the way around also gets you back where you started: $(1, \frac{\pi}{4})$, $(1, \frac{9\pi}{4})$, and $(1, -\frac{7\pi}{4})$ are all the same point. Every point has infinitely many names in polar coordinates.

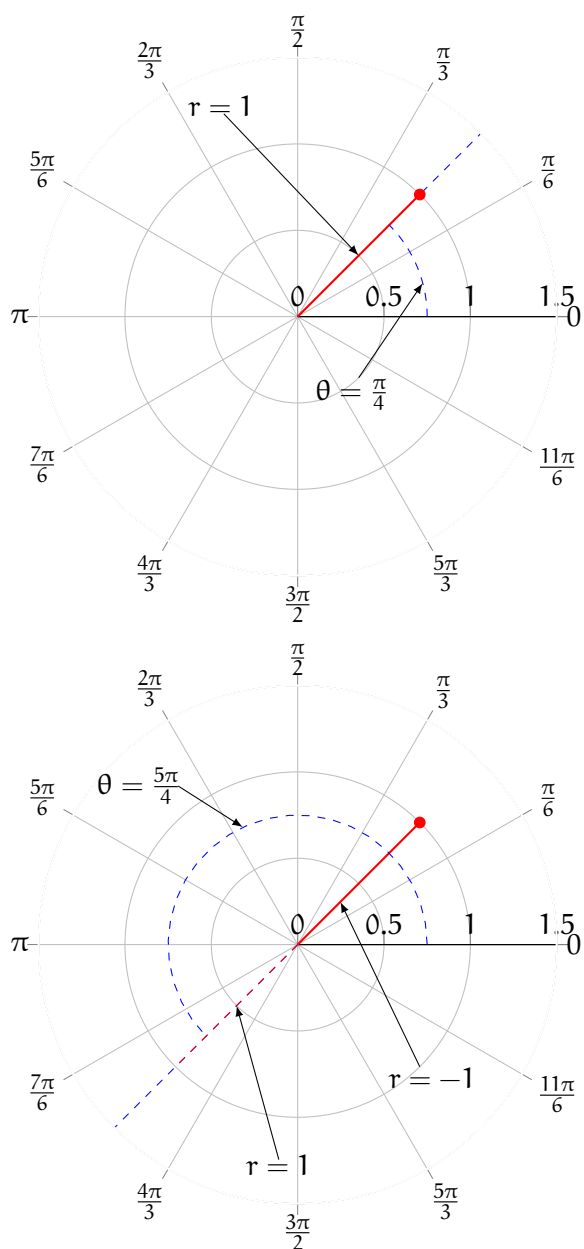


Figure 6.5: The polar coordinates points $(1, \frac{\pi}{4})$ and $(-1, \frac{5\pi}{4})$ are the same location on a polar axis

6.3 Changing coordinate systems

6.3.1 Cartesian to Polar

From the example above, you should see that a given Cartesian coordinate, (x, y) , can also be expressed as a polar coordinate, (r, θ) , where r is the distance from the origin and θ is the angle of rotation from the horizontal. (Note: Polar functions are generally given as r defined in terms of θ , which means the *dependent* variable is listed first in the coordinate pair, unlike Cartesian coordinates.) Additionally,

$$r = \sqrt{x^2 + y^2}$$

$$\theta = \arctan \frac{y}{x}$$

Example: Express the Cartesian point $(-3, 4)$ in polar coordinates.

Solution: Taking $x = -3$ and $y = 4$, we find that:

$$r = \sqrt{(-3)^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5$$

We follow the convention of only taking the positive solution to the square root. Finding θ :

$$\theta = \arctan \frac{4}{-3}$$

When you evaluate the arctan with a calculator, you are likely to get back $\theta = -0.928$. Recall that $\tan \theta = \tan(\theta \pm n\pi)$, where n is an integer. We know our Cartesian point, $(-3, 4)$, is in the II quadrant, while the angle -0.928 radians would fall in the IV quadrant. So, clearly, -0.928 radians is not correct. Most calculators restrict the output of arctan to angles between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, because there are actually multiple angles where $\tan \theta = -\frac{4}{3}$. Since $\tan \theta = \tan(\theta \pm n\pi)$, every angle of the form

$$\theta = -0.928 + n\pi$$

has $\tan \theta = -\frac{4}{3}$.

Another possible θ is $-0.928 + \pi \approx 2.214$, which does fall in the appropriate quadrant. This means the polar coordinates $(5, 2.214)$ are the same as the Cartesian coordinates $(-3, 4)$. *Note:* It is standard practice to express angles in radians, and not degrees, when using polar coordinates.

6.3.2 Polar to Cartesian

We can also leverage our knowledge of right triangles to convert polar coordinates to Cartesian coordinates. Take the polar coordinate $(2, \frac{\pi}{4})$ (see figure 6.6). We can draw a right triangle with legs parallel to the x and y axes (not shown in the figure) and a hypotenuse that goes from the origin to the polar coordinate $(2, \frac{\pi}{4})$.

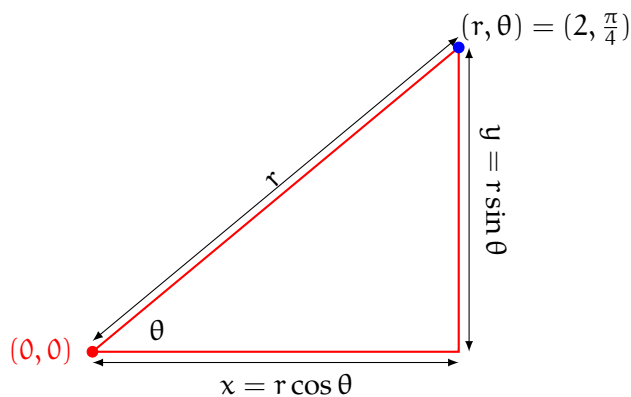


Figure 6.6: To convert from polar to Cartesian coordinates, use the identities $x = r \cos \theta$ and $y = r \sin \theta$

Recall from trigonometry that:

$$\sin \theta = \frac{\text{opposite leg}}{\text{hypotenuse}}$$

We know that the hypotenuse of this triangle has a length of r . The opposite leg is vertical and is the same length as the distance of the polar coordinate from the x -axis. Therefore, the length of the vertical leg represents the y value of that same polar coordinate if it were expressed in Cartesian coordinates. So, we can say that:

$$\sin \theta = \frac{y}{r}$$

And therefore:

$$y = r \sin \theta$$

By a similar process, we also see that:

$$x = r \cos \theta$$

This is easy to visualize and understand for $0 \leq \theta \leq \frac{\pi}{2}$, but it also holds for other values of θ .

Example: Express the polar coordinate $(\frac{3}{2}, \frac{2\pi}{3})$ in Cartesian coordinates.

Solution: From the polar coordinate, we see that $\theta = \frac{2\pi}{3}$ and $r = \frac{3}{2}$. Therefore:

$$x = r \cos \theta = \frac{3}{2} \cdot \cos \frac{2\pi}{3} = \frac{3}{2} \cdot -\frac{1}{2} = -\frac{3}{4}$$

$$y = r \sin \theta = \frac{3}{2} \cdot \sin \frac{2\pi}{3} = \frac{3}{2} \cdot \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{4}$$

The Cartesian coordinate $(-\frac{3}{4}, \frac{3\sqrt{3}}{4})$ has the same location as the given polar coordinate.

Exercise 21

Convert the following polar coordinates to Cartesian coordinates:

1. $(2, \frac{3\pi}{2})$
2. $(\sqrt{2}, \frac{3\pi}{4})$
3. $(3, -\frac{\pi}{4})$
4. $(-3, -\frac{\pi}{3})$
5. $(2, -\frac{\pi}{2})$

Working Space

Answer on Page 94

Exercise 22

Convert the following Cartesian coordinates to polar coordinates. Restrict θ to $0 \leq \theta < 2\pi$.

1. $(-4, 4)$
2. $(3, 3\sqrt{3})$
3. $(\sqrt{3}, -1)$
4. $(-6, 0)$

Working Space

5. $(-2, -2)$

Answer on Page 95

In Python, `math.atan2(y, x)` does the quadrant bookkeeping for you. `math.atan2(4, -3)` returns 2.214 directly.

Exercise 23 Air Traffic Control

A radar dish sits at the origin, with east along the positive x -axis and north along the positive y -axis. Distances are in km.

Working Space

1. A plane shows up at $(12, \frac{5\pi}{6})$. How far east (or west) and north (or south) of the radar is it?
2. A drone is 5 km west and 12 km south of the radar. Give its polar coordinates with $0 \leq \theta < 2\pi$.

Answer on Page 95

6.4 Circles in Polar Coordinates

Many conic sections, including circles, are simpler to express as polar functions than as Cartesian functions. Consider a circle with a radius of 2 centered about the origin. The polar function for this is $r = 2$ for all θ . Let us write a Cartesian function for the same circle.

We know that for every point on the circle, the distance to the origin is 2. This means that, by the Pythagorean theorem,

$$r^2 = x^2 + y^2$$

(see figure 6.7)

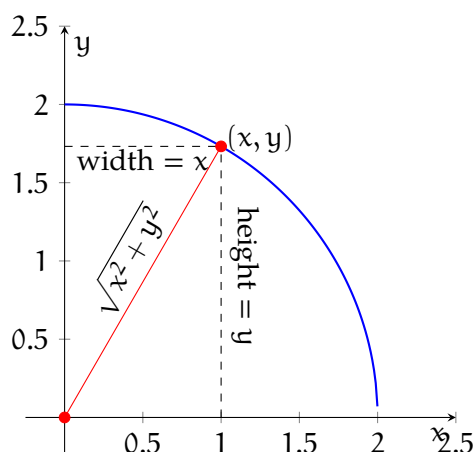


Figure 6.7: All (x, y) pairs on the circle are the same distance from the origin

We can solve this equation for y , given that $r = 2$ (in this case):

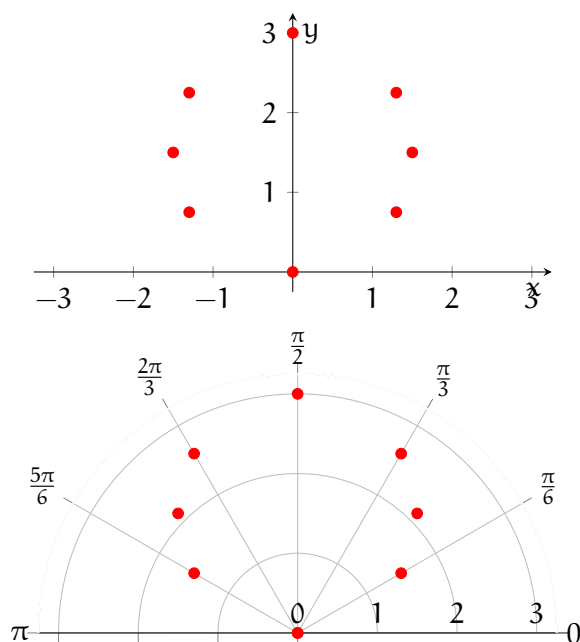
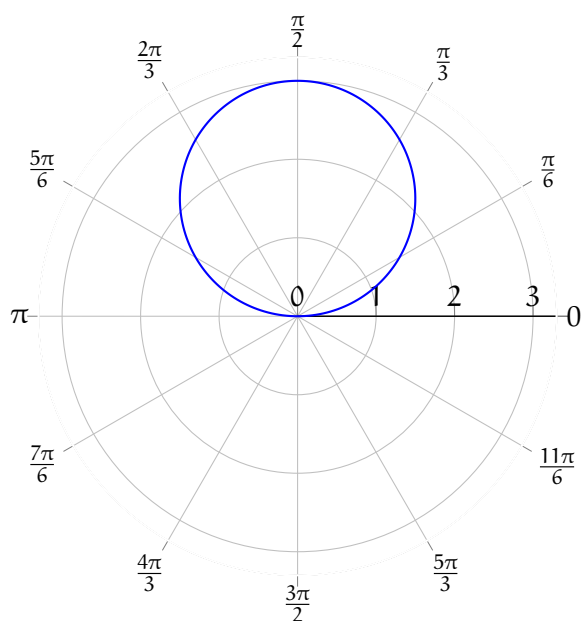
$$y = \pm \sqrt{2^2 - x^2}$$

Notice that this is really two equations: $y = \sqrt{2^2 - x^2}$ and $y = -\sqrt{2^2 - x^2}$. This is more complex than the polar equation, $r = 2$.

As seen above, the equation of a circle with radius R centered on the origin is simply $r = R$ in polar coordinates. What if we want a circle centered somewhere else? Polar coordinates are best when a circle is bisected by the x or y axis. Consider the polar equation $r = 3 \sin \theta$. Let us use a table to find some points and plot the function:

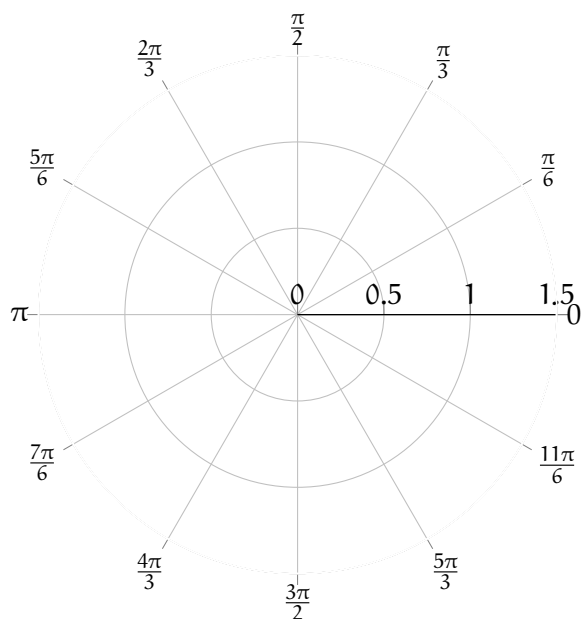
θ	$r = 3 \sin \theta$
0	0
$\frac{\pi}{6}$	$\frac{3}{2}$
$\frac{\pi}{4}$	$\frac{3\sqrt{2}}{2}$
$\frac{\pi}{3}$	$\frac{3\sqrt{3}}{2}$
$\frac{\pi}{2}$	3
$\frac{2\pi}{3}$	$\frac{3\sqrt{3}}{2}$
$\frac{3\pi}{4}$	$\frac{3\sqrt{2}}{2}$
$\frac{5\pi}{6}$	$\frac{3}{2}$
π	0

Here is how those points look plotted (see figures 6.8 and 6.9):

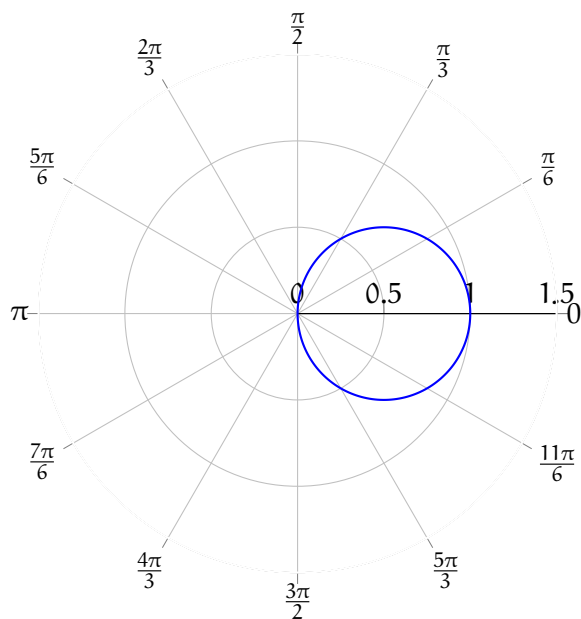
Figure 6.8: Several points for $r = 3 \sin \theta$ plotted on Cartesian and polar coordinate systemsFigure 6.9: $r = 3 \sin \theta$ plotted on a polar coordinate system

So, the polar equation $r = 3 \sin \theta$ gives a circle with radius $\frac{3}{2}$ centered at $(0, \frac{3}{2})$.

Example: Describe the graph of $r = \cos \theta$. Feel free to make a rough plot on the blank polar axis below:



Solution: This plot will look like a circle of radius 0.5 centered at the Cartesian point $(0.5, 0)$.

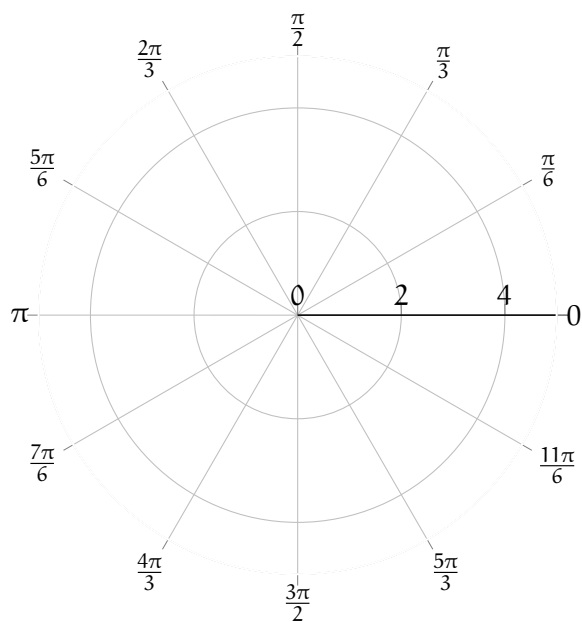


Exercise 24

Sketch the following polar functions on the provided polar axis for $0 \leq \theta < 2\pi$:

Working Space

1. $r = 3$
2. $\theta = \pi$
3. $r = 2 \cos \frac{\theta}{2}$
4. $r = -4 \sin \theta$
5. $r = \theta$



Answer on Page 95

6.5 Roses

Change $r = \cos \theta$ to $r = \cos(3\theta)$ and something lovely happens: a flower with three petals.

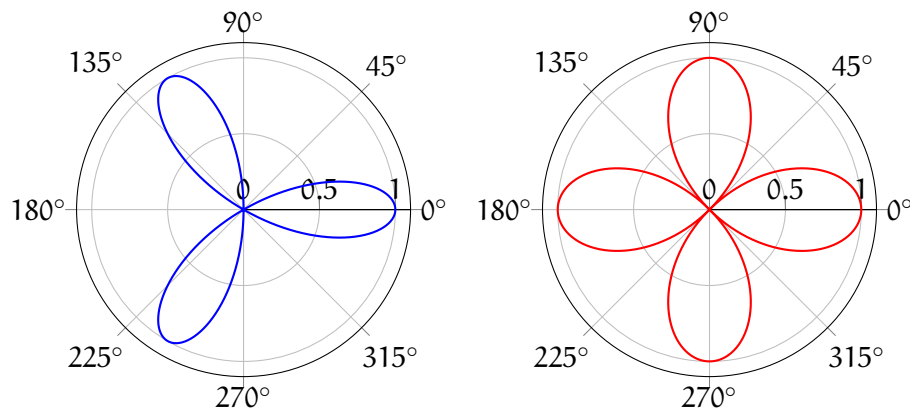


Figure 6.10: $r = \cos(3\theta)$ has three petals; $r = \cos(2\theta)$ has four.

Curves of the form $r = a \cos(n\theta)$ or $r = a \sin(n\theta)$ are called *roses*. Each petal is $|a|$ long. If n is odd, the rose has n petals. If n is even, it has $2n$ petals. (When n is odd, the negative values of r retrace petals you have already drawn; when n is even, they draw new ones.)

You can draw these in Python. Matplotlib has a polar mode:

```

1 import numpy as np
2 import matplotlib.pyplot as plt
3
4 theta = np.linspace(0, 2 * np.pi, 500)
5 r = np.cos(3 * theta)
6
7 ax = plt.subplot(projection='polar')
8 ax.plot(theta, r)
9 plt.show()
```

Polar graphs will be back when you learn to find areas with calculus: a petal is much easier to describe with r and θ than with x and y .

Exercise 25 Petals

How many petals does each rose have, and how long is each petal?

1. $r = 3 \cos(5\theta)$
2. $r = 2 \sin(4\theta)$

Working Space

3. $r = 4 \sin(2\theta)$

Answer on Page 98

The Definition of the Derivative

Glance at a car's speedometer and it says 60 km/h. But what does that mean? Speed is distance divided by time, and at a single instant the car covers no distance in no time. Zero divided by zero is not 60. Making sense of "speed right now" took Newton and Leibniz, and the tool they used is the *derivative*. You already have the key ingredient: limits (Chapter ??).

7.1 Differentiation

Finding the slope between two points on a graph, at $x = a$ and $x = b$, is easy:

$$m = \frac{f(b) - f(a)}{b - a}$$

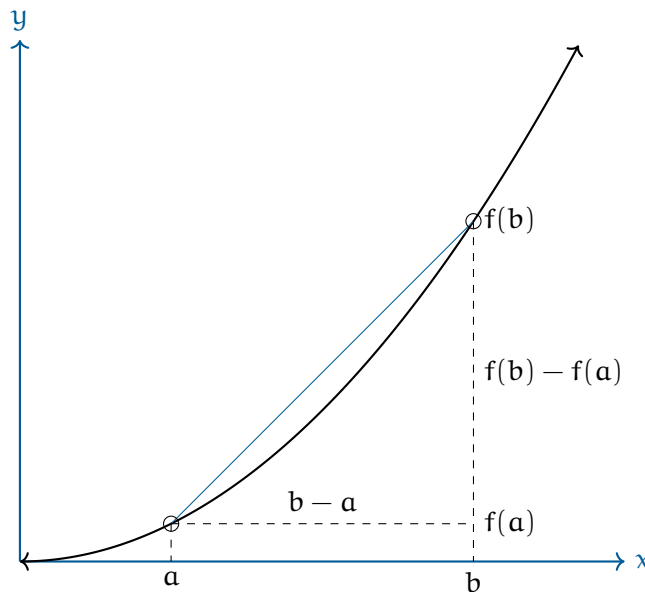


Figure 7.1: The slope of point a using points a and b.

If we want to find the slope at a , we take the limit of this as the b goes to a :

$$f'(a) = \lim_{b \rightarrow a} \frac{f(b) - f(a)}{b - a}$$

This idea is usually expressed using h as the difference between b and a :

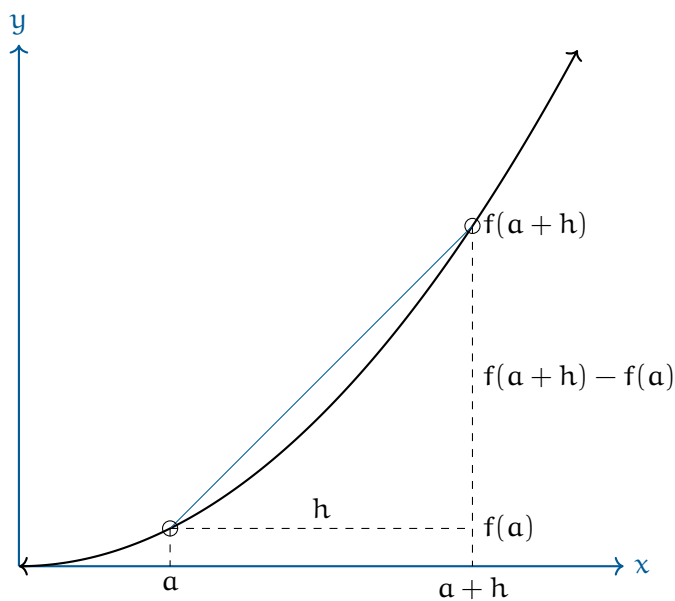
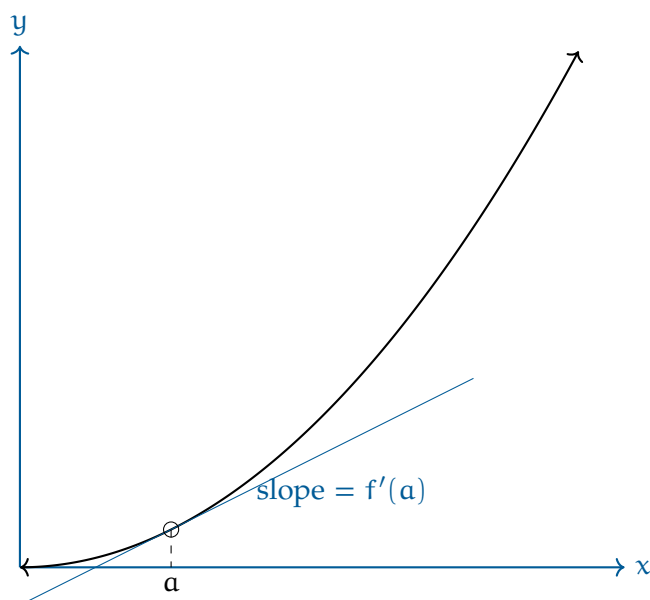


Figure 7.2: Limit of a point using h difference.

The formula then becomes:

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$$

As h decreases to very close to zero, almost infinitesimal, what we are finding is the slope of the tangent line to point a .

Figure 7.3: Slope at a as h approaches 0.

7.1.1 Differentiability

Warning: Not every function is differentiable everywhere. For example, if $f(x) = |x|$, you get a corner at zero.

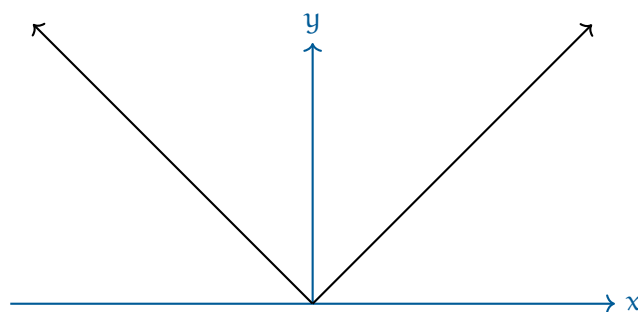


Figure 7.4: Absolute value function.

To the left of zero, the slope is -1 . To the right of zero, the slope is 1 . At zero? The derivative is not defined because the slopes are two different numbers.

If a function has a derivative everywhere, it is said to be *differentiable*. Generally, you can think of differentiable functions as smooth — their graphs have no corners or sharp turns. Another place where a function is not differentiable is at a vertical tangent, as the slope reaches $\pm\infty$.

It is important to note: if f is differentiable at a , it *must* also be continuous at a .

Likewise, if f is not continuous at a it is not differentiable. An example of this is a jump discontinuity.

Exercise 26

[This problem was originally presented as a no-calculator, multiple-choice question on the 2012 AP Calculus BC exam.]

Let f be the function defined by $f(x) = \sqrt{|x-2|}$ for all x . Classify each of the following statements as true or false.

1. f is continuous at $x = 2$.
2. f is differentiable at $x = 2$.
3. $\lim_{x \rightarrow 2} f(x) = 0$.
4. $x = 2$ is a vertical asymptote of the graph of $f(x)$.

Working Space

Answer on Page 98

7.1.2 Using the definition of derivative

Let us say that you want to know the slope of $f(x) = -3x^2$ at $x = 2$. Using the definition of the derivative, that would be:

$$\begin{aligned} f'(2) &= \lim_{h \rightarrow 0} \frac{f(2+h) - f(2)}{h} = \lim_{h \rightarrow 0} \frac{-3(2+h)^2 - (-3(2)^2)}{h} \\ &= \lim_{h \rightarrow 0} \frac{-12 - 12h - 3h^2 + 12}{h} = \lim_{h \rightarrow 0} \frac{-12h - 3h^2}{h} = \lim_{h \rightarrow 0} (-12 - 3h) = -12 \end{aligned}$$

Notice the trick: once the $f(a)$ terms cancel, every remaining term has an h in it. Divide out the h , and the limit is easy.

7.2 The Derivative

Put together, the derivative of a function $f(x)$ at a point x is the limit:

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \quad (7.1)$$

provided this limit exists. In words: the derivative is the rate at which f is changing at x , which is the slope of the graph there. We write it $f'(x)$ (read “ f prime of x ”) or $\frac{df}{dx}$.

Estimating the Derivative

Consider the function $f(x) = x^2$. Suppose we want to write an equation for a line that is tangent to the curve at $x = 2$ (see figure 7.5). We already have a point that the line passes through: $(2, 4)$. To write an equation for the tangent line, we would need to know its slope, m .

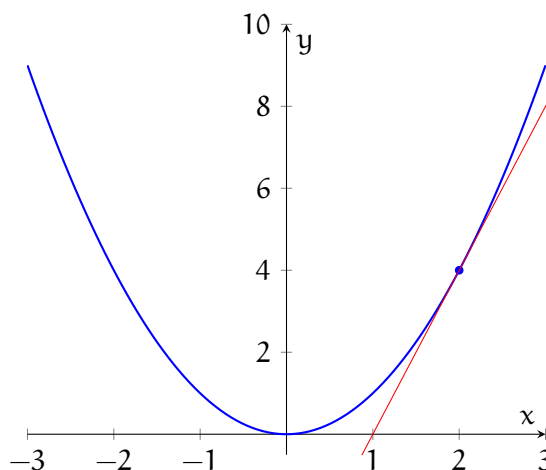


Figure 7.5: The red line is tangent to $f(x) = x^2$ at the point $(2, 4)$

We can estimate the slope by choosing points on either side of P , drawing a line through those points, and calculating the slope of that secant line (a *secant line* crosses the curve at two points). See figure 7.6 for a visualization.

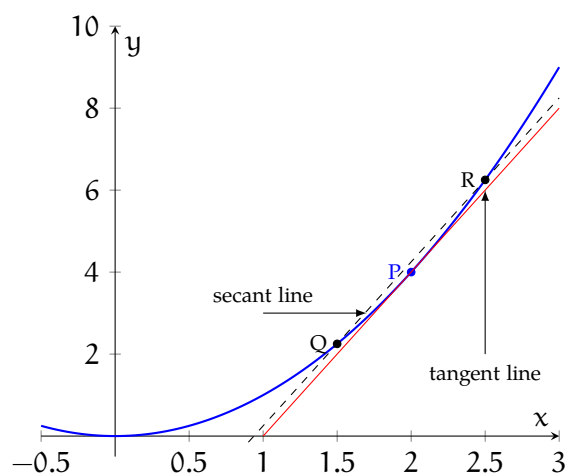


Figure 7.6: The slope of the secant line is approximately the slope of the tangent line

As the points Q and R get closer to P, the better the estimate becomes.

Much scientific data is not described as continuous functions, but rather as discrete data points. Consider the following data of a falling object:

time (seconds)	height (m)
0	50
0.5	48.775
1	45.1
1.5	38.975
2	30.4
2.5	19.375
3	5.9

A graph of the data is shown in figure 7.7.

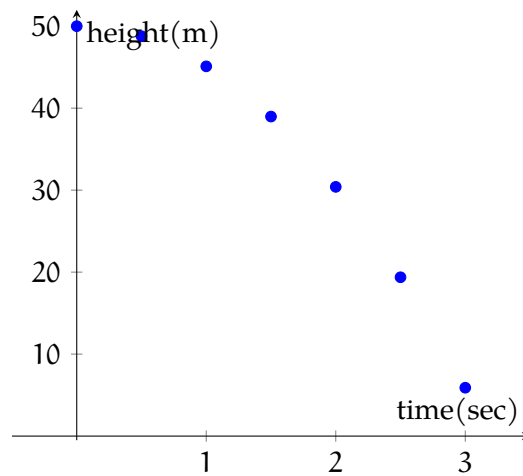
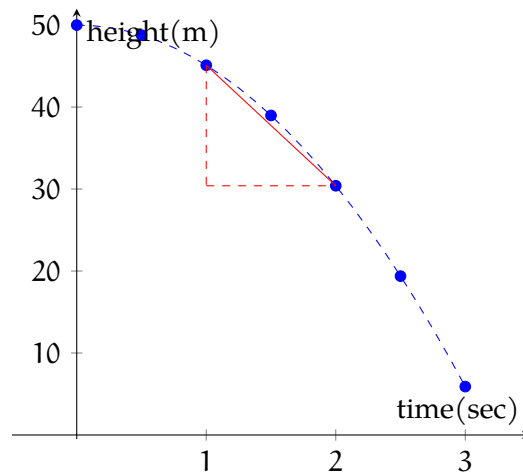


Figure 7.7: The height of a falling object over time

Suppose we wanted to estimate the velocity of the falling object at $t = 1.5$ s. Recall that velocity is given by the change in position divided by the change in time. We can select data points on either side of $t = 1.5$ s and use them to find the average velocity from $t = 1$ s and $t = 2$ s (see figure 7.8):

$$v = \frac{h_2 - h_1}{t_2 - t_1} = \frac{30.4\text{m} - 45.1\text{m}}{2\text{s} - 1\text{s}} = -14.7 \frac{\text{m}}{\text{s}}$$

Figure 7.8: The slope of the line connecting the data points on either side of $t = 1.5$ s is approximately the velocity of the falling object at $t = 1.5$ s

The object was dropped from 50 m, so its height is $h = 50 - 4.9t^2$, and its true velocity at $t = 1.5$ s is -14.7 m/s. Our estimate is spot on. (For a parabola, the slope between two points spaced evenly on either side always equals the slope in the middle. For other

curves it is just close.)

Example: A 1000-gallon tank drains from the bottom in 30 minutes. The volume left in the tank is recorded every 5 minutes, as shown in the data table below. Use the data to estimate $V'(15)$ and $V'(25)$, including appropriate units. At which time is the tank draining faster?

t (min)	V (gal)
5	694
10	444
15	250
20	111
25	28
30	0

Solution: To estimate $V'(15)$, we find the slope of the line connecting the data points on either side of $t = 15$:

$$V'(15) \approx \frac{111\text{gal} - 444\text{gal}}{20\text{min} - 10\text{min}}$$

$$V'(15) \approx \frac{-333\text{gal}}{10\text{min}}$$

$$V'(15) \approx -33.3 \frac{\text{gal}}{\text{min}}$$

And we can use the data at $t = 20$ and $t = 30$ to estimate $V'(25)$:

$$V'(25) \approx \frac{0\text{gal} - 111\text{gal}}{30\text{min} - 20\text{min}}$$

$$V'(25) \approx \frac{-111\text{gal}}{10\text{min}}$$

$$V'(25) \approx -11.1 \frac{\text{gal}}{\text{min}}$$

Both answers are negative because the tank is emptying, and the tank is draining faster at $t = 15$ than at $t = 25$.

Exercise 27

[This question was originally presented as a free-response, calculator-allowed question on the 2012 AP Calculus BC Exam.] The temperature of water in a tub at time t is modeled by a function,

Working Space

W , where $W(t)$ is measured in degrees Fahrenheit and t is measured in minutes. Values of $W(t)$ at selected times for the first 20 minutes are given in the table. Use the data in the table to estimate $W'(12)$. Show the computations that lead to your answer. Using correct units, interpret the meaning of your answer in the context of the problem.

t (minutes)	$W(t)$ (degrees Fahrenheit)
0	55.0
4	57.1
9	61.8
15	67.9
20	71.0

Answer on Page 99

7.2.1 The Derivative as a Function

We have seen how to estimate the value of a derivative at a specific point on a graph. Suppose we wanted to describe the slope of a graph everywhere. That is: can we find a function, $g(x)$ that describes the slope of another function, $f(x)$, over the domain of f ? Using the definition of a derivative, we can. Here is a shortcut you will use constantly in the next chapter: if $f(x) = x^n$, where n is a positive integer, then $f'(x) = nx^{n-1}$. For example, the derivative of x^5 is $5x^4$. This is called the *power rule*. Here is why it works:

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{(x+h)^n - x^n}{h}$$

In order to expand the polynomial, $(x+h)^n$, we will need to apply the Binomial Theorem, which tells us that:

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}b^3 + \cdots + nab^{n-1} + b^n$$

Substituting this into our limit definition of a derivative, we see that:

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{x^n + nx^{n-1}h + \frac{n(n-1)}{2!}x^{n-2}h^2 + \cdots + nxh^{n-1} + h^n - x^n}{h} \\ f'(x) &= \lim_{h \rightarrow 0} \frac{nx^{n-1}h + \frac{n(n-1)}{2!}x^{n-2}h^2 + \cdots + nxh^{n-1} + h^n}{h} \\ f'(x) &= \lim_{h \rightarrow 0} \left(nx^{n-1} + \frac{n(n-1)}{2!}x^{n-2}h + \cdots + nxh^{n-2} + h^{n-1} \right) \end{aligned}$$

Every term except the first has an h in it, so they all vanish as $h \rightarrow 0$:

$$f'(x) = nx^{n-1}$$

Example: Use the limit definition of a derivative to find $f'(x)$ if $f(x) = 2x^3 - x^2$.

Solution: According to the limit definition, f' is:

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{[2(x+h)^3 - (x+h)^2] - [2x^3 - x^2]}{h} \\ f'(x) &= \lim_{h \rightarrow 0} \frac{[2(x^3 + 3hx^2 + 3h^2x + h^3) - (x^2 + 2xh + h^2)] - 2x^3 + x^2}{h} \\ f'(x) &= \lim_{h \rightarrow 0} \frac{2x^3 - 2x^3 + 6hx^2 + 6h^2x + 2h^3 - x^2 + x^2 - 2xh - h^2}{h} \\ f'(x) &= \lim_{h \rightarrow 0} \frac{6hx^2 + 6h^2x + 2h^3 - 2xh - h^2}{h} \\ f'(x) &= \lim_{h \rightarrow 0} (6x^2 + 6hx + 2h^2 - 2x - h) = 6x^2 - 2x \end{aligned}$$

Therefore, if $f(x) = 2x^3 - x^2$, then $f'(x) = 6x^2 - 2x$.

Exercise 28 Finding Functions for Derivatives

Use the limit definition of a derivative to find an equation for $f'(x)$.

Working Space

1. $f(x) = mx + b$
2. $f(x) = \sqrt{16 - x}$

3. $f(x) = \frac{x^2-1}{2x-3}$

[Answer on Page 99](#)

7.3 Exercises

Exercise 29 Limit definition at a point

Use the limit definition of the derivative to find $f'(3)$ for $f(x) = x^2$.

[Working Space](#)[Answer on Page 100](#)

Exercise 30 Limit definition for a reciprocal

Use the limit definition to find $f'(2)$ for $f(x) = \frac{1}{x}$.

[Working Space](#)[Answer on Page 100](#)

Exercise 31 The Falling Object

The falling object in this chapter has height $y(t) = 50 - 4.9t^2$.

[Working Space](#)

1. Use the table to estimate its velocity at $t = 2.5$ s.
2. Use the limit definition to find $y'(t)$,

then compute $y'(2.5)$.

Answer on Page 100

Exercise 32 **Differentiability at a corner**

Is $f(x) = |x|$ differentiable at $x = 0$? Justify your answer with the definition.

Working Space

Answer on Page 100

Answers to Exercises

Answer to Exercise 1 (on page 8)

The correct answer is: The frequency would increase by a factor of $\sqrt{2}$.

The frequency of oscillation for a mass-spring system is given by the formula:

$$f = \frac{1}{2\pi} \sqrt{\frac{k}{m}}$$

where k is the spring constant and m is the mass attached to the spring. Since frequency is inversely proportional to the square root of the mass, if the mass is halved, the new frequency f' can be calculated as follows:

$$\begin{aligned} f' &= \frac{1}{2\pi} \sqrt{\frac{k}{m/2}} \\ &= \frac{1}{2\pi} \sqrt{\frac{2k}{m}} \\ &= \sqrt{2} \cdot \frac{1}{2\pi} \sqrt{\frac{k}{m}} \\ &= \sqrt{2} \cdot f \end{aligned}$$

Therefore, the frequency increases by a factor of $\sqrt{2}$ when the mass is halved.

Answer to Exercise 2 (on page 10)

First, we find the angular frequency ω using the formula:

$$\omega = 2\pi f = \sqrt{\frac{k}{m}} = \sqrt{\frac{400 \text{ N/m}}{4 \text{ kg}}} = \sqrt{100} = 10 \text{ rad/s}$$

Therefore, the general equation for the position of a mass-spring system undergoing simple harmonic motion is:

$$x(t) = A \cos(\omega t + \phi)$$

The block starts at its maximum displacement ($x = 6 \text{ cm}$, velocity 0), and $\cos 0 = 1$, so

$\phi = 0$ and $x(t) = 6 \cos(10t)$: option (A).

Option (C) is the same function in disguise. Writing it with sine, $x = 6 \sin(10t + \phi)$, we need $x(0) = 6$:

$$6 = 6 \sin(\phi) \implies \sin(\phi) = 1 \implies \phi = \frac{\pi}{2}$$

So the base equation becomes:

$$x(t) = 6 \sin(10t + \frac{\pi}{2})$$

which equals $6 \cos(10t)$, because $\sin(\theta + \frac{\pi}{2}) = \cos \theta$. So both (A) and (C) are correct. (Options (D) and (E) use 10π , which would be the wrong angular frequency.)

Answer to Exercise 3 (on page 10)

To find the speed of the ball as it leaves the spring, we can use the principle of conservation of mechanical energy. The net energy in the system remains constant, meaning that the potential energy stored in the compressed spring is converted into the kinetic energy of the ball when it is released. The potential energy stored in the spring when compressed is given by:

$$U_s = \frac{1}{2} kx^2 = \frac{1}{2} (500)(-0.24)^2 = 14.4 \text{ J}$$

The kinetic energy of the ball as it leaves the spring is the same amount, so we set the potential energy equal to the kinetic energy:

$$\frac{1}{2} mv^2 = U_s$$

Solving for v :

$$v = \sqrt{\frac{2U_s}{m}} = \sqrt{\frac{2(14.4)}{0.2}} = \sqrt{144} = 12 \text{ m/s}$$

The speed of the ball as it leaves the spring is 12 m/s.

Answer to Exercise 4 (on page 11)

$$\omega = \sqrt{\frac{200}{0.5}} = 20 \text{ rad/s, and } T = \frac{2\pi}{20} \approx 0.314 \text{ s.}$$

Answer to Exercise 5 (on page 14)

The restoring force along the arc is $F = -mg \sin \theta$, so the tangential acceleration is $a =$

$-g \sin \theta$. For small angles, $\sin \theta \approx \theta$, so $a \approx -g\theta$. The distance along the arc from the bottom is $x = L\theta$, so $\theta = \frac{x}{L}$ and

$$a \approx -\frac{g}{L}x \quad (2.1)$$

This is the signature of simple harmonic motion, $a = -\omega^2 x$, from the oscillations chapter, with

$$\omega = \sqrt{\frac{g}{L}} \quad (2.2)$$

Therefore, the period T of the pendulum is

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{L}{g}} \quad (2.3)$$

So only changes (b) and (d) affect the period: a longer string or weaker gravity makes it longer. The mass cancels out, and for small swings the amplitude does not matter.

Answer to Exercise 6 (on page 15)

The correct answer is (c): approximately 4.9 seconds.

The period T of a simple pendulum is given by the formula:

$$T = 2\pi\sqrt{\frac{L}{g}}$$

where L is the length of the pendulum and g is the acceleration due to gravity.

On Earth, the period is 2 seconds, so we can set up the equation:

$$2 = 2\pi\sqrt{\frac{L}{g_{\text{Earth}}}}$$

This shows that T is inversely proportional to the square root of g . We can rearrange this equation to solve for L . So if g decreases by a factor of 6, then T will increase by a factor of $\sqrt{6}$.

Now, on the Moon, the acceleration due to gravity is approximately $\frac{1}{6}g_{\text{Earth}}$. We can substitute this into the period formula:

$$T_{\text{Moon}} = 2\pi\sqrt{\frac{L}{g_{\text{Moon}}}} = 2\pi\sqrt{\frac{L}{\frac{1}{6}g_{\text{Earth}}}} = 2\pi\sqrt{\frac{6L}{g_{\text{Earth}}}} = \sqrt{6} \cdot 2\pi\sqrt{\frac{L}{g_{\text{Earth}}}} = \sqrt{6} \cdot T_{\text{Earth}}$$

Since our original period on Earth is 2 seconds, we have:

$$T_{\text{Moon}} = \sqrt{6} \cdot 2 \approx 4.9 \text{ seconds}$$

Which is equal to option (c).

Answer to Exercise 7 (on page 18)

$$T = 2\pi\sqrt{\frac{0.25}{9.8}} \approx 1.00 \text{ s.}$$

Answer to Exercise 8 (on page 21)

$$v = \sqrt{3.721(3.4 \times 10^6)} = 3,557 \text{ m/s}$$

The circular orbit is $2\pi(3.4 \times 10^6) = 21.4 \times 10^6$ meters in circumference.

The period of the orbit is $(21.4 \times 10^6)/3,557 \approx 6,000$ seconds.

Answer to Exercise 9 (on page 25)

Using equation 3.4:

1.

$$\begin{aligned} v_{\text{esc}} &= \sqrt{\frac{2GM}{R}} \\ &= \sqrt{\frac{2(6.674 \times 10^{-11} \text{ N m}^2/\text{kg}^2)(6.42 \times 10^{23} \text{ kg})}{3.4 \times 10^6 \text{ m}}} \\ &\approx 5,020 \text{ m/s} \end{aligned}$$

2. Now the radius is $3.4 \times 10^6 + 11,000 = 3.411 \times 10^6$ meters, because the satellite is 11

km above the surface. Recalculating:

$$\begin{aligned}
 v_{\text{esc}} &= \sqrt{\frac{2GM}{R}} \\
 &= \sqrt{\frac{2(6.674 \times 10^{-11} \text{ N m}^2/\text{kg}^2)(6.42 \times 10^{23} \text{ kg})}{3.411 \times 10^6 \text{ m}}} \\
 &\approx 5,013 \text{ m/s}
 \end{aligned}$$

Answer to Exercise 10 (on page 26)

$$v = \sqrt{\frac{2(6.674 \times 10^{-11})(6.42 \times 10^{23})}{3.39 \times 10^6}} \approx 5.0 \text{ km/s.}$$

Answer to Exercise 11 (on page 42)

In the two-moon simulation, the positions on day 100 move by only about 1 km. In the three-moon simulation, the 1 km nudge sends the moons to completely different places: on day 100, moon 3 ends up about 440,000 km from where it was before, and moon 1 moves by about 250,000 km. This sensitivity to starting conditions is the hallmark of chaos: to predict the far future, you would need to know the starting positions with impossible precision.

Answer to Exercise 12 (on page 43)

$$x = 1 + (-3)(0.1) = 0.7 \text{ m.}$$

Answer to Exercise 13 (on page 54)

The Earth and 1 kg on the surface would attract each other with a force of:

$$F_g = \frac{(6.67430 \times 10^{-17})(5.97219 \times 10^{24})(1)}{6,371^2} = \frac{3.98597 \times 10^8}{4.05896 \times 10^7} \approx 9.82 \text{ N}$$

Thus, if the Earth were still and alone in the universe, the oceans would form a perfect sphere.

Answer to Exercise 14 (on page 54)

$$F_c = \frac{(1)(465)^2}{6,371,000} \approx 0.0339 \text{ N}$$

So the spinning of the Earth is trying to throw you into space, but the force of gravity is about 289 times more powerful.

This centripetal force decreases as you move from the equator to the north pole. In fact, at the north pole, there is no centripetal force. Thus, the spinning of the Earth makes the oceans an oblate ellipsoid instead of a perfect sphere: the diameter going from pole-to-pole is shorter than a diameter measured at the equator.

You should feel a teensy-tiny bit lighter on your feet at the equator than you do at the north pole: 0.34% lighter.

Answer to Exercise 15 (on page 55)

Overhead, the moon is $384,467 - 6,371 = 378,096$ km from your 1 kg mass.

$$F_g = \frac{Gm_1m_2}{r^2} = \frac{(6.67430 \times 10^{-17})(7.347673 \times 10^{22})(1)}{378,096^2} = \frac{4.9040574 \times 10^6}{1.42956585216 \times 10^{11}} = 3.43058 \times 10^{-5} \text{ N}$$

This is a very small force: the force due to Earth's gravity is nearly three hundred thousand times stronger.

Underfoot, the moon is $384,467 + 6,371 = 390,838$

$$F_g = \frac{Gm_1m_2}{r^2} = \frac{(6.67430 \times 10^{-17})(7.347673 \times 10^{22})(1)}{390,838^2} = \frac{4.9040574 \times 10^6}{1.52754 \times 10^{11}} = 3.2103 \times 10^{-5} \text{ N}$$

The force due to the moon's gravity is about 7% stronger when the moon is overhead than when it is underfoot.

Answer to Exercise 16 (on page 56)

If we let r be the distance (in km) from the center of the Earth to the center of mass, the distance from the center of the mass to the center of the moon is $384,467 - r$.

To find the balance point, multiply each mass by how far it is from the center of mass:

$$(5.97219 \times 10^{24}) r = (7.347673 \times 10^{22}) (384,467 - r)$$

Solving for r :

$$r = \frac{4,730.15}{1 + 0.0123} = 4,673 \text{ km}$$

That is less than the Earth's radius of 6,371 km, so the barycenter is inside the Earth, about 1,700 km below the surface on the side facing the Moon.

Answer to Exercise 17 (on page 56)

27.3 days is 2,358,720 seconds, so $\omega = \frac{2\pi}{2,358,720} \approx 2.664 \times 10^{-6} \text{ rad/s}$, and

$$F_c = (1)(4,673,000)(2.664 \times 10^{-6})^2 \approx 3.32 \times 10^{-5} \text{ N}$$

The Moon's pull at the Earth's center is

$$\frac{(6.67430 \times 10^{-17})(7.347673 \times 10^{22})}{384,467^2} \approx 3.32 \times 10^{-5} \text{ N}$$

They match, as they must: the Moon's pull on the Earth is exactly what swings the Earth around the barycenter.

Answer to Exercise 18 (on page 57)

Closest to the Moon:

$$3.4305 \times 10^{-5} - 3.3177 \times 10^{-5} \approx 1.13 \times 10^{-6} \text{ N, toward the Moon}$$

Farthest from the Moon:

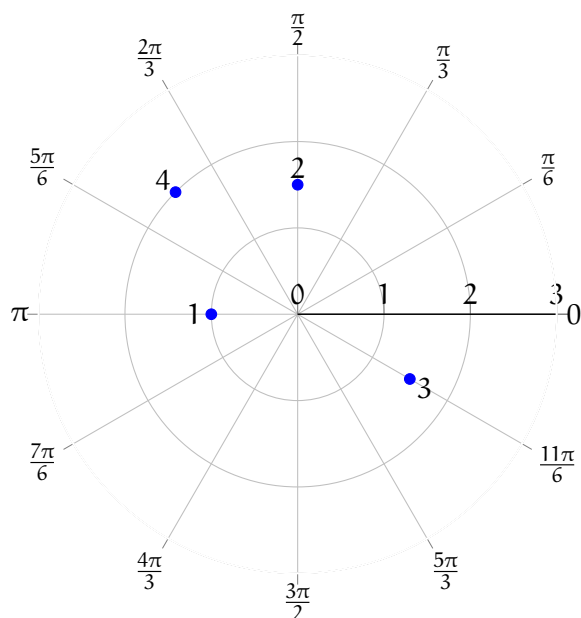
$$3.2104 \times 10^{-5} - 3.3177 \times 10^{-5} \approx -1.07 \times 10^{-6} \text{ N, away from the Moon}$$

The two leftover forces are nearly equal and point in opposite directions: one pulls the near-side water toward the Moon, the other lets the far-side water bulge away from it. That is why there are two nearly equal high tides every day. Both are tiny: Earth's own gravity, 9.82 N, is about 9 million times larger.

Answer to Exercise 19 (on page 57)

No. It is divisible by 100 but not by 400.

Answer to Exercise 20 (on page 62)



Answer to Exercise 21 (on page 67)

1. $(0, -2)$. $x = 2 \cdot \cos \frac{3\pi}{2} = 2 \cdot 0 = 0$ and $y = 2 \cdot \sin \frac{3\pi}{2} = 2 \cdot -1 = -2$.
2. $(-1, 1)$. $x = \sqrt{2} \cdot \cos \frac{3\pi}{4} = \sqrt{2} \cdot -\frac{\sqrt{2}}{2} = -\frac{2}{2} = -1$ and $y = \sqrt{2} \cdot \sin \frac{3\pi}{4} = \sqrt{2} \cdot \frac{\sqrt{2}}{2} = \frac{2}{2} = 1$.
3. $(\frac{3\sqrt{2}}{2}, -\frac{3\sqrt{2}}{2})$. $x = 3 \cdot \cos -\frac{\pi}{4} = 3 \cdot \frac{\sqrt{2}}{2} = \frac{3\sqrt{2}}{2}$ and $y = 3 \cdot \sin -\frac{\pi}{4} = 3 \cdot -\frac{\sqrt{2}}{2} = -\frac{3\sqrt{2}}{2}$.
4. $(-\frac{3}{2}, \frac{3\sqrt{3}}{2})$. $x = (-3) \cos (-\frac{\pi}{3}) = (-3) (\frac{1}{2}) = -\frac{3}{2}$ and $y = (-3) \sin (-\frac{\pi}{3}) = (-3) (-\frac{\sqrt{3}}{2}) = \frac{3\sqrt{3}}{2}$. (The negative radius flips the point from quadrant IV to quadrant II.)

5. $(0, -2)$. $x = 2 \cdot \cos -\frac{\pi}{2} = 2 \cdot 0 = 0$ and $y = 2 \cdot \sin -\frac{\pi}{2} = 2 \cdot -1 = -2$.

Answer to Exercise 22 (on page 67)

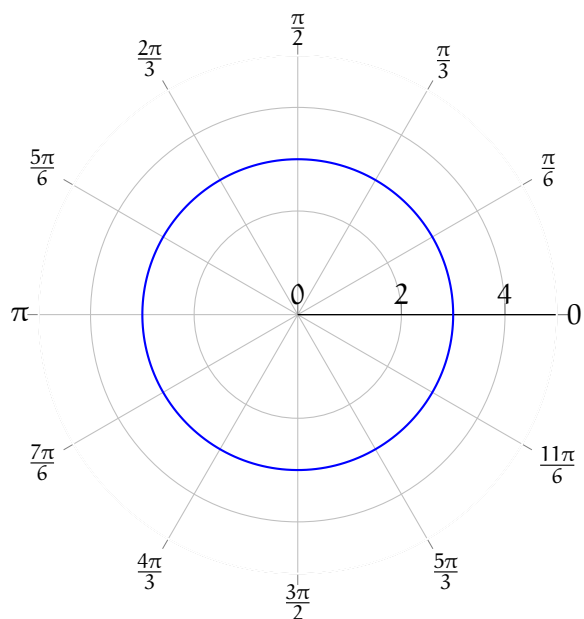
- $(4\sqrt{2}, \frac{3\pi}{4})$. $r = \sqrt{x^2 + y^2} = \sqrt{32} = 4\sqrt{2}$. $\arctan \frac{y}{x} = \arctan \frac{4}{-4} = \arctan -1 = -\frac{\pi}{4} + n\pi$. We take $\theta = \frac{3\pi}{4}$ to satisfy the domain restriction and be in the correct quadrant.
- $(6, \frac{\pi}{3})$. $r = \sqrt{3^2 + (3\sqrt{3})^2} = \sqrt{9 + 27} = \sqrt{36} = 6$. $\arctan \frac{3\sqrt{3}}{3} = \arctan \sqrt{3} = \frac{\pi}{3} + n\pi$. We take $\theta = \frac{\pi}{3}$ to satisfy the domain restriction and be in the correct quadrant.
- $(2, \frac{11\pi}{6})$. $r = \sqrt{\sqrt{3}^2 + (-1)^2} = \sqrt{3 + 1} = 2$. $\arctan \frac{-1}{\sqrt{3}} = -\frac{\pi}{6} + n\pi$. We take $\theta = \frac{11\pi}{6}$ to satisfy the domain restriction and have the point in the correct quadrant.
- $(6, \pi)$. $r = \sqrt{(-6)^2 + 0^2} = 6$. $\arctan \frac{0}{-6} = 0 + n\pi$. We take $\theta = \pi$ to satisfy the domain restriction.
- $(2\sqrt{2}, \frac{5\pi}{4})$. $r = \sqrt{(-2)^2 + (-2)^2} = \sqrt{8} = 2\sqrt{2}$. $\arctan \frac{-2}{-2} = \arctan 1 = \frac{\pi}{4} + n\pi$. We take $\theta = \frac{5\pi}{4}$ to satisfy the domain restriction and be in the correct quadrant.

Answer to Exercise 23 (on page 68)

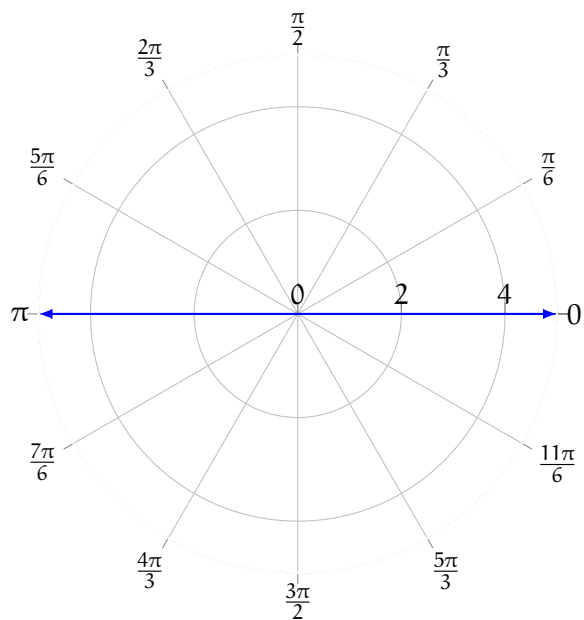
- $x = 12 \cos \frac{5\pi}{6} = 12 \cdot \left(-\frac{\sqrt{3}}{2}\right) = -6\sqrt{3} \approx -10.4$ and $y = 12 \sin \frac{5\pi}{6} = 6$. The plane is about 10.4 km west and 6 km north.
- The drone is at $(-5, -12)$. $r = \sqrt{25 + 144} = 13$ km. $\arctan \frac{-12}{-5} \approx 1.176$, but that is in quadrant I and the drone is in quadrant III, so $\theta \approx 1.176 + \pi \approx 4.318$. The drone is at $(13, 4.318)$.

Answer to Exercise 24 (on page 71)

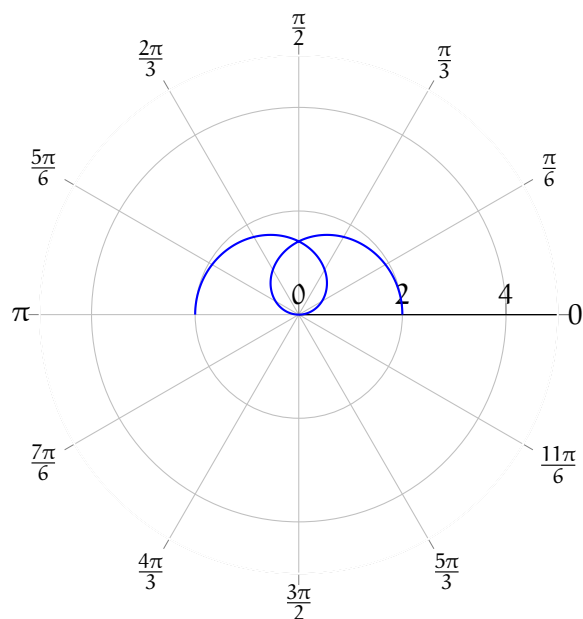
- $r = 3$



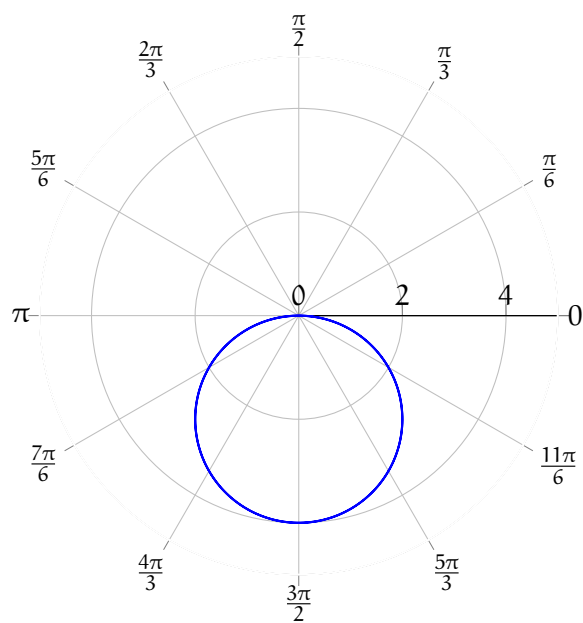
2. $\theta = \pi$ Because r includes all real numbers, negative r is possible and the line $\theta = \pi$ extends in both directions



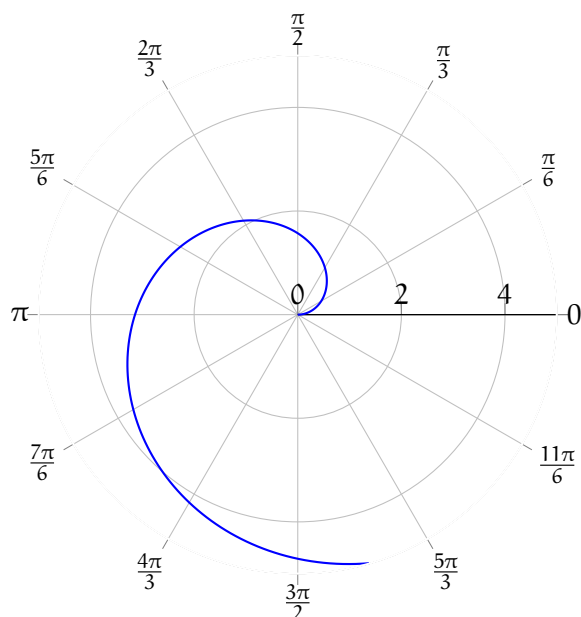
3. $r = 2 \cos \frac{\theta}{2}$



4. $r = -4 \sin \theta$



5. $r = \theta$ (The spiral continues, but is beyond the boundary of the graph)



Answer to Exercise 25 (on page 73)

1. 5 is odd: 5 petals, each 3 long.
2. 4 is even: 8 petals, each 2 long.
3. 2 is even: 4 petals, each 4 long.

Answer to Exercise 26 (on page 78)

1. True. $f(2)$ exists and $\lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2^-} f(x) = f(2) = 0$.
2. False. The graph has a vertical cusp at $x = 2$. On the right, $f'(x) \rightarrow +\infty$, and on the left, $f'(x) \rightarrow -\infty$. The derivative is unbounded there, so f is not differentiable at $x = 2$.
3. True. $\sqrt{|2-2|} = \sqrt{0} = 0$.
4. False. $f(2) = 0$ and $\lim_{x \rightarrow 2} f(x) = 0$. The function is finite at $x = 2$, so there is no vertical asymptote.

Answer to Exercise 27 (on page 82)

To estimate the slope at $t = 12$, we can use the data at $t = 9$ and $t = 15$. The slope of the line connecting those points is approximately the slope at $t = 12$.

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{67.9 - 61.8}{15 - 9} = \frac{6.1}{6} = 1.017$$

The units for the numerator are degrees Fahrenheit and for the denominator are minutes. Therefore, the estimated slope has units of degrees Fahrenheit per minute. This represents the change in temperature of the water in the tub. When $t = 12$, the water in the tub is increasing in temperature at about 1 degree Fahrenheit per minute.

Answer to Exercise 28 (on page 84)

1.

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{[m(x+h) + b] - [mx + b]}{h}$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{mx + mh + b - mx - b}{h} = \lim_{h \rightarrow 0} \frac{mh}{h} = \lim_{h \rightarrow 0} m = m$$

2.

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{\sqrt{16-x-h} - \sqrt{16-x}}{h}$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{\sqrt{16-x-h} - \sqrt{16-x}}{h} \cdot \frac{\sqrt{16-x-h} + \sqrt{16-x}}{\sqrt{16-x-h} + \sqrt{16-x}}$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{(16-x-h) - (16-x)}{h(\sqrt{16-x-h} + \sqrt{16-x})}$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{-h}{h(\sqrt{16-x-h} + \sqrt{16-x})} = \lim_{h \rightarrow 0} \frac{-1}{\sqrt{16-x-h} + \sqrt{16-x}}$$

$$f'(x) = \frac{-1}{\sqrt{16-x} + \sqrt{16-x}} = \frac{-1}{2\sqrt{16-x}}$$

3.

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{\frac{(x+h)^2-1}{2(x+h)-3} - \frac{x^2-1}{2x-3}}{h}$$

$$f'(x) = \lim_{h \rightarrow 0} \left(\frac{1}{h} \right) \left[\frac{x^2 + 2xh + h^2 - 1}{2x + 2h - 3} - \frac{x^2 - 1}{2x - 3} \right]$$

$$f'(x) = \lim_{h \rightarrow 0} \left(\frac{1}{h} \right) \left[\frac{x^2 + 2xh + h^2 - 1}{2x + 2h - 3} \left(\frac{2x - 3}{2x - 3} \right) - \frac{x^2 - 1}{2x - 3} \left(\frac{2x + 2h - 3}{2x + 2h - 3} \right) \right]$$

$$\begin{aligned}
f'(x) &= \lim_{h \rightarrow 0} \left(\frac{1}{h} \right) \left[\frac{(x^2 + 2xh + h^2 - 1)(2x - 3) - (x^2 - 1)(2x + 2h - 3)}{(2x - 3)(2x + 2h - 3)} \right] \\
f'(x) &= \lim_{h \rightarrow 0} \left(\frac{1}{h} \right) \left[\frac{2x^3 + 4x^2h + 2xh^2 - 2x - 3x^2 - 6xh - 3h^2 + 3 - (2x^3 + 2x^2h - 3x^2 - 2x - 2h + 3)}{(2x - 3)(2x + 2h - 3)} \right] \\
f'(x) &= \lim_{h \rightarrow 0} \left(\frac{1}{h} \right) \left[\frac{2x^2h + 2xh^2 - 6xh - 3h^2 + 2h}{(2x - 3)(2x + 2h - 3)} \right] \\
f'(x) &= \lim_{h \rightarrow 0} \frac{2x^2 + 2xh - 6x - 3h + 2}{(2x - 3)(2x + 2h - 3)} = \frac{2x^2 - 6x + 2}{(2x - 3)^2}
\end{aligned}$$

Answer to Exercise 29 (on page 85)

Compute $\frac{f(3+h)-f(3)}{h} = \frac{(3+h)^2-9}{h} = 6+h$. As $h \rightarrow 0$, this approaches 6, so $f'(3) = 6$.

Answer to Exercise 30 (on page 85)

$\frac{\frac{1}{2+h} - \frac{1}{2}}{h} = \frac{-1}{2(2+h)} \rightarrow -\frac{1}{4}$ as $h \rightarrow 0$. So $f'(2) = -\frac{1}{4}$.

Answer to Exercise 31 (on page 85)

1. $\frac{5.9-30.4}{3-2} = -24.5$ m/s.

2.

$$y'(t) = \lim_{h \rightarrow 0} \frac{[50 - 4.9(t+h)^2] - [50 - 4.9t^2]}{h} = \lim_{h \rightarrow 0} \frac{-9.8th - 4.9h^2}{h} = \lim_{h \rightarrow 0} (-9.8t - 4.9h) = -9.8t$$

So $y'(2.5) = -24.5$ m/s, matching the estimate.

Answer to Exercise 32 (on page 86)

No. The right-hand limit $\lim_{h \rightarrow 0^+} \frac{|h|}{h} = 1$ and the left-hand limit $\lim_{h \rightarrow 0^-} \frac{|h|}{h} = -1$ differ, so the limit does not exist.



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