



CONTENTS

1	Limits and Continuity	3
1.1	Introduction to Limits	3
1.2	Limits of Asymptotes	3
1.3	Continuity	12
1.3.1	Continuity Practice	14
2	Limit Laws, the Squeeze Theorem, and the Intermediate Value Theorem	17
2.1	Limits Rules	17
2.2	Squeeze Theorem	20
2.2.1	Squeeze Theorem Practice	22
2.3	Intermediate Value Theorem	23
3	Parametric Functions	25
3.1	Conversion to Rectangular Form	26
3.2	Sketching Parametric Equations	27
3.3	Using Python to Explore Parametric Curves	28
3.4	Looking Ahead	30
4	Uniform Circular Motion and Centripetal Force	31
4.1	Introduction to Uniform Circular Motion	31
4.2	The Flying Billiard Ball	32
4.3	Velocity	34
4.4	Acceleration	36
4.5	Centripetal force	37

4.5.1	Banked Turns	39
5	Circular Motion in Action	41
5.1	Modeling Circular Motion	41
5.2	Flying Pigs and Tension	43
5.3	Non-uniform circular motion	46
5.3.1	Roller Coaster Loop	46
5.3.2	At the bottom	46
5.3.3	At the top	47
5.3.4	At the sides	48
6	Torque and Equilibrium	51
6.1	Rotational Kinematics	51
6.2	Rigid Bodies	51
6.3	Torque	52
6.3.1	Torque directions	55
7	Moment of Inertia and Angular Momentum	59
7.1	Moment of Inertia and Rotational $F = ma$	59
7.1.1	Relating $\tau = I\alpha$ and other Torque Def'ns	60
7.2	Parallel-Axis Theorem	61
7.3	Rotational Kinetic Energy and Rolling Motion	62
7.4	Flywheel Energy Storage, Work, and Angular Momentum	64
7.4.1	Angular Momentum	65
7.4.2	Work done by Torque	67
7.5	Summary	67
A	Answers to Exercises	69
	Index	85

Limits and Continuity

1.1 Introduction to Limits

To begin talking about limits, let us imagine a scenario:

I flip a coin 10 times, and count the number of times it lands on heads. Let us say we get the following sequence

H, T, T, H, T, H, T, T, H, T

the fraction of heads to total flips is

$$\frac{\text{number of heads}}{\text{number of flips}} = \frac{4}{10} = 0.4$$

Now, if we were to flip the coin 100 times instead, and, we got 46 heads. Our sample proportion gets closer to 0.5, which is our *expected value* or *mean*:

$$\frac{\text{number of heads}}{\text{number of flips}} = \frac{46}{100} = 0.46$$

As we the number flips increases, the fraction of heads to total flips approaches the true probability of heads. Flipping the coin 10,000 times would produce a result even closer to the expected value, such as 5,023 heads, a fraction of 0.5023. While all of the flips are individual events, the long running average stabilizes closer to the true probability.

This long-run stabilization is one version of the *Law of Large Numbers*, an important idea in statistics.

We will revisit this in our statistics and sequence converge chapters. For now, what does this tell us? The law of large numbers provides insight into the central idea of a limit: what value something approaches as an input or process continues.

1.2 Limits of Asymptotes

A function has a **vertical asymptote** at $x = a$ when the function values grow without bound as x approaches a . A function's **end behavior** describes how the function values behave as $x \rightarrow \infty$ or $x \rightarrow -\infty$; if the function approaches a fixed value L , we say it has a **horizontal asymptote** at $y = L$.

The asymptotic behavior we see in rational functions suggests that we need to expand our

vocabulary of function characteristics. We examined vertical asymptotes and end behavior through graphs and tables, and discussed them in English. The language of limits enables us to discuss these attributes mathematically and with greater efficiency.

Let us revisit an example from the previous chapter. This function has a hole at $x = 1$, a vertical asymptote at $x = 3$, and a horizontal asymptote of $y = 1$.

$$f(x) = \frac{x^2 - 3x + 2}{x^2 - 4x + 3} = \frac{(x-1)(x-2)}{(x-1)(x-3)}$$

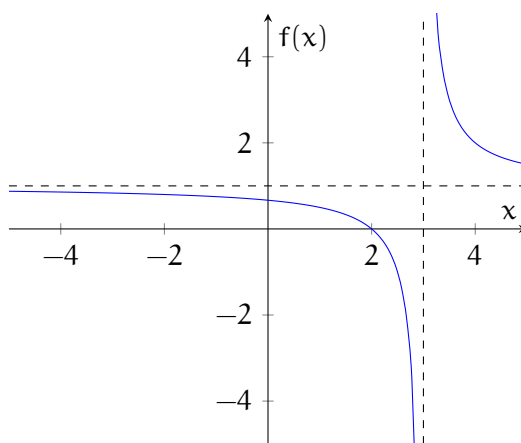


Figure 1.1: Graph of $f(x) = \frac{x^2-3x+2}{x^2-4x+3}$ with asymptotes

First, consider the vertical asymptote. We see that the graph goes down as it hugs the left side of the vertical asymptote, and goes up as it hugs the right side. We can describe these behaviors as the left- and right-hand limits, respectively. We say that the left-hand limit of f at $x = 3$ is negative infinity. Another way of communicating this is to say that as x approaches 3 from the left, the function approaches negative infinity. Symbolically, we summarize this as

$$\lim_{x \rightarrow 3^-} f(x) = -\infty$$

The little negative sign in $x \rightarrow 3^-$ indicates we are approaching $x = 3$ from the left (the negative side of the axis).

Similarly, the right-hand limit of f at $x = 3$ is positive infinity. In other words, as x approaches 3 from the right, the function approaches positive infinity. Symbolically, we write

$$\lim_{x \rightarrow 3^+} f(x) = \infty$$

This time, the little $+$ indicates we are approaching the x -value from the right (positive) side of the axis.

The limit of a function at a particular x -value is the y -value that the function approaches as it approaches the given x -value. In the previous example, we could only specify the left- and right-hand limits, because they were different. In cases where the left- and right-hand limits are equal, we can say that the function has a limit there. The hole in our function f is one such value. We see that as we approach the hole from both the left and right, the function takes on values near $\frac{1}{2}$. This is more apparent numerically:

x	0.9	0.99	0.999	1	1.001	1.01	1.1
$f(x)$	0.5238	0.5025	0.5003	undefined	0.4998	0.4975	0.4737

We can also see this by zooming in on the graph (see Figure 1.2):

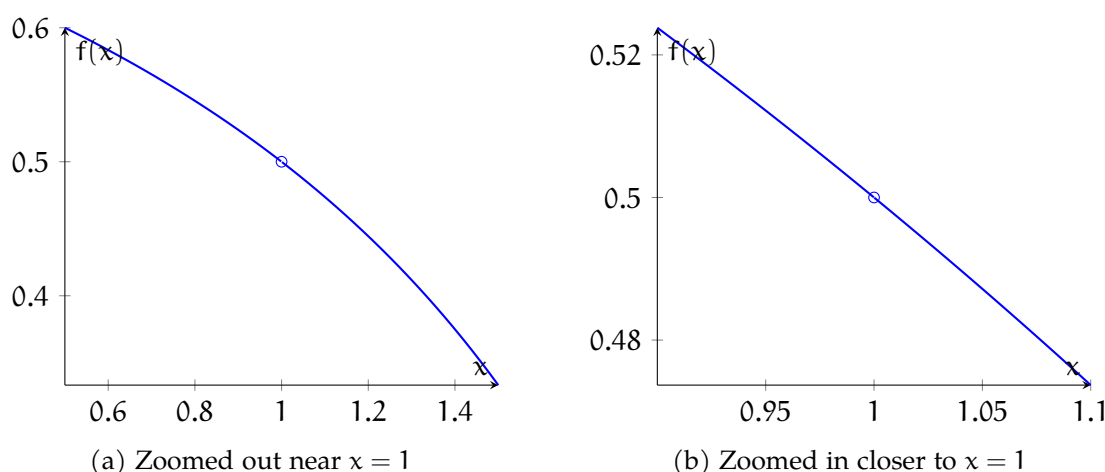


Figure 1.2: Two graphs of $f(x) = \frac{x^2 - 3x + 2}{x^2 - 4x + 3}$ zoomed in about $x = 1$

The left-hand and right-hand limits of f at 1 are both $\frac{1}{2}$. Since they are equal, we can also say that the limit of f at 1 is $\frac{1}{2}$. This allows us to efficiently discuss the behavior of f at 1, even though the function is not defined there, as substituting 1 into the function gives division by zero.

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1} f(x) = \frac{1}{2}$$

We can also talk about limits at x -values where nothing weird is happening (that is, no hole or vertical asymptote). For example, as x approaches 4 from the left and right, y approaches 2.

x	3.9	3.99	3.999	4	4.001	4.01	4.1
$f(x)$	2.1111	2.0101	2.0010	2	1.9990	1.9901	1.9091

In this case, since nothing weird is happening, the limit is equal to the function value. This is an example of continuity, which we will discuss in more detail later in this chapter. By contrast, at the vertical asymptote $x = 3$, since the left- and right-hand limits are not equal, we say the function does not have a limit, or the limit does not exist.

Finally, let us consider the horizontal asymptote of f . The graph hugs the line $y = 1$ as x goes far to the left and far to the right. We say that as x approaches negative infinity, f approaches 1; likewise, that as x approaches positive infinity, f approaches 1. We write these symbolically as $\lim_{x \rightarrow -\infty} f(x) = 1$ and $\lim_{x \rightarrow \infty} f(x) = 1$.

Exercise 1 Limits Practice 1

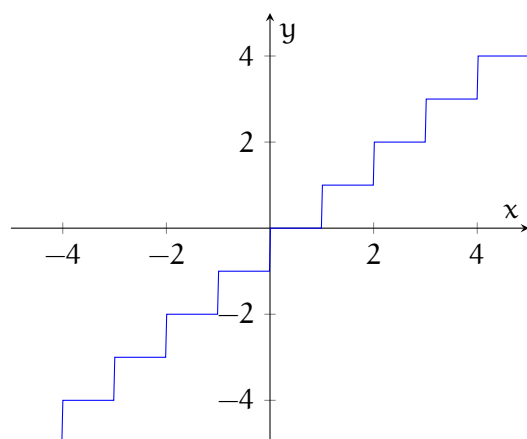
Determine the left- and right-hand limits of the function as x approaches the given values. At x -values where the limit exists, determine it.

1. $p(x) = \frac{x+3}{x^2+9x+18}$, $x = -6, -5, -3, \infty$

Working Space

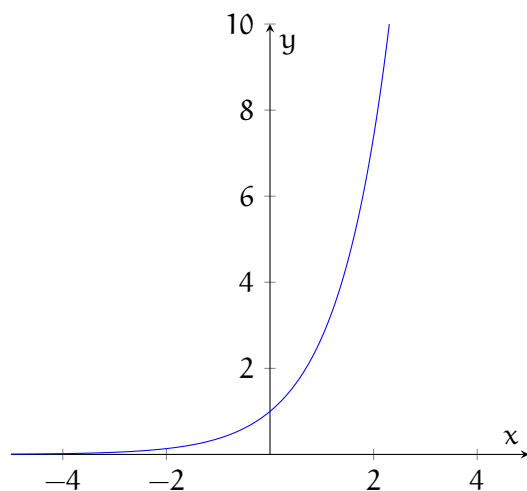
Answer on Page 69

We have seen two weird behaviors of rational functions at certain x -values: holes and vertical asymptotes. Now, we will examine another type of weird behavior: jumps. This is a characteristic of some piecewise-defined functions. In piecewise-defined functions, the domain is divided into two or more pieces, and a different expression is used to give the y -value depending on which piece contains the x -value. One common piecewise-defined function is the floor function (shown in figure 1.3), sometimes denoted $\lfloor x \rfloor$. The standard floor function rounds any real number down to the nearest integer. So, for a price quoted in dollars and cents, the floor would just be the number of dollars.

Figure 1.3: Graph of $y = \lfloor x \rfloor$

When x is exactly 1, the function value is 1: the number of dollars in a price of \$1.00. When x is any number greater than 1 but less than 2, the function value is still 1. Also, $\lfloor 1.01 \rfloor$, $\lfloor 1.5 \rfloor$, and $\lfloor 1.99999 \rfloor$ are all 1. As we continue to look to the right, once x equals exactly 2, the function jumps up to the value 2. So, $\lim_{x \rightarrow 2^-} \lfloor x \rfloor = 1$, while $\lim_{x \rightarrow 2^+} \lfloor x \rfloor = 2$.

Besides rational and piecewise defined functions, there are other functions with interesting limits. Consider the standard exponential function, $y = e^x$ (shown in figure 1.4).

Figure 1.4: Graph of $y = e^x$

As x increases, y increases without bound; that is, $\lim_{x \rightarrow \infty} e^x = \infty$. However, looking far to the left, we see that y hugs the x -axis. This is because raising e to a large negative exponent is the same as 1 divided by e raised to a large positive exponent; that is, 1 divided by a very large number, which yields a very small positive number. In limit notation, $\lim_{x \rightarrow -\infty} e^x = 0$. This example illustrates that horizontal asymptotes need not

model end behavior in both directions. Note that this reasoning holds for $y = b^x$ for any $b > 1$, so all such functions have the same horizontal asymptote, $y = 0$.

We know that the natural logarithm function, $y = \ln x$, is the inverse of $y = e^x$. Since inverse functions swap the role of x and y , it stands to reason that a horizontal asymptote in one function corresponds with a vertical asymptote in the other function, and that is indeed the case (see figure 1.5).

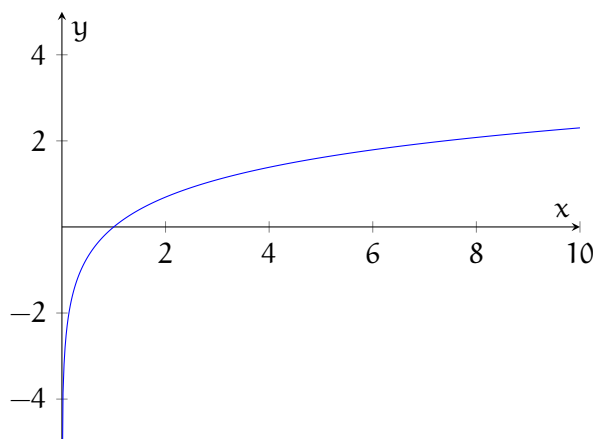


Figure 1.5: Graph of $y = \ln x$

An untransformed logarithm function is defined only for positive inputs. That is because it is not possible to find an exponent of a positive number that will yield a negative or zero result. What type of exponent on a positive number yields a number near zero? That would be a large-magnitude negative number. So, on the logarithm graph, large negative y -values correspond with x -values only slightly greater than zero. So, $\ln x$ (and $\log_2 x$, and indeed $\log_b x$ for any $b > 1$) approaches negative infinity as x approaches 0 from the right. There is no left-hand limit at 0, however. In limit notation, $\lim_{x \rightarrow 0^+} \ln x = -\infty$.

Exercise 2 Limits Practice 2

State the asymptotes of the following transformed exponential and logarithmic functions. Give the limit statement which describes the behavior of the function along the asymptote.

1. $y = 3^x + 1$
2. $y = \log_2(x - 4)$
3. $y = 2^{1-x}$

Working Space

4. $y = \log_{10}(-2x)$

Answer on Page 69

We next consider two functions that each have two horizontal asymptotes. These two seemingly obscure functions are quite important in data science.

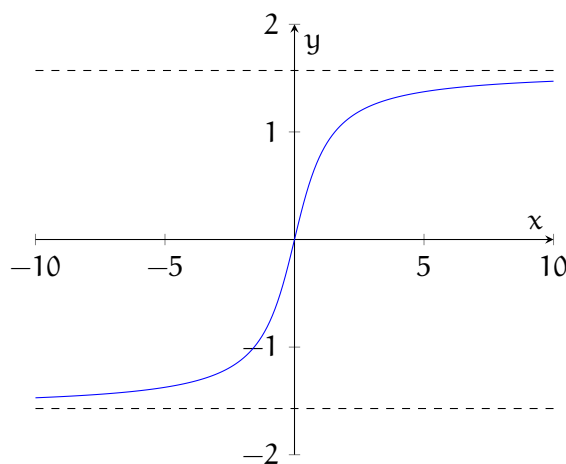
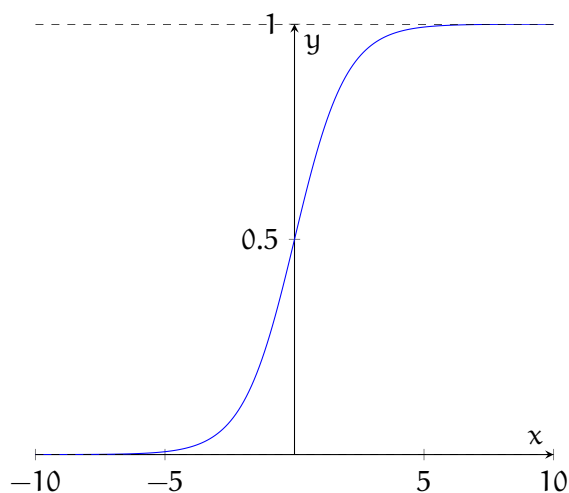


Figure 1.6: Graph of $y = \arctan x$

We know that the arctangent, or inverse tangent, function is the inverse of the piece of the tangent function which passes through the origin. The vertical asymptotes bounding this piece become horizontal asymptotes when the function is inverted.

Here are the equation and graph of the logistic function:

Figure 1.7: Graph of the logistic function, $y = \frac{1}{1+e^{-x}}$

For large magnitude negative values of x , the exponential term in the denominator becomes a very large positive value. The fraction thus becomes a positive number very close to zero. For large magnitude positive values of x , that exponential term becomes a very small positive number. Adding it to 1 yields a denominator just barely greater than 1. Dividing 1 by this number therefore yields a function value just barely less than 1. So, the logistic function yields values between 0 and 1, though never equaling either of these values exactly. It is precisely this characteristic which makes the logistic function so useful.

Exercise 3 Limits Practice 3

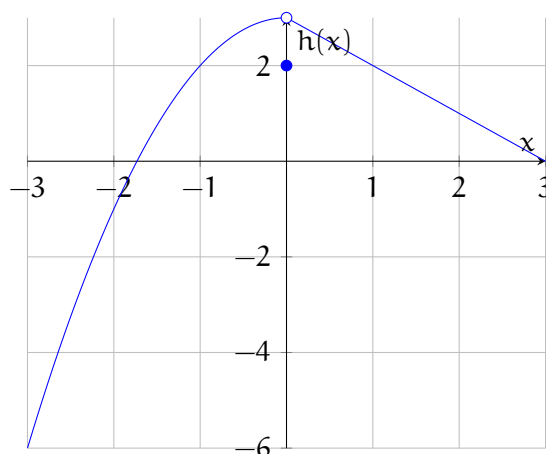
Using limit notation, state the limits as x approaches negative and positive infinity for the inverse tangent and logistic functions given above.

Working Space

Answer on Page 69

As seen above, the limit of a function from the left may be different from the limit of the function from the right. Additionally, the actual *value* of the function may be different from the limit. Consider the piecewise function $h(x)$:

$$h(x) = \begin{cases} -x^2 + 3, & \text{if } x < 0 \\ 2, & x = 0 \\ -x + 3, & \text{if } x > 0 \end{cases}$$

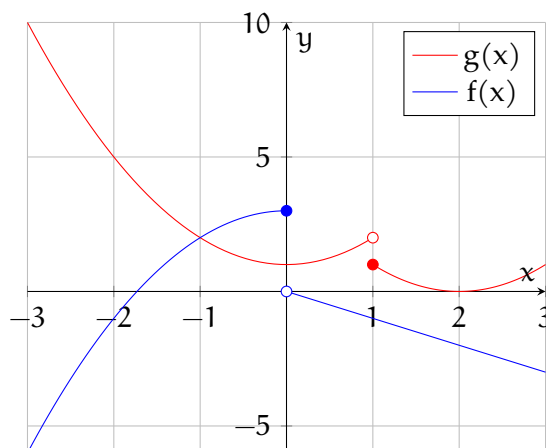

 Figure 1.8: Graph of the piecewise function, $h(x)$

From examining the graph, we see that

$$\lim_{x \rightarrow 0^-} h(x) = \lim_{x \rightarrow 0^+} h(x) = 3$$

However, $h(0) = 2 \neq 3$. So, does this limit exist? It does! The limit of a function describes the *behavior* of the function around a particular value, not the value of the function itself. In order for a limit to exist, the limits from the left and right must be equal to each other, but not necessarily the actual value of the function.

Use the graph of $h(x)$ above and the graphs of $f(x)$ and $g(x)$ below to complete the following exercise.


 Figure 1.9: Piecewise functions $f(x)$ and $g(x)$

Exercise 4 Limits Practice 4

Determine the limit from the left and the right for each function at the given value(s). State the limit at that value, if it exists.

1. $h(x), x = -1, 0, 1$
2. $f(x), x = -1, 0, 2$
3. $g(x), x = -2, 0, 1, 2$

Working Space

Answer on Page 69

1.3 Continuity

A note about continuity:

In order to be able to talk more about limits and know when we can apply certain rules and theorems, we first must discuss continuity. A function is continuous if there are no "jumps" or "gaps" in the graph of the function. For example, the function $f(x) = x^2$ is continuous for all real values of x . On the other hand, the function $g(x) = \tan(x)$ has many discontinuities, including at $x = \frac{\pi}{2}$. Let us examine the graph of each of these functions:

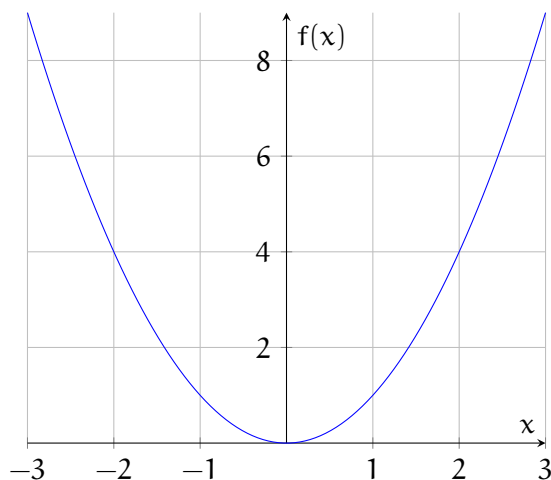


Figure 1.10: Graph of $f(x) = x^2$

If you wanted, you could trace your finger along the graph of $f(x)$ from $x = -3$ to $x = 3$ without ever picking up your finger. This means the function is continuous in the domain from $-3 \leq x \leq 3$. In this case, the domain of continuity *includes* the end points ($x = 3$ and $x = -3$). This is called a closed interval. In other cases, the function will be continuous right up to, but not including, the endpoints, as with the domains of continuity for our other example, $g(x) = \tan x$. This is called an open interval. Let us learn more about intervals of continuity by examining $g(x) = \tan x$.

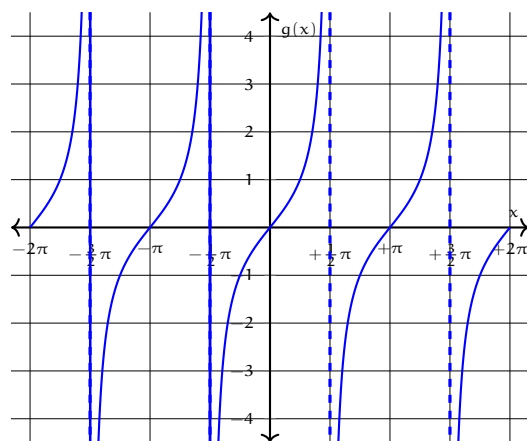


Figure 1.11: Graph of $g(x) = \tan x$

As you can see, if you trace your finger along the graph of the function starting at $x = 0$, you can continue without lifting your finger to $x = \frac{\pi}{2}$. As you approach $x = \frac{\pi}{2}$ from the left, the value of $g(x)$ approaches ∞ . In order to continue tracing the function PAST $x = \frac{\pi}{2}$, you have to lift your finger and bring it down to $-\infty$. The function then continues continuously again until $x = \frac{3\pi}{2}$.

In the case of $g(x) = \tan x$, the function is continuous on *open intervals*, including the open interval $\frac{\pi}{2} < x < \frac{3\pi}{2}$.

There is a shorter way to represent open and closed domain intervals. We can represent that $f(x) = x^2$ is continuous on the closed interval $-3 \leq x \leq 3$ in the following way:

$$x \in [-3, 3]$$

This reads as “ x is in the interval from -3 to 3 , inclusive.” That is, all the values from -3 to 3 , including the endpoints. The inclusion of the endpoints is implied by the use of *brackets*. For open intervals, we use parentheses to communicate that the interval goes up to, but does not include, the endpoints. For $g(x) = \tan x$, we can use parentheses:

$$x \in \left(-\frac{3\pi}{2}, -\frac{\pi}{2} \right)$$

because the $g(x) = \tan x$ is not continuous at $x = \frac{-3\pi}{2}$ or at $x = \frac{-\pi}{2}$.

Formally, a function $f(x)$ is continuous at $x = a$ if $\lim_{x \rightarrow a} f(x)$ exists *and* $\lim_{x \rightarrow a} f(x) = f(a)$. In other words, the limit is equal to the actual value of the function. Re-examine the graph of $h(x)$. We have already seen that $\lim_{x \rightarrow 0} h(x)$ exists and is equal to 3. However, $h(0) = 2 \neq \lim_{x \rightarrow 0} h(x)$. So $h(x)$ is not continuous at $x = 0$. Both pieces, $-x^2 + 3$ and $-x + 3$, are polynomials, which have no jumps or gaps, so $h(x)$ is continuous everywhere *except* $x = 0$. We can represent this mathematically by saying $h(x)$ is continuous on the domain $x \in (-\infty, 0) \cup (0, \infty)$. We use parentheses for $\pm\infty$ because we can never actually reach ∞ . Additionally, the function is continuous up to, but not including 0, and the use of parentheses excludes $x = 0$ from the domain of continuity.

1.3.1 Continuity Practice

Exercise 5

[This problem was originally presented as a calculator-allowed, multiple-choice question on the 2012 AP Calculus BC exam.] Suppose a function f is continuous at $x = 3$. Classify the following statements as always true, sometimes true, or never true. Explain your answers.

1. $f(3) < \lim_{x \rightarrow 3} f(x)$
2. $\lim_{x \rightarrow 3^+} f(x) \neq \lim_{x \rightarrow 3^-} f(x)$
3. $f(3) = \lim_{x \rightarrow 3^+} f(x) = \lim_{x \rightarrow 3^-} f(x)$

Working Space

Answer on Page 70

Exercise 6 Limits Practice 5

State the location of discontinuities (if any) and explain why the function is discontinuous at that location:

1. $f(x) = \frac{3x^2 - 8x - 3}{x - 3}$

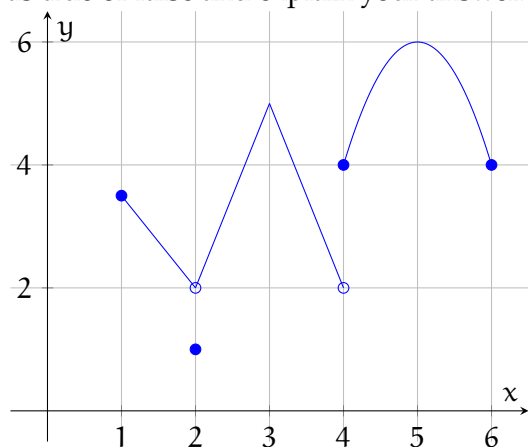
Working Space

2. $f(x) = \begin{cases} \frac{2}{x^4}, & \text{if } x \neq 0 \\ 2, & \text{if } x = 0 \end{cases}$
3. $f(x) = \begin{cases} \frac{3x^2-8x-3}{x-3}, & \text{if } x \neq 3 \\ 1, & \text{if } x = 3 \end{cases}$

Answer on Page 70

Exercise 7

The graph of a function, $h(x)$, is shown. Classify each of the following statements as true or false and explain your answer.



1. $\lim_{x \rightarrow 2} h(x)$ exists
2. $\lim_{x \rightarrow 3} h(x)$ does not exist
3. $\lim_{x \rightarrow 4} h(x)$ exists
4. $h(x)$ is continuous at $x = 5$
5. $h(x)$ is not continuous at $x = 4$
6. $\lim_{x \rightarrow 2} h(x) = h(2)$

Working Space

Answer on Page 71

Limit Laws, the Squeeze Theorem, and the Intermediate Value Theorem

2.1 Limits Rules

In the last chapter, you found limits by staring at graphs and tables. That works, but it is slow, and graphs can fool you. A few simple rules, the limit laws, let you compute limits of complicated functions by breaking them into simple pieces.

The following laws are true given that c is a constant, $\lim_{x \rightarrow a} f(x)$ exists, and $\lim_{x \rightarrow a} g(x)$ exists.

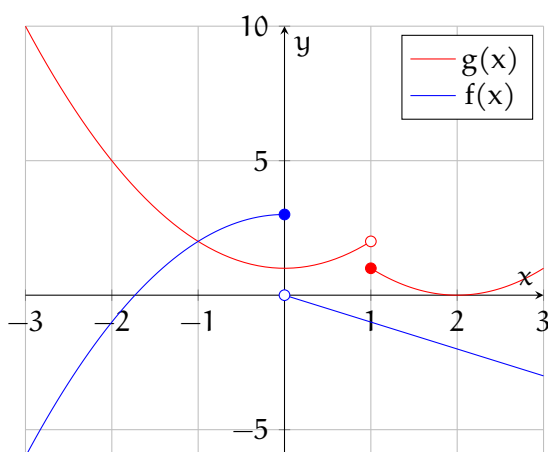
1. Sum Law $\lim_{x \rightarrow a} [f(x) + g(x)] = \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x)$
2. Difference Law $\lim_{x \rightarrow a} [f(x) - g(x)] = \lim_{x \rightarrow a} f(x) - \lim_{x \rightarrow a} g(x)$
3. Constant Multiple Law $\lim_{x \rightarrow a} [cf(x)] = c \cdot \lim_{x \rightarrow a} f(x)$
4. Product Law $\lim_{x \rightarrow a} [f(x)g(x)] = \lim_{x \rightarrow a} f(x) \cdot \lim_{x \rightarrow a} g(x)$
5. Quotient Law $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)}$ given that $\lim_{x \rightarrow a} g(x) \neq 0$

These laws are fairly obvious — the limit of the sum of two functions is equal to the sum of the limits of each function individually. The only tricky one is the last: The limit of the quotient of two functions is equal to the quotient of the limits if and only if the limit of the function in the denominator does not equal zero. This makes sense, as we know dividing by zero yields an undefined result.

Let us practice applying these laws to evaluate the limits of the functions $f(x)$, shown in blue below, and $g(x)$, shown in red below:

$$f(x) = \begin{cases} -x^2 + 3, & \text{if } x \leq 0 \\ -x, & \text{if } x > 0 \end{cases}$$

$$g(x) = \begin{cases} x^2 + 1, & \text{if } x < 1 \\ (x - 2)^2, & \text{if } x \geq 1 \end{cases}$$

Figure 2.1: Graphs of the piecewise functions $f(x)$ and $g(x)$

We can use these laws to evaluate limits involving $f(x)$ and $g(x)$ (shown on the graph above). Here are some examples: Use the graphs of $f(x)$ and $g(x)$ given above to evaluate each limit, if it exists. If the limit does not exist, explain why. Two examples are given first:

Example 1: Evaluate $\lim_{x \rightarrow 0} f(x) \cdot g(x)$

Solution 1: From the Product Law, we know that:

$$\lim_{x \rightarrow 0} f(x) \cdot g(x) = \lim_{x \rightarrow 0} f(x) \cdot \lim_{x \rightarrow 0} g(x)$$

Looking at the graph, we can see that

$$\lim_{x \rightarrow 0} g(x) = 1$$

and there is a discontinuity in $f(x)$ at $x = 0$. Therefore,

$$\lim_{x \rightarrow 0} f(x) = \text{DNE}$$

Substituting this, we get:

$$\lim_{x \rightarrow 0} f(x) \cdot \lim_{x \rightarrow 0} g(x) = \text{DNE} \cdot 1 = \text{DNE}$$

Therefore, *the limit does not exist.*

Example 2: Evaluate $\lim_{x \rightarrow 2} f(x) - g(x)$

Solution 2: Applying the Difference Law, we see that:

$$\lim_{x \rightarrow 2} [f(x) - g(x)] = \lim_{x \rightarrow 2} f(x) - \lim_{x \rightarrow 2} g(x)$$

Examining the graph, we see that

$$\lim_{x \rightarrow 2} f(x) = -2$$

and

$$\lim_{x \rightarrow 2} g(x) = 0$$

Substituting these values, we get:

$$\lim_{x \rightarrow 2} f(x) - g(x) = -2 - 0 = -2$$

Exercise 8 Limits Practice 6

Working Space

1. $\lim_{x \rightarrow -3} \frac{f(x)}{g(x)}$
2. $\lim_{x \rightarrow 2} [f(x) + 5g(x)]$
3. $\lim_{x \rightarrow -1} \frac{3g(x)}{f(x)}$
4. $\lim_{x \rightarrow 0} f(x) \cdot 5g(x)$
5. $\lim_{x \rightarrow -1} f(x) - 3g(x)$

Answer on Page 71

Recall that exponents represent repeated multiplication. Therefore, if we apply the Product Law multiple times, we obtain the Power Law for limits:

6. Power Law $\lim_{x \rightarrow a} [f(x)]^n = [\lim_{x \rightarrow a} f(x)]^n$ where n is a positive integer

There are two special limits that will be useful to us and are intuitively obvious, but we

will not formally prove here.

7. $\lim_{x \rightarrow a} c = c$

8. $\lim_{x \rightarrow a} x = a$

Combining Law 8 with the Power Law, we find that:

9. $\lim_{x \rightarrow a} x^n = a^n$

And similarly, for square roots:

10. $\lim_{x \rightarrow a} \sqrt[n]{x} = \sqrt[n]{a}$ (if n is even, we assume $a > 0$)

Direct substitution property: If f is a polynomial or rational function and a is in the domain for f , then

$$\lim_{x \rightarrow a} f(x) = f(a)$$

Often, rational functions can be simplified. In the limits chapter, we computed the limit by simplifying $f(x) = \frac{3x^2 - 8x - 3}{x - 3}$ to the simpler $g(x) = 3x + 1$. This is a valid strategy because $\frac{3x^2 - 8x - 3}{x - 3} = 3x + 1$ when $x \neq 3$. Remember: a limit describes how a function behaves *as it approaches* a , not its value/behavior when x *actually equals* a . This reveals the following useful rule:

If $f(x) = g(x)$ when $x \neq a$, then $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} g(x)$, provided the limit exists.

2.2 Squeeze Theorem

The Squeeze Theorem states that if $f(x) \leq g(x) \leq h(x)$ when x is near a (except at a) and

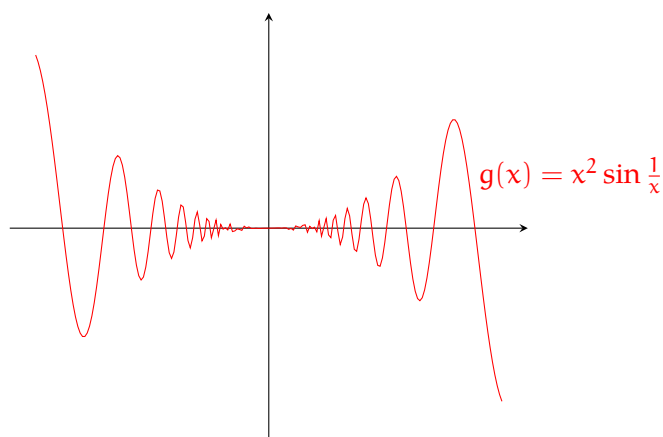
$$\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} h(x) = L$$

then

$$\lim_{x \rightarrow a} g(x) = L$$

In other words, if $g(x)$ is between $f(x)$ and $h(x)$ near a , and f and h have the same limit, L , then the limit of g must also be L .

Example: Let us examine the graph of $g(x) = x^2 \sin \frac{1}{x}$ and determine $\lim_{x \rightarrow 0} g(x)$:


 Figure 2.2: Graph of $g(x) = x^2 \sin \frac{1}{x}$

Solution: Because $\sin \frac{1}{x}$ is undefined at $x = 0$, we cannot compute the limit directly. However, from examining the graph, we can guess that $\lim_{x \rightarrow 0} g(x) = 0$. Feel free to confirm this with your calculator. We need to choose two functions: one that is larger than $g(x)$ near $x = 0$ and one that is smaller. Since $|\sin \frac{1}{x}| \leq 1$ (when $x \neq 0$), then

$$|x^2 \sin \frac{1}{x}| \leq x^2$$

and

$$-x^2 \leq x^2 \sin \frac{1}{x} \leq x^2$$

Let us confirm this by plotting $f(x) = -x^2$, $g(x)$, and $h(x) = x^2$ on the same graph:

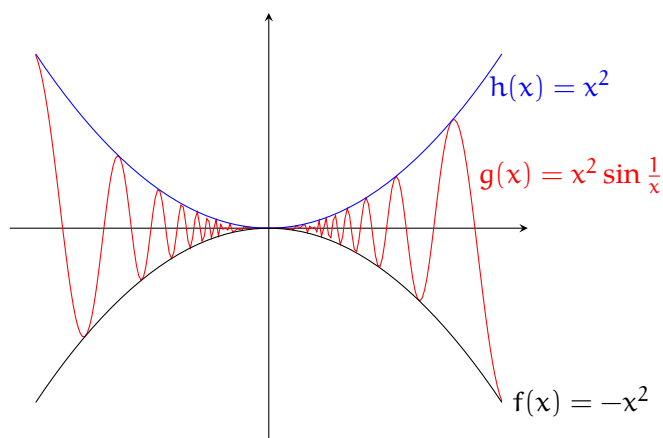


Figure 2.3: Squeeze Theorem example

As you can see, when x is near 0, $f(x) \leq g(x) \leq h(x)$. Because $f(x)$ and $h(x)$ are both

polynomials, their limits are straightforward:

$$\lim_{x \rightarrow 0} -x^2 = 0 \text{ and } \lim_{x \rightarrow 0} x^2 = 0$$

Then, by the Squeeze Theorem, we can say that:

$$\lim_{x \rightarrow 0} x^2 \sin \frac{1}{x} = 0$$

2.2.1 Squeeze Theorem Practice

Exercise 9 Squeeze Theorem 1

Use the Squeeze Theorem to show that $\lim_{x \rightarrow 0} \sqrt{x^3 + x^2} \cos \frac{1}{x} = 0$. Illustrate by graphing the functions you define as f , g , and h on the same plot.

Working Space

Answer on Page 73

Exercise 10 Squeeze Theorem 2

If $2x + 3 \leq f(x) \leq x^2 - 2x + 7$ for $x \geq 0$, find $\lim_{x \rightarrow 2} f(x)$

Working Space

Answer on Page 73

Exercise 11 Squeeze Theorem 3

Prove that $\lim_{x \rightarrow 0^+} \sqrt{x} e^{\sin \frac{\pi}{x}} = 0$

Working Space

Answer on Page 74

2.3 Intermediate Value Theorem

When considering functions that are continuous on a closed interval, the Intermediate Value Theorem can help us: Given a function, $f(x)$, that is continuous on the closed interval $[a, b]$ and $f(a) \neq f(b)$, there is at least one number c such that $f(c) = N$, where N is any number between $f(a)$ and $f(b)$. The theorem is illustrated in figures 2.4 and 2.5:

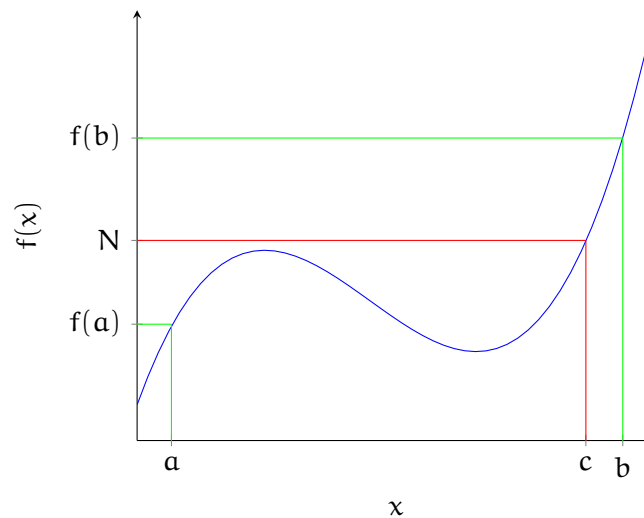


Figure 2.4: An example where one solution satisfies the Intermediate Value Theorem

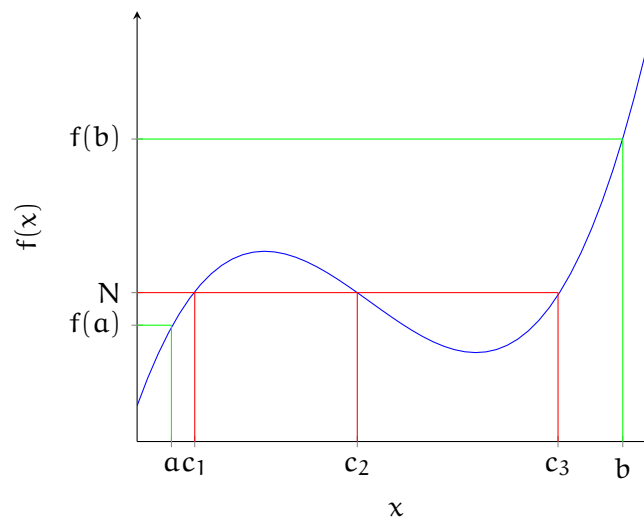


Figure 2.5: An example where more than one solution satisfies the Intermediate Value Theorem

Logically, we can think of the IVT this way: If a function is continuous on a closed interval, there are no gaps or breaks. If there are no gaps or breaks, then the function must pass through the line $y = N$, since it cannot jump over the line. Calculators and computers find roots this way: if f is negative at one end of an interval and positive at the other, a

root must lie between, so they cut the interval in half and keep the half where the sign still changes. You will write this *bisection method* yourself in the root-finding chapter.

Example: Show that there is at least one root of the equation $2x^3 - 6x^2 + 3x - 1 = 0$ between $x = 2$ and $x = 3$.

Solution: For IVT to apply, we must first check that the function is continuous on the closed interval $x \in [2, 3]$. We define $f(x) = 2x^3 - 6x^2 + 3x - 1$, which is continuous everywhere, because it is a polynomial function. *For more complex functions, always be sure to check the endpoints of an interval, since IVT only applies on closed intervals of continuity.* We will take $a = 2$, $b = 3$, and $N = 0$. We find the values of $f(x)$ at the endpoints:

$$f(2) = 2(2)^3 - 6(2)^2 + 3(2) - 1 = 16 - 24 + 6 - 1 = -3$$

$$f(3) = 2(3)^3 - 6(3)^2 + 3(3) - 1 = 54 - 54 + 9 - 1 = 8$$

Therefore, $f(2) < 0 < f(3)$ and according to IVT, there must exist some c such that $f(c) = 0$ and there is a root to the equation in the interval $x \in (2, 3)$.

Exercise 12 Intermediate Value Theorem Practice

Use the IVT to show there is a solution the given equation on the stated interval:

Working Space

1. $2x^4 + x - 12 = 0$, $(1, 2)$
2. $\ln(x) = 3x - 4\sqrt{x}$, $(2, 3)$
3. $2 \sin x = 3x^2 - 2x$, $(1, 2)$

Answer on Page 74

Parametric Functions

In a video game, a character runs, jumps, and loops through the air. At every instant, the game needs to know where it is: an x and a y , both changing with time. Describing a path this way, with time as the input and a point as the output, is the idea behind parametric equations, and it lets you draw curves that no single function $y = f(x)$ could.

Throughout your study of functions, you have seen functions that look like this:

$$y = f(x)$$

Sometimes, to describe certain shapes or curves on a graph, we must use more than one function to do so. This means a shape, like a circle on the graph, can be represented using

$$y = f(t) \quad \text{and} \quad x = g(t).$$

In this case, there are two functions, $x(t)$ and $y(t)$, that depend on a third variable t , and together they help us visualize and define a curve.

The central word you should think about as you go through this chapter is *parameter*. A parameter is a variable that both x and y depend on. The most common example of a parametric function is the circle. To represent a circle on a graph, we can use the equations

$$x = \cos(\theta) \quad \text{and} \quad y = \sin(\theta).$$

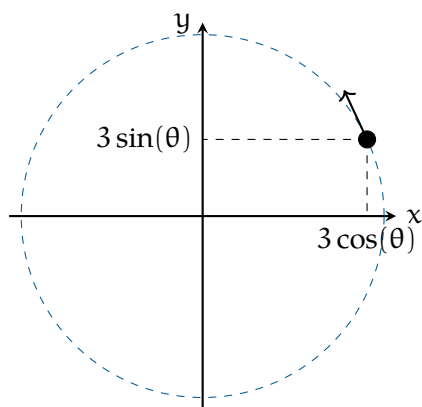


Figure 3.1: A diagram of a circle formed through a parametric functions.

In this case, the graph of the circle is determined by the value of the angle θ , the parameter.

So, why parametric functions? When you look up the function that describes a circle, you might see something like

$$x^2 + y^2 = r^2.$$

The equation $x^2 + y^2 = r^2$ does not describe y as a single function of x . You would need two functions (one for the top half and one for the bottom half), and many curves cannot be written as a single func-

tion at all. Parametric equations solve this problem by allowing us to describe the entire curve with one pair of equations, using a parameter t . This parameter can be denoted by other letters, but we will use t in this chapter.

Imagine a ball rolling across graph paper. At each moment, the ball is at a different location. If we label time with a variable t , then for every value of t the ball has an x -coordinate and a y -coordinate. What if we had two equations, one that told us its x -position at time t and one that told us its y -position? As t changes, these two equations would trace the entire path of the ball. This is the basic idea behind parametric equations.

3.1 Conversion to Rectangular Form

In many cases, a parametric curve can be rewritten as a single equation involving only x and y . This process is called eliminating the parameter, and the resulting equation is known as the rectangular form of the curve.

To eliminate the parameter, we solve one of the parametric equations for t and substitute into the other equation.

Example:

Consider the parametric equations

$$x = 2t + 1, \quad y = 3t - 4.$$

Solution: Solve the equation for x to isolate t :

$$t = \frac{x - 1}{2}.$$

Substitute into the equation for y :

$$y = 3 \left(\frac{x - 1}{2} \right) - 4.$$

Simplifying,

$$y = \frac{3}{2}x - \frac{11}{2}.$$

Example

Consider the parametric equations

$$x = \cos t, \quad y = \sin t.$$

Solution: In this case, solving for t directly is not convenient. Instead, we use the trigonometric identity

$$\cos^2 t + \sin^2 t = 1.$$

Substituting $x = \cos t$ and $y = \sin t$ gives

$$x^2 + y^2 = 1.$$

This rectangular equation describes a circle of radius 1 centered at the origin.

Note

Converting to rectangular form describes the shape of a curve, but it does not show the direction of motion or how many times the curve is traced. This information is contained in the parametric equations and the domain of the parameter!

3.2 Sketching Parametric Equations

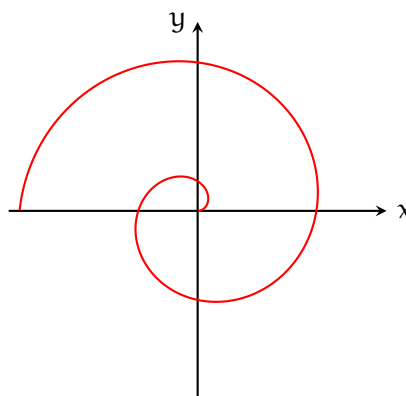
Parametric equations describe curves by expressing both x and y in terms of a third variable, usually t , called the parameter. When sketching a parametric curve, it is important to remember that the curve is traced as the parameter changes, rather than being drawn all at once as a function of x .

To sketch a parametric curve,

- choose several values of the parameter t ,
- compute the corresponding x - and y -values,
- plot the resulting points in the xy -plane,
- and indicate the direction in which the curve is traced as t increases.

Unlike rectangular graphs, parametric sketches also depend on the direction and speed of motion, which are determined by how $x(t)$ and $y(t)$ change as t varies.

Here is an example of an Archimedean Spiral, a spiral where the radius r is propor-



tional to the angle θ . In parametric form, it is represented as

$$x(t) = t \cos(t), \quad y(t) = t \sin(t)$$

Exercise 13

Working Space

Sketch the parametric equations

$$x = t + 1, \quad y = t^2.$$

In your sketch:

- create a table of values for t , x , and y ,
- plot the corresponding points,
- and indicate the direction of motion as t increases.

Answer on Page 75

3.3 Using Python to Explore Parametric Curves

Python can be used to visualize parametric curves and explore how the point $(x(t), y(t))$ moves as t changes. For example, we could write code that:

- plots points along a circle defined by $x = \cos(t)$ and $y = \sin(t)$,
- shows the distance from the origin,
- traces more complicated parametric curves like spirals or ellipses.

Below is a Python script that creates an animation for sketching parametric curves. There is a provision for changing the equations based on exercises and questions you come across.

```
1 import numpy as np
2 import matplotlib.pyplot as plt
```

```

3  from matplotlib.animation import FuncAnimation
4
5  t = np.linspace(0, 3, 50)
6
7  x = 4 - 2*t
8  y = 3 + 6*t - 4*t**2
9
10 fig, ax = plt.subplots()
11 ax.set_aspect("equal", "box")
12
13 ax.axhline(0, color="black", linewidth=0.5)
14 ax.axvline(0, color="black", linewidth=0.5)
15
16 ax.set_xlabel("x(t)")
17 ax.set_ylabel("y(t)")
18 ax.set_title("Parametric Equation of Curve")
19
20 ax.set_xlim([x.min() - 1, x.max() + 1])
21 ax.set_ylim([y.min() - 1, y.max() + 1])
22
23 (line,) = ax.plot([], [], "b--")
24 (point,) = ax.plot([], [], "ro")
25
26 def update(frame):
27     line.set_data(x[:frame + 1], y[:frame + 1])
28     point.set_data([x[frame]], [y[frame]])
29     return line, point
30
31 ani = FuncAnimation(fig, update, frames=len(t), interval=50, blit=True)
32 ani.save("parametric_animation.mp4")
33
34 plt.show()

```

Exercise 14

Working Space

Consider the parametric equations

$$x = 4 - 2t, \quad y = 3 + 6t - 4t^2,$$

for $0 \leq t \leq 3$.

- Sketch the curve by creating a table of values for t , x , and y .
- Use the provided Python animation code to visualize the curve.
- Describe the direction in which the

curve is traced as t increases.

- (d) Explain how the Python animation supports or clarifies your hand-drawn sketch.

Answer on Page 75

3.4 Looking Ahead

Because a parametric curve describes motion, it raises natural questions: How steep is the curve at a given moment? How fast is the point moving? How far does it travel in all? Answering those takes calculus. You will learn to find the slope of a parametric curve in the implicit differentiation chapter, and its length in the arc length chapter.

Exercise 15

How can two sets of parametric equations represent the same graph, but different curves?

Working Space

Answer on Page 76

Uniform Circular Motion and Centripetal Force

A Ferris wheel, a spinning washing machine, a car rounding a curve, the Moon going around Earth: circles are everywhere. So far, you have studied motion with position, velocity, acceleration, and force mostly along straight lines and parabolas. What changes when an object moves in a circle? Surprisingly, it turns out to be accelerating even when its speed never changes.

4.1 Introduction to Uniform Circular Motion

You may be surprised to know that we can use equations very similar to the straight-line kinematics equations to describe circular motion, provided you use the correct variables.

The angular displacement, θ , is the angle in *radians* that the object has traveled. The angular velocity, ω , is the rate of change of the angular displacement, and the angular acceleration, α , is the rate of change of the angular velocity.

Linear motion	Rotational motion
$v = v_0 + at$	$\omega = \omega_0 + \alpha t$
$x = x_0 + v_0 t + \frac{1}{2}at^2$	$\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2$
$v^2 = v_0^2 + 2a(x - x_0)$	$\omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0)$

Table 4.1: Kinematic equations for linear and rotational motion.

Linear velocity, v , is related to angular velocity by the equation:

$$v = r\omega$$

The *Period*, T , is the time it takes to complete one full rotation. The frequency, f , is the number of rotations per second. The two are related by:

$$T = \frac{2\pi}{\omega} = \frac{2\pi r}{v}$$

$$f = \frac{1}{T}$$

The *centripetal force*, usually labelled differently (such as tension or gravity), is given by the equation:

$$F = mr\omega^2 = \frac{mv^2}{r}$$

The *angular velocity*, ω , measures how fast the angle changes. It is measured in radians per second. Over a time interval Δt , the average angular velocity is

$$\omega_{\text{avg}} = \frac{\Delta\theta}{\Delta t}$$

and the angular velocity at one instant is this ratio for a very short time interval.

$$\omega = 2\pi f = \frac{2\pi}{T}$$

Angular acceleration measures how fast the angular velocity changes:

$$\alpha_{\text{avg}} = \frac{\Delta\omega}{\Delta t}$$

Note that all of the kinematic formulas still work as long as you stay in the same relative frame of reference (rotational or linear). You cannot “mix and match” the two sets of equations.

You may be asking, “If the object is moving at a constant speed, why is there acceleration?” Recall that acceleration is a vector quantity, meaning it has both magnitude and direction. Even though the speed (the magnitude of the velocity) is constant, the direction of the velocity vector is always changing. Since acceleration is the rate of change of velocity, any change in direction means there is acceleration. This acceleration is always directed toward the center of the circle, which is why it is called centripetal (center-seeking) acceleration.

Now that we have established the basic relationships for circular motion, let us apply them to a practical example.

4.2 The Flying Billiard Ball

Let us say you tie a 0.16 kg billiard ball to a long string and begin to swing it around in a circle above your head. The string is 3 meters long and stays horizontal, and the ball returns to where it started every 4 seconds. We will assume the ball moves at a constant speed. If you start your stopwatch as the ball crosses the x-axis, the coordinates of the position (x, y, z) of the ball at any time t given by:

$$\mathbf{p}(t) = \left[3 \cos\left(\frac{2\pi}{4}t\right), 3 \sin\left(\frac{2\pi}{4}t\right), 2 \right]$$

(This assumes that the ball would be going counter-clockwise if viewed from above. The spot you are standing on is considered the origin $[0, 0, 0]$.)

Notice that the height is a constant — 2 meters in this case. That is not very interesting, so we will talk just about the first two components. Figure 4.1 shows a visual of the situation:

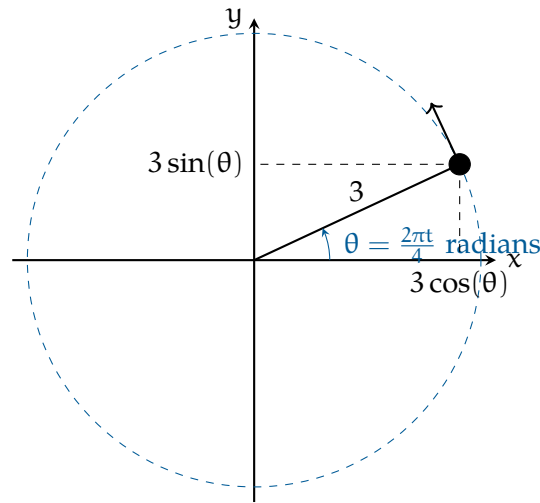


Figure 4.1: A diagram of our ball and string scenario.

In this case, the radius, r , is 3 meters. The period, T , is 4 seconds. In general, we say that circular motion is given by:

$$\mathbf{p}(t) = \left[r \cos \frac{2\pi t}{T}, r \sin \frac{2\pi t}{T} \right]$$

A common question is “How fast is it turning right now?” If you divide the 2π radians of a circle by the 4 seconds it takes, you get the answer “About 1.57 radians per second.” This is known as *angular velocity* and we typically represent it with the lowercase Omega: ω . (Yes, it looks a lot like a “w”.) To be precise, in our example, the angular velocity is $\omega = \frac{\pi}{2}$. Note that in this scenario, the angular velocity and linear *speed* are constant. However, the *velocity* vector is not constant; as we will see in the next section, the direction of the velocity vector is always changing.

Notice that this is different from the question “How fast is it going? (referring to *linear velocity*)” This ball is traveling the circumference of $6\pi \approx 18.85$ meters every 4 seconds. This means the speed of the ball is about 4.71 meters per second.

It is very important to distinguish between angular velocity and linear velocity. Angular velocity is how fast the angle is changing and is referred to by ω , while linear velocity, v , is how fast the object is moving along its path. While linear velocity has a constant speed, it has always changing direction. The angular velocity is constant, but the linear velocity is not.

4.3 Velocity

The velocity of the ball is a vector, and we can find it with two facts about circles.

Its size. The ball travels the circumference $2\pi r$ every period T , so its speed is $\frac{2\pi r}{T} = r\omega$. In our example, that is $3 \cdot \frac{\pi}{2} \approx 4.71$ m/s.

Its direction. At every instant, the ball is moving along the circle, so its velocity is tangent to the circle. A tangent to a circle is perpendicular to the radius. The radius points in the direction $[\cos \omega t, \sin \omega t]$. Turning a vector $[a, b]$ by 90° counterclockwise gives $[-b, a]$, so the velocity points in the direction $[-\sin \omega t, \cos \omega t]$.

Putting the size and the direction together, in our example, the velocity of the ball at any time t is given by:

$$\mathbf{v}(t) = \left[-\frac{3(2\pi)}{4} \sin \frac{2\pi t}{4}, \frac{3(2\pi)}{4} \cos \frac{2\pi t}{4}, 0 \right]$$

Notice that the velocity vector is perpendicular to the position vector. It has a constant magnitude.

In general, an object traveling in a circle at a constant speed has the velocity vector:

$$\mathbf{v}(t) = [-r\omega \sin \omega t, r\omega \cos \omega t]$$

where $t = 0$ is the time that it crosses the x axis. If ω is negative, that means the motion would be clockwise when viewed from above. (When you learn derivatives, you will be able to get this same velocity in one line, by differentiating each component of the position.)

The magnitude of the velocity vector is $r\omega$. See Figure 4.2 for a visual of the velocity vector. Note that the z -coordinate is constant, so it does not change at all, and it is eliminated from our vector for simplification.

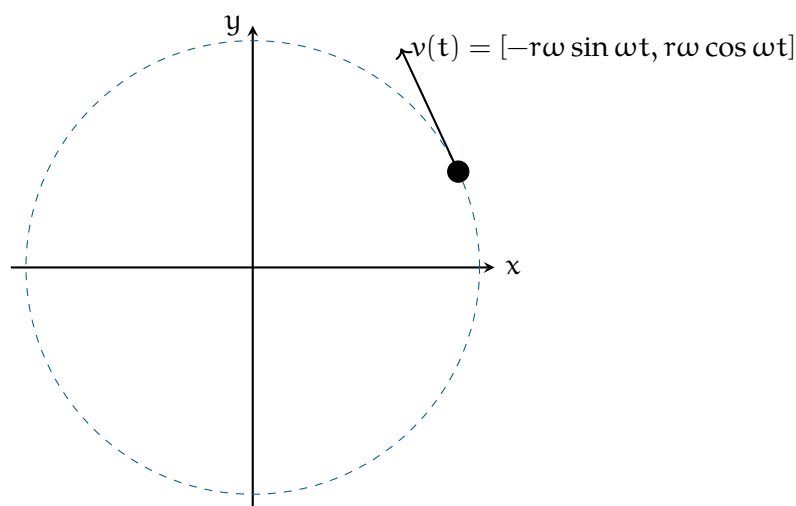


Figure 4.2: Velocity vector of the ball in circular motion.

Exercise 16 A Particle in Motion

[This question was originally presented as two multiple-choice problems on the 2012 AP Physics C exam.] The x and y coordinates of a particle as it moves in a circle are given by:

$$x = 5 \cos(3t) \quad y = 5 \sin(3t)$$

What is the radius of the particle's circular path? What is the particle's *speed*? Based on your answers, how long does it take the particle to complete the circular path?

Working Space

Answer on Page 76

4.4 Acceleration

Acceleration is to velocity what velocity is to position: it is the rate at which the velocity vector changes. Here is a neat trick for finding it. Draw every velocity vector starting from the same point. The velocity has the constant size $r\omega$ and turns at the same rate ω as the ball, so its tip moves around a circle of radius $r\omega$. That is circular motion again! So we can reuse what we just learned: the rate of change of a vector that moves in a circle of radius R has size $R\omega$ and points 90° ahead of it. Here $R = r\omega$, so the acceleration has size $r\omega^2$, and turning the velocity direction $[-\sin \omega t, \cos \omega t]$ by another 90° gives $[-\cos \omega t, -\sin \omega t]$:

$$\mathbf{a}(t) = [-r\omega^2 \cos \omega t, -r\omega^2 \sin \omega t] = -r\omega^2 [\cos \omega t, \sin \omega t]$$

Notice that the acceleration vector points **toward the center** of the circle it is traveling on. That is, when an object is traveling on a circle at a constant (uniform) speed (notice that coefficients r and ω are constant), its only acceleration is toward the center of the circle. The acceleration does not come from the motion itself, it comes from the constantly changing direction. This is known as *centripetal acceleration*.

The magnitude of the acceleration vector is $a_c = r\omega^2$, or more commonly, $a_c = \frac{v^2}{r}$. This gives us the common calculation for whatever force acts as the centripetal force: $F_c = ma_c = \frac{mv^2}{r}$.¹

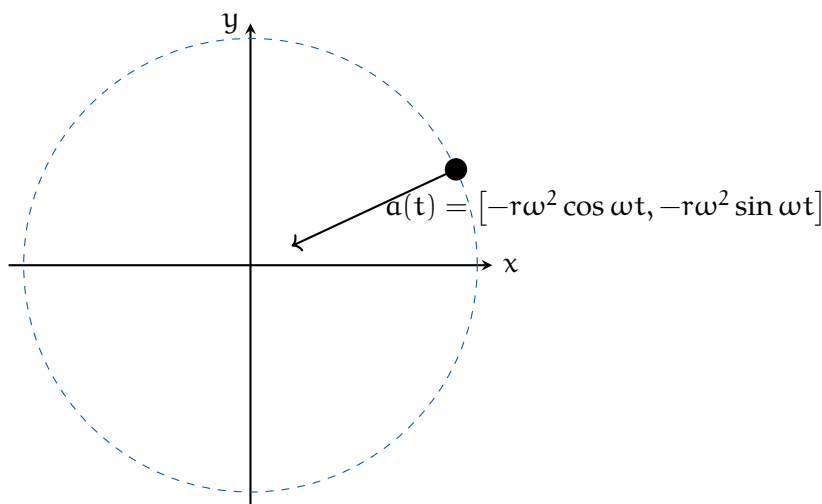


Figure 4.3: Acceleration vector of the particle in circular motion.

Let us compare the velocity and acceleration paths from figures 4.2 and 4.3. Notice that the velocity vector is always tangent to the circle, while the acceleration vector is always

¹You will see centripetal acceleration noted as a_c or a_r , standing for centripetal or radial, respectively. Note that both refer to the same acceleration.

pointing toward the center of the circle.

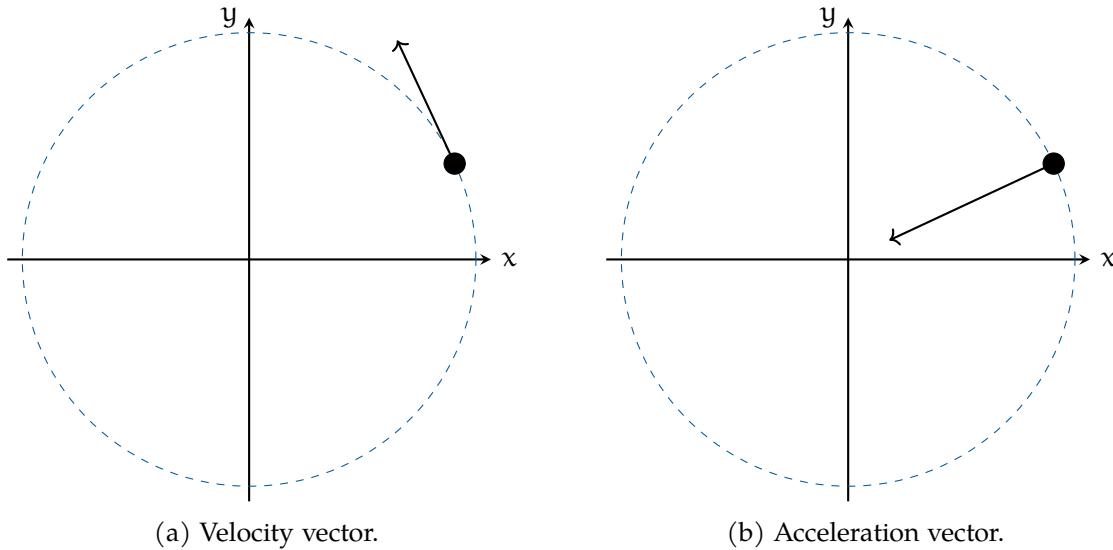


Figure 4.4: Uniform circular motion: velocity (tangent) and acceleration (centripetal).

4.5 Centripetal force

What would happen if there were no force pulling the ball toward the center of the circle? It would fly off in a straight line, according to Newton's First Law.

The force you are exerting on the string is what causes it to accelerate toward the center of the circle. We call this the *centripetal force*. It makes sense that the centripetal force and centripetal acceleration are in the same direction, as they are proportional.

Recall that $F = ma$. The magnitude of the acceleration is $r\omega^2 = 3 \left(\frac{2\pi}{4}\right)^2 \approx 7.4 \text{ m/s}^2$. The mass of the ball is 0.16 kg. So, the force pulling against your hand is about 1.18 newtons.

The general rule is that when something is traveling in a circle at a constant speed, the centripetal force needed to keep it traveling in a circle is:

$$F = mr\omega^2$$

If you know the radius r and the speed v of the object, here is the rule:

$$F = \frac{mv^2}{r}$$

We did not introduce centripetal force as a type of force because it is not its own force, like friction or gravity. Rather, calling a force “centripetal” tells us what the force is doing. A centripetal force is any force that causes circular motion, a force which acts in the *radial* direction. For a satellite, the centripetal force is gravity. In the opening example with the billiard ball, the centripetal force is the tension in the string.

Example: A child is sitting on a merry-go-round as it spins. What provides the centripetal force?

Solution: In this case, the child is rotating horizontally, so the centripetal force must also be horizontal. Therefore, the centripetal force is not the child’s weight or the normal force. It is the *friction* between the child and the merry-go-round that keeps the child turning. Here is a free body diagram:

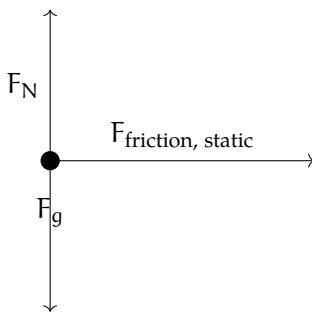


Figure 4.5: The static friction is what causes the centripetal force.

Example: If the child is 1.2 meters from the center of the merry-go-round and it spins at 0.25 rotations/second, what is the coefficient of friction between the child and the merry-go-round?

Solution: Using our FBD and applying Newton’s Second Law, we see that:

$$F_N - F_g = 0 \rightarrow F_N = m_{\text{child}}g$$

$$F_f = mr\omega^2 = \mu F_N = \mu mg$$

Since the child is not accelerating vertically, we know that the normal force equals the child’s weight. The horizontal force, friction, must equal the mass of the child times the acceleration. Additionally, from the vertical component, we know the friction is equal to μmg , since the normal force is equal to the child’s weight. Looking at the second equation, we see that:

$$mr\omega^2 = \mu mg$$

We were given ω in rotations per second, but we need radians per second:

$$\frac{0.25\text{rotation}}{1\text{second}} = \frac{1\text{rotation}}{4\text{seconds}} = \frac{2\pi\text{radians}}{4\text{seconds}} = \frac{\pi\text{ rad}}{2\text{ s}}$$

We can divide m from both sides of $m r \omega^2 = \mu m g$ (notice: we do not need to know the child's mass to determine the coefficient of friction!):

$$r \omega^2 = \mu g \rightarrow \mu = \frac{r \omega^2}{g} = \frac{(1.2\text{m}) \left(\frac{\pi}{2\text{s}}\right)^2}{9.8 \frac{\text{m}}{\text{s}^2}} = 0.302$$

Therefore, the coefficient of friction between the child and the merry-go-round is 0.302.

4.5.1 Banked Turns

Have you ever been driving on the highway and taken a turn where the road is at an angle? Or maybe you have seen a sign like Figure 4.6 on the road as you take a turn.

A banked curve is designed so that a car can safely turn without slipping, even in the rain, if the driver does not exceed the indicated speed. Engineers choose an angle such that the bank provides sufficient force to turn the car without friction. Let us look at a free body diagram for a car taking a banked turn (see figure 4.7).

In the past, we have split the vector for the force of gravity into components that are parallel and perpendicular to the ramp. This time, we will split the normal force into x and y components (see figure 4.8).



Figure 4.6: A banked turn sign.

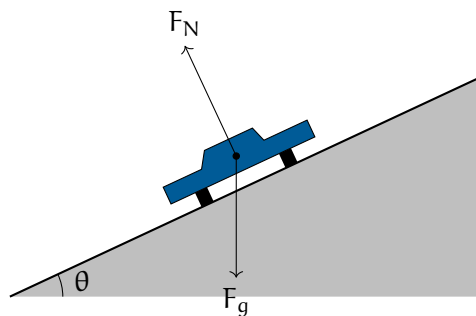


Figure 4.7: If there is no friction due to rain, then the only two forces acting on the car are gravity and the normal force.

Assuming the radius of the turn is 20 m, at what angle should the engineers build the banked turn? (Let us also assume the maximum speed is 25 mph,

like the traffic sign above.) First, let us apply Newton's Second Law in the x and y directions:

$$(1) F_N \cos(\theta) - F_g = 0$$

$$(2) F_N \sin(\theta) = \frac{mv^2}{r}$$

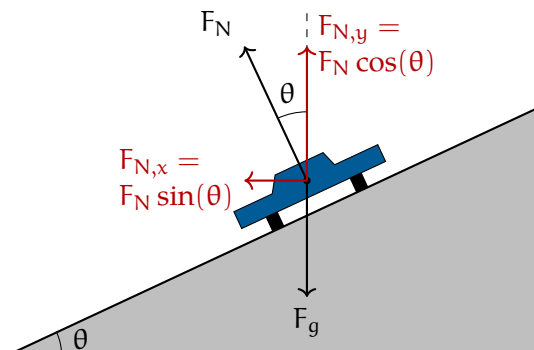


Figure 4.8: Geometrically, the vertical (y) component of the normal force is $F_N \cos(\theta)$ and the horizontal (x) component is $F_N \sin(\theta)$.

We know that $F_g = mg$. From this and equation (1) we see that:

$$F_N = \frac{mg}{\cos(\theta)}$$

Substituting for F_N into equation (2):

$$\left(\frac{mg}{\cos(\theta)} \right) \sin(\theta) = \frac{mv^2}{r}$$

The mass can be divided from both sides, and $\frac{\sin(\theta)}{\cos(\theta)} = \tan(\theta)$:

$$g \tan(\theta) = \frac{v^2}{r}$$

Solving for θ and substituting for the speed ($25 \text{ mph} \approx 11.18 \frac{\text{m}}{\text{s}}$) and radius:

$$\theta = \arctan\left(\frac{v^2}{gr}\right) = \arctan\left(\frac{(11.18 \frac{\text{m}}{\text{s}})^2}{(9.8 \frac{\text{m}}{\text{s}^2})(20\text{m})}\right)$$

$$\theta = \arctan(0.6377) \approx 32.53^\circ$$

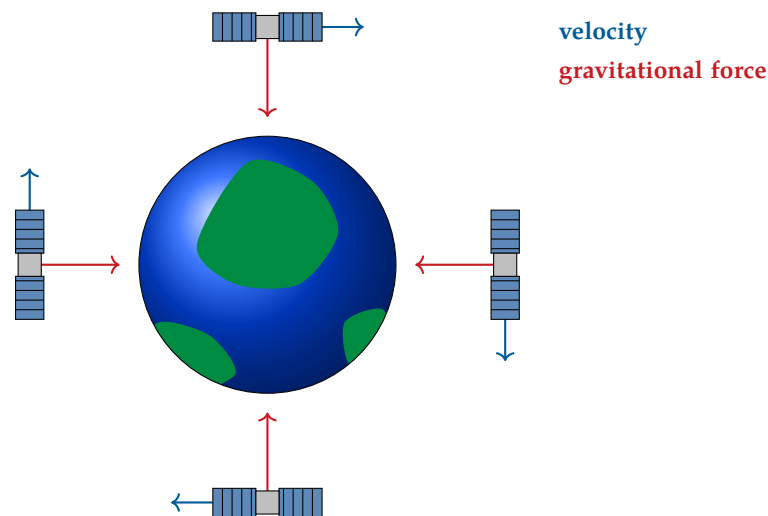
The next chapter, Circular Motion in Action, puts centripetal force to work on swinging objects and roller coaster loops.

Circular Motion in Action

Swing a bucket of water over your head fast enough, and not a drop spills. Ride a roller coaster through a loop, and you press into your seat at the bottom and feel nearly weightless at the top. In the Uniform Circular Motion and Centripetal Force chapter, you learned that anything moving in a circle needs a net force pointing toward the center. In this chapter, you will use that idea to model real situations: a pig on a string, and a coaster car whose speed changes as it goes around a loop.

5.1 Modeling Circular Motion

What causes circular motion is a constant force perpendicular to the motion. Consider a satellite circling the Earth. The only force acting on the satellite is gravity, yet the satellite does not fall to the Earth. Why? Let us look at the relative direction of motion and gravity for some different positions of the satellite (we will assume this satellite is moving clockwise from our point of view):



No matter what position the satellite is in, the velocity and gravity vectors are perpendicular.

Exercise 17 Circular Motion

Just as your car rolls onto a circular track

Working Space

with a radius of 200 m, you realize your 0.4 kg cup of coffee is on the slippery dashboard of your car. While driving 120 km/hour, you hold the cup to keep it from sliding.

What is the maximum amount of force you would need to use? (The friction of the dashboard helps you, but the max is when the friction is zero.)

Answer on Page 77

Exercise 18 Twirling a Whistle

The lifeguard at a local pool is twirling their whistle horizontally. You wonder if the lifeguard could spin the whistle fast enough to break the string. The string the whistle is attached to can hold a maximum mass of 20 kg before breaking. If the lifeguard's whistle string is 0.35 m long and the average whistle has a mass of 165 grams, what is the maximum tangential speed the lifeguard can spin the whistle? How many rotations per second would the whistle be spinning at? Based on this, do you think the lifeguard is capable of spinning the whistle fast enough to break the string?

Working Space

Answer on Page 77

Exercise 19 The Gravitron

The Gravitron is a carnival ride where riders "stick" to the wall of a spin-

Working Space

ning cylinder as the floor beneath them drops away. A video explanation is given here: <https://www.youtube.com/watch?v=ifAY5tbYDmQ>. Draw a free body diagram of a rider. If the coefficient of friction between a rider and the wall is 0.32 and the ride is 10 meters across, what angular velocity must the ride reach before the floor drops away?

Answer on Page 78

5.2 Flying Pigs and Tension

Take a look at the following two videos that involve flying pigs!

1. Flying Pig: Uniform Circular Motion and Centripetal Force
2. flying pig - circular motion [workthrough demo]

This involves a combination of circular motion and tension. Say the string makes an angle θ with the vertical and the string of length L , causing the pig to move in a circular path with radius r . We can draw the following two diagrams for our pig, as shown in Figure 5.1.

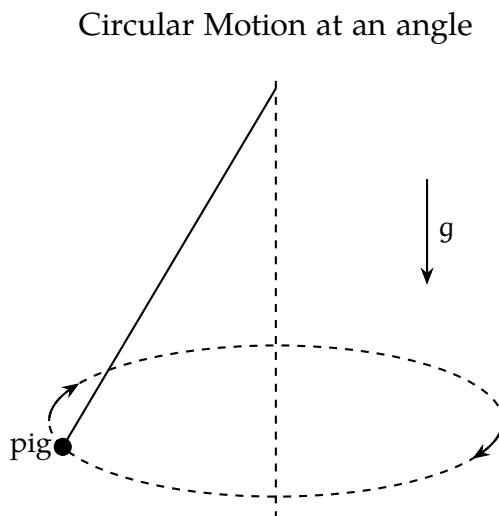


Figure 5.1: The uniform motion of a flying pig attached to a string.

If this is confusing, think of a tetherball from your local playground. It follows the same mechanics!

At any point, we can draw a free body diagram for the pig. Figure 5.2 shows the forces acting on the pig at all points.

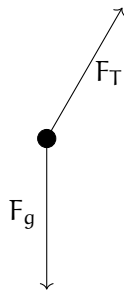


Figure 5.2: The FBD of the flying pig at any point.

To analyze the net forces, the tension force needs to be split into its x and y components. The y component of the tension force must balance out the weight of the pig (as it is not moving vertically), while the x component of the tension force provides the centripetal force to keep the pig moving in a circle.

$$\sum F_y = 0 \rightarrow F_{T_y} - F_g = 0 \rightarrow F_{T_y} = F_g = F_T \cos \theta = mg$$

The centripetal force is found by:

$$F_{T_x} = F_c = F_T \sin \theta = ma_r = \frac{mv^2}{r}$$

Dividing the second equation by the first equation gives:

$$\frac{F_T \sin(\theta)}{F_T \cos(\theta)} = \frac{\frac{mv^2}{r}}{mg} \rightarrow \tan(\theta) = \frac{v^2}{rg}$$

Solving for θ gives us $\theta = \arctan\left(\frac{v^2}{rg}\right)$. We can also solve for v :

$$v = \sqrt{rg \tan \theta} = \sqrt{Lg \sin \theta \tan \theta}$$

Since r can be solved for θ using $r = L \sin \theta$.

Explicitly, we can express our tension force as $F_T = \frac{mg}{\cos(\theta)}$ (coming from our net force in the y -direction).

We can also explicitly solve for ω :

$$\begin{aligned}\tan(\theta) &= \frac{v^2}{rg} \\ &= \frac{(r\omega)^2}{rg} = \frac{r\omega^2}{g} \\ \omega^2 &= \frac{g \tan(\theta)}{r} \\ \omega &= \sqrt{\frac{g \tan(\theta)}{r}}\end{aligned}$$

What does all of this tell us?

- The vertical component of tension balances the weight of the pig, so there is no vertical motion.
- The horizontal component of tension provides the centripetal force that keeps the pig moving in a circle.
- The faster the pig moves, the larger the angle θ and the greater the tension in the string.
- The tension is always greater than the pig's weight: $F_T = \frac{mg}{\cos \theta}$.
- The angular speed required for a given angle is $\omega = \sqrt{\frac{g \tan \theta}{r}}$. This increases with larger angles, and increases as the radius gets smaller.

Exercise 20 Playground Tetherball

Working Space

A tetherball of mass .5kg is attached to a string of length 2.5m. When the ball is swung around in a horizontal circle, the string makes an angle of 38° with the vertical. If the ball is in uniform circular motion, what is the speed of the ball?

Answer on Page 79

5.3 Non-uniform circular motion

We have talked enough about uniform circular motion, where the speed is constant. However, in many real-world scenarios, the speed is not constant. This is known as *non-uniform circular motion*. In this case, there are two components of acceleration: the centripetal acceleration $a_{\perp}$ (toward the center of the circle) and the tangential acceleration $a_{\parallel}$ (along the direction of motion). The tangential acceleration is responsible for changes in speed, while the centripetal acceleration is responsible for changes in direction. (A swinging pendulum is a good example: it moves along an arc of a circle, speeding up and slowing down. You will study it in the pendulums chapter.)

Let us look at a few examples of these concepts in action.

5.3.1 Roller Coaster Loop

Now we can talk about a roller coaster. A roller coaster loop is a great example of circular motion. As the roller coaster cart goes through the loop, it experiences both centripetal and tangential acceleration.

We will analyze the cart on a roller coaster at positions at the top, bottom, and sides of the loop, as shown in Figure 5.3

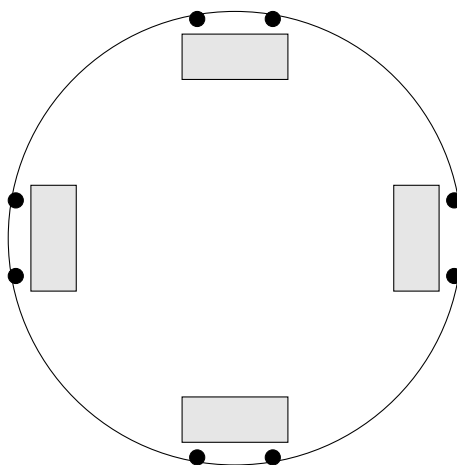


Figure 5.3: Carts at four positions around the loop with wheels on the track.

5.3.2 At the bottom

At the very bottom of the loop, the forces acting on the cart are gravity and the normal force from the track. Let us imagine the cart goes around the circular track infinitely without “exiting” the loop. There must, then, be a force keeping the cart moving in a circle.

This force is the centripetal force, which points toward the center of the loop. The only two forces acting on the cart are gravity (downward) and the normal force from the track (upward). We know that a centripetal force must satisfy the formula $F_{\text{net}} = F_N - F_g = \frac{mv^2}{r}$. Therefore, the normal force must be greater than the weight of the cart at the bottom of the loop: $F_N = \frac{mv^2}{r} + mg$. This is represented in Figure 5.4.

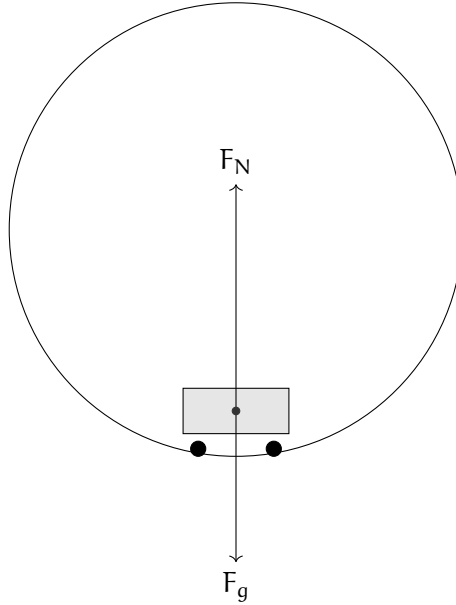


Figure 5.4: The FBD of the roller coaster at the bottom.

5.3.3 At the top

At the very top of the loop, the forces acting on the cart are gravity and the normal force from the track. Both forces point *downward* this time, toward the center of the loop. We can say that $F_{\text{net}} = F_N + F_g = \frac{mv^2}{r}$. Solving for the normal force gives: $F_N = \frac{mv^2}{r} - mg$.

Notice that the normal force at the top is smaller than at the bottom. The normal force can never be negative (the track can push, but not pull), so $\frac{mv^2}{r} \geq mg$, which means $v \geq \sqrt{rg}$. This is represented in Figure 5.5.

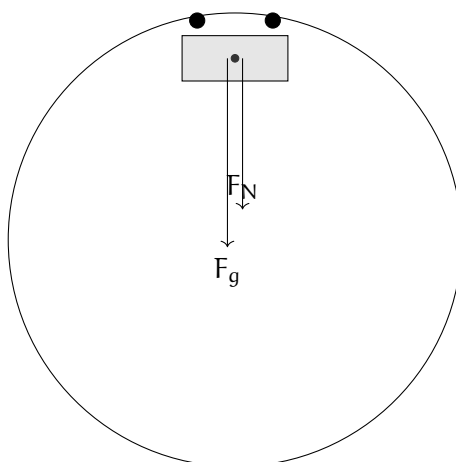


Figure 5.5: The FBD of the roller coaster at the top.

So at the top of the loop, the minimum speed to stay on the track is $v = \sqrt{rg}$. (At the bottom there is no minimum: the track pushes up harder the faster you go.) If the cart is slower than $\sqrt{rg}$ at the top, it will lose contact with the track, as there will not be enough normal force to provide the centripetal force needed to keep the cart moving in a circle. Think about where this observation comes from!

5.3.4 At the sides

Now at the sides of the loop (say 0 and π if we view the roller coaster as a unit circle), the net force is directed at an angle. The normal force from the track acts horizontally, pointing toward the center of the circle, while the gravitational force acts vertically downward. Since these two forces are perpendicular to each other, the net force is the vector sum of the normal and gravitational forces.

At the right side ($\theta = 0$), the normal force points leftward (toward the center), and gravity points downward, so the net force points diagonally down and to the left, toward the lower-left quadrant of the circle.

At the left side ($\theta = \pi$), the normal force points rightward (toward the center), and gravity still points downward, so the net force points diagonally down and to the right, toward the lower-right quadrant of the circle.

The normal force provides the centripetal part of the net force, and gravity provides a tangential part, along the track. Because the net force does not point straight at the center, the cart's speed changes: on the side where it is climbing, gravity slows it down; on the side where it is falling, gravity speeds it up. This is modeled in Figure 5.6.

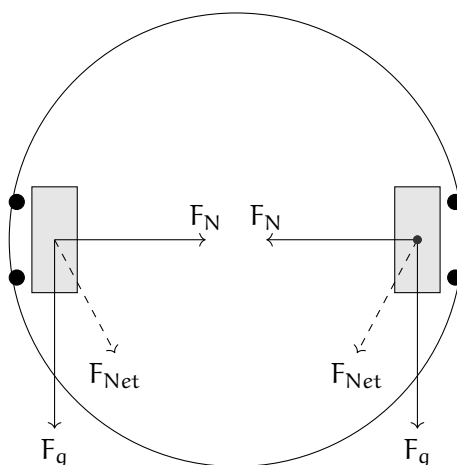


Figure 5.6: The FBD of the roller coaster at the sides.

Exercise 21 Roller Coaster Loop

This question is taken from the AP Physics C Review Book, 19th edition.

A roller-coaster car enters the circular loop portion of the ride. At the very top of the circle (where people are upside down), the car has a speed of 25 m/s, and the acceleration points straight down. If the diameter of the loop is 50 m and the total mass of the car plus passengers is 1,200 kg, find the magnitude of the normal force exerted by the track on the car at this point. Also find the normal force exerted by the track on the car when it is at the very bottom of the loop, given it is still traveling at 25 m/s. You may use $g = 10 \text{ m/s}^2$ for simpler calculations.

Working Space

Answer on Page 80

Our next chapter will explore oscillations, restoring forces, simple harmonic motion, and more concepts of harmonic motion.

Torque and Equilibrium

Why is a door handle on the side away from the hinges? Why is it easier to loosen a stuck bolt with a long wrench? How can a seesaw balance a small child against a large adult? To answer these, we need to know not just how things move in circles, but what makes them start and stop spinning. First, we need a way to describe rotation itself.

6.1 Rotational Kinematics

We have rotational kinematics, which describes the motion of objects rotating around a fixed axis. The key variables in rotational kinematics are angular displacement (θ), angular velocity (ω), and angular acceleration (α). These variables are analogous to linear displacement, velocity, and acceleration in linear kinematics.

Linear motion	Rotational motion
$v = v_0 + at$	$\omega = \omega_0 + \alpha t$
$x = x_0 + v_0 t + \frac{1}{2}at^2$	$\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2$
$v^2 = v_0^2 + 2a(x - x_0)$	$\omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0)$

Table 6.1: Kinematic equations for linear and rotational motion.

When an object such as a wheel or a disk rotates about an axis, each point on the object moves along a circular path centered on that axis. The total distance traveled by a point during one complete rotation is the circumference of its circular path, which is given by $C = 2\pi r$. The rotation of the object is described by an angle θ , usually measured in radians, where one full rotation corresponds to 2π radians.

6.2 Rigid Bodies

Although all points on a rotating disk pass through the same angular displacement θ in the same amount of time, they do not all travel the same distance. Points farther from the axis of rotation move along larger circles and therefore cover more distance during each rotation, even though the angular displacement is the same for all points. As a result, points farther from the axis have greater linear velocity and greater linear (tangential) acceleration than points closer to the center, as described by the relationships $v = r\omega$ and $a = r\alpha$. An object whose points all move together like this is called a *rigid body*.

A rigid body is an idealized object in which the distances between all points remain constant, even when forces are applied. All points along a radial line have the same angular displacement, angular velocity, and angular acceleration.

In other words, a rigid body does not deform: it does not stretch, compress, or bend. This assumption allows the object to be treated as a single system in which all parts move in a predictable way relative to one another.

In rotational motion, treating an object as a rigid body means that:

- All points in the object rotate together around a common axis.
- The angular velocity (ω) and angular acceleration (α) are the same for all points in the rigid body.
- The linear velocity (v) and linear acceleration (a) of points in the rigid body depend on their distance (r) from the axis of rotation, following the relationships $v = r\omega$ and $a = r\alpha$.

This idea maintains for many of the problems and concepts we will discuss in this chapter.

6.3 Torque

Let us think about pushing a box along a frictionless surface. There is a linear relationship between how hard you push and how much the box accelerates. What if, instead, you apply the same force to a merry-go-round fixed at a central pivot? Where can you push the merry-go-round for the highest rotation? Obviously you cannot push the merry-go-round linearly (as it is fixed to the ground), so how far will the merry-go-round rotate? All of these questions can be answered by the concept of *torque*.

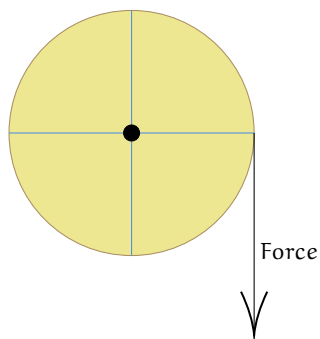


Figure 6.1: Merry-go-round with a singular applied force.

Torque is the concept of *rotational force*. Applying a torque to a surface, a *rigid body*, that has a fixed rotational pivot will cause the rigid body to *accelerate* around the pivot, ultimately

producing an *angular acceleration*.

Imagine pushing open a door that is bound to a hinge on one side of the door. Pushing near the hinge takes a large force to swing the door, while pushing at the handle, far from the hinge, takes much less force to produce the same torque. Try it!

Alternatively, think of a wrench rotating a large bolt. It will require more force from your arm closer towards the pivot, while pushing at the edge of the wrench will require less force to produce an equivalent torque. Take a look at this in action [here](#).

Torque

Torque, represented by the Greek letter τ (tau), is defined as

$$\tau = rF \quad (6.1)$$

where F is a force perpendicular to the line from the pivot, applied a distance r from the pivot. If the force acts at an angle θ to that line, then

$$\tau = rF \sin \theta \quad (6.2)$$

Notice that the torque around a pivot depends on the distance from the pivot, r , and the angle θ from the radius vector. The $\sin \theta$ indicates that only the *perpendicular* part of the force impacts the torque. This is equivalent to the cross product of the two vectors:

$$\tau = \mathbf{r} \times \mathbf{F} \quad (6.3)$$

Torque has units of Newton-meters, but in this case, it is not an energy form, so cannot be equated to Joules. Torque is an *vector*, not a scalar.

Let us do an example proving this.

Question: A rod with a fixed end is being pulled by a tension force of 50 N at a 40° angle from the horizontal, at a distance of 2 m. Find the Torque on the rod.

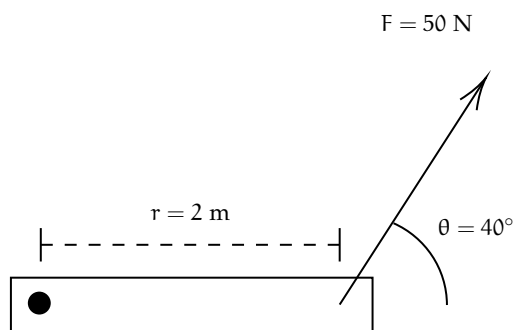


Figure 6.2: Main diagram.

Answer: Before just plugging in values, let us separate the force vector into components. The component that pulls perpendicular to the axis is $50 \sin 40^\circ$, while $50 \cos 40^\circ$ of the force is applied *pulling* the rod against the pivot, which produces no torque.

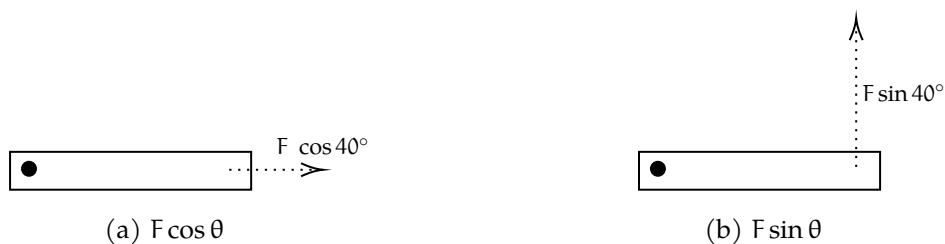


Figure 6.3: Components of the applied force.

Think back to the wrench example: you do not push or pull a wrench to loosen a bolt; you must apply a force *tangential* or *perpendicular* to the bolt to rotate it.

Since only the perpendicular component contributes to the torque, we calculate the torque to be $\tau = 50 \sin 40^\circ \text{ N} (2\text{m}) \approx 64.278 \text{ Newton-meters}$

Exercise 22 Door Torque

A door is connected to a wall by a hinge 0.5 m from the handle. A person opens the door with a tangential force of 38 N. What is the torque on the door?

Working Space

Answer on Page 81

Exercise 23 Torque but with a twist!

A uniform horizontal beam is hinged at the left end and held in place by a force applied at its right end. The beam has length $L = 2.0 \text{ m}$. A force of magnitude $F = 20 \text{ N}$ is applied at the right end of the beam, making an angle of 30° above the beam.

Working Space

If the pivot point is at the left end of the beam,

1. Find the magnitude of the torque about the hinge.
2. State whether the torque tends to rotate the beam clockwise or counterclockwise.

Answer on Page 81

6.3.1 Torque directions

What happens when two torques are applied in *opposite*? Let us take a look at the merry-go-round again, but this time applied with two torques.

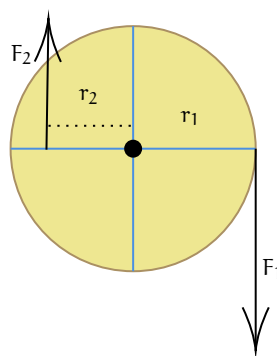


Figure 6.4: Merry-go-round with 2 forces in opposing directions.

Torques applied in opposition act *in opposite directions along the axis of rotation*. As torque is a vector defined by the *cross product equation 6.3*, so its direction is perpendicular to the plane formed by position vector and force which the plane live on. In problems, you will typically see the terms *out of* ($\odot$), or *into* ($\otimes$), the page, corresponding to the $+z$ or $-z$ directions, respectively. Remember these terms by the following visualizations: the out of symbol ($\odot$) looks like the tip of a feathered dart coming at you, while the into symbol ($\otimes$) looks like the feathers of a dart. Another way to think about this is using a drill and screw: rotating the screw counterclockwise causes it to advance toward you (out of the page), while rotating it clockwise causes it to retreat away from you (into the page).

To determine which direction a given torque points, one first identifies the sense of rotation the force would produce if acting alone: a force that tends to rotate the object counterclockwise in the plane produces a torque vector pointing out of the page, while a force that tends to rotate the object clockwise produces a torque vector pointing into the page. When two torques are opposite, one must therefore cause clockwise rotation

and the other counterclockwise rotation, leading to torque vectors that are equal in line of action but opposite in direction.

The *net torque* is the sum of all torques present, similar to how net force is the sum of all forces. Generally, counterclockwise torques are positive, while clockwise torques are negative.¹ Using our merry go round example, F_2 makes a $\odot$ torque while F_1 makes a $\otimes$ torque.

$$\begin{aligned}\tau_{\text{net}} &= \sum \tau \\ &= \tau_{\text{CCW}} - \tau_{\text{CW}} \\ &= +(F_2)(r_2) - (F_1)(r_1)\end{aligned}$$

Because we know $\tau_1 > \tau_2$, the net torque is going to be negative, characterized with $\otimes$, and clockwise.

Exercise 24 Opposing Torques on a Beam

A uniform horizontal beam of length 4.0 m and mass 20 kg is supported by a pivot located 1.0 m from the left end. Two forces act on the beam, producing opposing torques about the pivot.

Working Space

- A downward force of 50 N is applied at the left end of the beam.
- A downward force of unknown magnitude F is applied at the right end of the beam.
- The beam is in a state of static equilibrium.

Complete the following:

1. Find the Weight of the Beam
2. Write an equation for torque before solving for F
3. Solve for F , the unknown force acting

¹Note that other videos or professors may do this opposite, using CW - CCW, but as long as you are consistent and clear, both are equally valid.

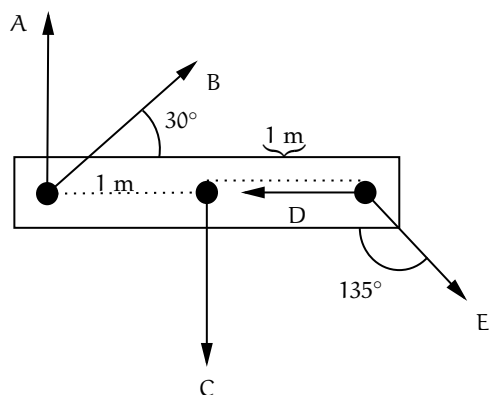
on the right end

Answer on Page 81

Exercise 25 More Complicated Opposing Torque

Create an equation for equilibrium of the following forces. Note that you may pick your own pivot point.

Working Space



Answer on Page 82

The next chapter, Moment of Inertia and Angular Momentum, asks how hard an object is to spin, and what rotating objects carry with them: rotational kinetic energy and angular momentum.

Moment of Inertia and Angular Momentum

Why does a figure skater spin faster when pulling in their arms? Why does a hollow ball roll downhill more slowly than a solid one? Why does a heavy flywheel keep spinning long after you stop pushing it? In the Torque and Equilibrium chapter, you learned what makes things start to turn. This chapter is about how hard they are to turn, and about the energy and momentum that rotating objects carry.

7.1 Moment of Inertia and Rotational $F = ma$

We know Newton's Second Law by heart now, $F_{\text{net}} = ma$. You probably could recite it in your sleep, but what about a rotational equivalent. Recall that from our first circular motion chapter, we have the equation $a = r\alpha$. Let us do some conversions to get this into an angular form:

$$\begin{aligned} F &= ma \\ F &= mr\alpha \\ rF &= mr^2\alpha \\ \tau &= mr^2\alpha \end{aligned} \tag{7.1}$$

From Equation 7.1, we can derive that the “mass equivalent” is mr^2 , since m is multiplied by a in the linear $F = ma$, while mr^2 is multiplied by α . Thus, we can state that for a point of distance r from the pivot (or axis of rotation), that singular point has *rotation inertia*, also called *moment of inertia*, is defined as mr^2 . For a whole object, the total moment of inertia is the sum of mr^2 over all its pieces.

Moment of Inertia

A mass's total moment of inertia is given by the sum of each mass multiplied the distance from the axis of rotation squared.

$$I = \sum mr^2 \tag{7.2}$$

For a solid object, think of it as millions of tiny pieces and add up mr^2 for every one. (In calculus, this kind of sum becomes an *integral*, which you will learn about later.)

From Equation 7.2, we see that mass farther from the axis matters much more than anything closer to the pivot. For two objects with the same mass and radius, one solid (mass spread evenly throughout) and the other a hollow cylinder or hoop (mass at the rim), the hollow one has the greater moment of inertia. That is part of why a figure skater spins faster after pulling their arms in: moving mass toward the axis shrinks the moment of inertia.

By finding the net rotational inertia of an object, we can create an equation for the rotational equivalent of Newton's Second Law:

$$\sum \tau = \tau_{\text{net}} = I\alpha \quad (7.3)$$

7.1.1 Relating $\tau = I\alpha$ and other Torque Def'ns

Since we have already defined angular acceleration, and have just discussed moment of inertia, we can now relate our two torque equations, $\tau = rF \sin \theta$ from the Torque and Equilibrium chapter, and Equation 7.3 by equating them, as they both equal net torque.

$$I\alpha = rF \sin \theta \quad (7.4)$$

This will be useful for solving rotational dynamics problems, and finding any of the variables I , α , r , F , or θ when the others are known.

Exercise 26 Angular Acceleration of a Wheel

A wheel has $I = 0.80 \text{ kg}\cdot\text{m}^2$. A force $F = 25 \text{ N}$ is applied at radius $r = 0.40 \text{ m}$ at an angle $\theta = 60^\circ$ to the radius.

Find α .

Working Space

Answer on Page 82

Exercise 27 Moment of Inertia from Multiple Torques

A rigid wheel is mounted on a frictionless axle. Two forces act on the wheel:

Working Space

- Force $F_1 = 30 \text{ N}$ is applied at radius

$r_1 = 0.50$ m at an angle $\theta_1 = 90^\circ$ to the radius, producing a counterclockwise torque.

- Force $F_2 = 20$ N is applied at radius $r_2 = 0.30$ m at an angle $\theta_2 = 45^\circ$ to the radius, producing a clockwise torque.

The wheel undergoes an angular acceleration of $\alpha = 8.0$ rad/s².

Find the moment of inertia I of the wheel.

Answer on Page 82

7.2 Parallel-Axis Theorem

The **parallel-axis theorem** allows us to calculate the moment of inertia of a rigid body about any axis that is parallel to an axis passing through the center of mass. The theorem states that

$$I_s = I_{\text{cm}} + Md^2 \quad (7.5)$$

where I_s is the moment of inertia about the axis of interest, I_{cm} is the moment of inertia about a parallel axis through the center of mass, M is the total mass of the object, and d is the perpendicular distance between the two axes.

In many practical situations, such as a door rotating about its hinges or a rod rotating about one end, the axis of rotation is offset from the center of mass, making direct calculation more difficult. That is where the parallel axis theorem comes in!

Exercise 28 Parallel Axis Theorem

Prove that the Moment of Inertia of the end of a rod is $I_{\text{end}} = \frac{1}{3}ML^2$ given that the moment of inertia of a rod about its center of mass is $I_{\text{CM}} = \frac{1}{12}ML^2$.

Working Space

Answer on Page 83

The parallel-axis theorem shows that the moment of inertia about an offset axis is always greater than the moment of inertia about the center-of-mass axis. The additional term Md^2 accounts for the translational motion of the object's center of mass relative to the new axis of rotation. As the distance between the axes increases, the moment of inertia increases accordingly.

If the mass of an object is not evenly distributed, the location of the center of mass shifts, but the parallel-axis theorem remains applicable. Once the center of mass is known, the theorem allows the moment of inertia about any parallel axis to be found by adding the contribution due to the displacement of the center of mass. In this way, the effects of mass distribution are fully captured by the center-of-mass term and the distance between the axes.

7.3 Rotational Kinetic Energy and Rolling Motion

Take a look at this video: [Rotational Inertia: The Race Between a Ring and a Disc](#).

Let us analyze why the solid cylinder reaches the bottom first.

When two objects roll down an incline without slipping, their acceleration depends not only on gravity, but also on how their mass is distributed. This is because a rolling object must both translate and rotate, so gravitational potential energy is split between translational and rotational kinetic energy. The fraction that goes into rotation depends on the object's moment of inertia.

When an object is rolling, it has both translational kinetic energy and rotational kinetic energy. The total kinetic energy of a rolling object is the sum of these two forms of kinetic energy.

The total rotational kinetic energy of a rolling object comes from all of its individual point masses rotating around the axis. Each point mass has its own rotational kinetic energy K_i , and the sum of all these individual kinetic energies gives the total rotational kinetic energy of the object:

$$\begin{aligned} KE_{\text{rotational}} &= \sum K_i \\ &= \sum \frac{1}{2} m_i r_i^2 \omega^2 \\ &= \frac{1}{2} \left(\sum m_i r_i^2 \right) \omega^2 \end{aligned}$$

This gives us the formula for rotational kinetic energy in terms of moment of inertia:

Rotational Kinetic Energy

The rotational kinetic energy of a rigid body rotating about a fixed axis is given by

$$KE_{\text{rotational}} = \frac{1}{2} I \omega^2 \quad (7.6)$$

where I is the moment of inertia of the object about the axis of rotation, and ω is the angular velocity. This expression applies to rigid bodies rotating about a fixed axis.

Going back to our rolling objects, we can now see that the total kinetic energy of a rolling object is the sum of its translational and rotational kinetic energies:

Total Kinetic Energy of a Rolling Object

The total kinetic energy of a rolling object is given by

$$KE_{\text{total}} = KE_{\text{translational}} + KE_{\text{rotational}} = \frac{1}{2} m v^2 + \frac{1}{2} I \omega^2 \quad (7.7)$$

where m is the mass of the object, v is its linear velocity, I is its moment of inertia, and ω is its angular velocity. For rolling without slipping, the translational and rotational motions are linked, but it is still useful to treat their kinetic energies separately.

The moment of inertia I varies depending on how the mass is distributed in the object, and is a measure of how much the object resists rotational motion. For example, a solid disc has a different moment of inertia than a hollow disc of the same mass and radius. This difference in moment of inertia affects how much of the total kinetic energy is rotational versus translational.

The solid disc has most of its mass concentrated closer to the center, resulting in a lower moment of inertia. This means that less of its total kinetic energy is rotational, allowing more energy to be available for translational motion, which leads to a higher linear velocity down the incline.

Relationship between translational velocity and moment

For an object of mass m and radius R rolling without slipping from rest down a height h , $\omega = \frac{v}{R}$, so energy conservation gives $mgh = \frac{1}{2} m v^2 + \frac{1}{2} I \frac{v^2}{R^2}$, and

$$v = \sqrt{\frac{2gh}{1 + \frac{I}{mR^2}}} \quad (7.8)$$

The larger $\frac{I}{mR^2}$ is, the slower the object rolls. A solid cylinder ($\frac{I}{mR^2} = \frac{1}{2}$) reaches $\sqrt{\frac{4}{3}gh}$, while a hoop ($\frac{I}{mR^2} = 1$) reaches only $\sqrt{gh}$, whatever their masses and sizes.

Each object has a unique moment of inertia based on its mass distribution, which directly affects its rolling behavior. Here is a reference table with common moments of inertia for various shapes:

Object and Axis of Rotation	Moment of Inertia
Solid cylinder or disk (symmetry axis)	$I = \frac{1}{2}MR^2$
Solid cylinder or disk (central diameter)	$I = \frac{1}{4}MR^2 + \frac{1}{12}ML^2$
Hoop (symmetry axis)	$I = MR^2$
Hoop (diameter)	$I = \frac{1}{2}MR^2$
Solid sphere (about diameter)	$I = \frac{2}{5}MR^2$
Thin spherical shell (about diameter)	$I = \frac{2}{3}MR^2$
Rod (about center, perpendicular to length)	$I = \frac{1}{12}ML^2$
Rod (about one end, perpendicular to length)	$I = \frac{1}{3}ML^2$

Table 7.1: Common Moments of Inertia

A visualization of each of these can be found at Georgia State University's HyperPhysics website: [HyperPhysics: Moments of Inertia](#).

7.4 Flywheel Energy Storage, Work, and Angular Momentum

A flywheel is a rigid body designed to store rotational energy. It typically consists of a heavy wheel or disk mounted on an axle, with much of its mass concentrated away from the axis of rotation. This mass distribution gives the flywheel a large moment of inertia, allowing it to resist changes in rotational speed. Its axle has very little friction, so once the flywheel is spinning, it keeps spinning for a long time with little energy loss. (We usually ignore that friction in problems.)

Because of its large moment of inertia, a flywheel can smooth out variations in rotational motion. When energy is added, the flywheel stores it as rotational kinetic energy. When energy is removed, the flywheel releases this stored energy gradually, helping maintain a more uniform angular velocity. For this reason, flywheels are commonly used in engines, generators, and mechanical systems that require steady rotation.

This relies on the principle of *Conservation of Angular Momentum*, which states that in the absence of *external torques*, the total angular momentum of a system remains constant. For a flywheel, this means that if no external torque acts on it, its angular momentum, and so its spin rate, will not change. A small torque from friction changes it only slowly, because of the large moment of inertia.

7.4.1 Angular Momentum

The connection between angular momentum and torque is direct: torque is the rate at which angular momentum changes. Applying a torque to a flywheel changes its angular momentum by increasing or decreasing its angular velocity. Because of the flywheel's large moment of inertia, a given torque produces only a gradual change in angular speed, contributing to the flywheel's stabilizing effect.

Angular Momentum and the Conservation of Angular Momentum

The angular momentum L of a rotating rigid body is given by

$$L = I\omega \quad (7.9)$$

where I is the moment of inertia of the body about the axis of rotation, and ω is its angular velocity. Angular momentum is a *vector* quantity, with both magnitude and direction.

Just as force is the rate of change of linear momentum ($F = \frac{\Delta p}{\Delta t}$), torque is the rate of change of angular momentum:

$$\tau = \frac{\Delta L}{\Delta t} \quad (7.10)$$

We can also derive this as a Newtonian cross product between $\mathbf{r}$ and $\mathbf{p}$, linear momentum. This works because angular momentum is the rotational equivalent of linear momentum, but depends on the choice of origin.

$$L = \mathbf{r} \times \mathbf{p} \quad (7.11)$$

Just like linear momentum, angular momentum is conserved in a system with no external torques. This is known as the *Conservation of Angular Momentum*:

$$L_{\text{initial}} = L_{\text{final}} \quad (\tau_{\text{net}} = 0) \quad (7.12)$$

This equation shows that torque is responsible for changing angular momentum. Because a flywheel has a large moment of inertia, a given applied torque produces only a small angular acceleration, allowing the flywheel to respond smoothly to changes in applied forces.

Angular Impulse

Just like in classical momentum, a change in angular momentum is called *angular impulse*. Angular Impulse is representative the cumulative effect of torque applied over given a

time interval.

Angular Impulse

For a constant torque τ applied for a time Δt , the angular impulse is

$$\Delta L = \tau \Delta t \quad (7.13)$$

If the torque changes over time, the angular impulse is the area under a *torque vs. time* graph.

Angular impulse has units Newton-meter-seconds, same as angular momentum:

$$\text{N} \cdot \text{m} \cdot \text{s} = \text{kg} \cdot \text{m}^2 \cdot \text{s}^{-1}$$

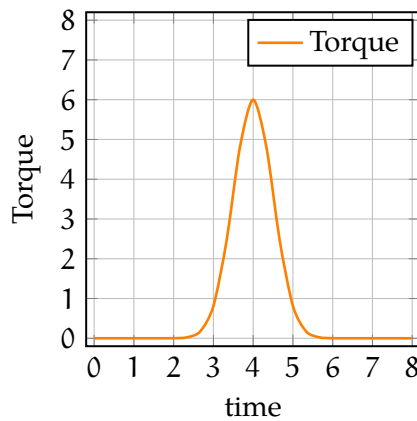
Angular impulse explains why sudden twists or torque have effects on rotational objects. A sudden kick to a door, a sharp tug on a wrench, and an off-center collision all deliver angular impulses. This impulse may be produced either by a large torque applied over a short time or by a smaller torque applied over a longer time. However, in many physical situations, applying a large torque over a near-instantaneous time interval is more effective and practical.

Exercise 29 Angular Impulse

Working Space

A kid rams a blue toy car into a red toy car, which is initially stationary. The collision causes the red toy car to start spinning in place.

Estimate, by counting the area under the curve, the angular impulse caused by a



toy car crash.

Answer on Page 83

7.4.2 Work done by Torque

Recall that work is a change in energy. Torque also does work when it causes rotation. For a constant torque acting on point mass through an angular displacement $\Delta\theta$ the work done is

$$W = Fr \Delta\theta = \tau \Delta\theta \quad (7.14)$$

This comes from the fact that linear work is defined as $W = F\Delta x$, and the arc length of a circle is defined as $s = r\theta$. Thus, we can substitute $r\theta$ for Δx in the linear work equation to get the rotational work equation.

7.5 Summary

Here is a summary of linear quantities in physics, and their now introduced rotational components. You will refer to it often, so keep it handy.

Table 7.2: Correspondence Between Linear and Rotational Motion

Linear Motion	Rotational Motion
Position x	Angle θ
Velocity v	Angular velocity ω
Acceleration a	Angular acceleration α
Mass m	Moment of inertia I
Force F	Torque τ
Momentum $p = mv$	Angular momentum $L = I\omega$
$F = \frac{\Delta p}{\Delta t}$	$\tau = \frac{\Delta L}{\Delta t}$
Work $W = F\Delta x$	Work $W = \tau\Delta\theta$
Kinetic Energy $\frac{1}{2}mv^2$	Rotational KE $\frac{1}{2}I\omega^2$
Power $P = Fv$	Power $P = \tau\omega$

Answers to Exercises

Answer to Exercise 1 (on page 6)

$$\begin{aligned}\lim_{x \rightarrow -6^-} p(x) &= -\infty, \lim_{x \rightarrow -6^+} p(x) = \infty \\ \lim_{x \rightarrow -5^-} p(x) &= \lim_{x \rightarrow -5^+} p(x) = \lim_{x \rightarrow -5} p(x) = 1 \\ \lim_{x \rightarrow -3^-} p(x) &= \lim_{x \rightarrow -3^+} p(x) = \lim_{x \rightarrow -3} p(x) = \frac{1}{3} \\ \lim_{x \rightarrow \infty} p(x) &= 0 \text{ called simply a limit, although it is a left-hand limit}\end{aligned}$$

Answer to Exercise 2 (on page 8)

$$\lim_{x \rightarrow -\infty} 3^x + 1 = 1; \lim_{x \rightarrow 4^+} \log_2(x - 4) = -\infty; \lim_{x \rightarrow \infty} 2^{1-x} = 0; \lim_{x \rightarrow 0^-} \log_{10}(-2x) = -\infty$$

Answer to Exercise 3 (on page 10)

$$\lim_{x \rightarrow -\infty} \tan^{-1} x = -\frac{\pi}{2}, \lim_{x \rightarrow \infty} \tan^{-1} x = \frac{\pi}{2}; \lim_{x \rightarrow -\infty} \frac{1}{1+e^{-x}} = 0, \lim_{x \rightarrow \infty} \frac{1}{1+e^{-x}} = 1$$

Answer to Exercise 4 (on page 12)

1. $\lim_{x \rightarrow -1^-} h(x) = 2$ and $\lim_{x \rightarrow -1^+} h(x) = 2$, therefore the limit exists and $\lim_{x \rightarrow -1} h(x) = 2$
 $\lim_{x \rightarrow 0^-} h(x) = 3$ and $\lim_{x \rightarrow 0^+} h(x) = 3$, therefore the limit exists and $\lim_{x \rightarrow 0} h(x) = 3$
 $\lim_{x \rightarrow 1^-} h(x) = 2$ and $\lim_{x \rightarrow 1^+} h(x) = 2$, therefore the limit exists and $\lim_{x \rightarrow 1} h(x) = 2$
2. $\lim_{x \rightarrow -1^-} f(x) = 2$ and $\lim_{x \rightarrow -1^+} f(x) = 2$, therefore the limit exists and $\lim_{x \rightarrow -1} f(x) = 2$.
 $\lim_{x \rightarrow 0^-} f(x) = 3$ and $\lim_{x \rightarrow 0^+} f(x) = 0$, and because $\lim_{x \rightarrow 0^-} f(x) \neq \lim_{x \rightarrow 0^+} f(x)$, the

limit does not exist.

$\lim_{x \rightarrow 2^-} f(x) = -2$ and $\lim_{x \rightarrow 2^+} f(x) = -2$, therefore the limit exists and $\lim_{x \rightarrow 2} f(x) = -2$.

3. $\lim_{x \rightarrow -2^-} g(x) = -1$ and $\lim_{x \rightarrow -2^+} g(x) = -1$, therefore the limit exists and $\lim_{x \rightarrow -2} g(x) = -1$.

$\lim_{x \rightarrow 0^-} g(x) = 1$ and $\lim_{x \rightarrow 0^+} g(x) = 1$, therefore the limit exists and $\lim_{x \rightarrow 0} g(x) = 1$

$\lim_{x \rightarrow 1^-} g(x) = 2$ and $\lim_{x \rightarrow 1^+} g(x) = 1$, and because $\lim_{x \rightarrow 1^-} g(x) = 2 \neq \lim_{x \rightarrow 1^+} g(x)$, the limit does not exist.

$\lim_{x \rightarrow 2^-} g(x) = 0$ and $\lim_{x \rightarrow 2^+} g(x) = 0$, therefore the limit exists and $\lim_{x \rightarrow 2} g(x) = 0$

Answer to Exercise 5 (on page 14)

1. Never true. If a function is continuous at a , then $f(a) = \lim_{x \rightarrow a} f(x)$.
2. Never true. If a function is continuous at a , then $\lim_{x \rightarrow a^+} f(x) = \lim_{x \rightarrow a^-} f(x)$
3. Always true. This is the definition of continuity.

(The original exam question also asked about derivatives, which you will meet in volume 19.)

Answer to Exercise 6 (on page 14)

1. $f(x)$ is not defined at $x = 3$. Therefore, it is also discontinuous at $x = 3$. As we learn about the continuity of polynomials, we will see why $f(x)$ is continuous everywhere else.
2. Here, $f(0) = 2$ is defined, so we need to check if $\lim_{x \rightarrow 0} f(x) = f(0)$. As x approaches 0 from either side, $\frac{2}{x^4}$ grows without bound. We write $\lim_{x \rightarrow 0} f(x) = \infty$, but infinity is not a number, so the limit does not exist as a real number, and it certainly does not equal $f(0) = 2$. The function is discontinuous at $x = 0$ (an *infinite discontinuity*).
3. In this function, $f(3)$ is defined, so we need to check if the limit equals the function value. The limit of $f(x)$ as x approaches 3 is:

$$\lim_{x \rightarrow 3} \frac{3x^2 - 8x - 3}{x - 3} = \lim_{x \rightarrow 3} \frac{(3x + 1)(x - 3)}{x - 3} = \lim_{x \rightarrow 3} 3x + 1 = 10$$

So the limit exists, but $\lim_{x \rightarrow 3} f(x) \neq f(3)$, and we see that the function is discontinuous at $x = 3$.

Answer to Exercise 7 (on page 15)

1. True, $h(x)$ approaches 2 from the left and right, therefore the limit exists
2. False, $h(x)$ approaches 5 from the left and right, therefore the limit exists
3. False, $h(x)$ approaches 2 from the left and 4 from the right, therefore the limit does not exist
4. True, $\lim_{x \rightarrow 5} h(x) = h(5)$, therefore $h(x)$ is continuous at $x = 5$
5. True, $\lim_{x \rightarrow 4} h(x)$ does not exist, therefore $h(x)$ is discontinuous at $x = 4$
6. False, $h(2) = 1 \neq 2 = \lim_{x \rightarrow 2} h(x)$

Answer to Exercise 8 (on page 19)

1. From the quotient law, we know that:

$$\lim_{x \rightarrow -3} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow -3} f(x)}{\lim_{x \rightarrow -3} g(x)}$$

From the graph (or the formulas), we see that:

$$\lim_{x \rightarrow -3} f(x) = -(-3)^2 + 3 = -6$$

and that:

$$\lim_{x \rightarrow -3} g(x) = (-3)^2 + 1 = 10$$

Substituting these values, we get:

$$\lim_{x \rightarrow -3} \frac{f(x)}{g(x)} = \frac{-6}{10} = -\frac{3}{5}$$

2. From the Sum Law, we know that:

$$\lim_{x \rightarrow 2} [f(x) + 5g(x)] = \lim_{x \rightarrow 2} f(x) + \lim_{x \rightarrow 2} 5g(x)$$

and applying the Constant Multiple Law, we see that:

$$\lim_{x \rightarrow 2} [f(x) + 5g(x)] = \lim_{x \rightarrow 2} f(x) + 5 \lim_{x \rightarrow 2} g(x)$$

Examining the graph of $f(x)$ and $g(x)$, we can determine that

$$\lim_{x \rightarrow 2} f(x) = -2$$

and

$$\lim_{x \rightarrow 2} g(x) = 0$$

Substituting these values, we get:

$$\lim_{x \rightarrow 2} [f(x) + 5g(x)] = -2 + 5 \cdot 0 = -2$$

3. From the quotient law, we see that:

$$\lim_{x \rightarrow -1} \frac{3g(x)}{f(x)} = \frac{\lim_{x \rightarrow -1} 3g(x)}{\lim_{x \rightarrow -1} f(x)}$$

Applying the Constant Multiple Law, we get:

$$\lim_{x \rightarrow -1} \left[\frac{3g(x)}{f(x)} \right] = \frac{3 \lim_{x \rightarrow -1} g(x)}{\lim_{x \rightarrow -1} f(x)}$$

From the graph, we see that:

$$\lim_{x \rightarrow -1} f(x) = 2$$

and

$$\lim_{x \rightarrow -1} g(x) = 2$$

Substituting, we get:

$$\lim_{x \rightarrow -1} \left[\frac{3g(x)}{f(x)} \right] = \frac{3 \cdot 2}{2} = 3$$

4. Applying the Product and Constant Multiple Laws, we get:

$$\lim_{x \rightarrow 0} [f(x) \cdot 5g(x)] = \lim_{x \rightarrow 0} f(x) \cdot 5 \cdot \lim_{x \rightarrow 0} g(x)$$

Examining the graphs, we see that $\lim_{x \rightarrow 0} f(x)$ does not exist and $\lim_{x \rightarrow 0} g(x) = 1$. Because $\lim_{x \rightarrow 0} f(x)$ does not exist, $\lim_{x \rightarrow 0} f(x) \cdot 5 \cdot \lim_{x \rightarrow 0} g(x)$ also does not exist.

5. Applying the Difference and Constant Multiple Laws, we see that:

$$\lim_{x \rightarrow -1} [f(x) - 3g(x)] = \lim_{x \rightarrow -1} f(x) - 3 \cdot \lim_{x \rightarrow -1} g(x)$$

Examining the graphs, we see that:

$$\lim_{x \rightarrow -1} f(x) = 2$$

and

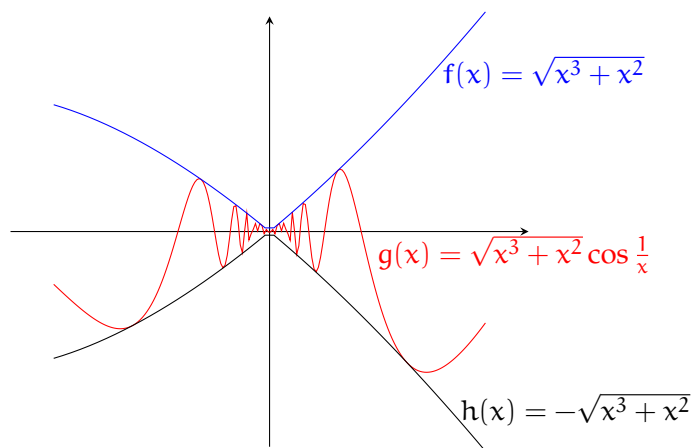
$$\lim_{x \rightarrow -1} g(x) = 2$$

Substituting, we get that:

$$\lim_{x \rightarrow -1} [f(x) - 3g(x)] = 2 - 3 \cdot 2 = 2 - 6 = -4$$

Answer to Exercise 9 (on page 22)

Let $f(x) = -\sqrt{x^3 + x^2}$ and $h(x) = \sqrt{x^3 + x^2}$. Near 0, $f(x) \leq \sqrt{x^3 + x^2} \cos \frac{1}{x} \leq h(x)$. Additionally, $\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} h(x) = 0$. Therefore, by the Squeeze Theorem, we can state that $\lim_{x \rightarrow 0} \sqrt{x^3 + x^2} \cos \frac{1}{x} = 0$. Plotting all three functions, we can confirm our answer:



Answer to Exercise 10 (on page 22)

The question tells us that the function in question, $f(x)$, is between two other functions.

$\lim_{x \rightarrow 2} 2x + 3 = 2(2) + 3 = 7$ and $\lim_{x \rightarrow 2} x^2 - 2x + 7 = 2^2 - 2(2) + 7 = 7$. Since the limits are equal, by Squeeze Theorem we can also know that $\lim_{x \rightarrow 2} f(x) = 7$.

Answer to Exercise 11 (on page 22)

Note that we can only evaluate the limit from the right, as the domain for this function is $x \geq 0$. Since the range of the sine function is $[-1, 1]$, we can state that

$$-1 \leq \sin \frac{\pi}{x} \leq 1$$

and therefore

$$\frac{1}{e} \leq e^{\sin \frac{\pi}{x}} \leq e$$

. Because we assume the positive root, it is also true that

$$\frac{\sqrt{x}}{e} \leq \sqrt{x} e^{\sin \frac{\pi}{x}} \leq \sqrt{x} e$$

. Taking the limits of the border functions, we see that

$$\lim_{x \rightarrow 0^+} \frac{\sqrt{x}}{e} = \lim_{x \rightarrow 0^+} \sqrt{x} e = 0$$

Therefore,

$$\lim_{x \rightarrow 0^+} \sqrt{x} e^{\sin \frac{\pi}{x}} = 0$$

Answer to Exercise 12 (on page 24)

1. define $f(x) = 2x^4 + x - 12$, $a = 1$, $b = 2$, and $N = 0$. Calculate $f(a)$ and $f(b)$:

$$f(1) = -9 \text{ and } f(2) = 22$$

Since $f(x)$ is a polynomial, it is continuous on the interval $x \in [1, 2]$ and we see that $f(a) < 0 < f(b)$. Therefore, there exists some $c \in [1, 2]$ such that $f(c) = 0$.

2. First, we can rearrange the equation we are considering and define $f(x) = \ln x - 3x + 4\sqrt{x}$, and realize we are looking for values where $f(x) = 0$. Both $\ln x$ and $\sqrt{x}$ are only continuous for $x > 0$. The interval we are interested in, $x \in [2, 3]$, is in the domain of continuity for both $\ln x$ and $\sqrt{x}$. Defining $a = 2$, $b = 3$, and $N = 0$, we find that $f(a) = 0.35 > N > -0.973 = f(b)$. Since $N \in [f(b), f(a)]$, there must exist some c such that $f(c) = N = 0$ and there is a solution to the equation $\ln x = 3x - 4\sqrt{x}$ on the interval $x \in (2, 3)$.

3. Similar to above, define $f(x) = 2 \sin x - 3x^2 + 2x$, $a = 1$, $b = 2$, and $N = 0$. Calculate that $f(a) = 0.683$ and $f(b) = -6.181$. Since $f(x)$ is continuous on the interval $x \in [1, 2]$ and $f(b) < N < f(a)$, there exists some $c \in (1, 2)$ such that $f(c) = 0$. Therefore, there is a solution to $2 \sin x = 3x^2 - 2x$ on the interval $x \in (1, 2)$.

Answer to Exercise 13 (on page 28)

To sketch the curve, we begin by choosing several values of t and computing the corresponding x - and y -values.

t	x	y
-2	-1	4
-1	0	1
0	1	0
1	2	1
2	3	4

Plotting these points produces a parabola opening upward. As t increases, the x -values increase and the curve is traced from left to right.

Although the rectangular equation of the curve is

$$y = (x - 1)^2,$$

the parametric equations determine the direction in which the curve is drawn, which cannot be seen from the rectangular form alone.

Answer to Exercise 14 (on page 29)

To sketch the curve, we begin by evaluating the parametric equations at several values of t .

t	x	y
0	4	3
1	2	5
2	0	-1
3	-2	-15

Plotting these points produces a curve that opens downward. As t increases, the x -values decrease, indicating that the curve is traced from right to left.

The Python animation confirms this behavior by showing the curve being drawn progressively as t increases. The moving point illustrates both the direction of motion and how the curve is traced over time, reinforcing the hand-drawn sketch.

Answer to Exercise 15 (on page 30)

Two sets of parametric equations can represent the same graph but describe different curves because the parameter controls how the graph is traced, not just the shape of the graph itself.

Different parametric equations may generate the same set of points in the plane while differing in one or more of the following ways:

- the direction of motion along the curve,
- the speed at which the curve is traced,
- the starting point of the curve,
- or the number of times the curve is traced.

Answer to Exercise 16 (on page 35)

The radius is 5 m, because the coefficients of both the x and y functions is 5.

The angle is $3t$, so the angular velocity is $\omega = 3$ radians per second. Using the velocity formula from this section, with $r = 5$ and $\omega = 3$:

$$v_x(t) = -15 \sin(3t), \quad v_y(t) = 15 \cos(3t)$$

The overall *speed* can be found from the x and y components of the velocity:

$$\begin{aligned} |v| &= \sqrt{v_x^2 + v_y^2} \\ &= \sqrt{[-15 \sin(3t)]^2 + [15 \cos(3t)]^2} \\ &= \sqrt{15^2 [\sin^2(3t) + \cos^2(3t)]} \\ &= \sqrt{15^2} = 15 \frac{\text{m}}{\text{s}} \end{aligned}$$

Notice that this is the coefficient for both components of the velocity.

To complete the path, the particle must travel 10π m. If the speed is $15\frac{\text{m}}{\text{s}}$, then the time it takes is:

$$t = \frac{d}{v} = \frac{10\pi \text{ m}}{15\frac{\text{m}}{\text{s}}} = \frac{2\pi}{3} \text{ s} \approx 2.09 \text{ s}$$

Answer to Exercise 17 (on page 41)

$$\frac{120 \text{ km}}{1 \text{ hour}} = \frac{1000 \text{ m}}{1 \text{ km}} \frac{120 \text{ km}}{1 \text{ hour}} \frac{1 \text{ hour}}{3600 \text{ seconds}} = 33.3 \text{ m/s}$$

$$F = \frac{mv^2}{r} = \frac{0.4(33.3)^2}{200} = 2.2 \text{ newtons}$$

Answer to Exercise 18 (on page 42)

Givens:

$$T_{\text{max}} = (20\text{kg}) \cdot \left(9.8\frac{\text{m}}{\text{s}^2}\right) = 196\text{N}$$

$$r = 0.35\text{m}$$

$$m_{\text{whistle}} = 165\text{g} = 0.165\text{kg}$$

Unknown:

$$v = ?$$

$$f = ?$$

Equation(s):

$$F = ma$$

$$a = \frac{v^2}{r}$$

$$v = 2\pi rf$$

Solution: First, we find the tangential speed if the tension in the string is the maximum tension:

$$T_{\text{max}} = m_{\text{whistle}}a = m_{\text{whistle}}\frac{v^2}{r}$$

$$v = \sqrt{\frac{T_{\text{max}}r}{m_{\text{whistle}}}}$$

$$v = \sqrt{\frac{196 \text{ N} \cdot 0.35 \text{ m}}{0.165 \text{ kg}}} \approx 20.4 \frac{\text{m}}{\text{s}}$$

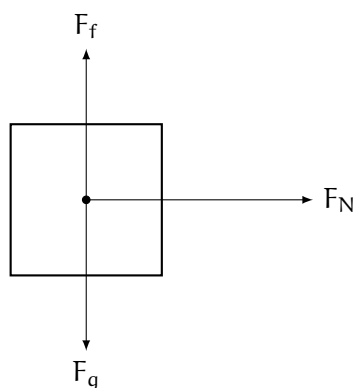
Therefore, the maximum tangential speed of the whistle before the string breaks is about 20.4 m/s. The frequency is the number of revolutions per second, which is the speed divided by the circumference $2\pi r$:

$$f = \frac{v}{2\pi r} = \frac{20.4 \text{ m/s}}{2\pi(0.35 \text{ m})} \approx 9.3 \text{ Hz}$$

So the lifeguard would have to swing the whistle at about 9 revolutions per second to break the string. That is very hard to do by hand, so the string is probably safe.

Answer to Exercise 19 (on page 42)

There are three forces acting on the rider: gravity, friction with the wall, and the normal force with the wall:



The FBD lets us write equations for Newton's Second Law in each dimension:

$$(1) \quad ma_x = F_N$$

$$(2) \quad ma_y = F_f - F_g = \mu F_N - mg$$

Because the rider does not fall down, we know that $a_y = 0 \frac{\text{m}}{\text{s}^2}$ and therefore equation (2) becomes:

$$\mu F_N - mg = 0 \rightarrow F_N = \frac{mg}{\mu}$$

Having solved for F_N , we substitute for it into equation (1):

$$ma_x = \frac{mg}{\mu} \rightarrow a_x = \frac{g}{\mu}$$

Since we know g and μ , we can calculate a_x :

$$a_x = \frac{9.8 \frac{\text{m}}{\text{s}^2}}{0.32} = 30.625 \frac{\text{m}}{\text{s}^2}$$

This is the minimum acceleration needed to keep the rider from slipping down. We can now use the relationship between centripetal acceleration, tangential velocity, and the radius to find the angular velocity:

$$a = \frac{v^2}{r} = \frac{(\omega r)^2}{r} = \omega^2 r$$

The ride is 10 meters across, so its radius is 5 meters:

$$\omega = \sqrt{\frac{a}{r}} = \sqrt{\frac{30.625 \frac{\text{m}}{\text{s}^2}}{5 \text{ m}}} \approx 2.47 \frac{\text{rad}}{\text{s}}$$

That is about 0.4 rotations per second, one turn every 2.5 seconds.

Answer to Exercise 20 (on page 45)

Givens:

- Mass of tetherball: $m = 0.5 \text{ kg}$
- Length of string: $L = 2.5 \text{ m}$
- Angle with vertical: $\theta = 38^\circ$

The components of the tension force are separated as follows:

$$T \cos \theta = mg, \quad T \sin \theta = \frac{mv^2}{r}$$

Dividing the second equation by the first equation gives:

$$\tan \theta = \frac{v^2}{rg} \rightarrow v = \sqrt{rg \tan \theta}$$

But we are not given r . We can find r using trigonometry:

$$r = L \sin \theta$$

Now we can find v :

$$v = \sqrt{(L \sin \theta)g \tan \theta}$$

Solving this gives us:

$$v = \sqrt{(2.5 \text{ m} \sin 38^\circ)(9.8 \frac{\text{m}}{\text{s}^2}) \tan 38^\circ} \approx 3.4 \frac{\text{m}}{\text{s}}$$

The speed of the tetherball is approximately $3.4 \frac{\text{m}}{\text{s}}$.

Answer to Exercise 21 (on page 49)

- At the top of the loop: At the top of the loop, both the gravitational force and the normal force point downward. Therefore, we can write:

$$F_{\text{net}} = F_N + F_g = \frac{mv^2}{r}$$

Solving for the normal force gives:

$$F_N = \frac{mv^2}{r} - F_g$$

Substituting for $F_g = mg$ (using $g = 10 \text{ m/s}^2$ for this problem) and $r = \frac{d}{2} = 25 \text{ m}$:

$$\begin{aligned} F_N &= \frac{(1200 \text{ kg})(25 \frac{\text{m}}{\text{s}})^2}{25 \text{ m}} - (1200 \text{ kg})(10 \frac{\text{m}}{\text{s}^2}) \\ F_N &= 30000 \text{ N} - 12000 \text{ N} = 18000 \text{ N} \end{aligned}$$

- At the bottom of the loop: At the bottom of the loop, the gravitational force points downward while the normal force points upward. Therefore, we can write:

$$F_{\text{net}} = F_N - F_g = \frac{mv^2}{r}$$

Solving for the normal force gives:

$$F_N = \frac{mv^2}{r} + F_g$$

Substituting for $F_g = mg$ (using $g = 10 \text{ m/s}^2$ for this problem) and $r = \frac{d}{2} = 25 \text{ m}$:

$$\begin{aligned} F_N &= \frac{(1200 \text{ kg})(25 \frac{\text{m}}{\text{s}})^2}{25 \text{ m}} + (1200 \text{ kg})(10 \frac{\text{m}}{\text{s}^2}) \\ F_N &= 30000 \text{ N} + 12000 \text{ N} = 42000 \text{ N} \end{aligned}$$

Answer to Exercise 22 (on page 54)

The torque on the door is equal to $rF \sin \theta$. Since our force is tangential, the sin component is at its max: $\sin 90 = 1$, our torque simplifies to $\tau = rF = 0.5 \cdot 38 = 19$ Newton-Meters.

Answer to Exercise 23 (on page 54)

1. Using $\tau = rF \sin \theta$, the torque is $\tau = (2.0)(20) \sin(30^\circ) = 40(0.5) = 20 \text{ N} \cdot \text{m}$.
2. Because the force has an upward component at the right end, so the beam tends to rotate counterclockwise.

Answer to Exercise 24 (on page 56)

1. Since the beam center is 2.0 m from the left end, and the pivot is at 1.0 m from the left end, so the weight is $W = mg = 20(9.8)$,

$$W = 196 \text{ N, acting 1.0 m right of pivot}$$

2. We can choose our own torque direction, so let us set counterclockwise be positive. The left end is 1.0 m to the left of the pivot and right end is 3.0 m to the right of the pivot. The beam's weight acts 1.0 m to the right of the pivot.

Since there are no angles and all pivots forces are perpendicular, there are three torques:

$$\tau_{50} = (1.0)(50) = +50 \text{ N} \cdot \text{m} \quad \text{CCW}$$

$$\tau_W = -(1.0)(196) = -196 \text{ N} \cdot \text{m} \quad \text{CW}$$

$$\tau_F = -(3.0)(F) \text{ N} \cdot \text{m} \quad \text{CW}$$

So equilibrium is solved by:

$$50 - 196 - 3F = 0$$

- 3.

$$50 - 196 - 3F = 0$$

$$-146 - 3F = 0$$

$$-3F = 146$$

$$F = -48.7 \text{ N}$$

The negative informs us that the torques act opposite the direction we assumed, so the force acts $F = 48.7 \text{ N}$ upward at the right end.

Answer to Exercise 25 (on page 57)

You can pick your own pivot, so let us choose the point that A and B stem from as our pivot. This eliminates those forces, as the r component of the torque would equal zero. Recall that the net torque is $\sum \tau = \tau_{\text{CCW}} - \tau_{\text{CW}}$. Analyzing the remaining torques:

- C is 1 m from the pivot, so the torque is $\tau_C = (1)C \sin -90^\circ$. It acts downwards, so our torque is clockwise.
- D acts directly parallel the rotating rod, adding no additional torque on the rod.
- E is 2 m from the pivot, and acting at a -135° angle from the horizontal. Our torque can be calculated from $\tau_E = (2)E \sin -135^\circ$. Note that using -45° provides the same answer. This torque is also clockwise.

Our net torque is $C \sin -90^\circ + (2)E \sin -135^\circ = 0$

Answer to Exercise 26 (on page 60)

Using Equation 7.4, we can rearrange to solve for α as $\alpha = \frac{rF \sin \theta}{I}$.

$$\alpha = \frac{rF \sin \theta}{I} = \frac{0.40 \cdot 25 \sin 60}{0.8} \approx 10.825 \text{ rad / s}^2$$

Answer to Exercise 27 (on page 60)

We choose counterclockwise torques to be positive. The net torque is

$$\sum \tau = r_1 F_1 \sin \theta_1 - r_2 F_2 \sin \theta_2$$

Using the rotational form of Newton's second law,

$$\sum \tau = I\alpha$$

Solving for I ,

$$I = \frac{r_1 F_1 \sin \theta_1 - r_2 F_2 \sin \theta_2}{\alpha}$$

Substituting values,

$$I = \frac{(0.50)(30) \sin 90^\circ - (0.30)(20) \sin 45^\circ}{8.0}$$

$$I = \frac{15 - 4.24}{8.0} \approx 1.35 \text{ kg}\cdot\text{m}^2$$

Answer to Exercise 28 (on page 61)

$$\begin{aligned} I_{\text{end}} &= I_{\text{CM}} + MD^2 \\ &= \frac{1}{12}ML^2 + M\left(\frac{L}{2}\right)^2 \\ &= \frac{1}{12}ML^2 + M\left(\frac{L^2}{4}\right) \\ &= \left(\frac{1}{12} + \frac{1}{4}\right)ML^2 \\ &= \left(\frac{4}{12}\right)ML^2 \\ &= \frac{1}{3}ML^2 \end{aligned}$$

Answer to Exercise 29 (on page 66)

As we can see, each unit square is equivalent to 1 torque $\cdot$ time. Counting the partial and full squares on $t = [2, 6]$, we get approximately 8 unit squares. This comes from the (approximately whole) 4 squares under $y = 2$, combining all squares from $y = [2, 4]$ to form 2 squares, and the additional partial area forming approximately another 2 squares, resulting in an angular impulse of approximately $8 \text{ N} \cdot \text{m} \cdot \text{s}$.

Counting squares is a first step toward *integration*, a calculus method for finding areas under curves exactly. When you learn it, you will find that the exact area here is about $7.52 \text{ N} \cdot \text{m} \cdot \text{s}$, pretty close to our estimate.



INDEX

- angular acceleration, 32
- angular impulse, 65
- angular momentum, 65
- angular velocity, 33
- angular velocity,, 32
- Archimedean spiral, 27
- asymptote
 - horizontal, 3
 - vertical, 3
- banked turn, 39
- centripetal
 - acceleration, 36
 - force, 37
- centripetal acceleration, 36
- centripetal force, 32, 37
- circular motion, 32
 - equations, 31
 - velocity, 34
- Conservation of Angular Momentum, 64
- continuity, 12
- eliminating the parameter, 26
- end behavior, 3
- expected value, 3
- floor function, 6
- flywheels, 64
- frequency, 31
- Intermediate Value Theorem, 23
- Law of Large Numbers, 3
- limit, 3
 - one-sided, 4
- Linear velocity, 31
- linear velocity, 33
- logistic function, 9
- mean, 3
- moment of inertia, 59
 - table of common, 64
- net torque, 56
- non-uniform circular motion, 46
- parallel-axis theorem, 61
- parameter, 25
- parametric curve
 - sketching, 27
- parametric equations, 25
- Period, 31
- period, 31
- rigid body, 51
- roller coaster loop, 46
- rolling motion, 62
- rotational kinetic energy, 63
- Squeeze Theorem, 20
- torque, 52, 53
 - vector, 53
- torques, 53

applied in opposite directions, [55](#)

work

by torque, [67](#)