Contents

Chapter 1

Circular Motion, continued

1.1 Review of Circular Motion Basics

We have previously discussed circular motion in the circular motion basics chapter, so let's do a quick review of the important concepts before we move on to rotational dynamics.

We have rotational kinematics, which describes the motion of objects rotating around a fixed axis. The key variables in rotational kinematics are angular displacement (θ), angular velocity (ω), and angular acceleration (α). These variables are analogous to linear displacement, velocity, and acceleration in linear kinematics.

Linear motion	Rotational motion
$v = v_0 + at$	$\omega = \omega_0 + \alpha t$
$x = x_0 + v_0 t + \frac{1}{2}at^2$	$\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2$
$v^2 = v_0^2 + 2a(x - x_0)$	$\omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0)$

Table 1.1: Kinematic equations for linear and rotational motion.

When an object such as a wheel or a disk rotates about an axis, each point on the object moves along a circular path centered on that axis. The total distance traveled by a point during one complete rotation is the circumference of its circular path, which is given by $C = 2\pi r$. The rotation of the object is described by an angle θ, usually measured in radians, where one full rotation corresponds to 2π radians.

1.2 Rigid Bodies

Although all points on a rotating disk pass through the same angular displacement θ in the same amount of time, they do not all travel the same distance. Points farther from the axis of rotation move along larger circles and therefore cover more distance during each rotation, even though the angular displacement is the same for all points. As a result, points farther from the axis have greater linear velocity and greater linear (tangential) acceleration than points closer to the center, as described by the relationships $v = r\omega$ and $a = r\alpha$. We call this points part of a *rigid body*.

A rigid body is an idealized object in which the distances between all points remain constant, even when forces are applied. All points along a radial line have the same angular displacement, angular velocity, and angular acceleration.

In other words, a rigid body does not deform: it does not stretch, compress, or bend. This assumption allows the object to be treated as a single system in which all parts move in a predictable way relative to one another.

In rotational motion, treating an object as a rigid body means that:

- All points in the object rotate together around a common axis.
- The angular velocity (ω) and angular acceleration (α) are the same for all points in the rigid body.
- The linear velocity (v) and linear acceleration (a) of points in the rigid body depend on their distance (r) from the axis of rotation, following the relationships $v = r\omega$ and $a = r\alpha$.

This idea maintains for many of the problems and concepts we will discuss in this chapter.

1.3 Torque

Let's think about pushing a box along a frictionless surface. There is a linear relationship between how hard you push and how much the box accelerates. What if, instead, you apply the same force to a merry-go-round fixed at a central pivot? Where can you push the merry-go-round for the highest rotation? Obviously you cannot push the merry-go-round linearly (as it is fixed to the ground), so how far will the merry-go-round rotate? All of these questions can be answered by the concept of *torque*.

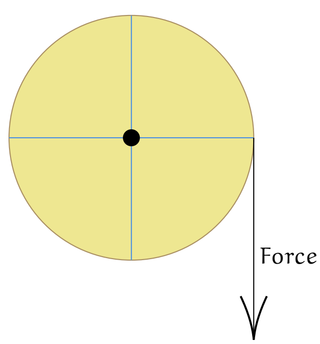

Figure 1.1: Merry-go-round with a singular applied force.

Torque is the concept of *rotational force*. Applying a torque to a surface, a *rigid body*, that has a fixed rotational pivot will cause the rigid body to *accelerate* around the pivot, ultimately producing an *angular acceleration*.

Imagine pushing open a door that is bound to a hinge on one side of the door. Pushing open the door at a point close to the hinge will require large amount of Torque, while pushing closer to the door handle and farther away from the hinge is much easier to do, and requires less Torque.

Alternatively, think of a wrench rotating a large bolt. It will require more force from your arm closer towards the pivot, while pushing at the edge of the wrench will require less force to produce an equivalent torque. Take a look at this in action here.

Torque

Torque, represented through the greek letter τ(tau), is defined as

$$\tau = rF \tag{1.1}$$

where F is a force perpendicular to

$$\tau = rF\sin\theta \tag{1.2}$$

Notice that the torque around a pivot depends on the distance from the pivot, r, and the angle from the radius vector θ. The $\sin\theta$ indicates that only the *perpendicular* part of the force impacts the torque. This is equivalent to the cross product of the two vectors:

$$\tau = r \times F \tag{1.3}$$

Torque has units of Newton-meters, but in this case, it is not an energy form, so cannot be equated to Joules. Torque is an *vector*, not a scalar.

Let's do an example proving this.

Question: A rod with a fixed end is being pulled by a tension force of $50\,\text{N}$ at a $40°$ angle from the horizontal, at a distance of $2\,\text{m}$. Find the Torque on the rod.

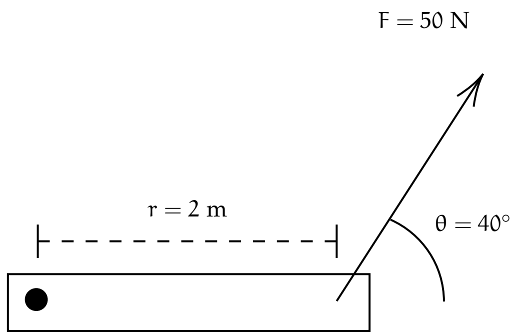

Figure 1.2: Main diagram.

Answer: Before just plugging in values, let's seperate the force vector into components. The component that pulls perpendicular to the axis is $50\sin 40°$, while $50\cos 40°$ of the force is applied *pulling* the rod against the pivot, which produces no torque.

(a) $F \cos \theta$ (b) $F \sin \theta$

Figure 1.3: Components of the applied force.

Think back to the wrench example: you don't push or pull a wrench to loosen a bolt; you must apply a force *tangential* or *perpendicular* to the bolt to rotate it.

Since only the perpendicular component contributes to the torque, we calculate the torque to be $\tau = 50 \sin 40° \text{ N} (2\text{m}) \approx 64.278$ Newton-meters

Exercise 1 Door Torque

A door is connected to a wall by a hinge 0.5 m from the handle. A person opens the door with a tangential force of 38 N. What is the torque on the door?

Working Space

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Exercise 2 Torque but with a twist!

A uniform horizontal beam is hinged at the left end and held in place by a force applied at its right end. The beam has length $L = 2.0$ m. A force of magnitude $F = 20$ N is applied at the right end of the beam, making an angle of $30°$ above the beam.

If the pivot point is at the left end of the beam,

1. Find the magnitude of the torque about the hinge.
2. State whether the torque tends to rotate the beam clockwise or counterclockwise.

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1.3.1 Torque directions

What happens when two torques are applied in *opposite*? Let's take a look at the merry-go-round again, but this time applied with two torques.

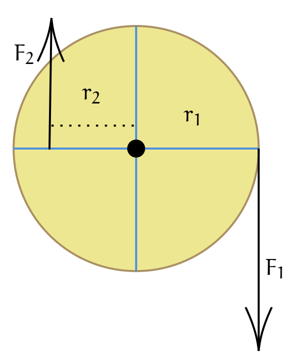

Figure 1.4: Merry-go-round with 2 forces in opposing directions.

Torques applied in opposition act *in opposite directions along the axis of rotation*. As torque is a vector defined by the *cross product equation* 1.3, so its direction is perpendicular to the plane formed by position vector and force which the plane live on. In problems, you will typically see the terms *out of* ($\odot$), or *into* ($\otimes$), the page, corresponding to the $+z$ or $-z$ directions, respectively. Remember these terms by the following visualiations: the out of symbol ($\odot$) looks like the tip of a feathered dart coming at you, while the into symbol ($\otimes$) looks like the feathers of a dart. Another way to think about this is using a drill and screw: rotating the screw counterclockwise causes it to advance toward you (out of the page), while rotating it clockwise causes it to retreat away from you (into the page).

To determine which direction a given torque points, one first identifies the sense of rotation the force would produce if acting alone: a force that tends to rotate the object counterclockwise in the plane produces a torque vector pointing out of the page, while a force that tends to rotate the object clockwise produces a torque vector pointing into the page. When two torques are opposite, one must therefore cause clockwise rotation and the other counterclockwise rotation, leading to torque vectors that are equal in line of action but opposite in direction.

The *net torque* is the sum of all torques present, similar to how net force is the sum of all forces. Generally, counterclockwise torques are positive, while clockwise torques are negative.[1] Using our merry go round example, we have F_2 is $\odot$while F_1 is $\otimes$.

$$\begin{aligned}\tau_{\text{net}} &= \sum \tau \\ &= \tau_{\text{CCW}} - \tau_{\text{CW}} \\ &= +(F_2)(r_2) - (F_1)(r_1)\end{aligned}$$

Because we know $\tau_1 > \tau_2$, the net torque is going to be negative, characterized with $\otimes$, and clockwise.

[1]Note that other videos or professors may do this opposite, using CW - CCW, but as long as you are consistent and clear, both are equally valid.

Exercise 3 Opposing Torques on a Beam

A uniform horizontal beam of length 4.0 m and mass 20 kg is supported by a pivot located 1.0 m from the left end. Two forces act on the beam, producing opposing torques about the pivot.

- A downward force of 50 N is applied at the left end of the beam.
- A downward force of unknown magnitude F is applied at the right end of the beam.
- The beam is in a state of static equilibrium.

Complete the following:

1. Find the Weight of the Beam
2. Write an equation for torque before solving for F
3. Solve for F, the unknown force acting on the right end

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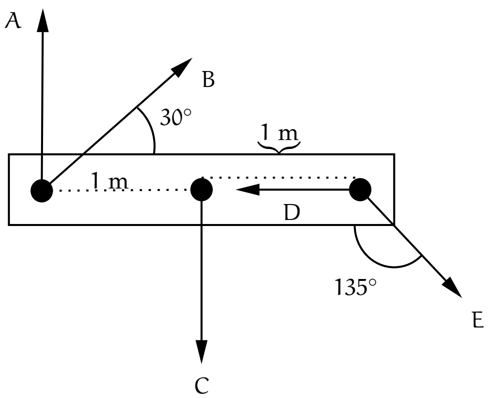
Exercise 4 More Complicated Opposing Torque

Create an equation for equilibrium of the following forces. Note that you may pick your own pivot point.



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1.4 Moment of Inertia and Rotational $F = ma$

We know Newton's Second Law by heart now, $F_{net} = ma$. You probably could recite it in your sleep, but what about a rotational equivalent. Recall that from our first circular motion chapter, we have the equation $a = r\alpha$. Let's do some conversions to get this into an angular form:

$$\begin{aligned} F &= ma \\ F &= mr\alpha \\ rF &= mr^2\alpha \\ \tau &= mr^2\alpha \end{aligned} \tag{1.4}$$

From Equation 1.4, we can derive that the "mass equivalent" is mr^2, since m is multiplied by a in the linear $F = ma$, while mr^2 is muliplied by α. Thus, we can state that for a point of distance r from the pivot (or axis of rotation), that singular point has *rotation inertia*, also called *moment of inertia*, is defined as mr^2. For all point masses on an object, the total inertia is the sum of all point masses of the object.

Moment of Inertia

A mass's total moment of inertia is given by the sum of each mass multiplied the distance from the axis of rotation squared.

$$I = \sum mr^2 \tag{1.5}$$

for continuous, solid objects, this becomes

$$I = \int r^2 \, dm \tag{1.6}$$

From Equations 1.5 and 1.6, we see that mass farther from the axis matters much more than anything closer to the pivot. Or, for two objects, one solid (mass spread evenly throughout) and the other a hollow cylinder or hoop (mass confined to the edges), the hollow cylinder will have a greater moment of inertia.

By finding the net rotational inertia of an object, we can create an equation for the rotational equivalent of Newton's Second Law:

$$\sum \tau = \tau_{net} = I\alpha \tag{1.7}$$

1.4.1 Relating $\tau = I\alpha$ and other Torque Def'ns

Since we have already defined angular acceleration, and have just discussed moment of inertia, we can now relate our two torque Equations 1.2 and 1.7 by equating them, as they both equal net torque.

$$I\alpha = rF\sin\theta \tag{1.8}$$

This will be useful for solving rotational dynamics problems, and finding any of the variables I, α, r, F, or θ when the others are known.

Exercise 5 Angular Acceleration of a Wheel

A wheel has $I = 0.80$ kg·m^2. A force $F = 25$ N is applied at radius $r = 0.40$ m at an angle $\theta = 60°$ to the radius.

Find α.

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Exercise 6 Moment of Inertia from Multiple Torques

A rigid wheel is mounted on a frictionless axle. Two forces act on the wheel:

- Force $F_1 = 30$ N is applied at radius $r_1 = 0.50$ m at an angle $\theta_1 = 90°$ to the radius, producing a counterclockwise torque.
- Force $F_2 = 20$ N is applied at radius $r_2 = 0.30$ m at an angle $\theta_2 = 45°$ to the radius, producing a clockwise torque.

The wheel undergoes an angular acceleration of $\alpha = 8.0$ rad/s^2.

Find the moment of inertia I of the wheel.

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1.5 Parallel-Axis Theorem

The **parallel-axis theorem** allows us to calculate the moment of inertia of a rigid body about any axis that is parallel to an axis passing through the center of mass. The theorem states that

$$I_s = I_{cm} + Md^2 \tag{1.9}$$

where I_s is the moment of inertia about the axis of interest, I_{cm} is the moment of inertia about a parallel axis through the center of mass, M is the total mass of the object, and d is the perpendicular distance between the two axes.

In many practical situations, such as a door rotating about its hinges or a rod rotating about one end, the axis of rotation is offset from the center of mass, making direct calculation more difficult. That's where the parallel axis theorem comes in!

Exercise 7 Parallel Axis Theorem

Prove that the Moment of Inertia of the end of a rod is $I_{end} = \frac{1}{3}ML^2$ given that the center of mass of the moment of inertia of a rod is $I_CM = \frac{1}{12}ML^2$.

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The parallel-axis theorem shows that the moment of inertia about an offset axis is always greater than the moment of inertia about the center-of-mass axis. The additional term Md^2 accounts for the translational motion of the object's center of mass relative to the new axis of rotation. As the distance between the axes increases, the moment of inertia increases accordingly.

If the mass of an object is not evenly distributed, the location of the center of mass shifts, but the parallel-axis theorem remains applicable. Once the center of mass is known, the theorem allows the moment of inertia about any parallel axis to be found by adding the contribution due to the displacement of the center of mass. In this way, the effects of mass distribution are fully captured by the center-of-mass term and the distance between the axes.

1.6 Rotational Kinetic Energy and Rolling Motion

Take a look at this video: Rotational Inertia: The Race Between a Ring and a Disc.

Let's analyze why the solid cylinder reaches the bottom first.

When two objects roll down an incline without slipping, their acceleration depends not only on gravity, but also on how their mass is distributed. This is because a rolling object must both translate and rotate, so gravitational potential energy is split between translational and rotational kinetic energy. The fraction that goes into rotation depends on the object's moment of inertia.

When an object is rolling, it has both translational kinetic energy and rotational kinetic energy. The total kinetic energy of a rolling object is the sum of these two forms of kinetic energy.

The total rotational kinetic energy of a rolling object comes from all of its individual point masses rotating around the axis. Each point mass has its own rotational kinetic energy

K_i, and the sum of all these individual kinetic energies gives the total rotational kinetic energy of the object:

$$\begin{aligned} KE_{rotational} &= \sum K_i \\ &= \sum \tfrac{1}{2} m_i r_i^2 \omega^2 \\ &= \tfrac{1}{2}\left(\sum m_i r_i^2\right)\omega^2 \end{aligned}$$

This gives us the formula for rotational kinetic energy in terms of moment of inertia:

Rotational Kinetic Energy

The rotational kinetic energy of a rigid body rotating about a fixed axis is given by

$$KE_{rotational} = \tfrac{1}{2} I\omega^2 \tag{1.10}$$

where I is the moment of inertia of the object about the axis of rotation, and ω is the angular velocity. This expression applies to rigid bodies rotating about a fixed axis.

Going back to our rolling objects, we can now see that the total kinetic energy of a rolling object is the sum of its translational and rotational kinetic energies:

Total Kinetic Energy of a Rolling Object

The total kinetic energy of a rolling object is given by

$$KE_{total} = KE_{translational} + KE_{rotational} = \tfrac{1}{2} m v^2 + \tfrac{1}{2} I\omega^2 \tag{1.11}$$

where m is the mass of the object, v is its linear velocity, I is its moment of inertia, and ω is its angular velocity. For rolling without slipping, the translational and rotational motions are linked, but it is still useful to treat their kinetic energies separately.

The moment of inertia I varies depending on how the mass is distributed in the object, and is a measure of how much the object resists rotational motion. For example, a solid disc has a different moment of inertia than a hollow disc of the same mass and radius. This difference in moment of inertia affects how much of the total kinetic energy is rotational versus translational.

The solid disc has most of its mass concentrated closer to the center, resulting in a lower moment of inertia. This means that less of its total kinetic energy is rotational, allowing more energy to be available for translational motion, which leads to a higher linear velocity down the incline.

Relationship between translational velocity and moment

For rolling objects of the same mass and radius starting from the same height, translational velocity is inversely related to moment of inertia. This relationship can be expressed as:

$$v \propto \frac{1}{I} \tag{1.12}$$

Explicitly, translational velocity is inversely related to moment of inertia. Objects with larger moments of inertia require more energy to rotate, leaving less energy available for translational motion.

Each object has a unique moment of inertia based on its mass distribution, which directly affects its rolling behavior. Here is a reference table with common moments of inertia for various shapes:

Object and Axis of Rotation	**Moment of Inertia**
Solid cylinder or disk (symmetry axis)	$I = \frac{1}{2}MR^2$
Solid cylinder or disk (central diameter)	$I = \frac{1}{4}MR^2 + \frac{1}{12}ML^2$
Hoop (symmetry axis)	$I = MR^2$
Hoop (diameter)	$I = \frac{1}{2}MR^2$
Solid sphere (about diameter)	$I = \frac{2}{5}MR^2$
Thin spherical shell (about diameter)	$I = \frac{2}{3}MR^2$
Rod (about center, perpendicular to length)	$I = \frac{1}{12}ML^2$
Rod (about one end, perpendicular to length)	$I = \frac{1}{3}ML^2$

Table 1.2: Common Moments of Inertia

A visualization of each of these can be found at Georgia State University's HyperPhysics website: HyperPhysics: Moments of Inertia.

1.7 Flywheel Energy Storage, Work, and Angular Momentum

A flywheel is a rigid body designed to store rotational energy. It typically consists of a heavy wheel or disk mounted on an axle, with much of its mass concentrated away from the axis of rotation. This mass distribution gives the flywheel a large moment of inertia, allowing it to resist changes in rotational speed. There is very little friction in the axle, so once the flywheel is spinning, it can maintain its angular velocity for a long time with minimal energy loss. We often eliminate friction in flywheel friction problems or calculations.

Because of its large moment of inertia, a flywheel can smooth out variations in rotational motion. When energy is added, the flywheel stores it as rotational kinetic energy. When energy is removed, the flywheel releases this stored energy gradually, helping maintain a more uniform angular velocity. For this reason, flywheels are commonly used in engines, generators, and mechanical systems that require steady rotation.

This relies on the principle of *Conservation of Angular Momentum*, which states that in the absence of *external torques*, the total angular momentum of a system remains constant. For a flywheel, this means that if no external torque acts on it, its angular momentum will not change, allowing it to maintain its rotational speed even when external forces try to slow it down.

1.7.1 Angular Momentum

The connection between angular momentum and torque is direct: torque is the rate at which angular momentum changes. Applying a torque to a flywheel changes its angular momentum by increasing or decreasing its angular velocity. Because of the flywheel's large moment of inertia, a given torque produces only a gradual change in angular speed, contributing to the flywheel's stabilizing effect.

Angular Momentum and the Conservation of Angular Momentum

The angular momentum L of a rotating rigid body is given by

$$L = I\omega \tag{1.13}$$

where I is the moment of inertia of the body about the axis of rotation, and ω is its angular velocity. Angular momentum is a *vector* quantity, with both magnitude and direction.

Just as linear momentum can be written as $F = \frac{dp}{dt}$, torque can be expressed as the time rate of change of angular momentum:

$$\tau = \frac{dL}{dt} \tag{1.14}$$

We can also derive this as a Newtonian cross product between r and $\mathbf{p}$, linear momentum. This works because angular momentum is the rotational equivalent of linear momentum, but depends on the choice of origin.

$$L = r \times p \tag{1.15}$$

Just like linear momentum, angular momentum is conserved in a system with no external torques. This is known as the *Conservation of Angular Momentum*:

$$L_{\text{initial}} = L_{\text{final}} \quad (\boldsymbol{\tau}_{\text{net}} = 0) \tag{1.16}$$

This equation shows that torque is responsible for changing angular momentum. Because a flywheel has a large moment of inertia, a given applied torque produces only a small angular acceleration, allowing the flywheel to respond smoothly to changes in applied forces.

Angular Impulse

Just like in classical momentum, a change in angular momentum is called *angular impulse*. Angular Impulse is representative the cumulative effect of torque applied over given a time interval.

Angular Impulse

Angular impulse is defined by

$$\Delta \mathbf{L} = \int_{t_1}^{t_2} \boldsymbol{\tau}\, dt \tag{1.17}$$

Graphically, this is area under a *torque vs. time* graph.

Angular impulse has units Newton-meter-seconds, same as angular momentum:

$$N \cdot m \cdot s^{-1} = kg \cdot m^2 \cdot s^{-1}$$

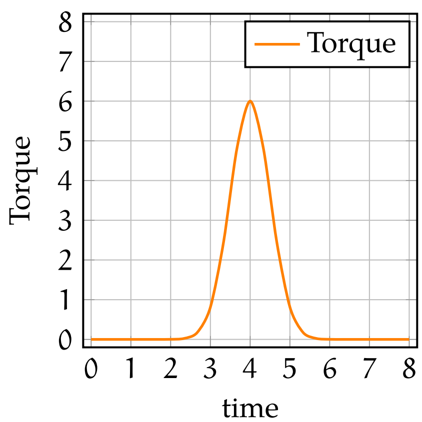
Angular impulse explains why sudden twists or torque have effects on rotational objects. A sudden hit to a door, instantaneous wrench forces, or off-center collisions cause a much better impulses. This impulse may be produced either by a large torque applied over a short time or by a smaller torque applied over a longer time. However, in many physical situations, applying a large torque over a near-instantaneous time interval is more effective and practical.

Exercise 8 Angular Impulse

Working Space

A kid rams a blue toy car into a red toy car, which is initially stationary. The collision causes the red toy car to start spinning in place.

Estimate, by counting the area under the curve, the angular impulse caused by a toy car crash.



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1.7.2 Work done by Torque

Recall that work is a change in energy. Torque also does work when it causes rotation. For a constant torque acting on point mass through an angular displacement $\Delta\theta$ the work done is

$$W = Fr\theta = \tau\Delta\theta \tag{1.18}$$

This comes from the fact that linear work is defined as $W = F\Delta x$, and the arc length of a circle is defined as $s = r\theta$. Thus, we can substitute $r\theta$ for Δx in the linear work equation to get the rotational work equation.

1.8 Summary

Here is a summary of linear quantities in physics, and their now introduced rotational components. You may need to refer to this often to so keep it handy.

Table 1.3: Correspondence Between Linear and Rotational Motion

Linear Motion	**Rotational Motion**
Position x	Angle θ
Velocity v	Angular velocity ω
Acceleration a	Angular acceleration α
Mass m	Moment of inertia I
Force F	Torque τ
Momentum $p = mv$	Angular momentum $L = I\omega$
$F = \frac{dp}{dt}$	$\tau = \frac{dL}{dt}$
Work $W = F\Delta x$	Work $W = \tau\Delta\theta$
Kinetic Energy $\frac{1}{2}mv^2$	Rotational KE $\frac{1}{2}I\omega^2$
Power $P = Fv$	Power $P = \tau\omega$

CHAPTER 2

Rocketry

A rocket propels hot gases, which creates an equal and opposite reaction that pushes it forwards, even in a vacuum.

Even without anything to push against, the rocket can still move forward thanks to Newton's Third Law.

Imagine a spacecraft with a bowling ball attached to the back. If that spacecraft exerts a force to throw the bowling ball backwards, the ball will exert a force on the ship, moving it forwards.

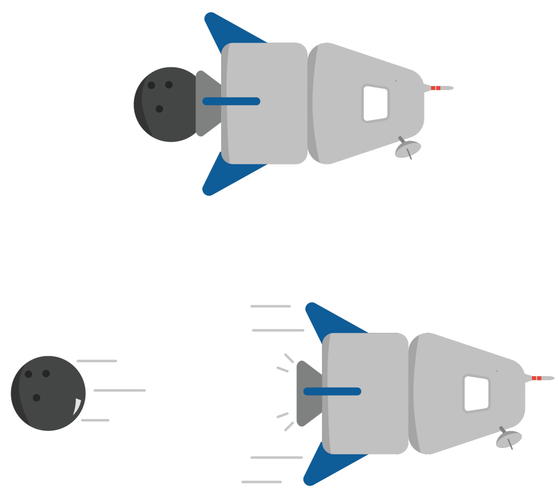


Figure 2.1: Newton's Third Law in action .

Instead of a bowling ball, real-life rockets usually "throw" particles of hot gas at very high speeds. Rockets carry their own oxidizer to provide oxygen to allow fuel to burn.

2.1 Types of rocket fuels

There are two main types of chemical rockets.

One type is a *Solid Fuel Rocket*, which ignites a solid fuel-oxidizer mix. Once the solid fuel is ignited, it can't be stopped until all of the fuel is exhausted.

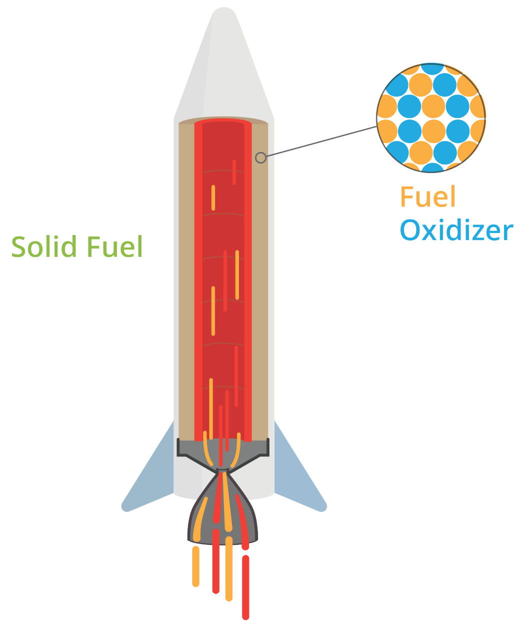


Figure 2.2: A solid fuel-oxidizer rocket.

The other main type of chemical rocket is called a *Liquid Fuel Rocket*. Liquid fuel rockets contain separate tanks for liquid fuel and liquid oxygen. Fuel pumps bring them both to a combustion chamber where they ignite and exit the rocket. Most liquid fuel engines can control their thrust.

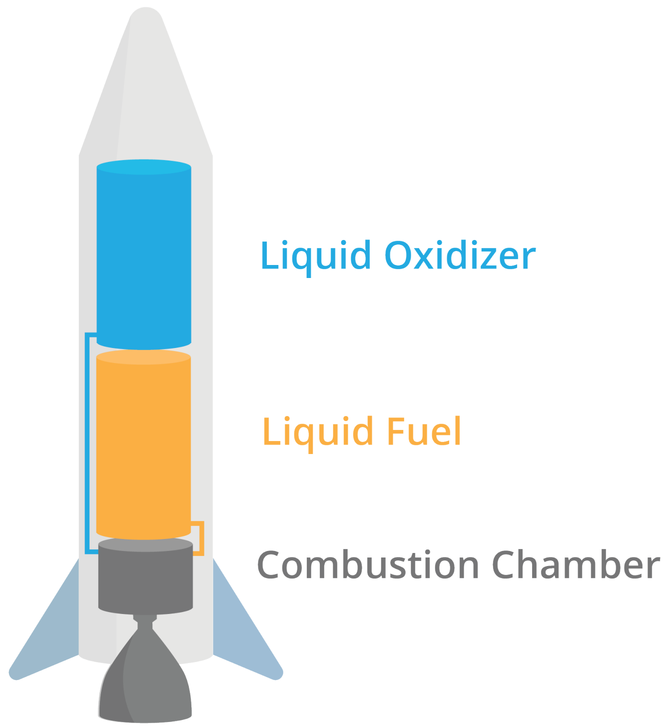


Figure 2.3: A liquid chemical phase.

2.2 Tyranny of the rocket equation

Chemical rockets can only burn the fuel that they bring with them. However, the more fuel you carry, the heavier the vehicle will be.

One way to help reduce this weight is by using *staging*.

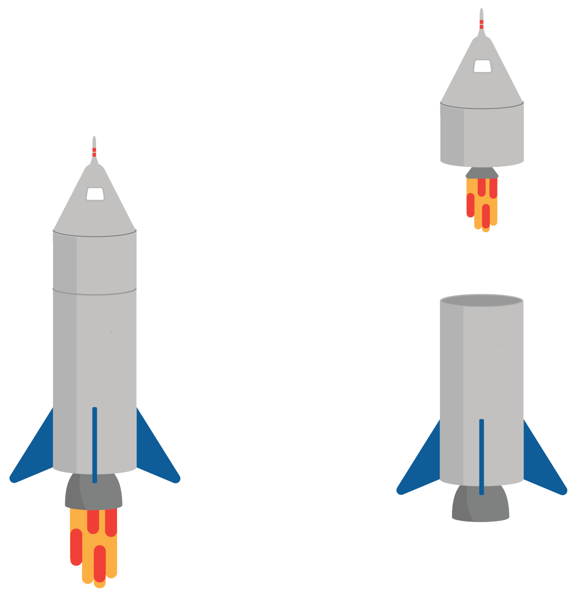


Figure 2.4: A dual stage rocket eliminates one of its stages throughout flight.

Staging allows rockets to drop unnecessary structural mass once they've used up a certain amount of fuel.

2.3 Control in atmosphere

There are several common ways that engineers have managed to control rockets' direction in the atmosphere. Usually, on-board sensors detect the orientation of the rocket, and can automatically adjust these controls to keep the rocket going the correct direction.

One method is using *movable fins*. The fins work similarly to control surfaces that we covered in the airplanes chapter.

Another method of control uses a *gimbaled engine*. A gimbaled engine is an engine that can pivot in a certain direction. By pivoting the engine, the rocket can change its direction of thrust, which changes the direction of the rocket. This is called *thrust vectoring*.

A more outdated method is using *vernier engines*, which are two smaller engines that control attitude. However, this adds a large amount of weight to the rocket, so they are less frequently used today.

Pros and Cons of Rocket Control Systems in Atmosphere

- **Movable fins**
  - **Pros:** Simple, reliable, lightweight, effective at high speeds in atmosphere.
  - **Cons:** Ineffective in vacuum, limited control at low speeds or thin atmosphere.
- **Gimbaled engine (Thrust vectoring)**
  - **Pros:** Precise control, works in both atmosphere and vacuum, allows for rapid directional changes.
  - **Cons:** Mechanically complex, heavier, more expensive, potential failure points.
- **Vernier engines**
  - **Pros:** Provides fine attitude control, redundancy.
  - **Cons:** Adds significant weight, less efficient, rarely used in modern rockets.

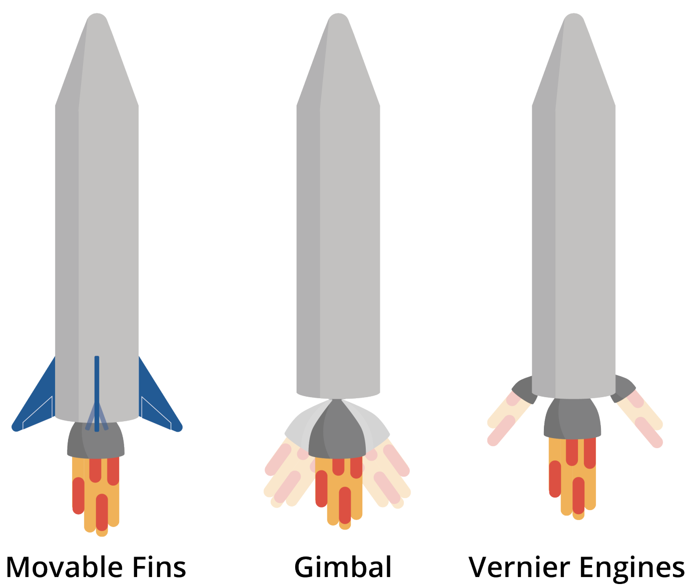


Figure 2.5: Three different methods of engines and their movement in outer space.

2.4 Control in space

The previous section describes ways that engineers control rockets in the atmosphere, but most rockets will end up in the vacuum of space. There are several common ways to adjust the orientation in space.

One method is using *RCS thrusters*. An RCS, or reaction control system, is a series of small thrusters that are used to change the direction and position of a spacecraft. RCS thrusters are usually small, and placed in blocks of 3-4 allowing for precise control in all three axes. They are often used for docking maneuvers, attitude control, and orbital adjustments.

Another method called *reaction wheels* uses angular momentum to rotate the spacecraft. By accelerating and decelerating wheels on three axes, the spacecraft can rotate in any direction.

A third common attitude control technology is *magnetorquer*. Magnetorquers use electromagnets and the earth's magnetic field to adjust the orientation of the spacecraft. Magnetorquers are common in small satellites, but they are not as common in larger spacecraft. To manipulate large spacecraft with magnetorquers effectively, you need either prohibitively high current or an external magnetic field more powerful than the Earth's magnetic field.

Pros and Cons of Rocket Control Systems in Space

- **RCS Thrusters**
  - **Pros:** Precise control, enables translation and rotation, reliable, effective for docking and attitude adjustments.
  - **Cons:** Consumes propellant, limited by onboard fuel, adds weight and complexity.
- **Reaction Wheels**
  - **Pros:** No propellant required, precise attitude control, long operational life, quiet operation.
  - **Cons:** Limited torque, can saturate and require desaturation (often using RCS), mechanical failure risk.
- **Magnetorquers**
  - **Pros:** No propellant required, lightweight, simple design, effective for small satellites in low Earth orbit.
  - **Cons:** Only works near planetary magnetic fields, limited control authority, not suitable for deep space or large spacecraft.

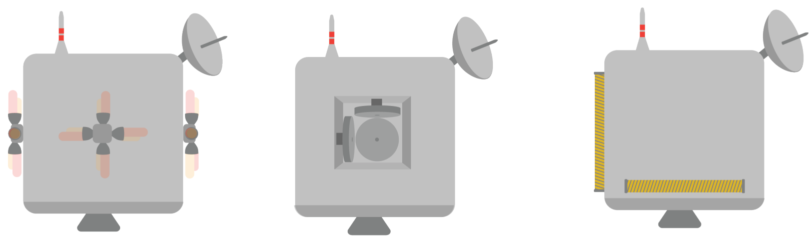


Figure 2.6: Control options for rockets in space.

2.5 Alternative propulsion

One type of alternate propulsion is called a *solar sail*. Solar sails have lightweight reflective surfaces that use photons in space to propel the spacecraft without on-board fuel.

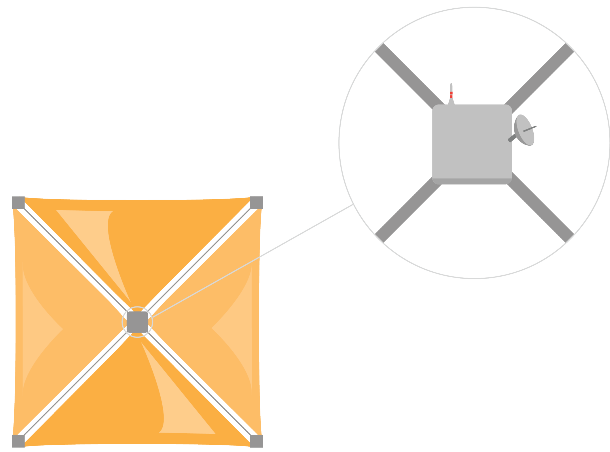


Figure 2.7: Solar sails propel the rocket with photon reflection.

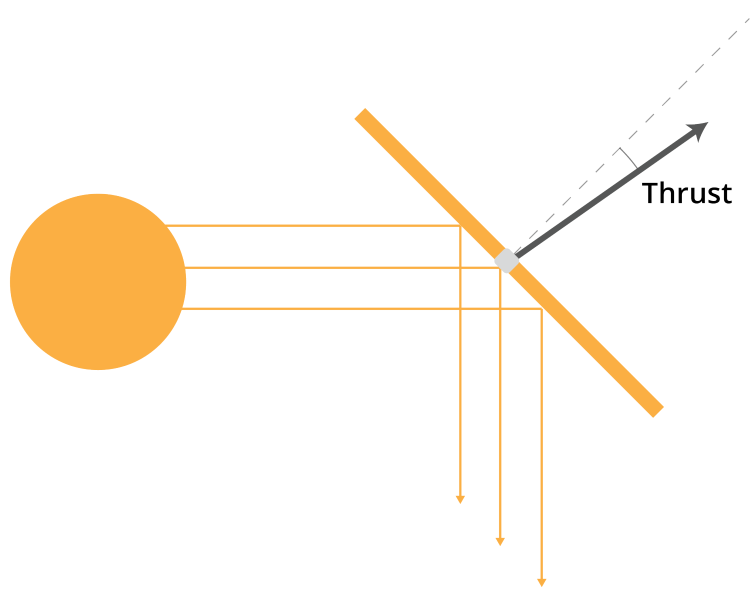


Figure 2.8: A zoomed in version of solar sails.

Photons reflect off of the surface of the sail. However, since the surface is not perfectly reflective, some of those photons are absorbed, and they produce a horizontal equal and opposite reaction. That small force causes the net thrust to be slightly skewed away from a right angle to the sail.

Ion propulsion is a form of electric propulsion that accelerates ions to generate thrust. There are two main types: electrostatic and electromagnetic. Electrostatic ion propulsion uses electric fields to accelerate ions, typically xenon, through grids to produce thrust. Electrostatic thrusters use the Coulomb force to accelerate ions, whereas electromagnetic ion propulsion uses both electric and magnetic fields through the Lorentz force to accelerate ions. Both methods are highly efficient and suitable for long-duration space missions, but they produce low thrust compared to chemical rockets.

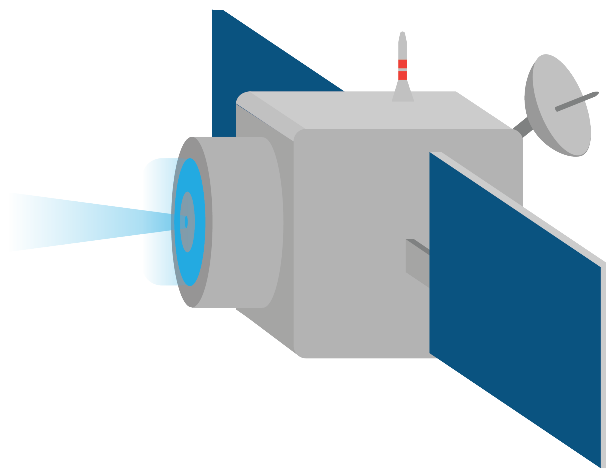


Figure 2.9: An ion thruster

Chapter 3

Simulation with Vectors

In an earlier chapter, you wrote a Python program that simulated the flight of a hammer to predict its altitude. Your simulation dealt only with scalars. Now, you are ready to create simulations of positions, velocities, accelerations, and forces as vectors.

In this chapter, you are going to simulate two moons that, as they wandered through the vast universe, get caught in each other's gravity well. We will assume there are no other forces acting upon the moons.

3.1 Force, Acceleration, Velocity, and Position

We talked about the magnitude of a gravitational attraction between two masses:

$$F = G\frac{m_1 m_2}{r^2}$$

where F is the magnitude of the force in newtons, m_1 and m_2 are the masses in kg, r is the distance between them in meters, and g is the universal gravitational constant: 6.67430×10^{-11}.

What is the direction? For the two moons, the force on moon 1 will pull toward moon 2. Likewise, the force on moon 2 will pull toward moon 1.

Of course, if something is big (like the sun), you need to be more specific: The force points directly at the center of mass of the object that is generating the force.

Each of the moons will start off with a velocity vector. That velocity vector will change over time as the moon is accelerated by the force of gravity. If you have a mass m with an initial velocity vector of $\vec{v_0}$ that is being accelerated with a constant force vector $\vec{F}$, at time t, the new velocity vector will be:

$$\vec{v}_t = \vec{v}_0 + \frac{t}{m}\vec{F}$$

If an object is at an initial position vector of $\vec{p_0}$ and moves with a constant velocity vector $\vec{v}$ for time t, the new position will be given by

$$\vec{p}_t = \vec{p}_0 + t\vec{v}$$

3.2 Simulations and Step Size

As two moons orbit each other, the force, acceleration, velocity, and position are changing smoothly and continuously. It is difficult to simulate truly continuous things on a digital computer.

However, think about how a movie shows you many frames each second. Each frame is a still picture of the state of the system. The more frames per second, the smoother it looks.

We do a similar trick in simulations. We say "We are going run our simulation in 2 hour steps. We will assume that the acceleration and velocity were constant for those two hours. We will update our position vectors accordingly, then we will recalculate our acceleration and velocity vectors."

Generally, as you make the step size smaller, your simulation will get more accurate and take longer to execute.

3.3 Make a Text-based Simulation

To start, you are going to write a Python program that simulates the moons and prints out their position for every time step. Later, we will add graphs and even animation.

We are going to assume the two moons are traveling the same plane so we can do all the math and graphing in 2 dimensions.

Each moon will be represented by a dictionary containing the state of the moon:

- Its mass in kilograms
- Its position — A 2-dimensional vector represent x and y coordinates of the center of the moon.
- Its velocity — A 2-dimensional vector
- Its radius — Each moon has a radius so we know when the centers of the two moons are so close to each other that they must have collided.
- Its color — We will use that when do the plots and animations. One moon will be red, the other blue.

There will then be a loop where we will update the positions of the moons and then recalculate the acceleration and velocities.

How much time will be simulated? 100 days or until the moons collide, whichever comes first.

We will use numpy arrays to represent our vectors.

Create a file called `moons.py`, and type in this code:

```
import numpy as np

# Constants
G = 6.67430e-11              # Gravitational constant (Nm^2/kg^2)
SEC_PER_DAY = 24 * 60 * 60   # How many seconds in a day?
MAX_TIME = 100 * SEC_PER_DAY # 100 days
TIME_STEP = 2 * 60 * 60      # Update every two hours

# Create the inital state of Moon 1
m1 = {
    "mass": 6.0e22,  # kg
    "position": np.array([0.0, 200_000_000]),  # m
    "velocity": np.array([100.0, 25.0]),  # m/s
    "radius": 1_500_000.0,  # m
    "color": "red" # For plotting
}

# Create the inital state of Moon 2
m2 = {
    "mass": 11.0e22,  # kg
    "position": np.array([0.0, -150_000_000]),  # m
    "velocity": np.array([-45.0, 2.0]),  # m/s
    "radius": 2_000_000.0,  # m
    "color": "blue" # For plotting
}

# Lists to hold positions and time
position1_log = []
position2_log = []
time_log = []

# Start at time zero seconds
current_time = 0.0

# Loop until current time exceed Max Time
while current_time <= MAX_TIME:

    # Add time and positions to log
    time_log.append(current_time)
    position1_log.append(m1["position"])
    position2_log.append(m2["position"])
```

```
    # Print the current time and positions
    print(f"Day {current_time/SEC_PER_DAY:.2f}:")
    print(f"\tMoon 1:({m1['position'][0]:,.1f},{m1['position'][1]:,.1f})")
    print(f"\tMoon 2:({m2['position'][0]:,.1f},{m2['position'][1]:,.1f})")

    # Update the positions based on the current velocities
    m1["position"] = m1["position"] + m1["velocity"] * TIME_STEP
    m2["position"] = m2["position"] + m2["velocity"] * TIME_STEP

    # Find the vector from moon1 to moon2
    delta = m2["position"] - m1["position"]

    # What is the distance between the moons?
    distance = np.linalg.norm(delta)

    # Have the moons collided?
    if distance < m1["radius"] + m2["radius"]:
        print(f"*** Collided {current_time:.1f} seconds in!")
        break

    # What is a unit vector that points from moon1 toward moon2?
    direction = delta / distance

    # Calculate the magnitude of the gravitational attraction
    magnitude = G * m1["mass"] * m2["mass"] / (distance**2)

    # Acceleration vector of moon1 (a = f/m)
    acceleration1 = direction * magnitude / m1["mass"]

    # Acceleration vector of moon2
    acceleration2 = (-1 * direction) * magnitude / m2["mass"]

    # Update the velocity vectors
    m1["velocity"] = m1["velocity"] + acceleration1 * TIME_STEP
    m2["velocity"] = m2["velocity"] + acceleration2 * TIME_STEP

    # Update the clock
    current_time += TIME_STEP

print(f"Generated {len(position1_log)} data points.")
\end{verbatim}

When your run the simulation,  you will see the positions of the moons for 100
↪ days:
\begin{verbatim}
> python3 moons.py
Day 0.00:
        Moon 1:(0.0,200,000,000.0)
        Moon 2:(0.0,-150,000,000.0)
Day 0.08:
        Moon 1:(720,000.0,200,180,000.0)
        Moon 2:(-324,000.0,-149,985,600.0)
Day 0.17:
```

```
        Moon 1:(1,439,990.7,200,356,896.1)
        Moon 2:(-647,995.0,-149,969,507.0)
...
Day 100.00:
        Moon 1:(119,312,305.5,283,265,313.5)
        Moon 2:(17,393,287.9,-60,319,261.9)
Generated 1201 data points.
```

Look over the code. Make sure you understand what every line does.

3.4 Graph the Paths of the Moons

Now, you will use the matplotlib to graph the paths of the moons. Add this line to the beginning of `moons.py`.

```
import matplotlib.pyplot as plt
```

Add this code to the end of your `moons.py`:

```
# Convert lists to np.arrays
positions1 = np.array(position1_log)
positions2 = np.array(position2_log)

# Create a figure with a set of axes
fig, ax = plt.subplots(1, figsize=(7.2, 10))

# Label the axes
ax.set_xlabel("x (m)")
ax.set_ylabel("y (m)")
ax.set_aspect("equal", adjustable='box')

# Draw the path of the two moons
ax.plot(positions1[:, 0], positions1[:, 1], m1["color"], lw=0.7)
ax.plot(positions2[:, 0], positions2[:, 1], m2["color"], lw=0.7)

# Save out the figure
fig.savefig("plotmoons.png")
```

When you run it, your `plotmoons.png` should look like this:

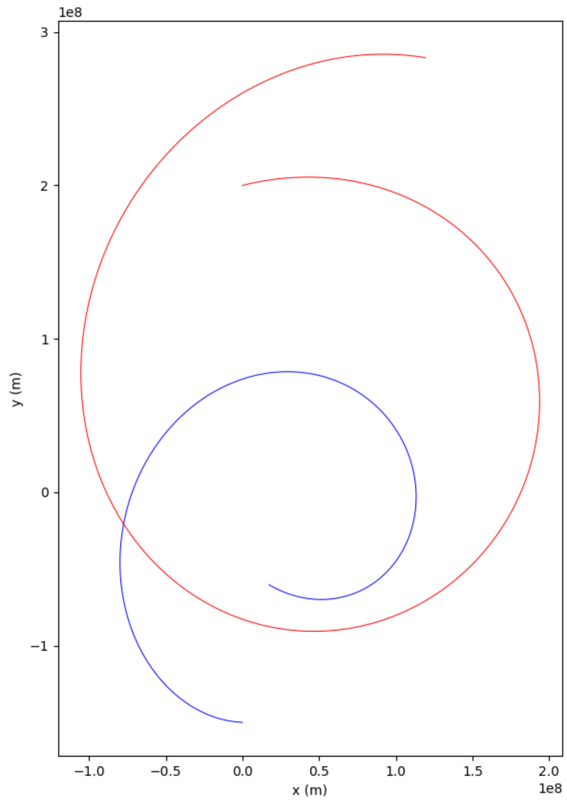


Figure 3.1: Output of moons_01.py.

It is nifty to see the paths, but we don't know where each moon was at a particular time. In fact, it is difficult to figure out which end of each curve was the beginning and which was the ending.

What if we added some lines and labels every 300 steps to put a sense of time into the plot? Add one more constant after the import statements:

```
PAIR_LINE_STEP = 300  # How time steps between pair lines
```

Immediately before you save the figure to the file, add the following code:

```
# Draw some pair lines that help the
# viewer understand time in the graph
i = 0
while i < len(positions1):

    # Where are the moons at the ith entry?
    a = positions1[i, :]
    b = positions2[i, :]
    ax.plot([a[0], b[0]], [a[1], b[1]], "--", c="gray", lw=0.6, marker=".")
```

```
    # What is the time at the ith entry?
    t = time_log[i]

    # Label the location of moon 1 with the day
    ax.text(a[0], a[1], f"{t/SEC_PER_DAY:.0f} days")
    i += PAIR_LINE_STEP
```

When you run it, your plot should look like this:

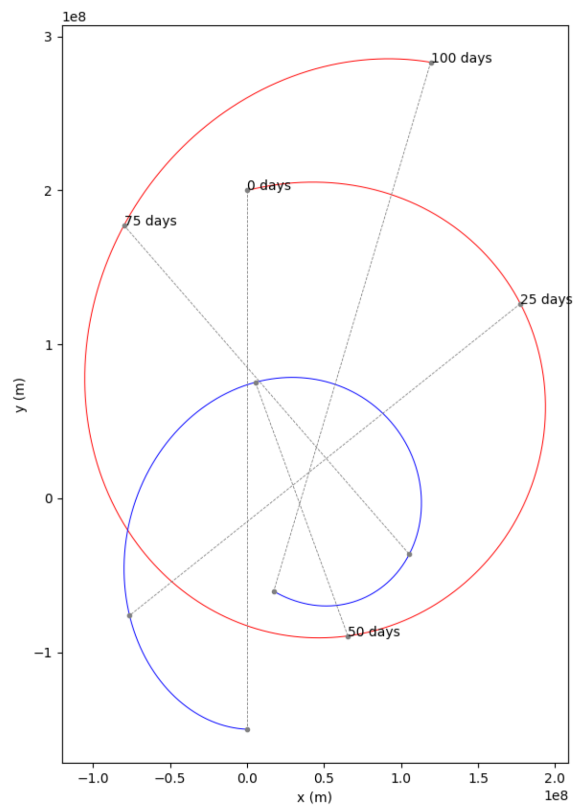


Figure 3.2: Plot moons_02.py.

Now you can get a feel for what happened. The moons were attracted to each other by gravity and started to circle each other. The heavier moon accelerates less quickly, so it makes a smaller loop.

Maybe we will get a better feel for what is happening if we look at more time. Let's increase it to 400 days. Change the relevant constant:

```
MAX_TIME = 400 * SEC_PER_DAY # 100 days
```

Now it should look like this:

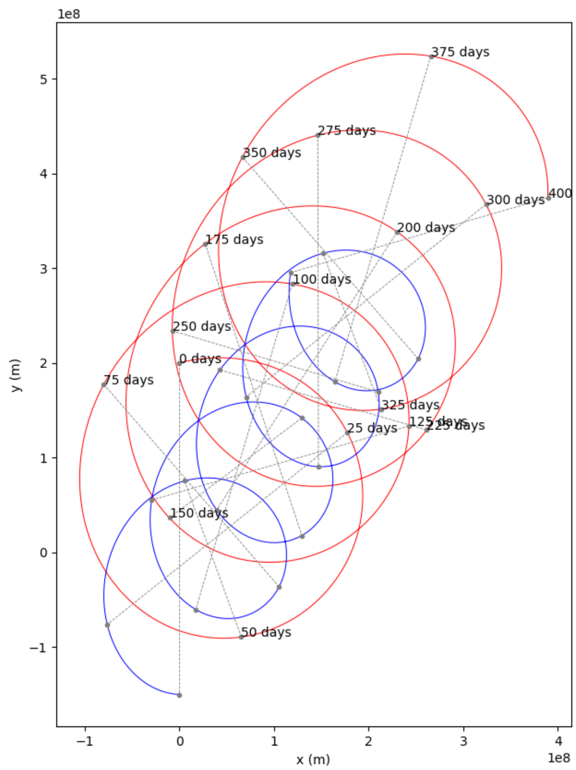


Figure 3.3: Output of plot moons_03.py.

Now you can see the pattern. They are rotating around each other and the pair is gradually migrating up and to the right.

3.5 Conservation of Momentum

Recall the idea of momentum being conserved in a system. In these programs, you are observing an extra important idea: *the momentum of a system will be conserved*. That is, absent forces from outside the system, the velocity of the center of mass will not change.

We can compute the initial center of mass and its velocity. In both cases, we just do a weighted average using the mass of the moon as the weight.

Immediately after you initialize the state of two moons, calculate the initial center of mass and its velocity:

```
# Calculate the initial position and velocity of the center of mass
tm = m1["mass"] + m2["mass"]  # Total mass
cm_position = (m1["mass"] * m1["position"] + m2["mass"] * m2["position"]) / tm
cm_velocity = (m1["mass"] * m1["velocity"] + m2["mass"] * m2["velocity"]) / tm
```

Let's record the center of mass for each time. Before the loop starts, create a list to hold them:

```
cm_log = []
```

Inside the loop (before any calculations), append the current center of mass position to the log:

```
    cm_log.append(cm_position)
```

Anywhere later in the loop (after you update the positions of the moon), update `cm_position`:

```
    # Update the center of mass
    cm_position = cm_position + cm_velocity * TIME_STEP
```

Now, let's look at the positions of the moons relative to the center of mass. Before you do any plotting, convert the list to a numpy array and subtract it from the positions:

```
cms = np.array(cm_log)

# Make positions relative to the center of mass
positions1 = positions1 - cms
positions2 = positions2 - cms
```

When you run it, you can really see what is happening:

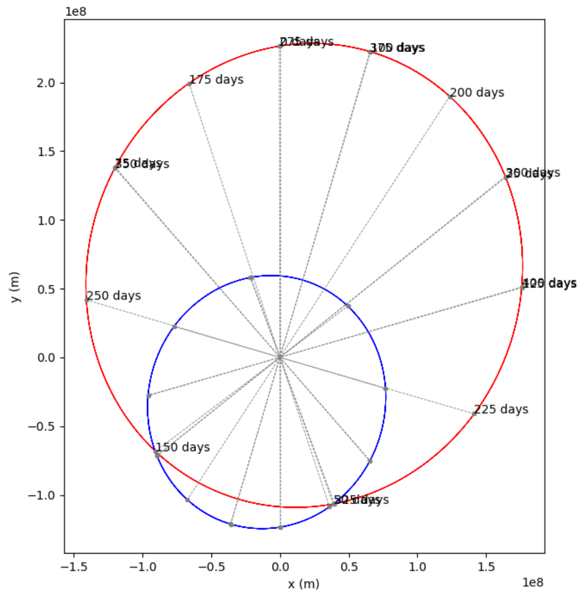


Figure 3.4: Output of plotmoons_04.py.

The moons are tracing elliptical paths. The center of mass is the focus point for both of them.

3.6 Animation

One of the features of matplotlib that not a lot of people understand is how to make animations with it. This seems like a really great opportunity to make an animation showing the position, velocity, acceleration of the moons. We will also show the center of mass.

The trick to animations is that you create a bunch "artist" objects. You create a function that updates the artists. matplotlib will call your functions, tell the artists to draw themselves, and make a movie out of that.

Make a copy of `moons.py` called `animate_moons.py`.

Edit it to look like this:

```
import numpy as np
import matplotlib.pyplot as plt

# Import animation support and artists
from matplotlib.animation import FuncAnimation
from matplotlib.patches import Circle, FancyArrow
```

```
from matplotlib.text import Text

# Constants
G = 6.67430e-11  # Gravitational constant (Nm^2/kg^2)
SEC_PER_DAY = 24 * 60 * 60  # How many seconds in a day?
MAX_TIME = 400 * SEC_PER_DAY  # 100 days
TIME_STEP = 12 * 60 * 60  # Update every 12 hours
FRAMECOUNT = MAX_TIME / TIME_STEP  # How many frames in animation
ANI_INTERVAL = 1000 / 50  # ms for each frame in animation

# The velocity and acceleration vectors are invisible
# unless we scale them up.  A lot.
VSCALE = 140000.0
ASCALE = VSCALE * 800000.0

# Create the inital state of Moon 1
m1 = {
    "mass": 6.0e22,  # kg
    "position": np.array([0.0, 200_000_000]),  # m
    "velocity": np.array([100.0, 25.0]),  # m/s
    "radius": 1_500_000.0,  # m
    "color": "red",  # For plotting
}

# Create the inital state of Moon 2
m2 = {
    "mass": 11.0e22,  # kg
    "position": np.array([0.0, -150_000_000]),  # m
    "velocity": np.array([-45.0, 2.0]),  # m/s
    "radius": 2_000_000.0,  # m
    "color": "blue",  # For plotting
}

# Calculate the initial position and velocity of the center of mass
tm = m1["mass"] + m2["mass"]  # Total mass
cm_position = (m1["mass"] * m1["position"] + m2["mass"] * m2["position"]) / tm
cm_velocity = (m1["mass"] * m1["velocity"] + m2["mass"] * m2["velocity"]) / tm

# Start at time zero seconds
current_time = 0.0

# Create the figure and axis
fig, ax = plt.subplots(1, figsize=(7.2, 10))

# Set up the axes
ax.set_xlabel("x (m)")
ax.set_xlim((-1.2e8, 4e8))
ax.set_ylabel("y (m)")
ax.set_ylim((-1.6e8, 5.5e8))
ax.set_aspect("equal", adjustable="box")
fig.tight_layout()
```

```
# Create artists that will be edited in animation
time_text = ax.add_artist(Text(0.03, 0.95, "", transform=ax.transAxes))
circle1 = ax.add_artist(Circle((0, 0), radius=m1["radius"], color=m1["color"]))
circle2 = ax.add_artist(Circle((0, 0), radius=m2["radius"], color=m2["color"]))
circle_cm = ax.add_artist(Circle((0, 0), radius=m2["radius"], color="purple"))
varrow1 = ax.add_artist(FancyArrow(0, 0, 0, 0, color="green", head_width=m1["radius"]))
varrow2 = ax.add_artist(FancyArrow(0, 0, 0, 0, color="green", head_width=m2["radius"]))
acc_arrow1 = ax.add_artist(
    FancyArrow(0, 0, 0, 0, color="purple", head_width=m1["radius"])
)
acc_arrow2 = ax.add_artist(
    FancyArrow(0, 0, 0, 0, color="purple", head_width=m2["radius"])
)


# This function will get called for every frame
def animate(frame):

    # Global variables needed in scope from the model
    global cm_position, cm_velocity, current_time, m1, m2

    # Global variables needed in scope from the artists
    global time_text, varrow1, varrow2, acc_arrow1, acc_arrow2, circle1, circle2, circle_cm

    print(f"Updating artists for day {current_time/SEC_PER_DAY:.1f}.")

    # Update the positions based on the current velocities
    m1["position"] = m1["position"] + m1["velocity"] * TIME_STEP
    m2["position"] = m2["position"] + m2["velocity"] * TIME_STEP

    # Update day label
    time_text.set_text(f"Day {current_time/SEC_PER_DAY:.0f}")

    # Update positions of circles
    circle1.set_center(m1["position"])
    circle2.set_center(m2["position"])

    # Update velocity arrows
    varrow1.set_data(
        x=m1["position"][0],
        y=m1["position"][1],
        dx=VSCALE * m1["velocity"][0],
        dy=VSCALE * m1["velocity"][1],
    )
    varrow2.set_data(
        x=m2["position"][0],
        y=m2["position"][1],
        dx=VSCALE * m2["velocity"][0],
        dy=VSCALE * m2["velocity"][1],
    )

    # Update the center of mass
```

```
cm_position = cm_position + cm_velocity * TIME_STEP
circle_cm.set_center(cm_position)

# Find the vector from moon1 to moon2
delta = m2["position"] - m1["position"]

# What is the distance between the moons?
distance = np.linalg.norm(delta)

# Have the moons collided?
if distance < m1["radius"] + m2["radius"]:
    print(f"*** Collided {current_time:.1f} seconds in!")

# What is a unit vector that points from moon1 toward moon2?
direction = delta / distance

# Calculate the magnitude of the gravitational attraction
magnitude = G * m1["mass"] * m2["mass"] / (distance**2)

# Acceleration vector of moons (a = f/m)
acceleration1 = direction * magnitude / m1["mass"]
acceleration2 = (-1 * direction) * magnitude / m2["mass"]

# Update the acceleration arrows
acc_arrow1.set_data(
    x=m1["position"][0],
    y=m1["position"][1],
    dx=ASCALE * acceleration1[0],
    dy=ASCALE * acceleration1[1],
)
acc_arrow2.set_data(
    x=m2["position"][0],
    y=m2["position"][1],
    dx=ASCALE * acceleration2[0],
    dy=ASCALE * acceleration2[1],
)

# Update the velocity vectors
m1["velocity"] = m1["velocity"] + acceleration1 * TIME_STEP
m2["velocity"] = m2["velocity"] + acceleration2 * TIME_STEP

# Update the clock
current_time += TIME_STEP

# Return the artists that need to be redrawn
return (
    time_text,
    varrow1,
    varrow2,
    acc_arrow1,
    acc_arrow2,
    circle1,
```

```
        circle2,
        circle_cm,
    )


# Make the rendering happen
animation = FuncAnimation(
    fig,
    animate,
    np.arange(FRAMECOUNT),
    interval=ANI_INTERVAL
)

# Save the rendering to a video file
animation.save("moonmovie.mp4")
```

When you run this, it will take longer than the previous versions. You should have a video file that shows a simulation of the moons tracing their elliptical paths around their center of mass:

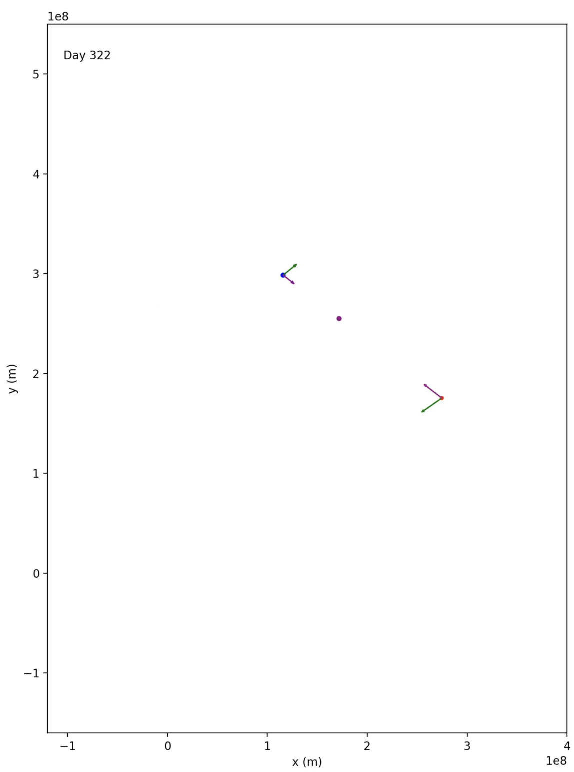


Figure 3.5: A still of the animation produced.

3.7 Challenge: The Three-Body Problem

It is time to stretch a little as a physicist and programmer: You are going to make a new version of `moons.py` that handles three moons instead of just two.

This is known as ”The Three-Body Problem”, and people have tried for centuries to come up with a way to figure out (from the initial conditions) where the three moons would be at time t without doing a simulation. And no one has.

For a lot of problems, the outcome is not very sensitive to the initial conditions. For example, consider the flight of a cannonball: If it leaves the muzzle of the cannon a little faster, it will go a little farther.

For the three-body problem, the outcome can be radically different even if the initial conditions are very similar.

(There is a whole field of mathematics studying systems that are very sensitive to initial conditions. It is known as *dynamical systems* or *chaos theory*.)

Copy `moons.py` to `3moons.py`. Here is a reasonable initial state for your third moon:

```
m3 = {
    "mass": 4.0e22,  # kg
    "position": np.array([50_000_000, 80_000_000]),  # m
    "velocity": np.array([-30.0, -35.0]),  # m/s
    "radius": 1_700_000.0,  # m
    "color": "green"
}
```

If we run that simulation for 100 days, we get a plot like this:

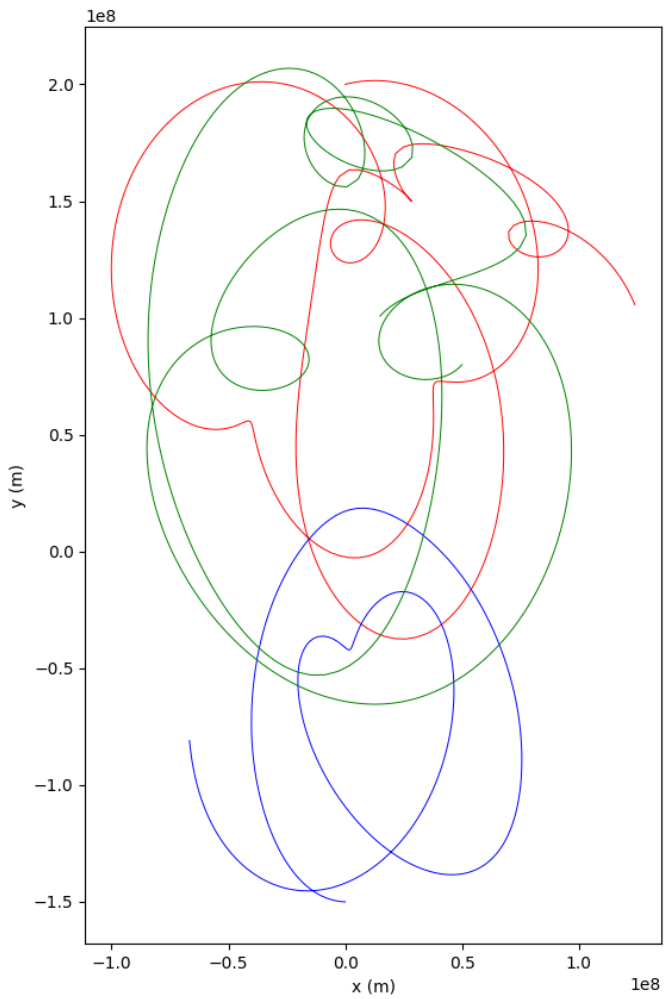


Figure 3.6: The Three Body Problem.

Visibly, you can see this is very different from the two-body problem that just traced ellipses around the center of mass.

Chapter 4

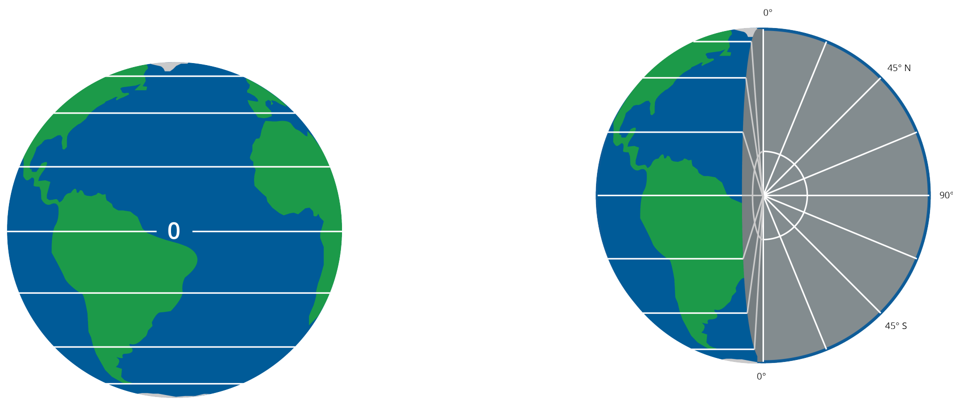
Longitude and Latitude

4.1 Longitude and Latitude

The Earth can be represented as a sphere, and the position of a point on its surface can be described using two coordinates: latitude and longitude.

Figure 4.1: A diagram of latitude and longitude.

Latitude is a measure of a point's distance north or south of the equator, expressed in degrees. It ranges from $-90°$ at the South Pole to $+90°$ at the North Pole, with $0°$ representing the Equator. (See figures 4.2a and 4.2b)

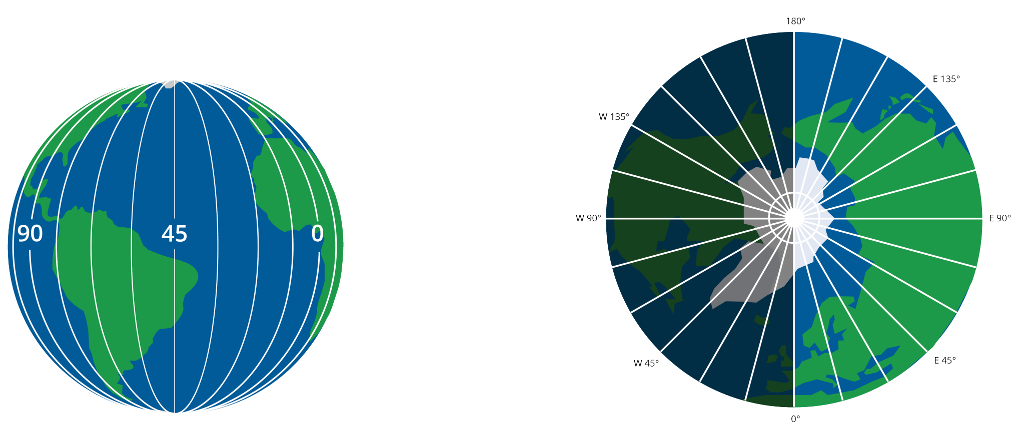


(a) Latitude drawn on the earth.

(b) A cross-section of the earth showing latitude.

Figure 4.2: Latitude illustrations

Longitude, on the other hand, measures a point's distance east or west of the Prime Meridian (which passes through Greenwich, England). It ranges from $-180°$ to $+180°$, with the Prime Meridian represented as $0°$. (See figures 4.3a and 4.3b)



(a) Longitudinal lines drawn on the earth.

(b) A cross-section of the earth showing longitude.

Figure 4.3: Longitude illustrations.

The coordinates follow a system of latitude and then longitude (in the majority of contexts). Here is the Longitude and Latitude of four different large cities:

- **London, England**: 51° N 0° W
- **New York City, New York, USA**: 40° N 74° W

- **Sydney, New South Whales, Australia**: 33° S 150° E
- **Rio de Janerio, Brazil**: 22° S 43° W

These coordinates may be broken down further into minute and second components as well, for futher geometric accuracy. Each degree (°) is divided into 60 minutes (′). Each minute (′) is divided into 60 seconds ("). Just like in time. Note some do not

- **London, England**: 51° 30‘ 26” N 0° 7‘ 39” W
- **New York City, New York, USA**: 40° 42‘ 46” N 74° 0‘ 22” W
- **Sydney, New South Whales, Australia**: 33° 52‘ 0” S 150° 12‘ 0” E
- **Rio de Janerio, Brazil**: 22° 54‘ 0” S 43° 12‘ 0” W

4.2 Nautical Mile

A nautical mile is a unit of measurement used primarily in aviation and maritime contexts. It is based on the circumference of the Earth, and is defined as one minute (1/60°) of latitude; in other words, **one minute of latitude is equal to 1 nautical mile.** This makes it directly related to the Earth's geometry, unlike a kilometer or a mile, which are arbitrary in nature. The exact value of a nautical mile can vary slightly depending on which type of latitude you use (e.g., geodetic, geocentric, etc.), but for practical purposes, it is often approximated as 1.852 kilometers or 1.15078 statute miles.

4.3 Haversine Formula

The haversine formula is an important equation in navigation for giving great-circle distances between two points on a sphere from their longitudes and latitudes. It is especially useful when it comes to calculating distances between points on the surface of the Earth, which we represent as a sphere for simplicity. See Figure 4.4

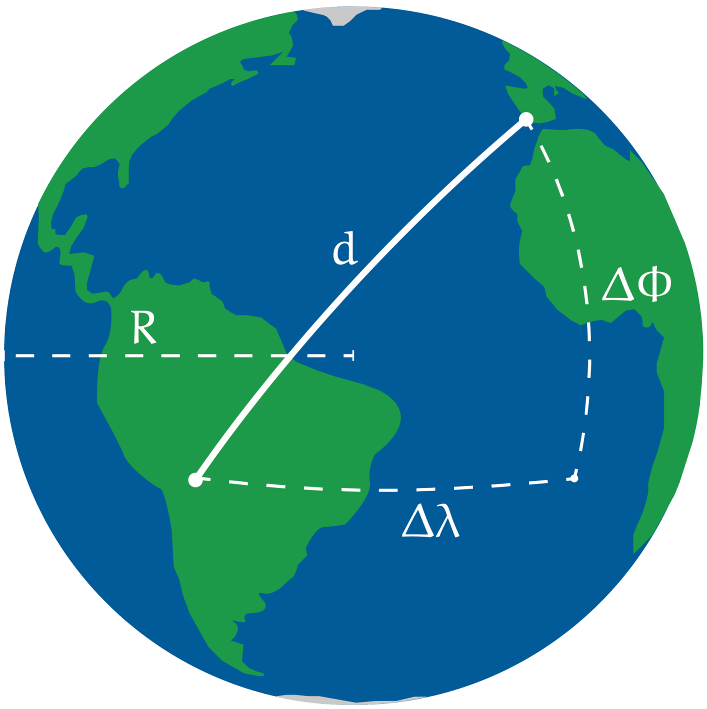


Figure 4.4: A diagram of the haversine formula.

In its basic form, the haversine formula is as follows:

The angle found using haversine haversine$(\theta) = sin^2\left(\frac{\theta}{2}\right)$:

$$a = \sin^2\left(\frac{\Delta\phi}{2}\right) + \cos(\phi_1)\cos(\phi_2)\sin^2\left(\frac{\Delta\lambda}{2}\right)$$

Compute c which determines the angular distance using $atan2$, a computer science tangential function with 2 arguments equivalent to $2\arcsin(\sqrt{a})$[1]:

$$c = 2 \cdot \text{atan2}\left(\sqrt{a}, \sqrt{1-a}\right)$$

Find d, the true distance along the sphere:

[1]Note that this function is not available on calculators, but commonly used in computer science programs. You may use the singular argument equivalent in calculations.

$$d = R \cdot c$$

In one line, this is:

$$d = 2R \cdot \arcsin\left(\sqrt{\sin^2\left(\frac{\Delta\phi}{2}\right) + \cos(\phi_1)\cos(\phi_2)\sin^2\left(\frac{\Delta\lambda}{2}\right)}\right)$$

Here, ϕ_1 and ϕ_2 represent the latitudes of the two points (in radians), $\Delta\phi$ and $\Delta\lambda$ represent the differences in latitude and longitude (also in radians), and R is the radius of the Earth (using whatever unit of distance you desire). The result, d, is the distance between the two points along the surface of the sphere. Notice that d is equal to the formula for the *arc length* of a sector. A sphere is just a bunch of circles with radius r, so it follows that we can use formulas for circles in our spherical calculations.

This is why planes fly arc-shaped paths rather than straight paths from points $A \rightarrow B$. Planes follow the shortest path between two points on a sphere, called a *great-circle route*. On a map, like plane flight-progress maps or the Mercator projection, the straight path looks very arc shaped, but this is just a geographical illusion. This straight distance can be calculated using the haversine formula, ultiamtely minimizing distance and fuel using the 'as-the-crow-flies' path.

Chapter 5

Tides and Eclipses

Your life on earth involves many orbital paths:

- The Earth is spinning. If you are standing at the equator, you are traveling at 1,674 km per hour around the center of the planet. We are all spinning east, which is why the sun comes up in the east and sets in the west.
- The Earth is orbiting the sun. It takes 365.242 days for the Earth to go once around the sun. This is why different constellations appear at different times during the year — we only see the stars at night and the direction of night shifts as the Earth moves around the sun.
- The moon is orbiting the Earth. The moon travels once around the Earth once every 27.3 days.

You can see the effects of these orbits on our planet. Let's go over a few.

5.1 Leap Years

Note that it takes 365.242 days for the Earth to go around the sun. If we declared "The calendar will *always* be 365 days per year!" then the seasons would gradually shift by 0.242 days every year. After a century, they would have migrated 24 days.

So, we made a rule: "Every fourth year, we will add an extra day to the calendar!" The years 2021, 2022,and 2023 get no February 29th, but 2024 does.

That got us a calendar with an average 365.25 days per year, so the seasons would not have migrated as quickly, but they still would have migrated about three days every four hundred years.

So, we made another rule: 'There will be no February 29th in the three century years (multiples of 100) that are not multiples of 400.' So the year 1900 had no Feb 29, but the year 2000 had one. Now, the average number of days per year is 365.2425.

5.2 Phases of the Moon

The Earth, the moon, and the sun form a triangle. If you were standing on the moon, you could measure the angle between the light coming from the sun and the the light going to the Earth. That angle would fluctuate between 0 degrees and 180 degrees.

- When the angle was close to 0, the people on Earth would see a full moon (fully illuminated disc).
- When the angle was close to 90 degrees, the people on Earth would see a half moon.
- When the angle was nearing 180 degrees, the people on Earth would see a slim crescent.
- When the angle was very close to 180 degrees, the moon would be dark. This is called a new moon (fully darked disc).

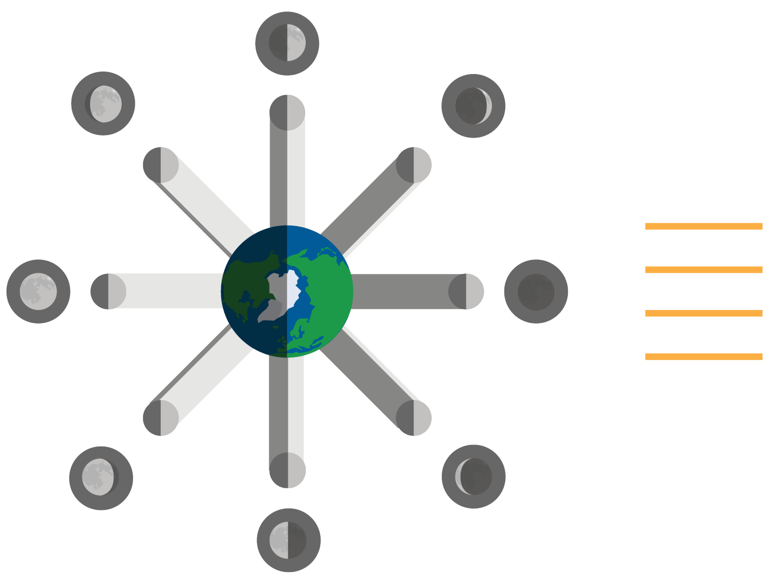


Figure 5.1: The moon revolves around the Earth in different phases every month.

Even though it takes 27.3 days for the moon to travel around the Earth once, it takes 29.5 days to get from one full moon to the next. Why? In the 27.3 days that it took the moon to travel around the Earth, the Earth has moved about 17 degrees around the sun. To get back into the same triangle configuration takes another 2.2 days.

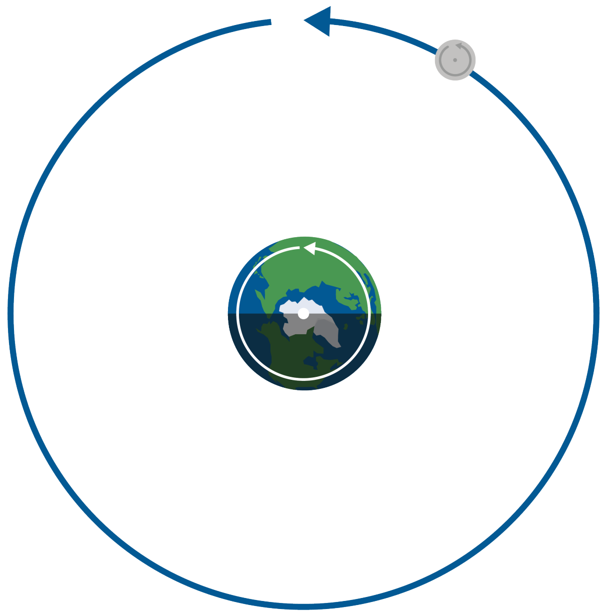


Figure 5.2: The moon and Earth rotate at the same time. Both revolve around its own axis at the same time.

Figure 5.3: A half moon circumstance in space.

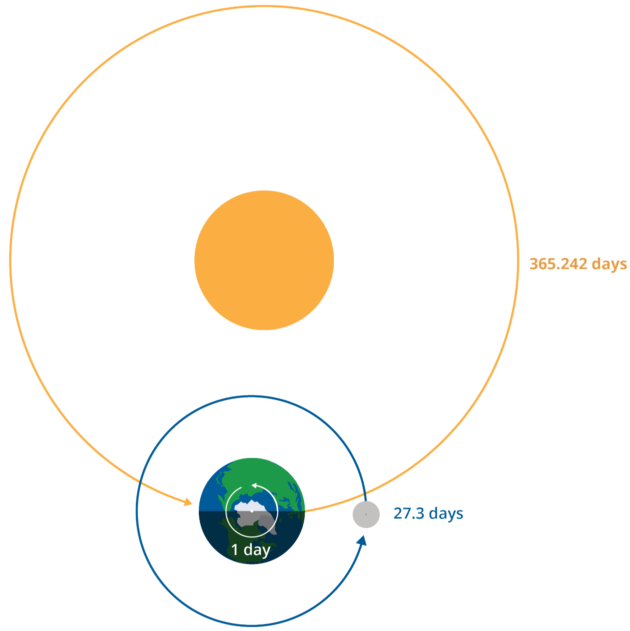


Figure 5.4: The earth, sun, and moon schedules of full rotations.

To explain why we often see a curve in the shadow of the moon, we can look at a ball that has one side painted yellow and the other red.

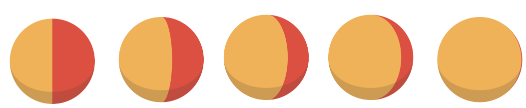


Figure 5.5: A rotating representation of the moon's phases.

As we rotate the ball, we can see that the straight color boundary between each hemisphere begins to look curved. The curve we see in the moon is due to this same basic principle of how the shading of spheres works.

FIXME: Add text about scale In all of these graphics, we have been using incorrect scale. Here is the true scale of the distance of the Earth and the moon with accurate radii:

Figure 5.6: The moon and Earth in their respective scale.

FIXME: Add text about tidal lock

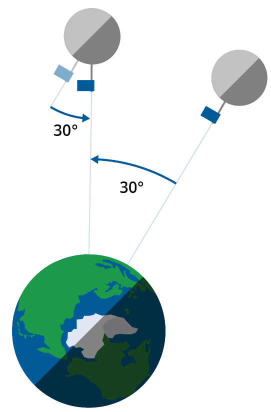


Figure 5.7: FIXME.

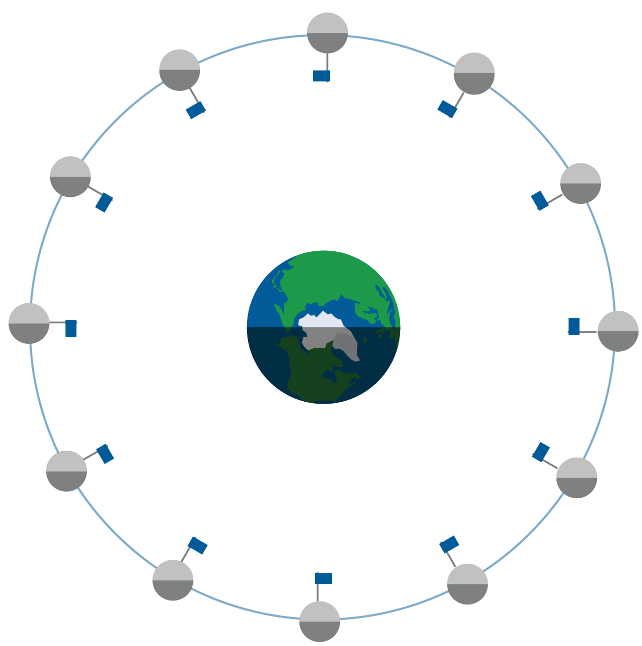


Figure 5.8: FIXME.

5.3 Eclipses

While the Earth orbits the sun and the moon orbits the Earth, the two orbits are *not in the same plane*. We call the plane that the Earth orbits the sun in the *ecliptic plane*. The plane of the moon's orbit is about 5° tilted from the ecliptic plane.

Note that the moon passes through the ecliptic plane only twice every 27.3 days. Imagine that the instant it passed through the ecliptic plane was also the precise instant of a full moon. The sun, the Earth, and the moon would be in a straight line! The earth would cast a shadow upon the moon — it would go from a bright full moon to a dark moon until the moon moved back out of the shadow of the earth. This is known as a *lunar eclipse*.

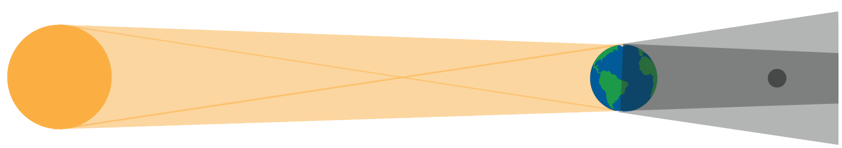


Figure 5.9: Lunar Eclipses happen when the Earth comes between the moon and the sun, causing the moon to be darkened by the Earth's shadow.

The diameter of the moon is a little more than a quarter the diameter of the Earth, so they don't have to be in perfect alignment for the moon to be darkened. Lunar eclipses actually happen once or twice per year.

Now, imagine that the instant the moon passed through the ecliptic plane was also the precise instant of a new moon. The sun, the moon, and the Earth would be in a straight line! The moon would cast a shadow upon some part of the Earth. To a person in that shadow, the sun disappear behind the moon. This is known as a *solar eclipse*.

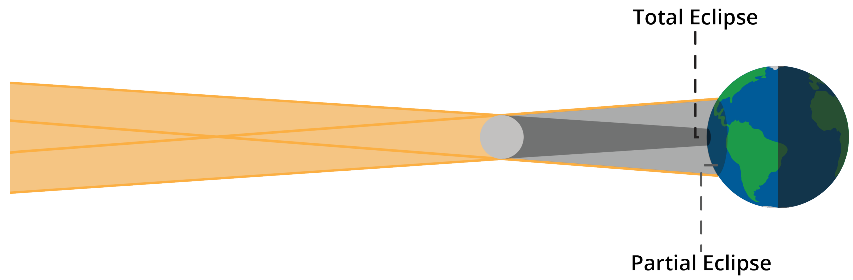


Figure 5.10: Partial and total solar eclipses.

The sun is pretty big, so if the moon blots out just part of it, we call it a *partial solar eclipse*, as seen in Figure 5.10. There are a few partial solar eclipses every year. Note that because the moon's shadow is too small to shade the whole Earth, only certain parts of the world will experience any solar eclipses.

Every 18 months or so, there is a total eclipse of the sun. Once again, only certain parts of the world experience it. You can expect to experience a total eclipse of the sun at your home about once every 375 years.

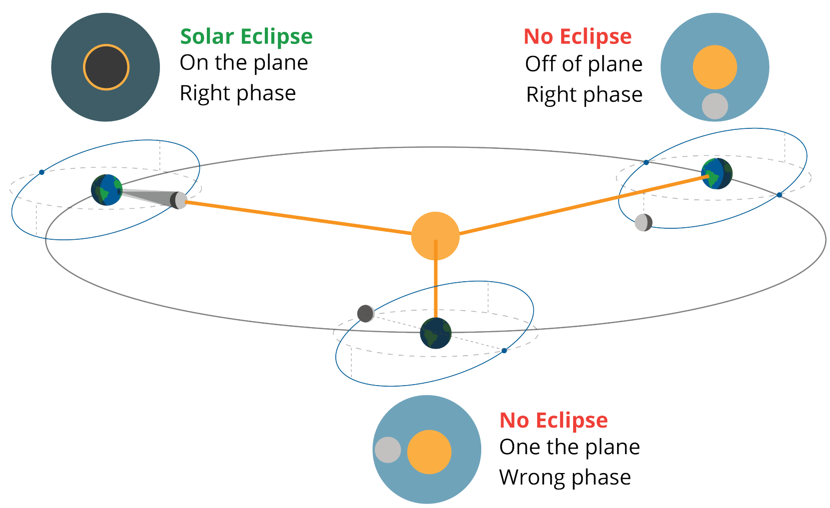


Figure 5.11: Different options for an eclipse.

5.4 The Far Side of the Moon

Like the earth, the moon spins on its axis. Due to Earth's gravity, the rotation of the moon slowed down until its spin matched the rate it orbits Earth. That is: we are always looking at the same side of the moon. Until we orbited the moon, we had no idea what the far side looked like.

Some people call it “The Dark Side of the Moon”, but it gets just as much sunshine as the side that faces Earth. The name comes from the fact that we lose communication with spacecraft (like the Apollo missions to the moon) when they are on the far side of the moon. When we lose communications with a craft, we often say “It went dark”.

5.5 Tides

When we say “The moon orbits the Earth”, that is a bit of an oversimplification. The force of gravity that pulls the moon toward the Earth, also pulls the Earth toward the moon. The Earth is about 81 times heavier than the moon, so the moon moves more, but the moon definitely moves the Earth.

The center of the moon and the center of the Earth rotate around each other. The point they rotate around is inside the the Earth, but it is closer to the surface of the earth than it is to the center of the Earth. This is referred to as a *barycenter*.

Orbits happen, remember, when the centripetal force is equal to gravitational force. So the centripetal force created by the Earth being swung by the moon is equal to the gravitational force that the moon exerts on all the mass on the moon.

However.

The parts of the Earth that are closer to the moon experience less centripetal force (away from the moon) and more gravitational force (toward the moon).

The parts of the Earth that are farther from the moon experience more centripetal force (away from the moon) and less gravitational force (away from the moon).

The effects are not big. For example, you won’t notice that you can jump higher when the moon is overhead. You will lose only about 1/200,000 of your weight.

But the ocean is huge: 1/200,000 of its weight is a lot of force.

The water in the oceans bulges a little both toward the moon and away from it.

The Earth is still rotating. If you are at the beach as your longitude slides into one of these bulges, you say “Hey, the tide is rising!” The peak of these bulges is known as “*high tide*”. Because there is a bulge on each side of the planet, high tide comes twice a day.

This is a *lunar tide* – because it is caused by the moon. There is a similar effect from the sun, but the sun is very, very far away: solar tidal forces are about half as powerful lunar tidal forces. When the sun and the moon work together, the tides are stronger. This is called a *spring tide*. Spring tides don’t happen in the spring time; they happen close to full moons and new moons.

When the moon and the sun are working against each other, the tides are weaker. This is called a *neap tide*. Neap tides happen when you see a half moon in the sky.

5.5.1 Computing the Forces

We are enumerating several forces that shape the water on the planet. All these forces are pulling on your body too. In these exercises, you are going to calculate how each force would effect a 1 kg mass on the surface of the earth.

Here are some numbers you will need:

- The mass of the Earth: 5.97219×10^{24} kg
- The mass of the sun: 1.9891×10^{30} kg
- The mass of the moon: 7.347673×10^{22} kg
- Radius of the Earth at the equator: $6,371$ km
- Average distance from the center of the Earth to the center of the sun: 149.6×10^{6} km
- Average distance from the center of the moon to the center of the Earth: $384,467$ km.

Exercise 9 Life Among the Orbits 1: Earth Gravity

Working Space

If the Earth were still and alone in the universe, there would still be the force of gravity. We have said that that a kilogram on the surface of the Earth is pulled toward the center of the earth with a force of 9.8 N.

Confirm that the gravity of the Earth pulls a 1kg mass on the surface of the planet with a force of about 9.8 N.

You will need the formula for gravitation:

$$F_g = \frac{g m_1 m_2}{r^2}$$

If we measure distance in km and mass in kg, the gravitation constant g is 6.67430×10^{-17}.

Answer on Page 70

Exercise 10 Life Among the Orbits 2: Earth Centripetal Force

Working Space

What if we add the spinning of the Earth? The spinning would try to throw the kg into space. The formula for centripetal force is

$$F_c = \frac{mv^2}{r}$$

Calculate the centripetal force on a 1 kg mass on the surface of the earth. It doesn't fly off into space, so the force due to gravity must be bigger. How many times bigger?

Assume that the mass is on the equator, thus rotating around the Earth at 465 m/s.

Does the centripetal force increase, decrease, or stay the same as you get closer to the north pole?

Answer on Page 70

Exercise 11 Life Among the Orbits 3: The Moon's Gravity

Working Space

Now we add the moon's gravitational force to our model.

When the moon is directly overhead, how strongly will it pull at the 1 kg mass on the equator?

When the moon is directly underfoot, how strongly will it pull at the 1 kg mass on the equator?

Is that a big difference?

Answer on Page 70

Exercise 12 Life Among the Orbits 4: The Swing of the Moon

Working Space

Now we add the moon's motion. The moon and the Earth swing each other around. This creates a centripetal force. They both travel in nearly a circle centered at their center of mass.

How far is the center of mass of the moon and the Earth from the center of the Earth? (You can imagine a see-saw with the center of the Earth on one end and the center of the moon on the other. Where would the balance point be?)

What point on the surface of the Earth is closest to the center of mass? How far is it?

What point on the surface of the Earth is farthest from the center of mass? How far is it?

Answer on Page 71

Exercise 13 Life Among the Orbits 5: Lunar Centripetal Force

Working Space

The moon swings us around that center of mass once every 27.3 days. (Forget about the spinning of the Earth for this part.) What is the largest and smallest centripetal forces on the surface of the Earth created by this swinging

What is the largest centripetal force on a 1 kg mass with the moon directly underfoot? (You need an answer from the previous question: There is a point on the surface of the Earth that is 11,044,000 m from the center of gravity.)

What is the resulting centripetal force on a 1 kg mass with the moon directly overhead? (You will need the other answer from the previous exercise: That point is 1,698,000 m from the center of mass of the moon and the Earth.)

For this problem is probably easier to use this formula for centripetal force:

$$F_c = mr\omega^2$$

Where m is mass in kg, r is radius in m, and ω is the angular velocity in radians per second.

Answer on Page 72

Exercise 14 Life Among the Orbits 6: Net Force

Working Space

Now add together the two forces at both the nearest point to the moon and the farthest.

Answer on Page 72

5.5.2 Solar Tidal Forces

The sun has a much larger gravitational effect on the Earth than the moon does:

- When the sun is overhead, it will pull on a 1 kg mass with a force of about 0.00593 N.
- When the moon is overhead, it will pull on a 1 kg mass with a force of about 0.0000343 N.

Why are lunar tides about twice as powerful solar tides?

Tides occur because the pull of gravity and the pull of the centripetal force are out of balance somewhere on the planet. The sun is so far away that the effects of gravitational and centripetal forces are very close to equal everywhere on Earth.

Appendix A

Answers to Exercises

Answer to Exercise 1 (on page 6)

The torque on the door is equal to $rF \sin\theta$. Since our force is tangential, the sin component is at its max: $\sin 90 = 1$, our torque simplifies to $\tau = rF = 0.5 \cdot 38 = 19$ Newton-Meters.

Answer to Exercise 2 (on page 7)

1. Using $\tau = rF \sin\theta$, we can state that the torque is $\tau = (2.0)(10)\sin(30^\circ) = 20(0.5) = 10$ N·m.
2. Because The force has an upward component at the right end, so the beam tends to rotate counterclockwise.

Answer to Exercise 3 (on page 9)

1. Since the beam center is 2.0 m from the left end, and the pivot is at 1.0 m from the left end, so the weight is $W = mg = 20(9.8)$,

   $$W = 196 \text{ N}, \quad \text{acting } 1.0 \text{ m right of pivot}$$

2. We can choose our own torque direction, so let's set counterclockwise be positive. The left end is 1.0 m to the left of the pivot and right end is 3.0 m to the right of the pivot. The beam's weight acts 1.0 m to the right of the pivot.

   Since there are no angles and all pivots forces are perpendicular, there are three torques:

   $$\tau_{50} = (1.0)(50) = +50 \text{ N} \cdot \text{m} \qquad \text{CCW}$$
   $$\tau_W = -(1.0)(196) = -196 \text{ N} \cdot \text{m} \qquad \text{CW}$$
   $$\tau_F = -(3.0)(F) \text{ N} \cdot \text{m} \qquad \text{CW}$$

   So equilibrium is solved by:

   $$50 - 196 - 3F = 0$$

3.

$$\begin{aligned}50 - 196 - 3F &= 0\\ -146 - 3F &= 0\\ -3F &= 146\\ F &= -48.7\ \text{N}\end{aligned}$$

The negative informs us that the torques act opposite the direction we assumed, so the force acts $F = 48.7$ N upward at the right end.

Answer to Exercise 4 (on page 10)

You can pick your own pivot, so let's choose the point that A and B stem from as our pivot. This eliminates those forces, as the r component of the torque would equal zero. Recall that the net torque is $\sum \tau = \tau_{CCW} - \tau_{CW}$. Analyzing the remaining torques:

- C is 1 m from the pivot, so the torque is $\tau_C = (1)C \sin -90°$. It acts downwards, so our torque is clockwise.
- D acts directly parallel the rotating rod, adding no additional torque on the rod.
- E is 2 m from the pivot, and acting at a $-135°$ angle from the horizontal. Our torque can be calculated from $\tau_E = (2)E \sin -135°$. Note that usign $-45°$ provides the same answer. This torque is also clockwise.

Our net torque is $C \sin -90° + (2)E \sin -135° = 0$

Answer to Exercise 5 (on page 11)

Using Equation 1.8, we can rearrange to solve for α as $\alpha = \frac{rF \sin \theta}{I}$.

$$\alpha = \frac{rF \sin \theta}{I} = \frac{0.40 \cdot 25 \sin 60}{0.8} \approx 10.825 \text{ rad / s}^2$$

Answer to Exercise ?? (on page 12)

We choose counterclockwise torques to be positive. The net torque is

$$\sum \tau = r_1 F_1 \sin \theta_1 - r_2 F_2 \sin \theta_2$$

Using the rotational form of Newton's second law,

$$\sum \tau = I\alpha$$

Solving for I,

$$I = \frac{r_1 F_1 \sin\theta_1 - r_2 F_2 \sin\theta_2}{\alpha}$$

Substituting values,

$$I = \frac{(0.50)(30)\sin 90^\circ - (0.30)(20)\sin 45^\circ}{8.0}$$

$$I = \frac{15 - 4.24}{8.0} \approx 1.35\ \text{kg·m}^2$$

Answer to Exercise 7 (on page 13)

$$\begin{aligned}
I_{end} &= I_{CM} + MD^2 \\
&= \frac{1}{2}ML^2 + M\left(\frac{L}{2}\right)^2 \\
&= \frac{1}{2}ML^2 + M\left(\frac{L^2}{4}\right) \\
&= \left(\frac{1}{12} + \frac{1}{4}\right)ML^2 \\
&= \left(\frac{4}{12}\right)ML^2 \\
&= \frac{1}{3}ML^2
\end{aligned}$$

Answer to Exercise 8 (on page 18)

As we can see, each unit square is equivalent to 1 torque · time. Counting the partial and full squares on $t = [2, 6]$, we get approximately 8 unit squares. This comes from the (approximately whole) 4 squares under $y = 2$, combining all squares from $y = [2, 4]$ to form 2 squares, and the additional partial area forming approximately another 2 squares, resulting in an angular impulse of approximately 8 $\text{N} \cdot \text{m} \cdot \text{s}^{-1}$.

This is a practice in simple integration, which is a method of finding areas under given functions. Integrating the given function from $x = [0, 8]$ gives us $7.51988482389 \ \text{N} \cdot \text{m} \cdot \text{s}^{-1}$, a value pretty close to our estimate.

Answer to Exercise 9 (on page 60)

The Earth and 1 kg on the surface would attract each other with a force of:

$$F_g = \frac{\left(6.67430 \times 10^{-17}\right)\left(5.97219 \times 10^{24}\right)(1)}{6,371^2} = \frac{3.98583 \times 10^8}{4.0590 \times 10^7} = 9.7987 \text{ N}$$

Thus, if the Earth were still and alone in the universe, the oceans would form a perfect sphere.

Answer to Exercise 10 (on page 61)

$$F_c = \frac{(1)(465)^2}{6,371,000} = 0.03373 \text{ N}$$

So the spinning of the Earth is trying to throw you into space, but the force of gravity is about 289 times more powerful.

This centripetal force decreases as you move from the equator to the north pole. In fact, at the north pole, there is no centripetal force. Thus, the spinning of the Earth makes the oceans an oblate ellipsoid instead of a perfect sphere: the diameter going from pole-to-pole is shorter than a diameter measured at the equator.

You should feel a teensy-tiny bit lighter on your feet at the equator than you do at the north pole: 0.34% lighter.

Answer to Exercise 11 (on page 62)

Overhead, the moon is $384,467 - 6,371 = 378,096$ km from your 1 kg mass.

$$F_g = \frac{g m_1 m_2}{r^2} = \frac{\left(6.67430 \times 10^{-17}\right)\left(7.347673 \times 10^{22}\right)(1)}{378,096^2} = \frac{4.9040574 \times 10^6}{1.42956585216 \times 10^{11}} = 3.43058 \times 10^{-5}\ \text{N}$$

This is a very small force: The force due to Earth's gravity is nearly three hundred thousand times stronger.

Underfoot, the moon is $384,467 + 6,371 = 390,838$

$$F_g = \frac{g m_1 m_2}{r^2} = \frac{\left(6.67430 \times 10^{-17}\right)\left(7.347673 \times 10^{22}\right)(1)}{390,838^2} = \frac{4.9040574 \times 10^6}{1.52754 \times 10^{11}} = 3.2103 \times 10^{-5}\ \text{N}$$

The force due to the moon's gravity is about 6% stronger when the the moon is overhead than when it is underfoot.

Answer to Exercise 12 (on page 63)

If we let r be the distance (in km) from the center of the Earth to the center of mass, the distance from the center of the mass to the center of the moon is $384,467 - r$.

To find the balance point, multiply each mass by how far it is from the center of mass:

$$\left(5.97219 \times 10^{24}\right) r = \left(7.347673 \times 10^{22}\right)(384,467 - r)$$

Solving for r:

$$r = \frac{4,730.15}{1 + 0.0123} = 4,673 \text{ km}$$

The point on the Earth closest to this? It is where the moon is directly overhead. The it is $6,371 - 4,673 = 1,698$ km from the center of mass.

The point on the Earth farthest from this? It is where the moon is directly underfoot. The it is $6,371 + 4,673 = 11,044$ km from the center of mass.

Answer to Exercise 13 (on page 64)

First, lets figure out ω. It travels through 2π radians in 27.3 days. 27.3 days = 2,358,720 seconds. $\omega = \frac{2\pi}{2{,}358{,}720} = 2.663811435 \times 10^{-6}$

$$F_c = (1)(11,044,000)(2.663811435 \times 10^{-6})^2 = 7.8365 \times 10^{-5}$$

Now the weakest:

$$F_c = (1)(1,698,000)(2.663811435 \times 10^{-6})^2 = 1.20512 \times 10^{-5}$$

Answer to Exercise 14 (on page 65)

Closest to the moon, the gravitational force of the moon and the centripetal forces are in the same direction: toward the moon.

$$F_{total} = 1.20488 \times 10^{-5} + 3.43045 \times 10^{-5} = 4.6356 \times 10^{-5}\ \mathrm{N}$$

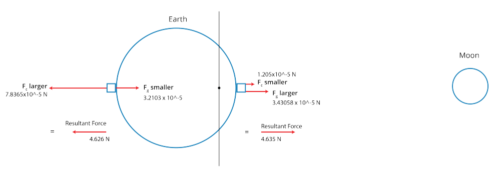
Farthest from the moon, the gravitational force of the moon and the centripetal forces are in opposite directions:

$$F_{total} = 7.8367 \times 10^{-5} - 3.2104 \times 10^{-5} = 4.62604^{-5}\mathrm{N}$$

This is great conclusion: The two forces are basically equal: one pulls the water closest to the moon toward the moon, the other pulls water farthest from the moon away from the moon.

Both forces are pretty small: The force due to Earth's gravity is about $211,000$ times more than either.

And that is why there are two basically equally large high tides every day.



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