Contents

Chapter 1

Alternating Current

We have discussed the voltage and current created by a battery. A battery pushes the electrons in one direction at a constant voltage; this is known as *Direct Current* or DC. A battery typically provides between 1.5 and 9 volts.

The electrical power that comes into your home on wires is different. If you plotted the voltage over time, it would look like this:

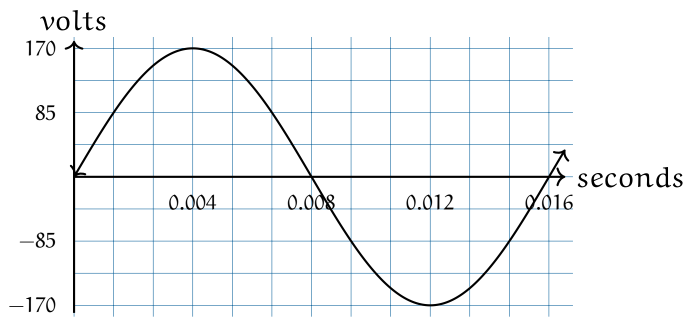


Figure 1.1: A graph of AC voltage over time.

The x axis here represents ground. When you insert a two-prong plug into an outlet, one is "hot" and the other is "ground". Ground represents 0 volts and should be the same voltage as the dirt under the building.

The voltage is a sine wave at 60Hz. Your voltage fluctuates between -170v and 170v. Think for a second what that means: The power company pushes electrons at 170v, then pulls electrons at 170v. It alternates back and forth this way 60 times per second.

1.1 Power of AC

Let's say you turn on your toaster, which has a resistance of 14.4 ohms. How much energy (in watts) does it change from electrical energy to heat? We know that $I = V/R$ and that watts of power are IV. So, given a voltage of V, the toaster is consuming V^2/R watts.

However, V is fluctuating. Let's plot the power the toaster is consuming:

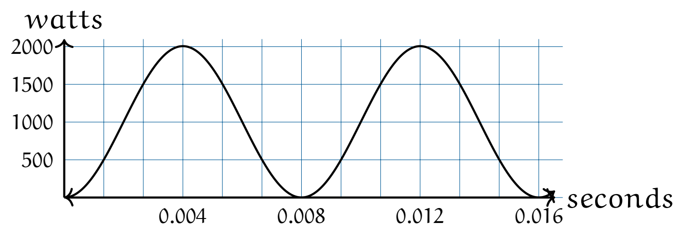


Figure 1.2: A plot of AC power over time.

Another sine wave! Here is a lesser-known trig identity: $(\sin(x))^2 = \frac{1}{2} - \frac{1}{2}\cos(2x)$

This is actually a cosine wave flipped upside down, scaled down by half the peak power, and translated up so that it is never negative. Note that it is also twice the frequency of the voltage sine wave.

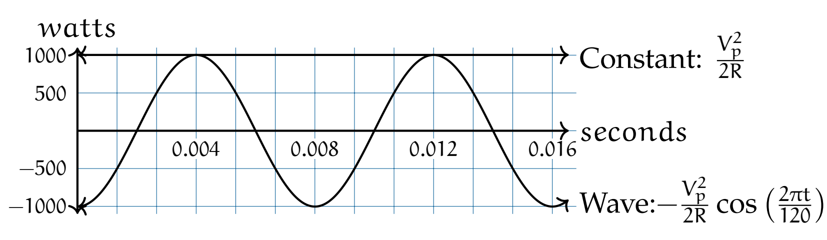
If we say the peak voltage is V_p and the resistance of the toaster is R, the power is given by

$$\frac{V_p^2}{2R} - \frac{V_p^2}{2R}\cos\left(\frac{2\pi t}{120}\right)$$

As a toaster user and as someone who pays a power bill, you are mostly interested in the average power. To get the average power, you take the area under the power graph and divide it by the amount of time.

We can think of the area under the curve as two easy-to-integrate quantities summed:

- A constant function of $y = \mathtt{frac}V_p^2 2R$
- A wave $y = -\frac{V_p^2}{2R}\cos\left(\frac{2\pi t}{120}\right)$



When we integrate that constant function, we get $\frac{tV_p^2}{2R}$.

When we integrate that wave for a complete cycle we get...zero! The positive side of the wave is canceled out by the negative side.

So, the average power is $\frac{V_p^2}{2R}$ watts.

Someone at some point said, "I'm used to power being V^2/R. Can we define a voltage measure for AC power such that this is always true?"

So we started using V_{rms} which is just $\frac{V_p}{\sqrt{2}}$. If you look on the back of anything that plugs into a standard US power outlet, it will say something like "For 120v". What they mean is, "For 120v RMS, we expect the voltage to fluctuate back and forth from 170v to -170v."

Notice that this is the same Root-Mean-Squared that we defined earlier, but now we know that if $y = \sin(x)$, the RMS of y is $1/\sqrt{2} \approx 0.707$.

For current, we do the same thing. If the current is AC, the power consumed by a resistor is $I_{RMS}^2 R$, where I_{RMS} is the peak current divided by $\sqrt{2}$.

1.2 Power Line Losses

A wire has some resistance. Thinner wires tend to have more resistance than thicker ones. Aluminum wires tend to have more resistance than copper wires.

Let's say that the power that comes to your house has to travel 20 km from the generator in a cable that has about 1Ω of resistance per km. Let's also say that your home is consuming 12 kilowatts of power. If that power is 120v RMS from the generator to your home, what percentage of the power is lost heating the power line?

10 amps RMS flow through your home. When that current goes through the wire, $I^2R = (100)(20) = 2000 watts$ is lost to heat. This means the power company would need to supply 14 kilowatts of power, knowing that 2 kilowatts would be lost on the wires.

What if the power company moved the power at 120,000 volts RMS? Now only 0.01 amps RMS flow through your home. When that current goes through the wire $I^2R = (0.0001)(20) = .002$ watts of power are lost on the power lines.

This is much, much more efficient. The only problem is that 120,000 volts would be incredibly dangerous. So the power company moves power long distances at very high voltages, like 765 kV. Before the power is brought into your home, it is converted into a lower voltage using a *transformer*.

1.3 Transformers

A transformer is a device that converts electrical power from one voltage to another. A good transformer is more than 95% efficient. The details of magnetic fields, flux, and inductance are beyond the scope of this chapter, so we are going to give a relatively simple (and admittedly incomplete) explanation for now.

A transformer is a ring with two sets of coils wrapped around it.

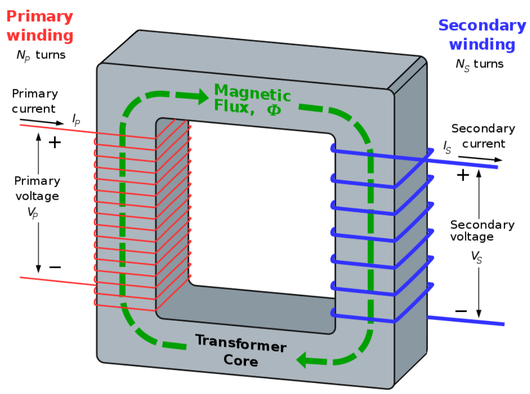


Figure 1.3: A diagram of a transformer from Wikipedia.

When alternating current is run through the primary winding, it creates magnetic flux in the ring. The magnetic flux induces current in the secondary winding. (This is called *induction*.)

If V_P is the voltage across the primary winding and V_S is the voltage across the secondary winding, they are related by the following equation:

$$\frac{V_P}{V_S} = \frac{N_P}{N_S}$$

where N_P and N_S are the number of turns in the primary and secondary windings.

There are usually at least two transformers between you and the very high voltage lines. There are transformers at the substation that make the voltage low enough to travel on regular utility poles. On the utility poles, you will see cans that contain smaller transformers. Those step the voltage down to the 120V RMS that your home uses.

1.4 Phase and 3-phase power

If two waves are "in sync", we say they have the same *phase*.

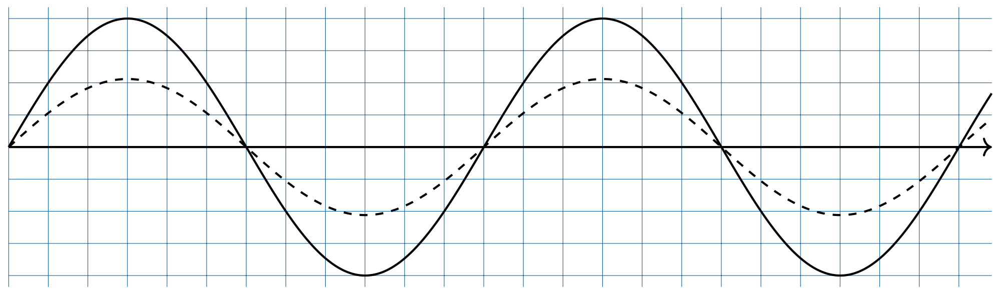


Figure 1.4: A diagram of two AC signals in sync.

If they are the same frequency, but are not in sync, we can talk about the difference in their phase.

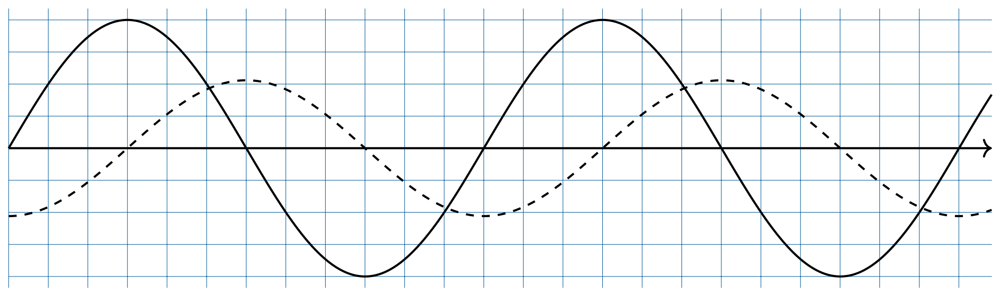


Figure 1.5: A diagram of two AC signals out of phase.

Here, we see that the smaller wave is lagging by $\pi/2$ or $90°$.

In most power grids, there are usually three wires carrying the power. The voltage on each is $2\pi/3$ out of phase with the other two:

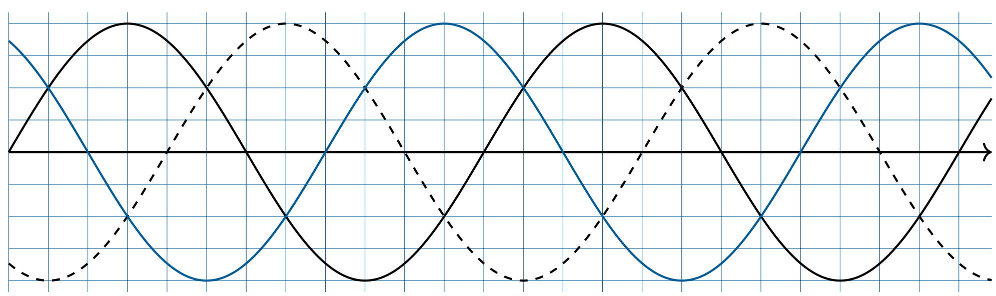


Figure 1.6: A diagram of three AC signals, each $2\pi/3$ radians out of phase.

This is nice in two ways:

- While the power in each wire is fluctuating, the total power is not fluctuating at all.

- While the power plant is pushing and pulling electrons on each wire, the total number number of electrons leaving the load is zero.

(Both these assume that there each wire is attached to a load with the same constant resistance.)

In big industrial factories, you will see all three wires enter the building. Large amounts of smooth power delivery means a great deal to an industrial user.

In residential settings, each home gets its power from one of the three wires. However, two wires typically carry power into the home. Each one carries 120V RMS, but they are out of phase by 180 degrees. Lights and small appliances are connected to one of the wires and ground, so they get 120V RMS. Large appliances, like air conditioners and washing machines, are connected across the two wires, so they get 240V RMS.

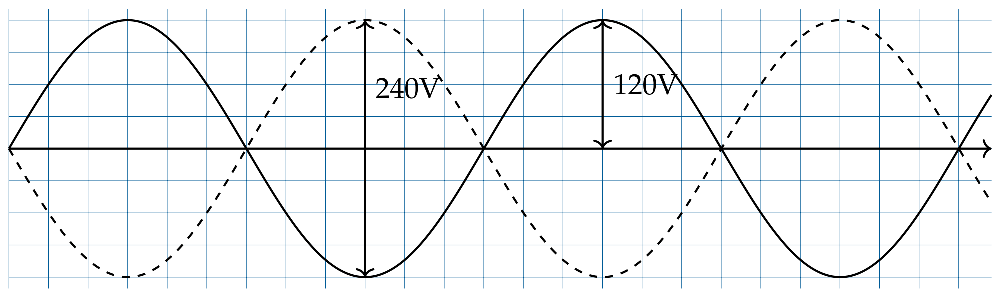


Figure 1.7: A diagram of AC out of phase in homes.

How do you get two circuits, 180 degrees out of phase, from one circuit? Using a center-tap transformer.

FIXME: Diagram here

Chapter 2

Electromagnetic Induction

Electromagnetic induction is a fundamental concept in physics that describes how a changing magnetic field can produce an electric current. The phenomenon was discovered by Michael Faraday in 1831 and is the principle behind many electrical devices, including generators and transformers.

2.1 Faraday's Law of Induction

It is important to note that induction is the process of generating a current in a conductor by **changing** the magnetic field around it. In other words, if a conductor is moving through a **constant** magnetic field, no current will be induced. However, if the magnetic field is changing or if the conductor is moving through a gradient, a current will be induced.

The equation that describes electromagnetic induction is known as *Faraday's Law of Induction*:

$$\mathcal{E} = -\frac{d\Phi_B}{dt}$$

where $\mathcal{E}$ is the induced electromotive force (emf) in volts, and Φ_B is the magnetic flux through the circuit in webers (Wb). The negative sign indicates the direction of the induced emf, as described by Lenz's Law. Note that the induced emf is proportional to the **rate of change** of the magnetic flux. This means that a faster change in the magnetic field will result in a larger induced emf.

2.1.1 Magnetic Flux

Magnetic flux (Φ_B) through a surface is defined as:

$$\Phi_B = \int \vec{B} \cdot d\vec{A}$$

where $\vec{B}$ is the magnetic field and $d\vec{A}$ is an infinitesimal area vector perpendicular to the surface. You'll learn more about vector calculus in later workbooks.

2.2 Lenz's Law

Lenz's Law gives the direction of the induced current. It states that the induced current will flow in such a way that its magnetic field opposes the change in magnetic flux that produced it. This is reflected in the negative sign in Faraday's Law.

2.3 Induction in a Coil

When a coil of wire is placed in a region where the magnetic field changes over time, an emf is induced in the coil according to Faraday's Law. If the coil has N turns, the total induced emf is given by:

$$\mathcal{E} = -N\frac{d\Phi_B}{dt}$$

This means that the induced emf is proportional to both the number of turns in the coil and the rate of change of the magnetic flux through each turn.

If the magnetic field changes uniformly and the area of the coil is constant, the change in flux can be written as:

$$\Delta\Phi_B = \Delta B \times A$$

where A is the area of the coil and ΔB is the change in magnetic field.

Therefore, the induced emf for a coil experiencing a uniform change in magnetic field is:

$$\mathcal{E} = -N\frac{A\Delta B}{\Delta t}$$

This principle is widely used in devices such as electric generators, where coils rotate in a magnetic field to produce electricity.

Exercise 1 Induced EMF in a Coil

A coil with 100 turns and area $0.2\,\text{m}^2$ experiences a change in magnetic field from 0 to $5\,\text{T}$ over $0.1\,\text{s}$. Calculate the induced emf.

Working Space

Answer on Page 63

2.4 Applications of Electromagnetic Induction

- **Electric Generators**: Convert mechanical energy into electrical energy using electromagnetic induction.
- **Transformers**: Change the voltage of alternating current (AC) electricity using induction between coils.
- **Induction Cooktops**: Use rapidly changing magnetic fields to heat cookware directly.

Chapter 3

Electric Motor

Electric motors are devices that convert electrical energy into mechanical energy. They operate based on the interaction between magnetic fields and electric currents, producing rotational motion. As mentioned previously, when an electric current flows through a conductor, it generates a magnetic field around the conductor. If you place a magnet near this conductor, the magnetic field will interact with the magnet's field, causing the conductor to experience a force. This force can be harnessed to create motion, which is the fundamental principle behind electric motors.

3.1 Basic Electric Motor

The most common type of electric motor is the DC (Direct Current) brushed motor. It consists of a coil of wire (armature) that rotates within a magnetic field created by permanent magnets or electromagnets. The armature is connected to a commutator, which reverses the direction of the current in the coil as it rotates, ensuring continuous rotation in one direction. The name ”brushed” refers to the use of brushes that make contact with the commutator to supply current to the armature.

TODO: Can there be a graphic of a basic electric motor? The generator image is similar, but the motor has a power source rather than a load.

Other types of electric motors include brushless DC motors, stepper motors, and AC (Alternating Current) motors. Each type has its own characteristics and applications, but they all operate on the same basic principles of electromagnetism.

3.2 Basic Electric Generator

Electric generators are the reverse of electric motors. They convert mechanical energy into electrical energy by using the principle of *electromagnetic induction*. When a conductor (such as a coil of wire) moves through a magnetic field, it induces an electric current in the conductor. This is covered in another chapter.

To construct a basic electric generator, all you need is an electric motor and a load.

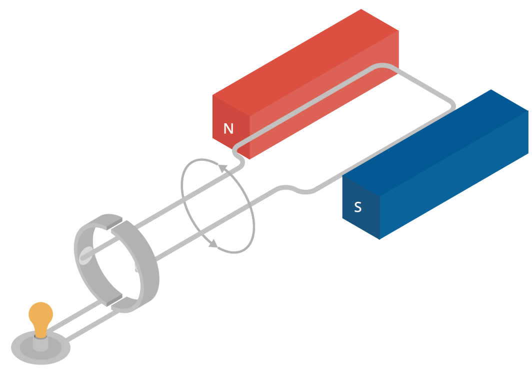


Figure 3.1: Basic Electric Generator Diagram

When you turn the motor with mechanical force, it will generate electricity that can be used to power a load. As the coil rotates within the magnetic field, the changing magnetic field induces a current in the coil, which can flow through the load, providing the necessary power.

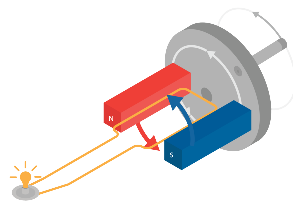


Figure 3.2: Basic Electric Generator in Motion with Handcrank

If you instead turn the motor with a wheel, you can generate electricity in any manner of ways. This simple principle is used all over the world to generate electricity. For example, in hydroelectric power plants, water is used to turn large turbines connected to generators, producing electricity on a massive scale. In wind power plants, wind turns the blades of a turbine, which is connected to a generator that produces electricity. In coal and natural gas power plants, steam is used to turn turbines connected to generators.

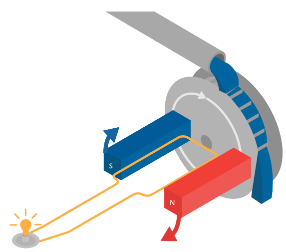


Figure 3.3: Basic Electric Generator in Motion with Wheel

Chapter 4

Drag

The very first computers were created to do calculations of how artillery would fly when shot at different angles. The calculations were similar to the ones you just did for the flying hammer, with two important differences:

- They were interested in two dimensions: the height and the distance across the ground.
- However, artillery flies a lot faster than a hammer, so they also had to worry about drag from the air.

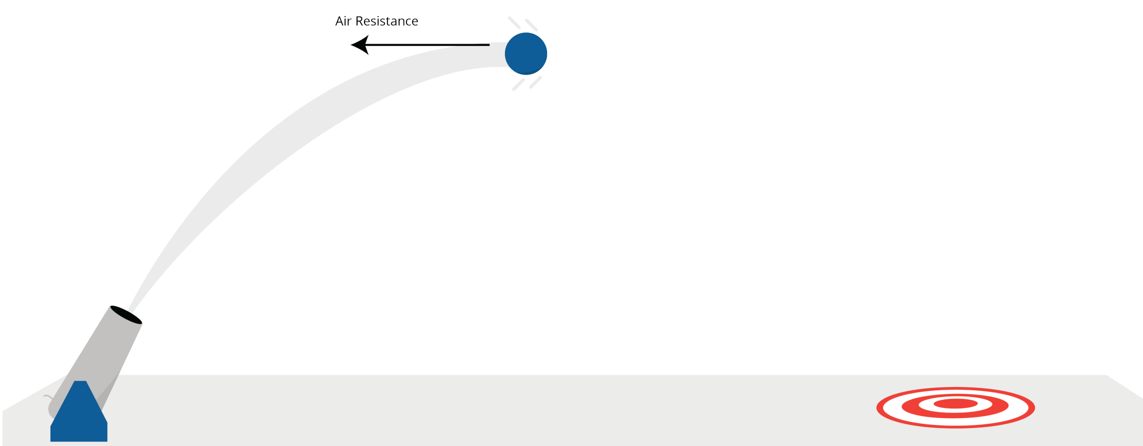


Figure 4.1: A cannonball shot has a parabolic trajectory.

4.1 Wind resistance

The first thing they did was put one of the shells in a wind tunnel. They measured how much force was created when they pushed 1 m/s of wind over the shell. Let's say it was 0.1 newtons.

One of the interesting things about the drag from the air (often called *wind resistance*) is that it increases with the *square* of the speed. Thus, if the wind pushing on the shell is 3 m/s, instead of 1 m/s, the resistance is $3^2 \times 0.1 = 0.9$ newtons.

(Why? Intuitively, three times as many air molecules are hitting the shell and each molecule is hitting it three times harder.)

So, if a shell is moving with the velocity vector v, the force vector of the drag points in the exact opposite direction. If μ is the force of wind resistance of the shell at 1 m/s, then the magnitude of the drag vector is $\mu|v|^2$ with μ being the wind resistance force.

4.2 Initial velocity and acceleration due to gravity

Let's say a shell is shot out of a tube at s m/s, and the tube is tilted θ radians above level. The initial velocity will be given by the vector $[s\cos(\theta), s\sin(\theta)]$

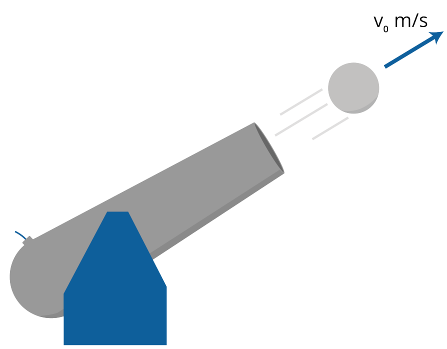


Figure 4.2: The initial velocity vector of the shell.

(The velocity of the shell is actually a 3-dimensional vector, but we are only going to worry about height and horizontal distance; we are assuming that the operator pointed it in the right direction.)

To figure out the path of the shell, we need to compute its acceleration. We remember that

$$F = ma$$

(Note that F and a are vectors.) Dividing both sides by m, we get:

$$a = \frac{F}{m}$$

Let's figure out the net force on the shell, so that we can calculate the acceleration vector.

If the shell has a mass of b, the force due to gravity will be in the downward direction, with a magnitude of 9.8b newtons.

To get the net force, we will need to add the force due to gravity with the force due to wind resistance.

4.3 Simulating artillery in Python

Create a file called `artillery.py`.

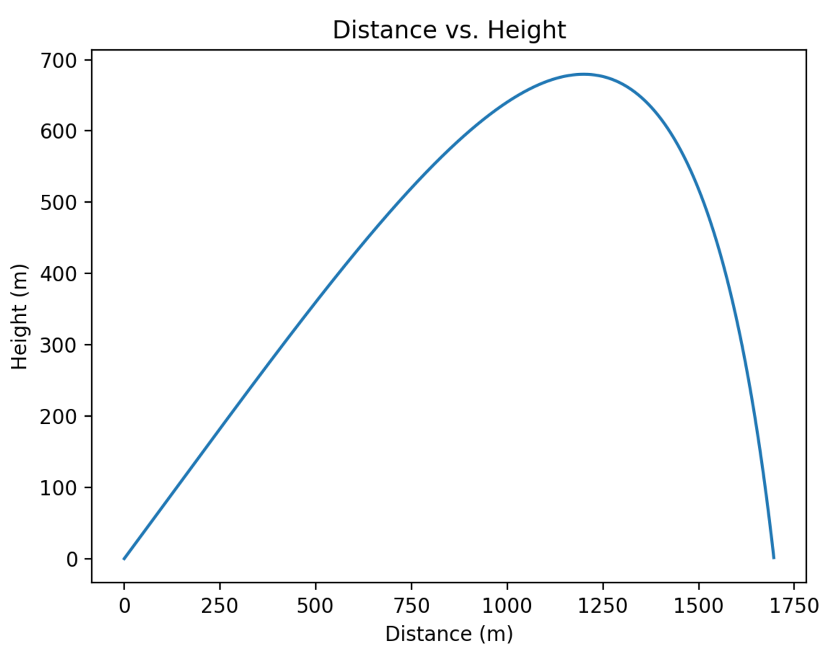
```
import numpy as np
import matplotlib.pyplot as plt

# Constants
mass = 45 # kg
start_speed = 300.0 # m/s
theta = np.pi/5 # radians (36 degrees above level)
time_step = 0.01 # s
wind_resistance = 0.05 # newtons in 1 m/s wind
force_of_gravity = np.array([0.0, -9.8 * mass]) # newtons

# Initial state
position = np.array([0.0, 0.0]) # [distance, height] in meters
velocity = np.array([start_speed * np.cos(theta), start_speed * np.sin(theta)])
time = 0.0 # seconds

# Lists to gather data
distances = []
heights = []
times = []

# While shell is aloft
while position[1] >= 0:
    # Record data
    distances.append(position[0])
    heights.append(position[1])
    times.append(time)

    # Calculate the next state
    time += time_step
    position += time_step * velocity

    # Calculate the net force vector
```

```
        force = force_of_gravity - wind_resistance * velocity**2

        # Calculate the current acceleration vector
        acceleration = force / mass

        # Update the velocity vector
        velocity += time_step * acceleration

    print(f"Hit the ground {position[0]:.2f} meters away at {time:.2f} seconds.")

    # Plot the data
    fig, ax = plt.subplots()
    ax.plot(distances, heights)
    ax.set_title("Distance vs. Height")
    ax.set_xlabel("Distance (m)")
    ax.set_ylabel("Height (m)")
    plt.show()
```

When you run it, you should get a message like:

```
Hit the ground 1696.70 meters away at 20.73 seconds.
```

You should also see a plot of the shell's path:



4.4 Terminal velocity

If you shot the shell very, very high in the sky, it would keep accelerating toward the ground until the force of gravity and the force of the wind resistance were equal. The speed at which this happens is called the *terminal velocity*. The terminal velocity of a falling human is about 53 m/s.

Note that kinematic equations do not apply to terminal velocity, because the acceleration is not constant. Instead, we can use the fact that at terminal velocity, the force of wind resistance equals the force of gravity.

Exercise 2 Terminal velocity

What is the terminal velocity of the shell described in our example?

Working Space

Answer on Page 63

Chapter 5

Parametric Functions

Throughout your study of functions, you have seen functions that look like this:

$$y = f(x)$$

Sometimes, to describe certain shapes or curves on a graph, we must use more than one function to do so. This means a shape, like a circle on the graph, can be represented using

$$y = f(t) \quad \text{and} \quad x = g(t).$$

In this case, there are two functions, $x(t)$ and $y(t)$, that depend on a third variable t, and together they help us visualize and define a curve.

The central word you should think about as you go through this chapter is *parameter*. A parameter is a variable that both x and y depend on. The most common example of a parametric function is the circle. To represent a circle on a graph, we can use the equations

$$x = \cos(\theta) \quad \text{and} \quad y = \sin(\theta).$$

FIXME: Insert graph of a circle showing parametric function

In this case, the graph of the circle is determined by the value of the angle θ, the parameter.

So, why parametric functions? When you look up the function that describes a circle, you might see something like

$$x^2 + y^2 = r^2.$$

The equation $x^2 + y^2 = r^2$ does not describe y as a single function of x. You would need two functions (one for the top half and one for the bottom half), and many curves cannot be written as a single function at all. Parametric equations solve this problem by allowing us to describe the entire curve with one pair of equations, using a parameter t. This parameter can be denoted by other letters, but we will use t in this chapter.

Imagine a ball rolling across graph paper. At each moment, the ball is at a different location. If we label time with a variable t, then for every value of t the ball has an x-coordinate and a y-coordinate. What if we had two equations, one that told us its x-position at time t and one that told us its y-position? As t changes, these two equations would trace the entire path of the ball. This is the basic idea behind parametric equations.

5.1 Conversion to Rectangular Form

In many cases, we can eliminate the parameter to convert a pair of parametric equations,

$$x = f(t), \quad y = g(t),$$

into a single equation relating x and y.

In many cases, a parametric curve can be rewritten as a single equation involving only x and y. This process is called eliminating the parameter, and the resulting equation is known as the rectangular form of the curve.

To eliminate the parameter, we solve one of the parametric equations for t and substitute into the other equation.

Example:

Consider the parametric equations

$$x = 2t + 1, \quad y = 3t - 4.$$

Solution: Solve the equation for x to isolate t:

$$t = \frac{x-1}{2}.$$

Substitute into the equation for y:

$$y = 3\left(\frac{x-1}{2}\right) - 4.$$

Simplifying,

$$y = \frac{3}{2}x - \frac{11}{2}.$$

Example

Consider the parametric equations

$$x = \cos t, \quad y = \sin t.$$

Solution: In this case, solving for t directly is not convenient. Instead, we use the trigonometric identity

$$\cos^2 t + \sin^2 t = 1.$$

Substituting $x = \cos t$ and $y = \sin t$ gives

$$x^2 + y^2 = 1.$$

This rectangular equation describes a circle of radius 1 centered at the origin.

Note

Converting to rectangular form describes the shape of a curve, but it does not show the direction of motion or how many times the curve is traced. This information is contained in the parametric equations and the domain of the parameter!

5.2 Sketching Parametric Equations

Parametric equations describe curves by expressing both x and y in terms of a third variable, usually t, called the parameter. When sketching a parametric curve, it is important to remember that the curve is traced as the parameter changes, rather than being drawn all at once as a function of x.

To sketch a parametric curve,

- choose several values of the parameter t,
- compute the corresponding x- and y-values,
- plot the resulting points in the xy-plane,
- and indicate the direction in which the curve is traced as t increases.

Unlike rectangular graphs, parametric sketches also depend on the direction and speed of motion, which are determined by how $x(t)$ and $y(t)$ change as t varies.

FIXME Sketched example

Exercise 3

Working Space

Sketch the parametric equations

$$x = t + 1, \quad y = t^2.$$

In your sketch:

- create a table of values for t, x, and y,
- plot the corresponding points,
- and indicate the direction of motion as t increases.

Answer on Page 63

5.3 Using Python to Explore Parametric Curves

Python can be used to visualize parametric curves and explore how the point $(x(t), y(t))$ moves as t changes. For example, we could write code that:

- plots points along a circle defined by $x = \cos(t)$ and $y = \sin(t)$,
- shows the distance from the origin,
- traces more complicated parametric curves like spirals or ellipses.

Below is a Python script that creates an animation for sketching parametric curves. There is a provision for changing the equations based on exercises and questions you come across.

```
import numpy as np
import matplotlib.pyplot as plt
from matplotlib.animation import FuncAnimation

t = np.linspace(0, 3, 50)

x = 4 - 2*t
y = 3 + 6*t - 4*t**2

fig, ax = plt.subplots()
ax.set_aspect("equal", "box")

ax.axhline(0, color="black", linewidth=0.5)
ax.axvline(0, color="black", linewidth=0.5)

ax.set_xlabel("x(t)")
ax.set_ylabel("y(t)")
ax.set_title("Parametric Equation of Curve")

ax.set_xlim([x.min() - 1, x.max() + 1])
ax.set_ylim([y.min() - 1, y.max() + 1])

(line,) = ax.plot([], [], "b--")
(point,) = ax.plot([], [], "ro")

def update(frame):
    line.set_data(x[:frame + 1], y[:frame + 1])
    point.set_data([x[frame]], [y[frame]])
    return line, point

ani = FuncAnimation(fig, update, frames=len(t), interval=50, blit=True)
ani.save("parametric_animation.mp4")

plt.show()
```

Exercise 4

Working Space

Consider the parametric equations

$$x = 4 - 2t, \quad y = 3 + 6t - 4t^2,$$

for $0 \leq t \leq 3$.

(a) Sketch the curve by creating a table of values for t, x, and y.

(b) Use the provided Python animation code to visualize the curve.

(c) Describe the direction in which the curve is traced as t increases.

(d) Explain how the Python animation supports or clarifies your hand-drawn sketch.

Answer on Page 64

5.4 Derivatives of Parametric Functions

Just like any other type of curve, you can be asked to determine the slope of the tangent to the point on a parametric curve. To do this, you should know how to find the derivative of the curve! You could do this by eliminating the variable t and solving for the derivative of the rectangular equation. However, there is an easier way!

The derivative of a parametric curve can be found by dividing $\frac{dy}{dt}$ by $\frac{dx}{dt}$:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}.$$

Derivatives of parametric functions are used to describe the slope of a curve at a given value of the parameter. This slope indicates how the curve is changing at that point. In some problems, the slope is used to find the equation of a tangent line to the curve. In other problems, the slope is used to analyze the behavior of the curve, such as determining where the curve has horizontal or vertical tangents.

Example

Find $\frac{dy}{dx}$ for the parametric equations

$$x = t^2 + 1, \quad y = 3t - 2.$$

Solution

First compute the derivatives with respect to t:

$$\frac{dx}{dt} = 2t, \quad \frac{dy}{dt} = 3.$$

Now divide:

$$\frac{dy}{dx} = \frac{3}{2t}.$$

As a check, eliminate the parameter. From $x = t^2 + 1$, we get

$$t = \sqrt{x - 1}.$$

Substitute into $y = 3t - 2$:

$$y = 3\sqrt{x - 1} - 2.$$

Differentiating gives

$$\frac{dy}{dx} = \frac{3}{2\sqrt{x-1}},$$

which matches the parametric result.

Exercise 5

Working Space

Consider the parametric equations

$$x = t^3 - 3t, \quad y = t^2.$$

(a) Compute $\frac{dx}{dt}$ and $\frac{dy}{dt}$.

(b) Find the values of t for which the curve has horizontal tangents.

(c) Find the values of t for which the curve has vertical tangents.

(d) Identify the points on the curve where these tangents occur.

Answer on Page 64

Exercise 6

Working Space

Consider the parametric equations

$$x = 2t + 1, \quad y = t^2.$$

(a) Find $\frac{dy}{dx}$ in terms of t.

(b) Evaluate $\frac{dy}{dx}$ at $t = 1$.

(c) Interpret your result as the slope of the tangent line at the corresponding point.

Answer on Page 65

FIXME Expand on section

5.5 Length of Curve Traced by Parametric Equations

In the Python animation, a point moves along the curve as the parameter t increases. This animation highlights an important idea: parametric equations do not only describe the shape of a curve, they describe motion along that curve.

At each moment in time, the point is located at $(x(t), y(t))$. As t changes, the point moves from one position to the next. Sometimes this movement covers a small distance, and sometimes it covers a larger distance, even though the point remains on the same curve. This means the point does not necessarily move at a constant rate.

Speed measures how fast the point is moving along the curve at a given value of t. Once the speed is known, we can determine how far the point travels over a time interval. The length of the curve traced by the parametric equations is the total distance traveled by the moving point.

Speed of a Parametric Curve

When a point moves along a parametric curve, its velocity is determined by the derivatives $x'(t)$ and $y'(t)$. The speed of the point is the magnitude of the velocity vector:

$$\text{Speed} = \sqrt{(x'(t))^2 + (y'(t))^2}.$$

Speed describes how fast the point is moving at a particular moment, but it does not tell us the total distance traveled along the curve.

Arc Length Formula

To find the length of the curve traced by a parametric equation from $t = a$ to $t = b$, we integrate the speed over that interval.

If a curve is defined parametrically by $x = x(t)$ and $y = y(t)$, where $x'(t)$ and $y'(t)$ are continuous on $[a, b]$, then the length of the curve is given by

$$L = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\, dt.$$

Example: Finding Speed

Given

$$x'(t) = 8t - t^2, \quad y'(t) = -t + \sqrt{t^{1.2} + 20},$$

the speed of the particle at time t is

$$\text{Speed} = \sqrt{(8t - t^2)^2 + \left(-t + \sqrt{t^{1.2} + 20}\right)^2}.$$

Evaluating at $t = 2$ gives a speed of approximately 12.3 cm/s.

Example: Length of a Curve

Consider the parametric equations

$$x = \cos t, \quad y = \sin t,$$

for $0 \leq t \leq 2\pi$.

First compute the derivatives:

$$x'(t) = -\sin t, \quad y'(t) = \cos t.$$

Substitute into the arc length formula:

$$L = \int_0^{2\pi} \sqrt{(-\sin t)^2 + (\cos t)^2}\, dt.$$

Using the identity $\sin^2 t + \cos^2 t = 1$, this simplifies to

$$L = \int_0^{2\pi} 1 \, dt = 2\pi.$$

Thus, the length of the curve traced is 2π, which matches the circumference of the unit circle.

Note

Speed describes how fast a point moves along a parametric curve at a given moment. Arc length measures the total distance traveled along the curve over an interval of time. For parametric curves, arc length is found by integrating the speed.

Exercise 7

How can two sets of parametric equations represent the same graph, but different curves?

Working Space

Answer on Page 65

Chapter 6

Vector-valued Functions

In the last chapter, you calculated the flight of the shell. For any time t, you could find a vector $[\mathtt{distance}, \mathtt{height}]$. This can be thought of as a function f that takes a number and returns a 2-dimensional vector. We call this a *vector-valued* function from $\mathbb{R} \to \mathbb{R}^2 \to \mathbb{R}^3$[1].

6.1 Vector-valued functions: position

We often make a vector-valued function by defining several real-valued functions. For example, if you threw a hammer with an initial upward speed of 12 m/s and a horizontal speed of 4 m/s along the x axis from the point $(1, 6, 2)$, its position at time t (during its flight) would be given by:

$$f(t) = [4t + 1,\ 6,\ -4.8t^2 + 12t + 2]$$

In other words, x is increasing with t, y is constant, and z is a parabola.

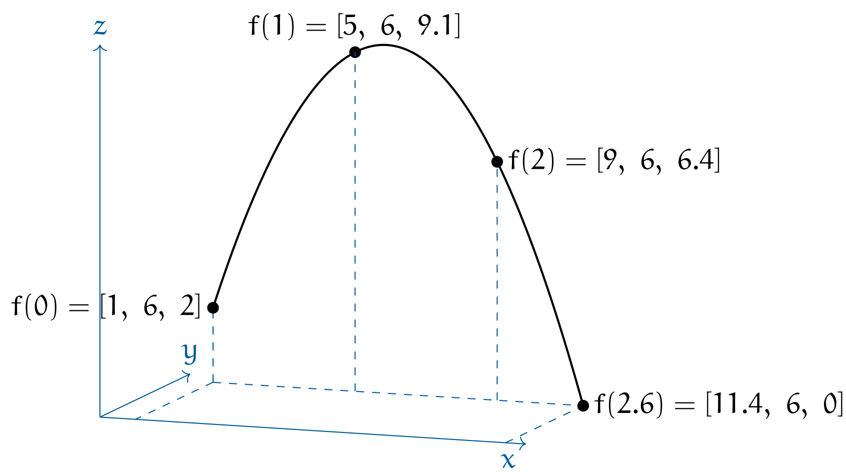


Figure 6.1: An example of a vector-valued function.

[1] the $\mathbb{R}$ symbol represents the set of all real numbers; the $\mathbb{R}^2$ symbol represents the set of all 2-dimensional vectors, and $\mathbb{R}^3$ represents the set of all 3-dimensional vectors

6.2 Finding the velocity vector

Now that we have its position vector, we can differentiate each component separately to get its velocity as a vector-valued function:

$$f'(t) = [4, 0, -9.8t + 12]$$

In other words, the velocity is constant along the x-axis, zero along the y-axis, and decreasing with time along the z axis.

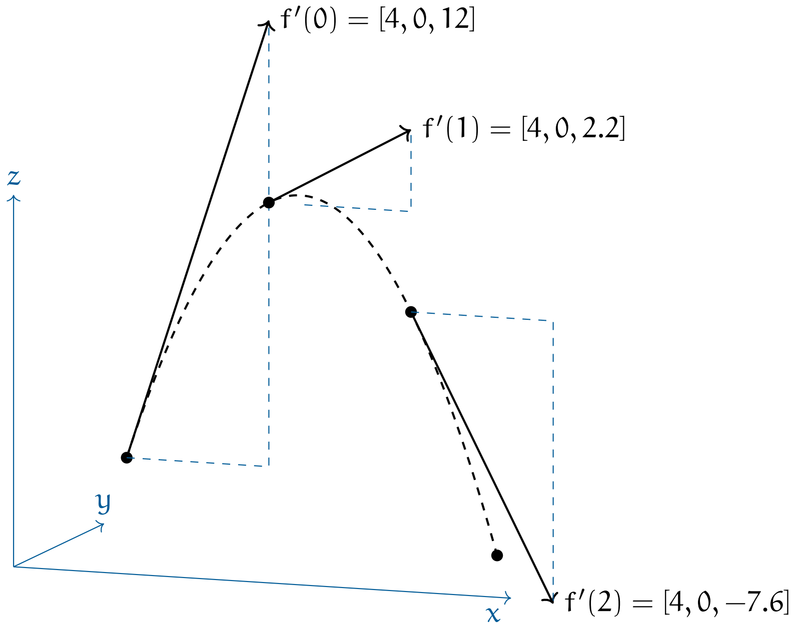


Figure 6.2: The derivatives of the position function (velocity) with respect to time.

6.3 Finding the acceleration vector

Now that we have its velocity, we can get its acceleration as a vector-valued function:

$$f''(t) = [0, 0, -9.8]$$

There is no acceleration along the x or y axes. It is accelerating down at a constant $9.8\text{m}/\text{s}^2$.

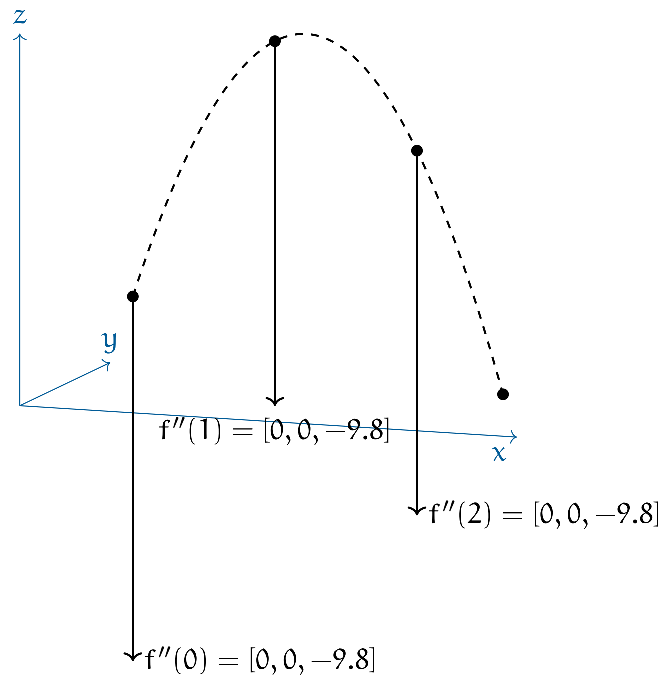


Figure 6.3: The acceleration vector is constant and points downward.

Chapter 7

Circular Motion

We can recall the motion of 2D objects, such as a projectile launched or a car driving along a road. They have components of linear motion, such as position, velocity, acceleration, and force.

But that only works for objects moving in a straight line. What about objects moving in a circle?

7.1 Introduction to Uniform Circular Motion

Recall that in linear motion, You may be surprised to know that we can use very similar equations to describe circular motion, provided you use the correct variables.

The angular displacement, θ, is the angle in *radians* that the object has traveled. The angular velocity, ω, is the rate of change of the angular displacement, and the angular acceleration, α, is the rate of change of the angular velocity.

Linear motion	Rotational motion
$v = v_0 + at$	$\omega = \omega_0 + \alpha t$
$x = x_0 + v_0 t + \frac{1}{2}at^2$	$\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2$
$v^2 = v_0^2 + 2a(x - x_0)$	$\omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0)$

Table 7.1: Kinematic equations for linear and rotational motion.

Linear velocity, v, is related to angular velocity by the equation:

$$v = r\omega$$

The *Period*, T, is the time it takes to complete one full rotation. The frequency, f, is the number of rotations per second. The two are related by:

$$T = \frac{2\pi}{\omega} = \frac{2\pi r}{v}$$

$$f = \frac{1}{T}$$

The *centripetal force*, usually labelled differently (such as tension or gravity), is given by the equation:

$$F = mr\omega^2 = \frac{mv^2}{r}$$

The *angular velocity,* ω is rotations with respect to time. It can be defined in the following ways:

$$\omega_{inst} = \frac{d\theta}{dt}, \qquad \omega_{avg} = \frac{\Delta\theta}{\Delta t}$$

$$\omega = 2\pi f = \frac{2\pi}{T}$$

Angular acceleration:

$$\alpha = \frac{d\omega}{dt}, \qquad \alpha_{avg} = \frac{\Delta\omega}{\Delta t}$$

Note that all of the kinematic formulas still work as long as you stay in the same relative frame of reference (rotational or linear). You cannot "mix and match" the two sets of equations.

You may be asking, "If the object is moving at a constant speed, why is there acceleration?" Recall that acceleration is a vector quantity, meaning it has both magnitude and direction. Even though the speed (the magnitude of the velocity) is constant, the direction of the velocity vector is always changing. Since acceleration is the rate of change of velocity, any change in direction means there is acceleration. This acceleration is always directed toward the center of the circle, which is why it is called centripetal (center-seeking) acceleration.

Now that we've established the basic relationships for circular motion, let's apply them to a practical example.

7.2 The Flying Billiard Ball

Let's say you tie a 0.16 kg billiard ball to a long string and begin to swing it around in a circle above your head. The string, perpendicular, is 3 meters long, and the ball returns to where it started every 4 seconds. We will assume the ball moves at a constant speed. If you start your stopwatch as the ball crosses the x-axis, the coordinates of the position (x, y, z) of the ball at any time t given by:

$$p(t) = \left[3\cos\left(\frac{2\pi}{4}t\right), 3\sin\left(\frac{2\pi}{4}t\right), 2\right]$$

(This assumes that the ball would be going counter-clockwise if viewed from above. The spot you are standing on is considered the origin $[0, 0, 0]$.)

Notice that the height is a constant — 2 meters in this case. That isn't very interesting, so we will talk just about the first two components. Figure 7.1 shows a visual of the situation:

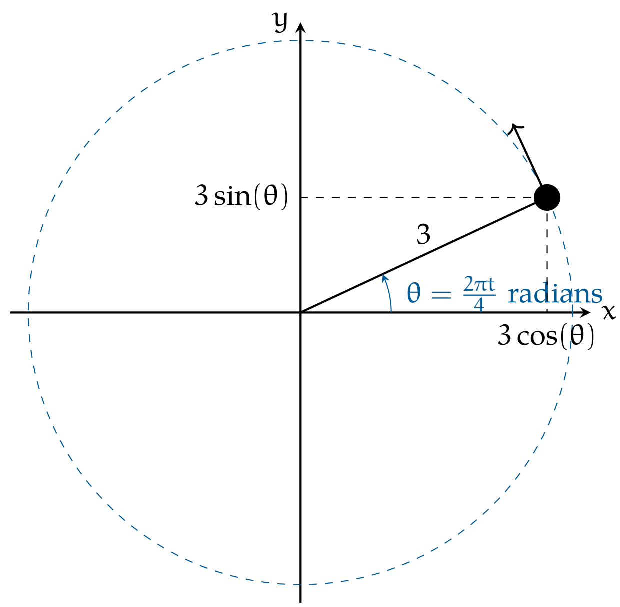


Figure 7.1: A diagram of our ball and string scenario.

In this case, the radius, r, is 3 meters. The period, T, is 4 seconds. In general, we say that circular motion is given by:

$$p(t) = \left[r\cos\frac{2\pi t}{T}, r\cos\frac{2\pi t}{T}\right]$$

A common question is "How fast is it turning right now?" If you divide the 2π radians of a circle by the 4 seconds it takes, you get the answer "About 1.57 radians per second." This is known as *angular velocity* and we typically represent it with the lowercase Omega: ω. (Yes, it looks a lot like a "w".) To be precise, in our example, the angular velocity is $\omega = \frac{\pi}{2}$. Note that in this scenario, the angular velocity and linear *speed* are constant. However, the *velocity* vector is not constant; as we will see in the next section, the direction of the velocity vector is always changing.

Notice that this is different from the question "How fast is it going? (referring to *linear velocity*)" This ball is traveling the circumference of $6\pi \approx 18.85$ meters every 4 seconds. This means the speed of the ball is about 4.71 meters per second.

It is very important to distinguish between angular velocity and linear velocity. Angular velocity is how fast the angle is changing and is referred to by ω, while linear velocity, v, is how fast the object is moving along its path. While linear velocity has a constant speed, it has always changing direction. The angular velocity is constant, but the linear velocity is not.

7.3 Velocity

The velocity of the ball is a vector, and we can find that vector by differentiating each component of the position vector.

For any constants a and b:

Expression $f(x)$	Derivative $f'(x)$
$a \sin bt$	$ab \cos bt$
$a \cos bt$	$-ab \sin bt$

Thus, in our example, the velocity of the ball at any time t is given by:

$$v(t) = \left[-\frac{3(2\pi)}{4} \sin \frac{2\pi t}{4}, \frac{3(2\pi)}{4} \cos \frac{2\pi t}{4}, 0\right]$$

Notice that the velocity vector is perpendicular to the position vector. It has a constant magnitude.

In general, an object traveling in a circle at a constant speed has the velocity vector:

$$v(t) = [-r\omega \sin \omega t, r\omega \cos \omega t]$$

where $t = 0$ is the time that it crosses the x axis. If ω is negative, that means the motion would be clockwise when viewed from above.

The magnitude of the velocity vector is $r\omega$. See Figure 7.4a for a visual of the velocity vector. Note that the z-coordinate is constant, so its derivative is zero, and it is eliminated from our vector for simplification.

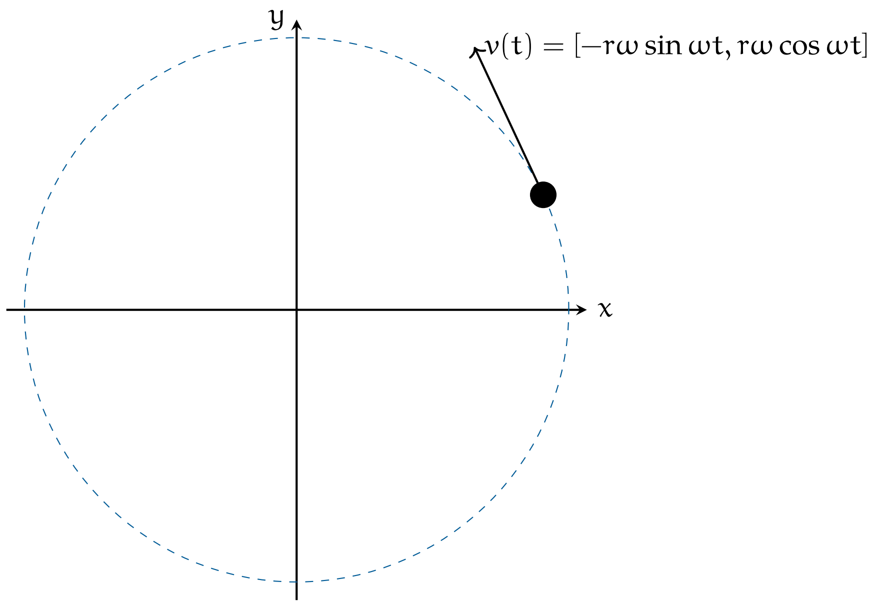


Figure 7.2: Velocity vector of the ball in circular motion.

Exercise 8 A Particle in Motion

[This question was originally presented as two multiple-choice problems on the 2012 AP Physics C exam.] The x and y coordinates of a particle as it moves in a circle are given by:

$$x = 5\cos(3t) y = 5\sin(3t)$$

What is the radius of the particle's circular path? What is the particle's *speed*? Based on your answers, how long does it take the particle to complete the circular path?

Working Space

Answer on Page 66

7.4 Acceleration

We can get the (linear) acceleration by differentiating the components of the velocity vector.

$$a(t) = \left[-r\omega^2 \cos \omega t, -r\omega^2 \sin \omega t\right] = -r\omega^2 [\cos \omega t, \sin \omega t]$$

Notice that the acceleration vector points **toward the center** of the circle it is traveling on. That is, when an object is traveling on a circle at a constant (uniform) speed (notice that coefficients r and ω are constant), its only acceleration is toward the center of the circle. The acceleration does not come from the motion itself, it comes from the constantly changing direction. This is known as *centripetal acceleration*.

The magnitude of the acceleration vector is $a_c = r\omega^2$, or more commonly, $a_c = \frac{v^2}{r}$. This gives us the common calculation for whatever force acts as the centripetal force: $F_c = ma_c = \frac{mv^2}{r}$.[1]

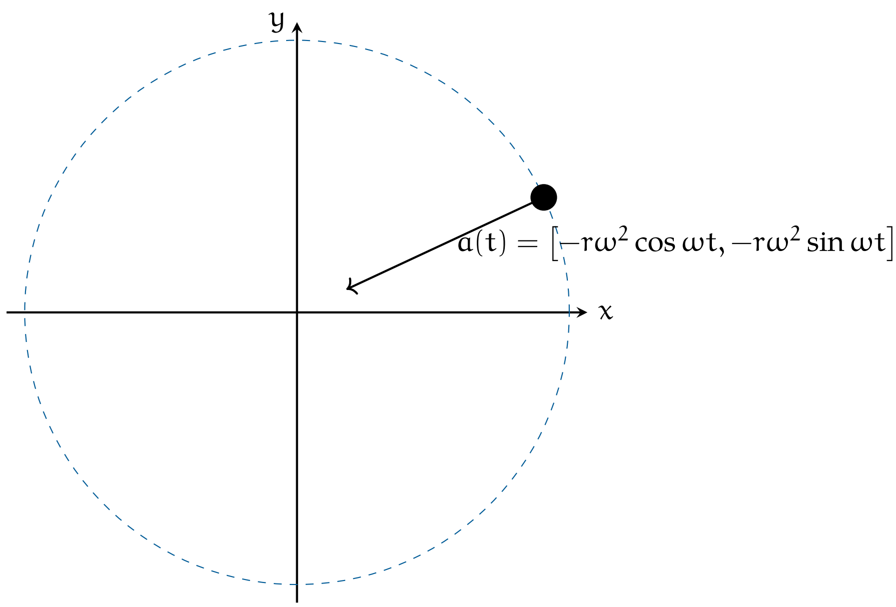
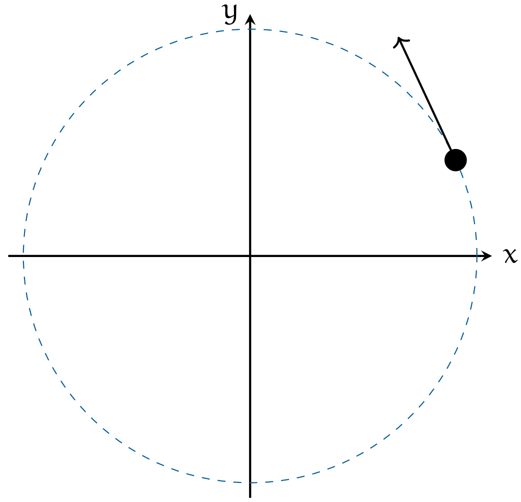


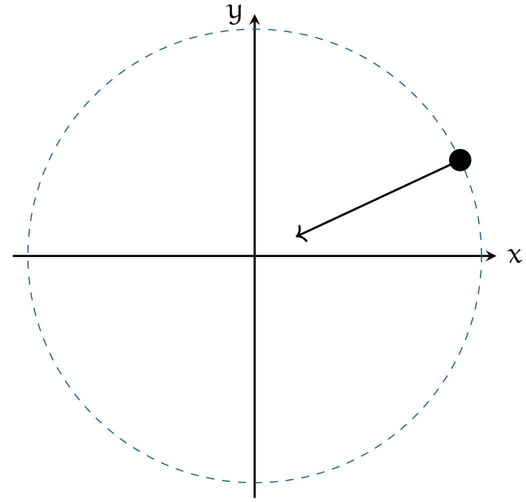
Figure 7.3: Acceleration vector of the particle in circular motion.

Let's compare the velocity and acceleration paths from figures 7.4a and 7.4b. Notice that the velocity vector is always tangent to the circle, while the acceleration vector is always pointing toward the center of the circle.

[1] You will see centripetal acceleration noted as a_c or a_r, standing for centripetal or radial, respectively. Note that both refer to the same acceleration.



(a) Velocity vector.



(b) Acceleration vector.

Figure 7.4: Uniform circular motion: velocity (tangent) and acceleration (centripetal).

7.5 Centripetal force

What would happen if there were no force pulling the ball toward the center of the circle? It would fly off in a straight line, according to Newton's First Law.

The force you are exerting on the string is what causes it to accelerate toward the center of the circle. We call this the *centripetal force*. It makes sense that the centripetal force and centripetal acceleration are in the same direction, as they are proportional.

Recall that $F = ma$. The magnitude of the acceleration is $r\omega^2 = 3\left(\frac{2\pi}{4}\right)^2 \approx 7.4\text{m/s}^2$. The mass of the ball is 0.16 kg. So, the force pulling against your hand is about 1.18 newtons.

The general rule is that when something is traveling in a circle at a constant speed, the centripetal force needed to keep it traveling in a circle is:

$$F = mr\omega^2$$

If you know the radius r and the speed v of the object, here is the rule:

$$F = \frac{mv^2}{r}$$

We didn't introduce centripetal force as a type of force because it isn't its own force, like friction or gravity. Rather, calling a force "centripetal" tells us what the force is doing. A centripetal force is any force that causes circular motion, a force which acts in the *radial* direction. For a satellite, the centripetal force is gravity. In the opening example with the billiard ball, the centripetal force is the tension in the string.

Example: A child is sitting on a merry-go-round as it spins. What provides the centripetal force?

Solution: In this case, the child is rotating horizontally, so the centripetal force must also be horizontal. Therefore, the centripetal force isn't the child's weight or the normal force. It is the *friction* between the child and the merry-go-round that keeps the child turning. Here is a free body diagram:

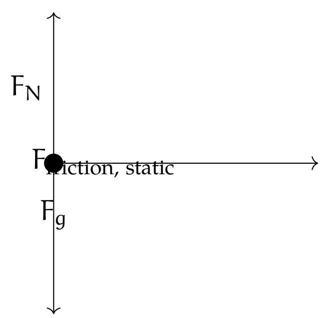


Figure 7.5: The static friction is what causes the centripetal force.

Example: If the child is 1.2 meters from the center of the merry-go-round and it spins at 0.25 rotations/second, what is the coefficient of friction between the child and the merry-go-round?

Solution: Using our FBD and applying Newton's Second Law, we see that:

$$F_N - F_g = 0 \rightarrow F_N = m_{child}g$$

$$F_f = mr\omega^2 = \mu F_N = \mu mg$$

Since the child isn't accelerating vertically, we know that the normal force equals the child's weight. The horizontal force, friction, must equal the mass of the child times the acceleration. Additionally, from the vertical component, we know the friction is equal to μmg, since the normal force is equal to the child's weight. Looking at the second equation, we see that:

$$mr\omega^2 = \mu mg$$

We were given ω in rotations per second, but we need radians per second:

$$\frac{0.25\text{rotation}}{1\text{second}} = \frac{1\text{rotation}}{4\text{seconds}} = \frac{2\pi\text{radians}}{4\text{seconds}} = \frac{\pi}{2}\frac{rad}{s}$$

We can divide m from both sides of $mr\omega^2 = \mu mg$ (notice: we don't need to know the child's mass to determine the coefficient of friction!):

$$r\omega^2 = \mu g \rightarrow \mu = \frac{r\omega^2}{g} = \frac{(1.2\text{m})\left(\frac{\pi}{2\text{s}}\right)^2}{9.8\frac{\text{m}}{\text{s}^2}} = 0.302$$

Therefore, the coefficient of friction between the child and the merry-go-round is 0.302.

7.5.1 Banked Turns

Have you ever been driving on the highway and taken a turn where the road is at an angle? Or maybe you've seen a sign like Figure 7.6 on the road as you take a turn.

A banked curve is designed so that a car can safely turn without slipping, even in the rain, if the driver does not exceed the indicated speed. Engineers choose an angle such that the bank provides sufficient force to turn the car without friction. Let's look at a free body diagram for a car taking a banked turn (see figure 7.7).

Figure 7.6: A banked turn sign on the highway.

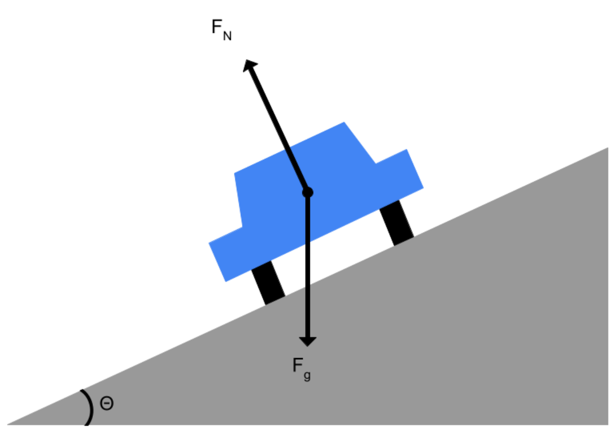


Figure 7.7: If there is no friction due to rain, then the only two forces acting on the car are gravity and the normal force.

In the past, we've split the vector for the force of gravity into components that are parallel and perpendicular to the ramp. This time, we will split the normal force into x and y components (see figure 7.8).

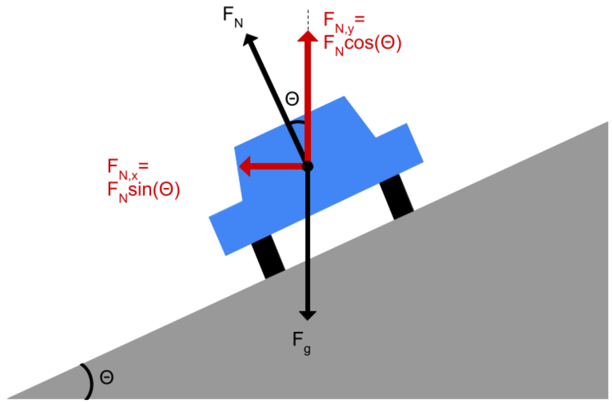


Figure 7.8: Geometrically, the x component of the normal force is given by $F_N \cos(\theta)$ and the y component by $F_N \sin(\theta)$.

Assuming the radius of the turn is 20 m, at what angle should the engineers build the banked turn? (Let's also assume the maximum speed is 25 mph, like the traffic sign above.) First, let's apply Newton's Second Law in the x and y directions:

$$(1)\, F_N \cos(\theta) - F_g = 0$$

$$(2)\, F_N \sin(\theta) = \frac{mv^2}{r}$$

We know that $F_g = mg$. From this and equation (1) we see that:

$$F_N = \frac{mg}{\cos(\theta)}$$

Substituting for F_N into equation (2):

$$\left(\frac{mg}{\cos(\theta)}\right) \sin(\theta) = \frac{mv^2}{r}$$

The mass can be divided from both sides, and $\frac{\sin(\theta)}{\cos(\theta)} = \tan(\theta)$:

$$g \tan(\theta) = \frac{v^2}{r}$$

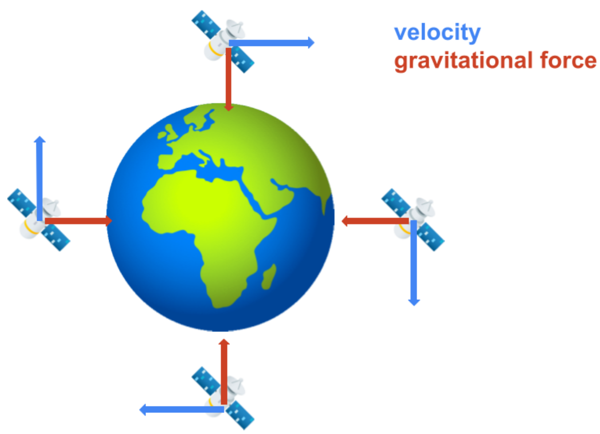
Solving for θ and substituting for the speed $(25 \text{ mph} \approx 11.18 \frac{m}{s})$ and radius:

$$\theta = \arctan\left(\frac{v^2}{gr}\right) = \arctan\left(\frac{(11.18\frac{m}{s})^2}{(9.8\frac{m}{s^2})(20m)}\right)$$

$$\theta = \arctan(0.6377) \approx 32.53°$$

7.6 Modeling Circular Motion

What causes circular motion is a constant force perpendicular to the motion. Consider a satellite circling the Earth. The only force acting on the satellite is gravity, yet the satellite does not fall to the Earth. Why? Let's look at the relative direction of motion and gravity for some different positions of the satellite (we'll assume this satellite is moving clockwise from our point of view):



No matter what position the satellite is in, the velocity and gravity vectors are perpendicular.

Exercise 9 Circular Motion

Just as your car rolls onto a circular track with a radius of 200 m, you realize your 0.4 kg cup of coffee is on the slippery dashboard of your car. While driving 120 km/hour, you hold the cup to keep it from sliding.

What is the maximum amount of force you would need to use? (The friction of the dashboard helps you, but the max is when the friction is zero.)

Working Space

Answer on Page 67

Exercise 10 Twirling a Whistle

The lifeguard at a local pool is twirling their whistle horizontally. You wonder if the lifeguard could spin the whistle fast enough to break the string. The string the whistle is attached to can hold a maximum mass of 20 kg before breaking. If the lifeguard's whistle string is 0.35 m long and the average whistle has a mass of 165 grams, what is the maximum tangential speed the lifeguard can spin the whistle? How many rotations per second would the whistle be spinning at? Based on this, do you think the lifeguard is capable of spinning the whistle fast enough to break the string?

Working Space

Answer on Page 67

Exercise 11 The Gravitron

The Gravitron is a carnival ride where riders "stick" to the wall of a spinning cylinder as the floor beneath them drops away. A video explanation is given here: https://www.youtube.com/watch?v=ifAY5tbYDmQ. Draw a free body diagram of a rider. If the coefficient of friction between a rider and the wall is 0.32 and the ride is 10 meters across, what angular velocity must the ride reach before the floor drops away?

Working Space

Answer on Page 68

7.7 Flying Pigs and Tension

Take a look at the following two videos that involve flying pigs!

1. Flying Pig: Uniform Circular Motion and Centripetal Force
2. flying pig - circular motion [workthrough demo]

This involves a combination of circular motion and tension. Let's say we have a pig at making angle θ with the vertical and the string of length L, causing the pig to move in a circular path with radius r. We can draw the following two diagrams for our pig, as shown in Figure 7.9.

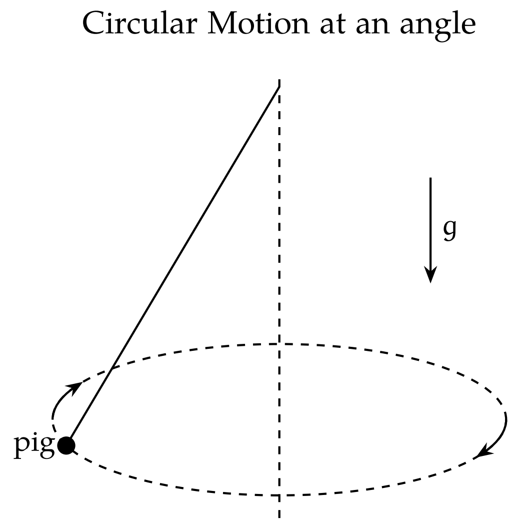


Figure 7.9: The uniform motion of a flying pig attached to a string.

If this is confusing, think of a tetherball from your local playground. It follows the same mechanics!

At any point, we can draw a free body diagram for the pig. Figure 7.10 shows the forces acting on the pig at all points.

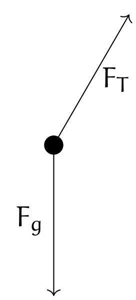


Figure 7.10: The FBD of the flying pig at any point.

To analyze the net forces, the tension force needs to be split into its x and y components. The y component of the tension force must balance out the weight of the pig (as it is not moving vertically), while the x component of the tension force provides the centripetal force to keep the pig moving in a circle.

$$\sum F_y = 0 \rightarrow F_{T_y} - F_g = 0 \rightarrow F_{T_y} = F_g = F_T \cos\theta = mg$$

The centripetal force is found by:

$$F_{T_x} = F_c = F_T \sin\theta = ma_r = \frac{mv^2}{r}$$

Dividing the second equation by the first equation gives:

$$\frac{F_T \sin(\theta)}{F_T \cos(\theta)} = \frac{\frac{mv^2}{r}}{mg} \rightarrow \tan(\theta) = \frac{v^2}{rg}$$

Solving for θ gives us $\theta = \arctan\left(\frac{v^2}{rg}\right)$. We can also solve for v:

$$v = \sqrt{rg\tan\theta} = \sqrt{Lg\sin\theta\tan\theta}$$

Since r can be solved for θ using $r = L\sin\theta$.

Explicitly, we can express our tension force as $F_T = \frac{mg}{\cos(\theta)}$ (coming from our net force in the y-direction).

We can also explicitly solve for ω:

$$\begin{aligned}
\tan(\theta) &= \frac{v^2}{rg} \\
&= \frac{(r\omega)^2}{rg} = \frac{r\omega^2}{g} \\
\omega^2 &= \frac{g\tan(\theta)}{r} \\
\omega &= \sqrt{\frac{g\tan(\theta)}{r}}
\end{aligned}$$

What does all of this tell us?

- The vertical component of tension balances the weight of the pig, so there is no vertical motion.
- The horizontal component of tension provides the centripetal force that keeps the pig moving in a circle.
- The faster the pig moves, the larger the angle θ and the greater the tension in the string.
- The tension is always greater than the pig's weight: $F_T = \frac{mg}{\cos\theta}$.

- The angular speed required for a given angle is $\omega = \sqrt{\frac{g \tan\theta}{r}}$. This increases with larger angles, and decreases with a shorter radius.

Exercise 12 **Playground Tetherball**

A tetherball of mass .5kg is attached to a string of length 2.5m. When the ball is swung around in a horizontal circle, the string makes an angle of 38° with the vertical. If the ball is in uniform circlular motion, what is the speed of the ball?

Working Space

Answer on Page 69

7.8 Non-uniform circular motion

We have talked enough about uniform circular motion, where the speed is constant. However, in many real-world scenarios, the speed is not constant. This is known as *non-uniform circular motion*. In this case, there are two components of acceleration: the centripetal acceleration $a_\perp$ (toward the center of the circle) and the tangential acceleration $a_\parallel$ (along the direction of motion). The tangential acceleration is responsible for changes in speed, while the centripetal acceleration is responsible for changes in direction. We will expand on this in the oscillations chapter, but this acceleration is caused by a *restoring force*, a force that tries to pull the object back toward equilibrium whenever it's displaced. We will introduce this concept here, but go into more detail in the oscillations chapter.

Let's look at a few examples of these concepts in action.

7.8.1 Roller Coaster Loop

Now we can talk about a roller coaster. A roller coaster loop is a great example of circular motion. As the roller coaster cart goes through the loop, it experiences both centripetal and tangential acceleration.

We will analyze the cart on a roller coaster at positions at the top, bottom, and sides of the loop, as shown in Figure 7.11

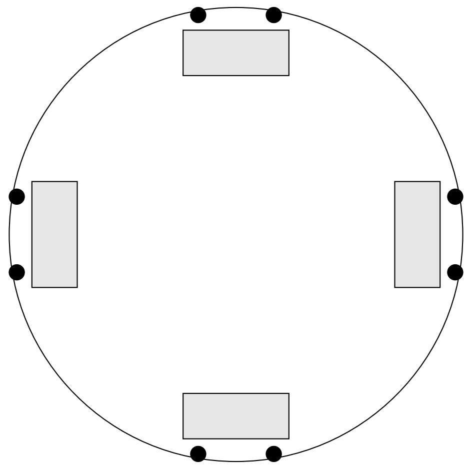


Figure 7.11: Carts at four positions around the loop with wheels on the track.

At the bottom

At the very bottom of the loop, the forces acting on the cart are gravity and the normal force from the track. Let's imagine the cart goes around the circular track infinitely without "exiting" the loop. There must, then, be a force keeping the cart moving in a circle. This force is the centripetal force, which points toward the center of the loop. The only two forces acting on the cart are gravity (downward) and the normal force from the track (upward). We know that a centripetal force must satisfy the formula $F_{net} = F_N - F_g = \frac{mv^2}{r}$. Therefore, the normal force must be greater than the weight of the cart at the bottom of the loop: $F_N = \frac{mv^2}{r} + mg$. This is represented in Figure 7.12.

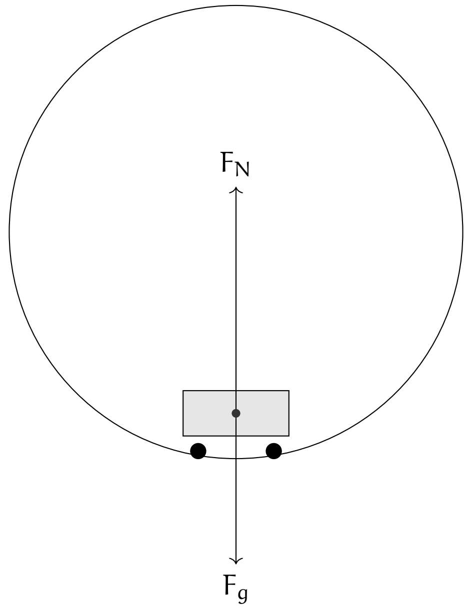


Figure 7.12: The FBD of the roller coaster at the bottom.

At the top

At the very top of the loop, the forces acting on the cart are gravity and the normal force from the track. Both forces point *downward* this time, toward the center of the loop. We can say that $F_{net} = F_N + F_g = \frac{mv^2}{r}$. Solving for the normal force gives: $F_N = \frac{mv^2}{r} - mg$.

Notice that the normal force is less than the weight of the cart at the top of the loop. This means that the centripetal acceleration would have to be greater than or equal mg. This is represented in Figure 7.13.

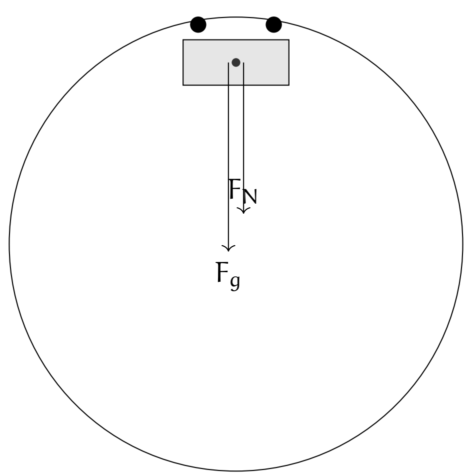


Figure 7.13: The FBD of the roller coaster at the top.

Note that for both the top and bottom of the loop, the minimum speed required to maintain contact with the track is given by $v = \sqrt{rg}$. If the cart goes any slower such that $v < \sqrt{rg}$, it will lose contact with the track, as there will not be enough normal force to provide the centripetal force needed to keep the cart moving in a circle. Think about where this observation comes from!

At the sides

Now at the sides of the loop (say 0 and π if we view the roller coaster as a unit circle), the net force is directed at an angle. The normal force from the track acts horizontally, pointing toward the center of the circle, while the gravitational force acts vertically downward. Since these two forces are perpendicular to each other, the net force is the vector sum of the normal and gravitational forces.

At the right side ($\theta = 0$), the normal force points leftward (toward the center), and gravity points downward, so the net force points diagonally down and to the left, toward the lower-left quadrant of the circle.

At the left side ($\theta = \pi$), the normal force points rightward (toward the center), and gravity still points downward, so the net force points diagonally down and to the right, toward the lower-left quadrant of the circle.

While the normal force provides the centripetal component of the net force, gravity introduces a tangential component, causing the cart to accelerate (speed up) along the track. Since the centripetal force is not directly towards the center of the circle, the cart must be speeding up. This is modeled in Figure 7.14.

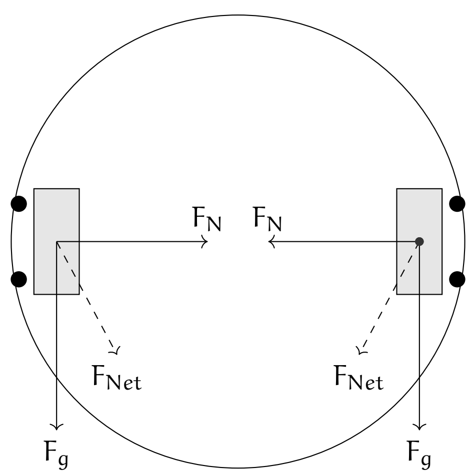


Figure 7.14: The FBD of the roller coaster at the sides.

Exercise 13 Roller Coaster Loop

This question is taken from the AP Physics C Review Book, 19th edition.

A roller-coaster car enters the circular loop portion of the ride. At the very top of the circle (where people are upside down), the car has a speed of 25 m/s, and the acceleration points straight down. If the diameter of the loop is 50 m and the total mass of the car plus passengers is $1,200$ kg, find the magnitude of the normal force exerted by the track on the car at this point. Also find the normal force exerted by the track on the car when it is at the very bottom of the loop, given it is still traveling at 25 m/s. You may use $g = 10 \text{ m/s}^2$ for simpler calculations.

Working Space

Answer on Page 70

Our next chapter will explore oscillations, restoring forces, simple harmonic motion, and more concepts of harmonic motion.

Appendix A

Answers to Exercises

Answer to Exercise 1 (on page 10)

The change in magnetic flux is:

$$\Delta\Phi_B = \Delta B \cdot A = (5\,\text{T} \times 0.2\,\text{m}^2) = 1\,\text{Wb}$$

The induced emf is:

$$\mathcal{E} = -N \cdot \frac{\Delta\Phi_B}{\Delta t} = -100 \cdot \frac{1\,\text{Wb}}{0.1\,\text{s}} = -1000\,\text{V}$$

The magnitude of the induced emf is $1000\,\text{V}$.

Answer to Exercise 2 (on page 21)

The force of gravity is $9.8 \times 45 = 441$ newtons.

At any speed s, the force of wind resistance is $0.05 \times s^2 = 0.05s^2$ newtons.

At terminal velocity, $0.05s^2 = 441$.

Solving for s, we get $s = \sqrt{\frac{441}{0.05}}$

Thus, terminal velocity should be about 94 m/s.

Answer to Exercise 3 (on page 26)

To sketch the curve, we begin by choosing several values of t and computing the corresponding x- and y-values.

t	x	y
-2	-1	4
-1	0	1
0	1	0
1	2	1
2	3	4

Plotting these points produces a parabola opening upward. As t increases, the x-values increase and the curve is traced from left to right.

Although the rectangular equation of the curve is

$$y = (x-1)^2,$$

the parametric equations determine the direction in which the curve is drawn, which cannot be seen from the rectangular form alone.

Answer to Exercise 4 (on page 28)

To sketch the curve, we begin by evaluating the parametric equations at several values of t.

t	x	y
0	4	3
1	2	5
2	0	-1
3	-2	-15

Plotting these points produces a curve that opens downward. As t increases, the x-values decrease, indicating that the curve is traced from right to left.

The Python animation confirms this behavior by showing the curve being drawn progressively as t increases. The moving point illustrates both the direction of motion and how the curve is traced over time, reinforcing the hand-drawn sketch.

Answer to Exercise 5 (on page 31)

First compute the derivatives:

$$\frac{dx}{dt} = 3t^2 - 3, \quad \frac{dy}{dt} = 2t.$$

Horizontal tangents occur when $\frac{dy}{dt} = 0$ and $\frac{dx}{dt} \neq 0$.

$$2t = 0 \Rightarrow t = 0.$$

At $t = 0$, the point on the curve is

$$x = 0, \quad y = 0.$$

Vertical tangents occur when $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} \neq 0$.

$$3t^2 - 3 = 0 \Rightarrow t = \pm 1.$$

At $t = 1$, the point is

$$(-2, 1),$$

and at $t = -1$, the point is

$$(2, 1).$$

Answer to Exercise 6 (on page 32)

First compute the derivatives with respect to t:

$$\frac{dx}{dt} = 2, \quad \frac{dy}{dt} = 2t.$$

The derivative of the curve is

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t}{2} = t.$$

At $t = 1$,

$$\frac{dy}{dx} = 1.$$

Thus, the slope of the tangent line at the corresponding point on the curve is 1.

Answer to Exercise 7 (on page 35)

Two sets of parametric equations can represent the same graph but describe different curves because the parameter controls how the graph is traced, not just the shape of the graph itself.

Different parametric equations may generate the same set of points in the plane while differing in one or more of the following ways:

- the direction of motion along the curve,
- the speed at which the curve is traced,
- the starting point of the curve,
- or the number of times the curve is traced.

Answer to Exercise 8 (on page 46)

The radius is 5 m, because the coefficients of both the x and y functions is 5.

Recall that the velocity in each direction is given by the derivative of the position functions:

$$v_x(t) = \frac{d}{dt}\left[5\cos(3t)\right] = -15\sin(3t)$$

$$v_y(t) = \frac{d}{dt}\left[5\sin(3t)\right] = 15\cos(3t)$$

The overall *speed* can be found from the x and y components of the velocity:

$$\begin{aligned} |v| &= \sqrt{v_x^2 + v_y^2} \\ &= \sqrt{[-15\sin(3t)]^2 + [15\cos(3t)]^2} \\ &= \sqrt{15^2\left[\sin(3t)^2 + \cos(3t)^2\right]} \\ &= \sqrt{15^2} = 15\frac{m}{s} \end{aligned}$$

Notice that this is the coefficient for both components of the velocity.

To complete the path, the particle must travel 10π m. If the speed is $15\frac{m}{s}$, then the time it takes is:

$$t = \frac{d}{v} = \frac{10\pi \text{ m}}{15\frac{m}{s}} = \frac{2\pi}{3}\text{s} \approx 2.09 \text{ s}$$

Answer to Exercise 9 (on page 53)

$$\frac{120 \text{ km}}{1\text{hour}} = \frac{1000 \text{ m}}{1 \text{ km}}\frac{120 \text{ km}}{1\text{hour}}\frac{1 \text{ hour}}{3600 \text{ seconds}} = 33.3 \text{ m/s}$$

$$F = \frac{mv^2}{r} = \frac{0.4(33.3)^2}{200} = 2.2 \text{ newtons}$$

Answer to Exercise 10 (on page 53)

Givens:

$$T_{max} = (20\text{kg}) \cdot \left(9.8\frac{m}{s^2}\right) = 196\text{N}$$

$$r = 0.35\text{m}$$

$$m_{whistle} = 165\text{g} = 0.165\text{kg}$$

Unknown:

$$v = ?$$

$$f = ?$$

Equation(s):

$$F = ma$$

$$a = \frac{v^2}{r}$$

$$v = rf$$

Solution: First, we find the tangential speed if the tension in the string is the maximum tension:

$$T_{max} = m_{whistle}a = m_{whistle}\frac{v^2}{r}$$

$$v = \sqrt{\frac{T_{max}r}{m_{whistle}}}$$

$$v = \sqrt{\frac{196\text{N}\,(0.35\text{m})}{0.165\text{kg}}} \approx 0.645\frac{m}{s}$$

Therefore, the maximum tangential speed of the whistle before the string breaks is $0.645\frac{m}{s}$.

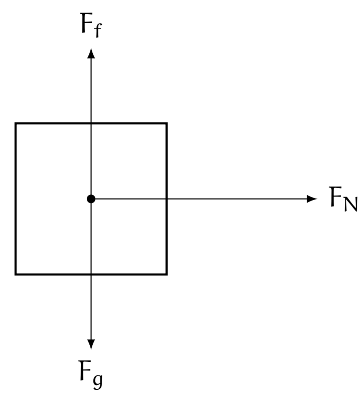
Now finding the equivalent frequency (rotations per second):

$$f = \frac{v}{r} = \frac{0.645\frac{m}{s}}{0.35m} \approx 1.842\text{Hz}$$

So, to break the string, the lifeguard would have to spin it at nearly 2 rotations per second, which is achievable. The lifeguard could possibly break the string.

Answer to Exercise 11 (on page 54)

There are three forces acting on the rider: gravity, friction with the wall, and the normal force with the wall:



The FBD lets us write equations for Newton's Second Law in each dimension:

$$(1)\ \ ma_x = F_N$$

$$(2)\ \ ma_y = F_f - F_g = \mu F_N - mg$$

Because the rider doesn't fall down, we know that $a_y = 0\frac{m}{s^2}$ and therefore equation (2) becomes:

$$\mu F_N - mg = 0 \rightarrow F_N = \frac{mg}{\mu}$$

Having solved for F_N, we substitute for it into equation (1):

$$ma_x = \frac{mg}{\mu} \rightarrow a_x = \frac{g}{\mu}$$

Since we know g and μ, we can calculate a_x:

$$a_x = \frac{9.8\frac{m}{s^2}}{0.32} = 30.625\frac{m}{s^2}$$

This is the minimum acceleration needed to keep the rider from slipping down. We can now use the relationship between centripetal acceleration, tangential velocity, and the radius to find the angular velocity:

$$a = \frac{v^2}{r} = \frac{(\omega r)^2}{r} = \omega^2 r$$

$$\omega = \sqrt{\frac{a}{r}} = \sqrt{\frac{30.625\frac{m}{s^2}}{10\ m}} \approx 1.75\frac{rad}{s}$$

Answer to Exercise 12 (on page 57)

Givens:

- Mass of tetherball: $m = 0.5$ kg
- Length of string: $L = 2.5$ m
- Angle with vertical: $\theta = 38°$

The components of the tension force are separated as follows:

$$T\cos\theta = mg, \qquad T\sin\theta = \frac{mv^2}{r}$$

Dividing the second equation by the first equation gives:

$$\tan\theta = \frac{v^2}{rg} \to v = \sqrt{rg\tan\theta}$$

But we are not given r. We can find r using trigonometry:

$$r = L\sin\theta$$

Now we can find v:

$$v = \sqrt{(L\sin\theta)g\tan\theta}$$

Solving this gives us:

$$v = \sqrt{(2.5\ m\sin 38°)(9.8\frac{m}{s^2})\tan 38°} \approx 3.4\frac{m}{s}$$

The speed of the tetherball is approximately $3.4\frac{m}{s}$.

Answer to Exercise 13 (on page 61)

- At the top of the loop: At the top of the loop, both the gravitational force and the normal force point downward. Therefore, we can write:

  $$F_{net} = F_N + F_g = \frac{mv^2}{r}$$

  Solving for the normal force gives:

  $$F_N = \frac{mv^2}{r} - F_g$$

  Substituting for $F_g = mg$ and $r = \frac{d}{2} = 25$ m:

  $$F_N = \frac{(1200 \text{ kg})(25\frac{m}{s})^2}{25 \text{ m}} - (1200 \text{ kg})(10\frac{m}{s^2})$$

  $$F_N = 30000 \text{ N} - 12000 \text{ N} = 18000 \text{ N}$$

- At the bottom of the loop: At the bottom of the loop, the gravitational force points downward while the normal force points upward. Therefore, we can write:

  $$F_{net} = F_N - F_g = \frac{mv^2}{r}$$

  Solving for the normal force gives:

  $$F_N = \frac{mv^2}{r} + F_g$$

  Substituting for $F_g = mg$ and $r = \frac{d}{2} = 25$ m:

  $$F_N = \frac{(1200 \text{ kg})(25\frac{m}{s})^2}{25 \text{ m}} + (1200 \text{ kg})(10\frac{m}{s^2})$$

  $$F_N = 30000 \text{ N} + 12000 \text{ N} = 42000 \text{ N}$$

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