# Contents

## Chapter 1

# Introduction to Data Visualization

It is difficult for the human mind to look at a list of numbers and identify the patterns in them, so we often use these numbers to make a picture. These pictures are called *graphs*, *charts*, or *plots*. Often, the right picture can make the meaning in the data obvious. *Data visualization* is the process of making pictures from numbers.

## 1.1 Common Types of Data Visualizations

Depending on the type of data and what you are trying to demonstrate about it, you will use different types of data visualizations. How many types of data visualizations are there? Hundreds, but we will concentrate on just four: The bar chart, the line graph, the pie chart, and the scatter plot.

### 1.1.1 Bar Chart

Here is an example of a bar chart.

Each bar represents the cookie sales of one person. For example, Charlie has sold 6 boxes of cookies, so the bar goes over Charlie's name and reaches to the number 6.

Looking at this chart, you probably think, "Wow, Debra has sold a lot more cookies than anyone else, and Francis has sold a lot fewer."

The same data could be in a table like this:

| Salesperson | Boxes Sold |
|---|---|
| Allison | 4 |
| Becky | 5 |
| Charlie | 6 |
| Debra | 12 |
| Elias | 5 |
| Francis | 1 |
| Glenda | 7 |

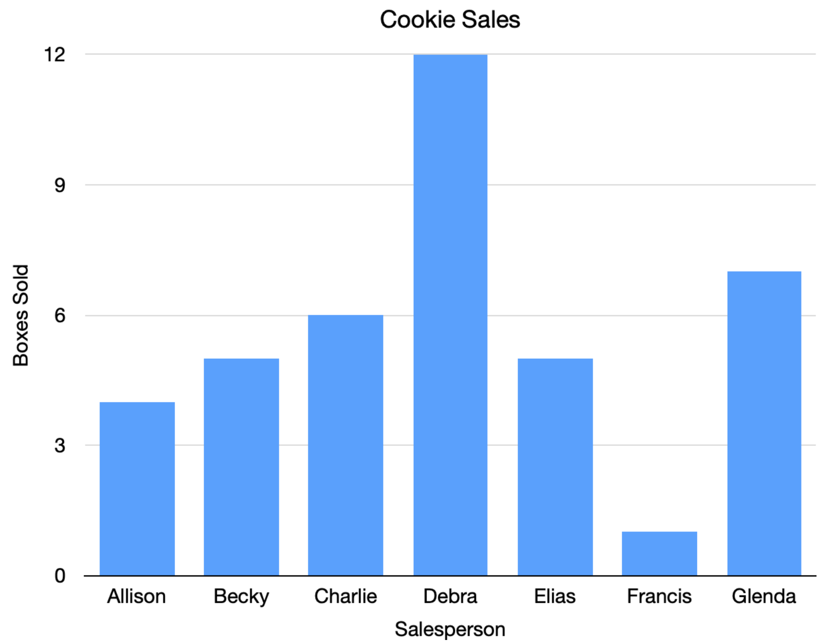



Figure 1.1: A bar chart showing cookie sales per person.

A table (especially a large table) is often just a bunch of numbers. A chart helps our brains understand what the numbers mean.

Bar charts can also go horizontally.

Sometimes we use colors to explain what contributed to the number.

This tells us that Becky sold more boxes of chocolate chip cookies than boxes of oatmeal cookies.

### 1.1.2 Line Graph

Here is a line graph:

These are often used to show trends over time. Here, for example, you can see that the number of shark attacks has been increasing over time.

You can have more than one line on a graph.

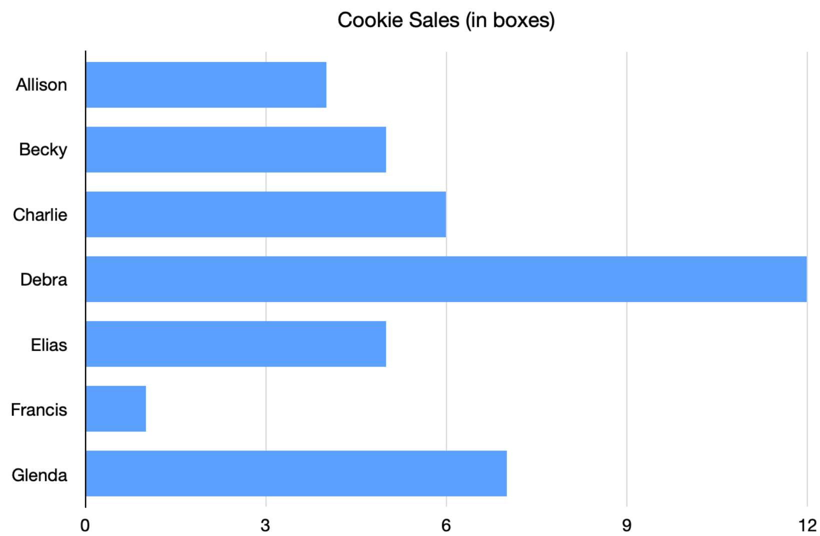



Figure 1.2: A horizontal bar chart showing the same cookie sales data.

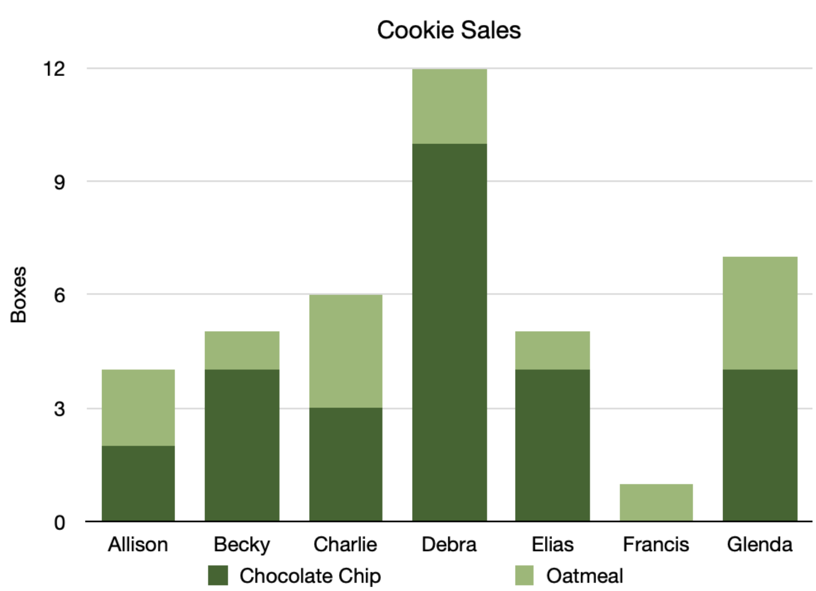



Figure 1.3: A bar chart with different colors showing types of cookes.

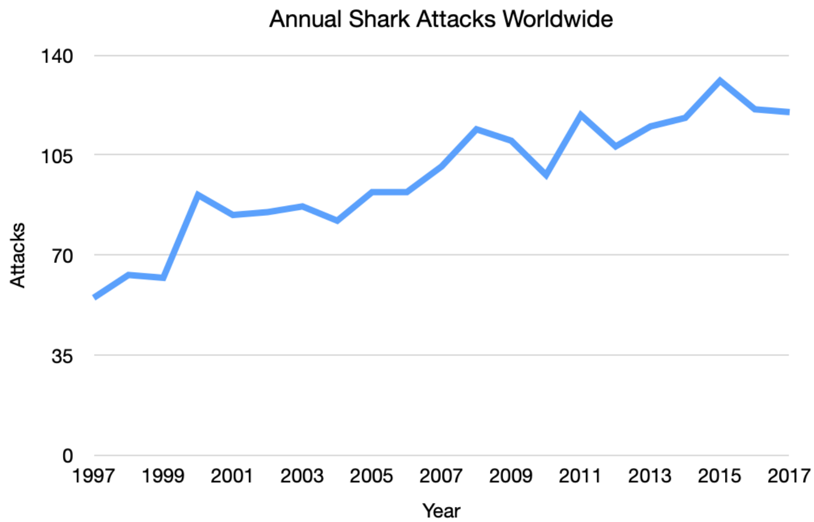



Figure 1.4: A line graph showing shark attacks per year over two decades.

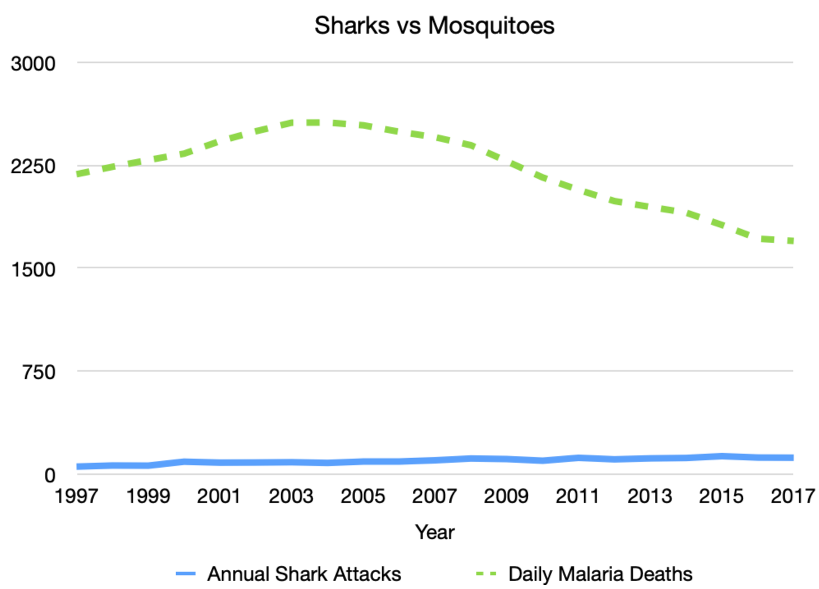



Figure 1.5: A line graph showing shark deaths versus mosquito deaths.

### 1.1.3 Pie Chart

You use a pie chart when you are looking at the comparative size of numbers. This is best for comparing percentages of a whole that sum to 100%. Here we can see that Nitrogen makes up 78% of the gases in the air.

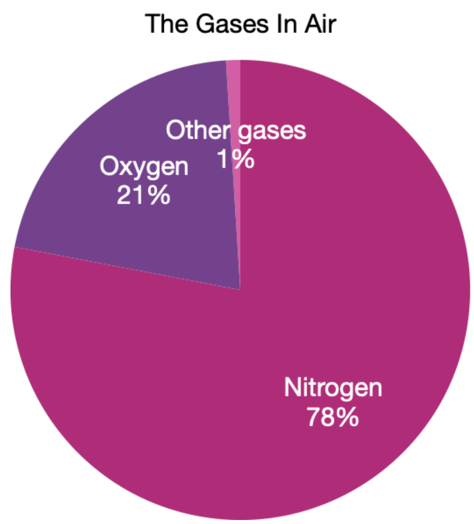



Figure 1.6: A pie chart of the various gases in the air.

### 1.1.4 Scatter Plot

Sometimes, you have a large number of data points with two values, and you are looking for a relationship between them. For example, maybe you write down the average temperature and the total sales for your lemonade stand on the 15th of every month:

| Date | Avg. Temp. | Total Sales |
|---|---|---|
| 15 January 2022 | 2.6º C | $183.85 |
| 15 February 2022 | -4.2º C | $173.56 |
| 15 March 2022 | 13.3º C | $195.22 |
| 15 April 2022 | 26.2º C | $207.61 |
| 15 May 2022 | 27.5º C | $210.88 |
| 15 June 2022 | 31.3º C | $214.18 |
| 15 July 2022 | 33.5º C | $215.23 |
| 15 Aug 2022 | 41.7º C | $224.07 |
| 15 September 2022 | 20.7º C | $198.94 |
| 15 October 2022 | 17.2º C | $196.10 |
| 15 November 2022 | 1.7º C | $185.10 |
| 15 December 2022 | 0.2º C | $188.70 |

You may wonder, "Do I sell more lemonade on hotter days?"

To figure this out, you might create a scatter plot. For each day, you put a mark that represents that temperature and the sales that day:

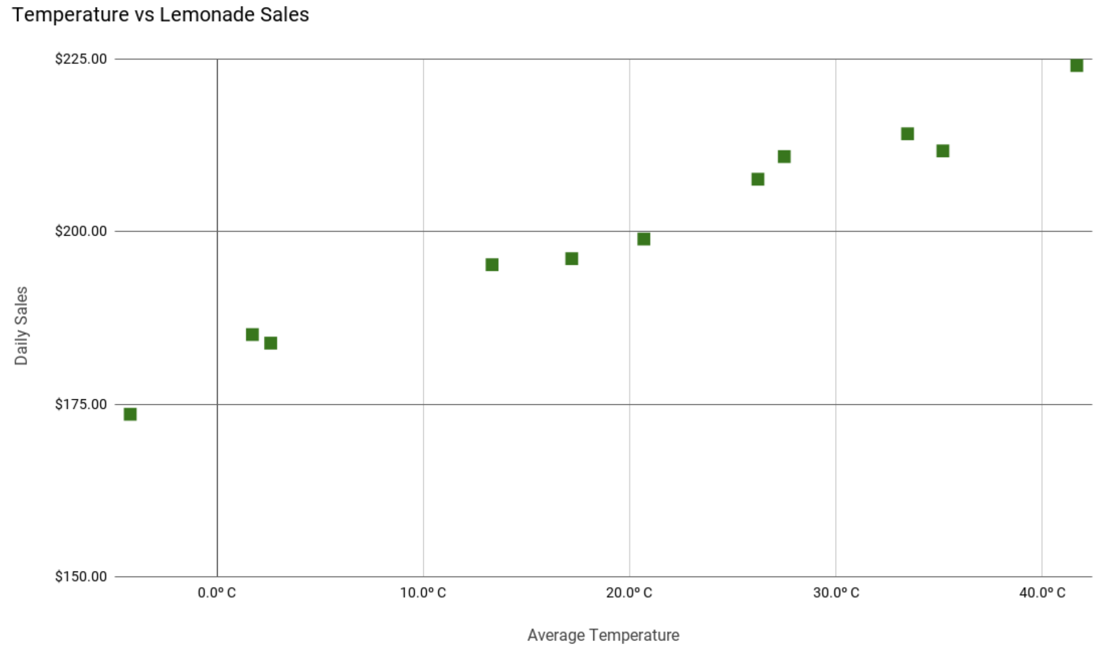



Figure 1.7: A scatterplot of temperature versus daily sales.

From this scatter plot, you can easily see that you do sell more lemonade as the temperature goes up. Drawing a *best-fit* line along the the points will give you a *correlation coefficient*. A positive correlation coefficient will give you a positively proportional relationship, while a negative coefficient will give you a inversely proportional relationship.

## 1.2 Make Bar Graph

Go back to your compound interest spreadsheet and make a bar graph that shows both balances over time:

The year column should be used as the x-axis. There are two series of data that come from C4:C16 and E4:E16. Tidy up the titles and legend as much as you like.

Looking at the graph, you can see the balances start the same, but balance of the account with the larger interest rate quickly pulls away from the account with the smaller interest rate.

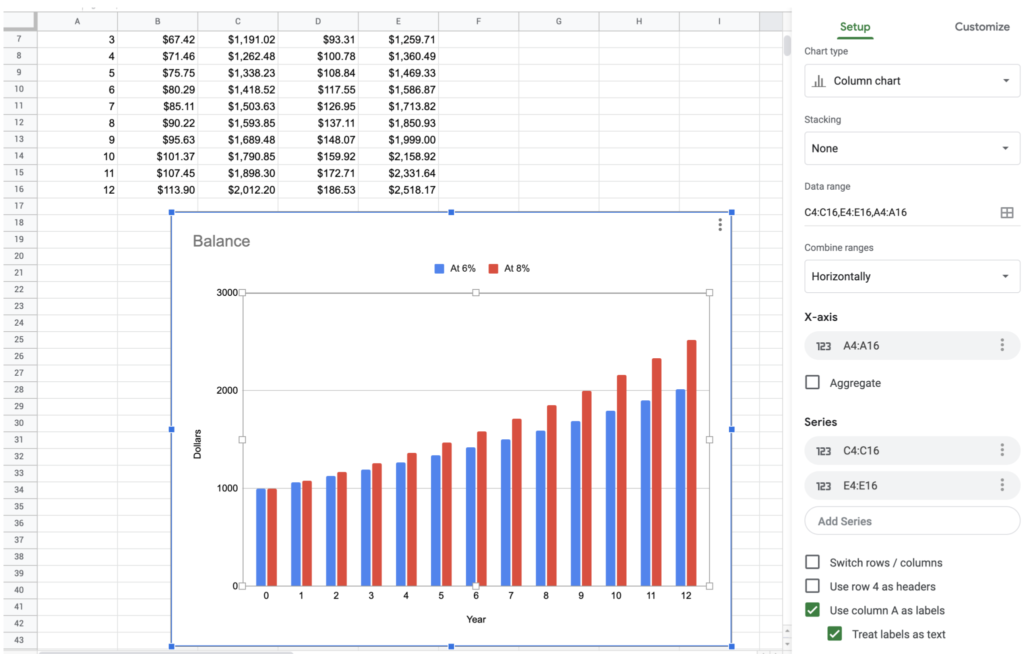



Figure 1.8: A bar graph made in google sheets showing interest.

## Chapter 2

# The Dot Product

If you have two vectors $\mathbf{u} = [u_1, u_2, \ldots, u_n]$ and $\mathbf{v} = [v_1, v_2, \ldots, v_n]$, we define the *dot product* $\mathbf{u} \cdot \mathbf{v}$ as

$$\mathbf{u} \cdot \mathbf{v} = (u_1 \times v_1) + (u_2 \times v_2) + \cdots + (u_n \times v_n)$$

The output of the dot product is a *scalar* quantity.

For example,

$$[2, 4, -3] \cdot [5, -1, 1] = 2 \times 5 + 4 \times -1 + -3 \times 1 = 3$$

This may not seem like a very powerful idea, but dot products are *incredibly* useful. The enormous GPUs (Graphics Processing Units) that let video games render scenes so quickly? They primarily function by computing huge numbers of dot products at mind-boggling speeds.

### *Exercise 1* Basic dot products

Compute the dot product of each pair of vectors:

- $[1, 2, 3]$, $[4, 5, -6]$
- $[\pi, 2\pi]$, $[2, -1]$
- $[0, 0, 0, 0]$, $[10, 10, 10, 10]$

*Working Space*

*Answer on Page 51*

## 2.1 Properties of the dot product

Sometimes we need an easy way to say "The vector of appropriate length is filled with zeros." We use the notation $\vec{0}$ to represent this. Then, for any vector $\mathbf{v}$, this is true:

$$\mathbf{v} \cdot \vec{0} = 0$$

The dot product is commutative:

$$\mathbf{v} \cdot \mathbf{u} = \mathbf{u} \cdot \mathbf{v}$$

The dot product of a vector with itself is its magnitude squared:

$$\mathbf{v} \cdot \mathbf{v} = |\mathbf{v}|^2$$

If you have a scalar $a$, then:

$$(\mathbf{v}) \cdot (a\mathbf{u}) = a(\mathbf{v} \cdot \mathbf{u})$$

So, if **v** and **w** are vectors that go in the same direction,

$$\mathbf{v} \cdot \mathbf{w} = |\mathbf{v}||\mathbf{w}|$$

If **v** and **w** are vectors that go in opposite directions,

$$\mathbf{v} \cdot \mathbf{w} = -|\mathbf{v}||\mathbf{w}|$$

If **v** and **w** are vectors that are perpendicular to each other, their dot product is zero:

$$\mathbf{v} \cdot \mathbf{w} = 0$$

## 2.2 Cosines and dot products

Furthermore, dot products' interaction with cosine makes them even more useful is what makes them so useful: If you have two vectors **v** and **u**,

$$\mathbf{v} \cdot \mathbf{u} = |\mathbf{v}||\mathbf{u}| \cos \theta$$

where $\theta$ is the angle between them.

So, for example, if two vectors **v** and **u** are perpendicular, the angle between them is $\pi/2$. The cosine of $\pi/2$ is 0. The dot product of any two perpendicular vectors is always 0. In fact, if the dot product of two non-zero vectors is 0, the vectors *must be* perpendicular (see figure 2.1 for an example of perpendicular 2-dimensional vectors).

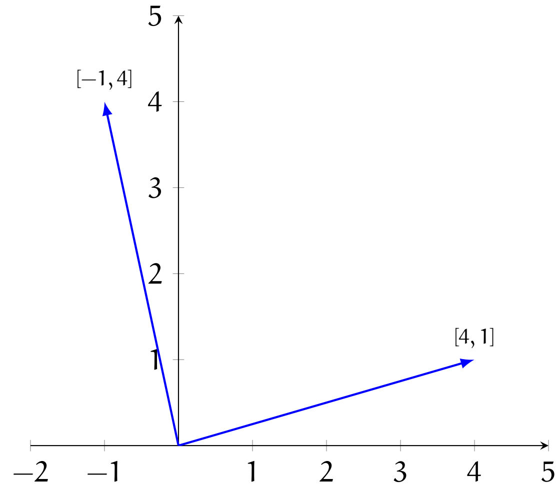



Figure 2.1: The dot product of any two perpendicular vectors is zero.

If you have two non-zero vectors **v** and **u**, you can always compute the angle between them:

$$\theta = \arccos\left(\frac{\mathbf{v} \cdot \mathbf{u}}{|\mathbf{v}||\mathbf{u}|}\right)$$

Arccos is short for arccosine, or $\cos^{-}1$, and it is a function that is the inverse of cosine. Cosine takes an angle and gives back the scaled $x$-component of the angle. Arccosine takes the $x$-component of an angle and returns an angle with that $x$-component. However, there is a limit to what arccos can return. Let's look at cosine and its inverse, arccos (see figures 2.2 and 2.3).

When you use a calculator to evaluate arccos, the calculator automatically restricts the results to between 0 and $\pi$. Let's look at an example of using the dot product to determine the angle between two vectors:

**Example**: What is the angle between $\mathbf{u} = \left[\sqrt{3}, 1\right]$ and $\mathbf{v} = [0, -1]$?

**Solution**: We know that $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}|\,|\mathbf{v}| \cos\theta$. Therefore, we also know that:

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|}$$

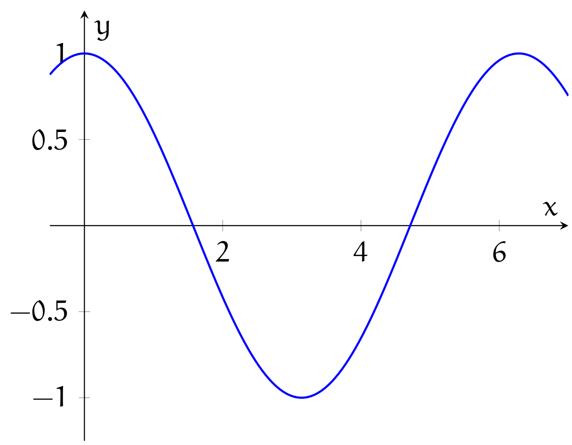



Figure 2.2: Cosine is a function: there is exactly one output for every input.

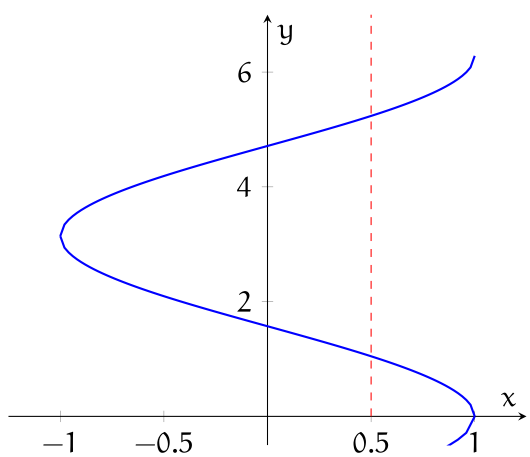



Figure 2.3: Arccos is not a function: there are many angles with the same $x$-component. Notice that one input value has many output values (see the red dashed line).

First, let's compute the dot product:

$$\mathbf{u} \cdot \mathbf{v} = \sqrt{3} \cdot 0 + 1 \cdot -1 = -1$$

And therefore:

$$\cos\theta = \frac{-1}{|\mathbf{u}|\,|\mathbf{v}|}$$

Now, let's find the magnitudes of both vectors:

$$|\mathbf{u}| = \sqrt{\left(\sqrt{3}\right)^2 + (1)^2} = 2$$

$$|\mathbf{v}| = \sqrt{(0)^2 + (-1)^2} = 1$$

Substituting for the magnitudes, we find that:

$$\cos\theta = \frac{-1}{2 \cdot 1} = \frac{-1}{2}$$

To solve for $\theta$, we take the arccos of both sides:

$$\arccos(\cos\theta) = \theta = \arccos\frac{-1}{2}$$

What angles have a cosine of $-1/2$? We know that $2\pi/3$, $4\pi/3$, $8\pi/3$, etc., all have a cosine of $-1/2$. Because the range of arccos is restricted to between 0 and $\pi$, our result is:

$$\theta = \arccos\frac{-1}{2} = \frac{2\pi}{3}$$

.

Therefore, the angle between **u** and **v** is $2\pi/3$ (or $120°$).

### *Exercise 2* Using dot products

What is the angle between these each pair of vectors:

- $[1, 0]$, $[0, 1]$
- $[3, 4]$, $[4, 3]$
- $[2, -1, 2]$, $[-1, 2, -2]$
- $[-5, 0, -1]$, $[2, 3, -4]$

*Working Space*

*Answer on Page 51*

## 2.3 Dot products in Python

NumPy will let you do dot products using the the symbol @. Open `first_vectors.py` and add the following to the end of the script:

```
# Take the dot product
d = v @ u
print("v @ u =", d)

# Get the angle between the vectors
a = np.arccos(d / (mv * mu))
print(f"The angle between u and v is {a * 180 / np.pi:.2f} degrees")
```

When you run it you should get:

```
v @ u = 4
The angle between u and v is 78.55 degrees
```

## 2.4 Work and Power

Earlier, we mentioned that mechanical work is the product of the force you apply to something and the amount it moves. For example, if you push a train with a force of 10 newtons as it moves 5 meters, you have done 50 joules of work.

What if you try to push the train sideways? It moves down the track 5 meters, but you push it as if you were trying to derail it — perpendicular to its motion. You have done no work, because the train didn't move at all in the direction you were pushing.

Now that you know about dot products: The work you do is the dot product of the force vector you apply and the displacement vector of the train. (The displacement vector is the vector that tells how the train moved while you pushed it.)

Similarly, we mentioned that power is the product of the force you apply and the velocity of the mass you are applying it to. It is actually the dot product of the force vector and the velocity vector.

For example, if you are pushing a sled with a force of 10 newtons and it is moving 2 meters per second, but your push is 20 degrees off, you aren't transferring 20 watts of power to the sled. You are transferring $10 \times 2 \times \cos(20 \text{ degrees}) \approx 18.8$ watts of power.

Chapter 3

# Manufacturing

If you try to think of any man-made object, whether it was made from woods, metals, or plastics, chances are it was produced through a manufacturing process.

Over time, these processes have been refined to be more efficient, cost-effective, and faster at producing the goods that we use on a daily basis.

New methods are also constantly being developed by engineers and scientists, and today the range of options available means that choosing the most appropriate manufacturing method involves finding the sweet spot between cost-effectiveness, yield, and time needed.

## 3.1 Woods and Metals Processes

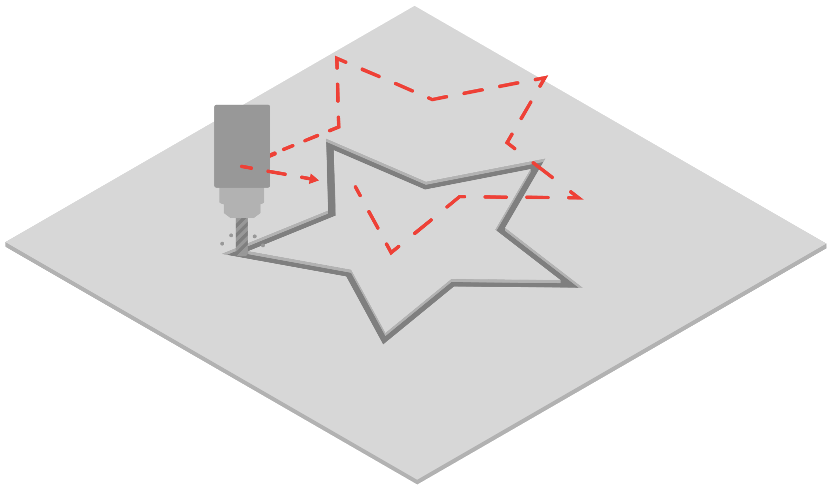
Manufacturing methods that are able to process woods and metals are typically the processes that are used to construct the vast majority of the built world around us.

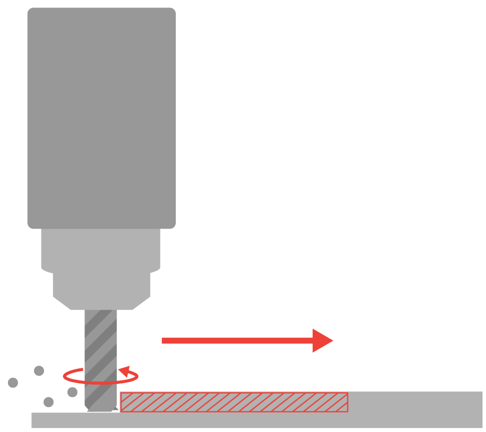
Infrastructure, transportation methods, and buildings would not exist without the advent of processes that allow us to accurately machine raw woods and metals into our desired forms.

The following sections will cover methods used to process both material types.

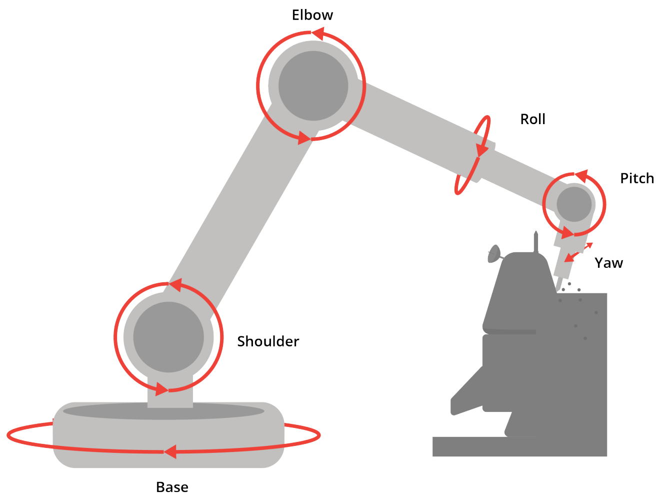
### 3.1.1 Milling

Mills are powerful tools that allow us to carve out complex shapes from blocks of raw material. A tool bit follows a path to remove the desired material, which makes it a *subtractive* manufacturing process. The tool bit rotates at a very high speed, which allows it to process harder materials such as woods and metals.





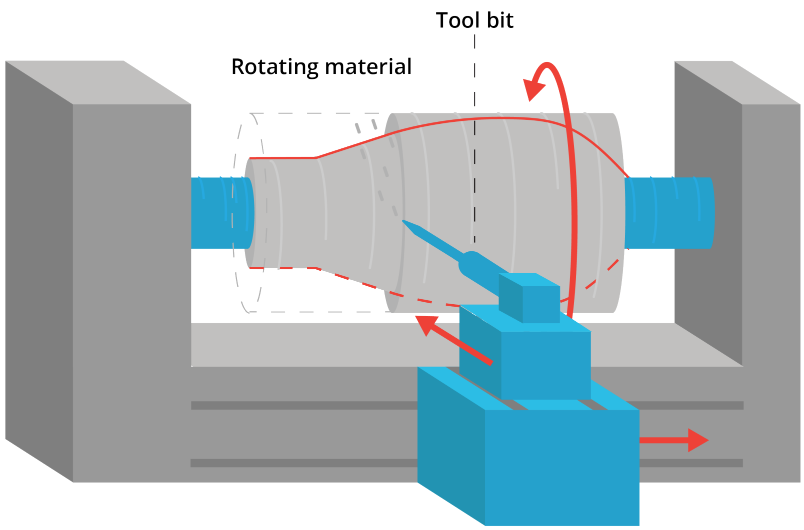
There are various types of mills, ranging from a 3-axis mill, which can cut out simple shapes in the X, Y, and Z axes, all the way to 6-axis mills, which can also rotate about those axes to create more complex curvatures.




In manufacturing settings where speed and repeatability is paramount, mills are often computer controlled. This functionality is referred to as *Computer Numerical Control*, or CNC. CNC mills are able to repeatedly follow a tool path, resulting in consistent and accurate parts.

### 3.1.2 Lathing

Lathes are tools that allow us to carve out complex cylindrical shapes from raw material. Like a mill, it also is a subtractive manufacturing process. However, lathes rotate the material itself at a high speed, rather than the tool bit. As the material rotates, the tool bit can be used to extract material layer by layer.




In the manufacturing industry, lathes are also often computer controlled. Alongside CNC mills, these CNC lathes are responsible for a majority of the objects that we interact with everyday.

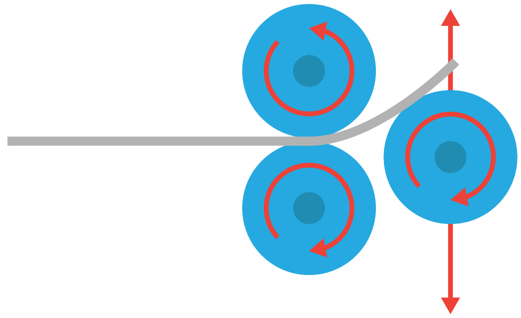
## 3.2 Metal-specific Processes

While mills and lathes are already able to cover the vast majority of manufacturing needs for woods and metals, there are certain processes that are specifically enabled by the unique properties of metals. More often than not, these processes leverage the malleability (ability to bend without breaking) of metals at room temperature or higher.

### 3.2.1 Sheet Metal Processes

While milling allows us to process blocks of metal to great effect, sometimes the application we need does not require material of such thickness.

This is where sheet metal comes in, as well as the methods we use to process it. One of the most common techniques in manufacturing is rolling, where a raw sheet is gradually rolled into the desired shape. This method is used to create many objects you may recognize, such as metal roofings, aircraft frames, and more.

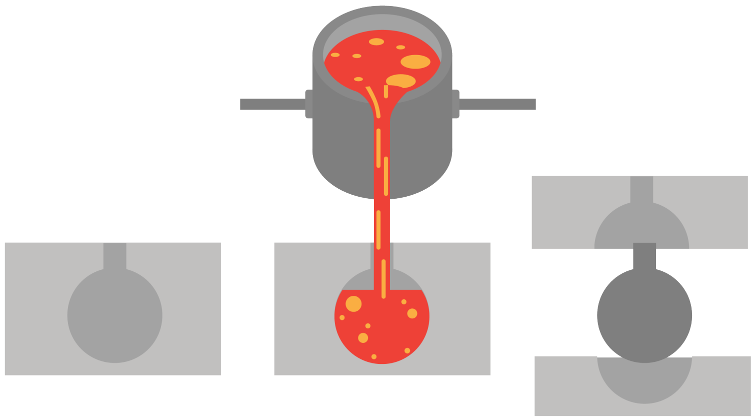


Another method is stamping, where a raw sheet is stamped into the desired shape. This allows us to create surfaces with complex geometries in an instant, and in large quantities. This method is often used for applications like the exterior panels of a car, where parts with compound curves are needed.

Lastly, there are also separate processes used to increase the strength of sheet metal parts. This falls under the category of sheet metal forming, and involves bending the edges of a part to form a reinforced flap. Almost all sheet metal parts are reinforced in this manner, as it is a relatively simple process and also helps to create a clean edge for a more finished look.

### 3.2.2 Casting

The last kind of metal-specific process we will cover is casting. Casting involves pouring molten metal into a mold, then letting the metal cool and set inside the mold. Smaller components with complex geometries and limited structural requirements (such as toys) are often cast, as it is an accurate and high-volume manufacturing method.



Casting also results in minimal material wastage, as it is not a subtractive manufacturing method where excess material is machined away, but rather only the specific amount of material required is poured in each time.

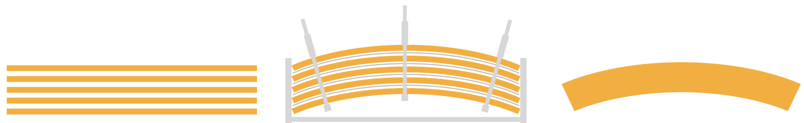
## 3.3 Wood-specific Processes

Similar to metals, there are also certain manufacturing processes that are enabled by the unique qualities of wood. These make use of the water content inherent in wood, and the flexibility it enables.

### 3.3.1 Bending

Turning raw wood into flat, workable pieces involves a variety of tools that you probably know of already, such as saws and drills. However, there are specific processes that help us create curved shapes with wood, in addition to the mill and the lathe mentioned earlier.

This is where bent lamination comes in. Bent lamination involves layering multiple thin veneers or strips of wood with adhesive, and clamping it to create the desired form while letting the glue dry. This method is often used for furniture production, enabling continuous curves in wood with tight radii.




Steam can also be used for bending, by helping soften the wood fibers to increase flexibility. Once the desired form is reached, the part can then cool down and harden. Unlike bent lamination, steam bending can be done without adhesives.

## 3.4 Plastic-specific Processes

Although plastics only came into prominence in the mid 20th century, they have changed manufacturing and, by extension, the world as we know it. Easily manufacturable, durable, and cost-effective, they have come to permeate almost everything we use on a daily basis.

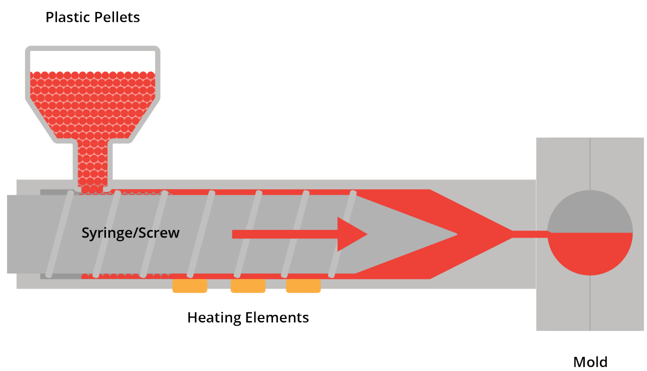
It must also be noted that these exact qualities have also resulted in the proliferation of plastics in our environment, and as such, usage of plastics should be well considered and limited. Alternative biodegradable materials are currently being trialed by scientists and would look to replicate many of the same qualities we expect from plastics, including its manufacturability.

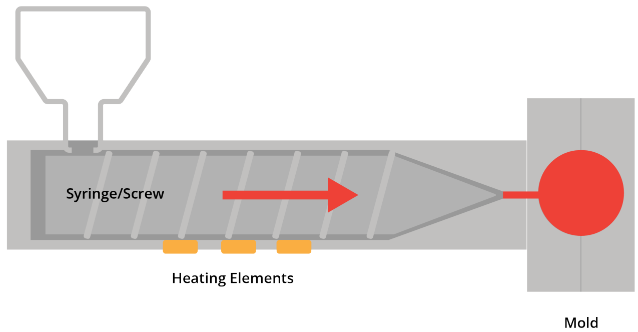
### 3.4.1 Injection Molding

Injection molding is responsible for the vast majority of plastic products that you interact with on a daily basis. It is extremely quick, highly accurate, and has minimal material wastage, making it a popular and cost effective method of manufacturing plastic goods.

Similar to casting, injection molding involves injecting molten plastic into a mold, then allowing the part to cool and set inside the cavity.

You can often tell when a part was produced through injection molding, with telltale signs such as the parting line. This is where the parts of the mold meet, forming a visible line on the surface of a part.

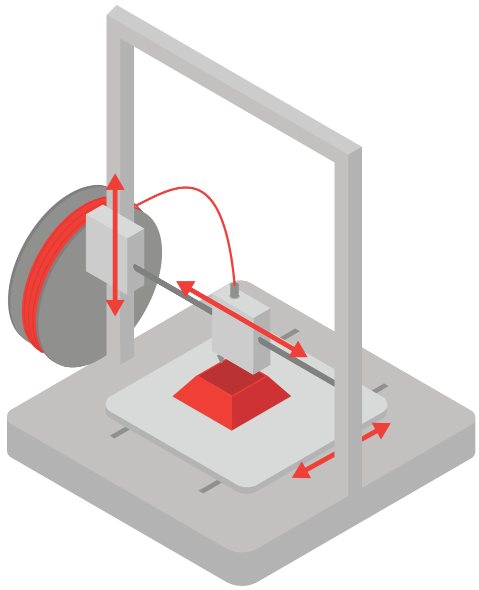



## 3.5 3D Printing

Injection molding has traditionally been the go-to technique for manufacturing plastic goods. However, new technologies result in the advent of new manufacturing methods. 3D printing is one such method, having come to prominence in the last few decades as a way to quickly prototype parts without having to create the molds needed for injection molding.

3D printing is an *additive* manufacturing process, where molten material is applied layer by layer to form the desired geometry. It allows for complex geometries, standardized batch production, and whilst the accuracy may currently lag behind traditional injection molding, it is also improving rapidly.

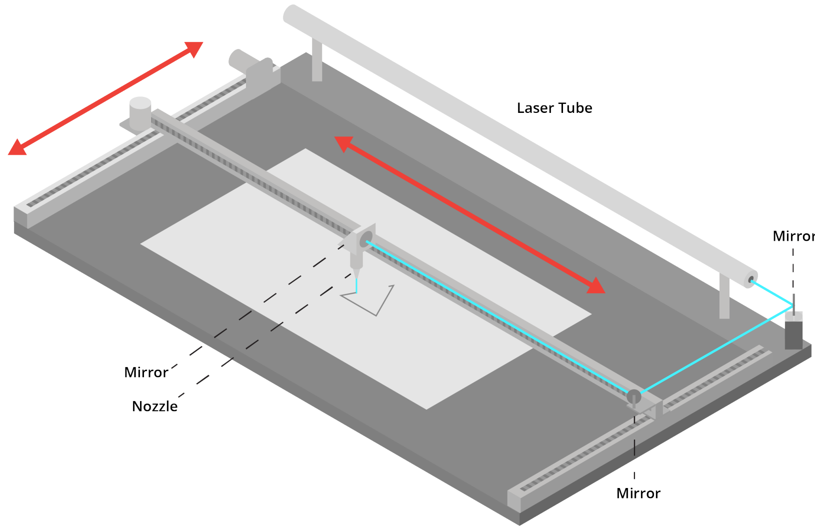


## 3.6 Laser Cutting and Water Jet

Similarly, another manufacturing method enabled by new technologies is laser cutting. Like 3D printing, it has come into prominence as a method to quickly prototype parts. However, it is not an additive manufacturing process.

Instead laser cutting uses a laser beam to cut and etch through sheets of plastic, however it can also be used for other materials such as fabrics and card stocks. Laser cutting is mostly limited by material thickness, and as such can only cut through thinner sheets of material.

Similarly, water jetting uses pressurized water to blast a stream of water through material (metal, wood, stone, or rubber usally). As the nozzle moves, the high pressure water traces a path throughout the material. Water jets are also limited to material thickness, as too thick of material may not easily be cut.


Laser Tube
Mirror
Mirror
Nozzle
Mirror

## Chapter 4

# Polar Coordinates

We have already seen how to plot a function with $(x, y)$ coordinates. For every $x$ that we put into a function, it returns a $y$. These pairs of coordinates tell us where on the $xy$-plane to graph the function. This coordinate system, where $x$ and $y$ are oriented horizontally and vertically, is called the *Cartesian* coordinate system. It can be used to describe 2D space, but it is not the only way.

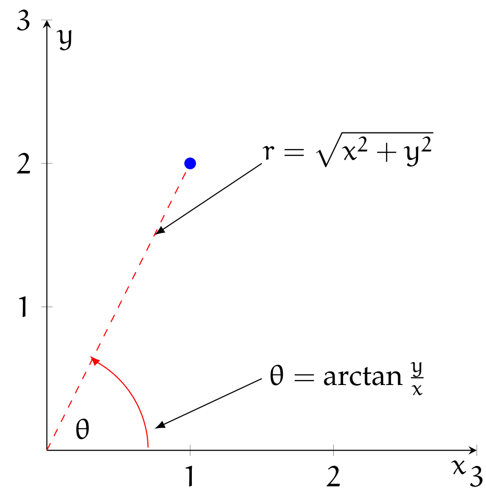



Figure 4.1: The point $(1, 2)$ is $\sqrt{5}$ units from the origin and approximately 1.107 radians counterclockwise from horizontal

Instead of thinking about the horizontal and vertical position, we could think about distance from the origin and rotation about the origin. Take the Cartesian coordinate point $(1, 2)$ (see figure 4.1). How far is $(1, 2)$ from the origin, $(0, 0)$? We can create a right triangle, where the legs are parallel to the $x$ and $y$ axes. This means the leg lengths are 1 and 2, and we can use the Pythagorean theorem to find the length of the hypotenuse (which is the distance from the origin to the point):

$$c^2 = a^2 + b^2$$

$$c^2 = 1^2 + 2^2 = 1 + 4 = 5$$

$$c = \sqrt{5}$$

Therefore, the Cartesian point $(1, 2)$ is $\sqrt{5}$ units from the origin. This is not enough to find our point: there are infinite points that are $\sqrt{5}$ from the origin (see 4.2). To identify a particular point that is a distance of $\sqrt{5}$ from the origin, we also need an *angle of rotation*. By convention, angles are measured from the positive $x$- axis. This means points on the positive $x$-axis have an angle of $\theta = 0$, points on the positive $y$-axis have an angle of $\theta = \frac{\pi}{2}$, and so on.

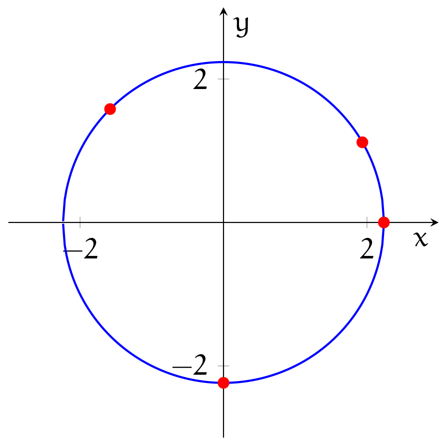



Figure 4.2: There are infinite points $\sqrt{5}$ from the origin, represented by the circle with a radius of $\sqrt{5}$ centered about the origin

We can use trigonometry to find the appropriate angle of rotation for our Cartesian point. There are many ways to do this, but using arctan is the most straightforward. Recall that:

$$\tan\theta = \frac{opposite}{adjacent}$$

That is, for a given angle in a right triangle, the tangent of that angle is given by the length of the opposite leg divided by the adjacent leg. In our case, the opposite leg is the vertical distance ($y$-value of the Cartesian point) and the adjacent leg is the horizontal distance ($x$-value of the Cartesian point), which means:

$$\tan\theta = \frac{2}{1}$$

$$\theta = \arctan 2 \approx 1.107 \text{ radians}$$

## 4.1 Plotting Polar Coordinate Points

How do we plot polar coordinate points? Begin by locating the angle given by the second coordinate (remember, the angle is measured counterclockwise from the horizontal). Your point will lie somewhere on this line. Next, move outwards along the angle by the radius given by the first coordinate.

**Example**: Plot the polar coordinate point $(2, \frac{\pi}{3})$.

**Solution**: Begin by locating $\theta = \frac{\pi}{3}$ (see figure 4.3)

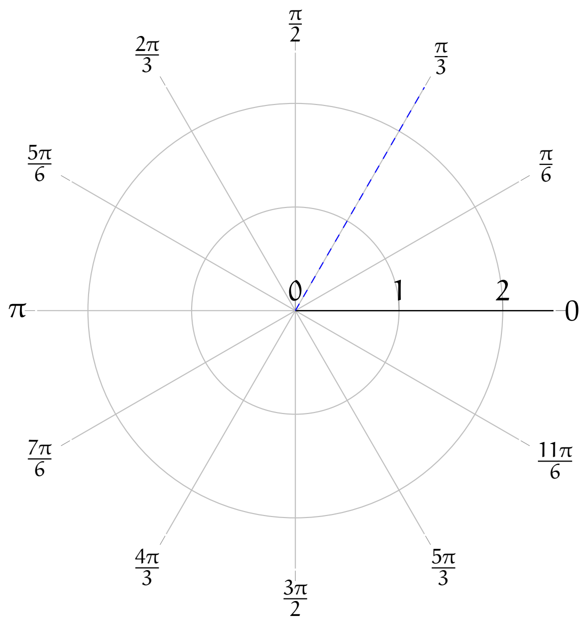



Figure 4.3: $\theta = \frac{\pi}{3}$

Then, move your finger or pencil along the line $\theta = \frac{\pi}{3}$ until you reach $r = 2$ (see figure 4.4).

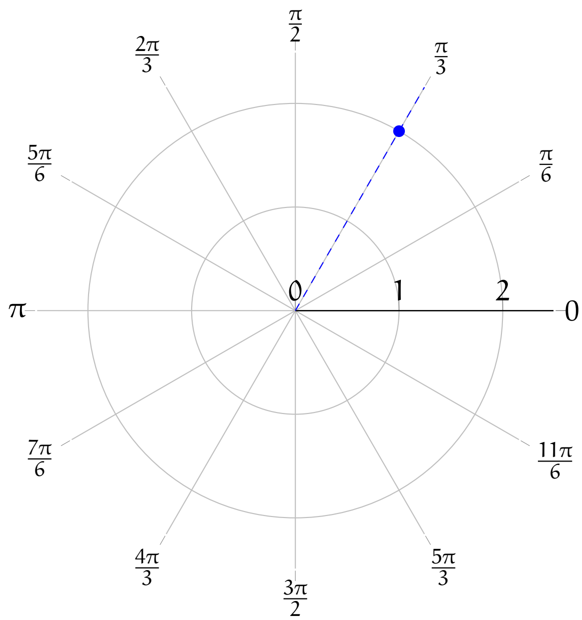
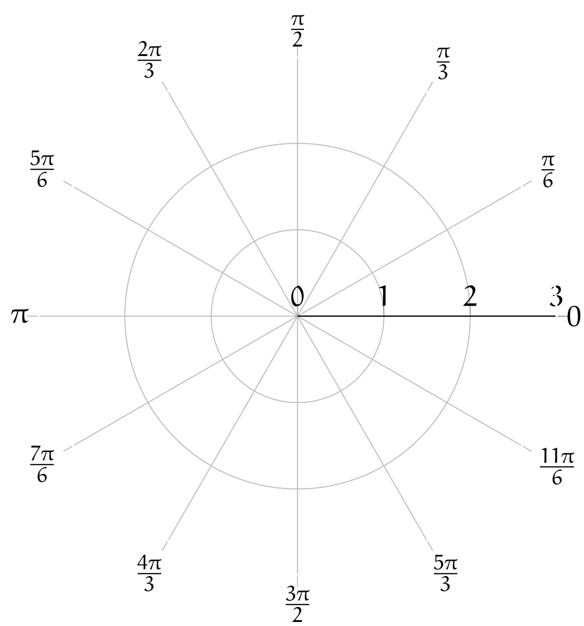



Figure 4.4: $(2, \frac{\pi}{3})$

### *Exercise 3*

Plot the following polar coordinate points on the provided polar axis (hint: negative angles are taken counterclockwise):

1. $(1, \pi)$
2. $(1.5, \frac{\pi}{2})$
3. $(1.5, -\frac{\pi}{6})$
4. $(2, \frac{3\pi}{4})$




*Working Space*

*Answer on Page 52*

## 4.2 Equivalent Points

Unlike the Cartesian coordinate system, two different coordinates may lie at the same location. Consider the points $(1, \frac{\pi}{4})$ and $(-1, \frac{5\pi}{4})$ (see figure 4.5). When a radius is negative, you move *backwards* back over the origin, like a mirror image.

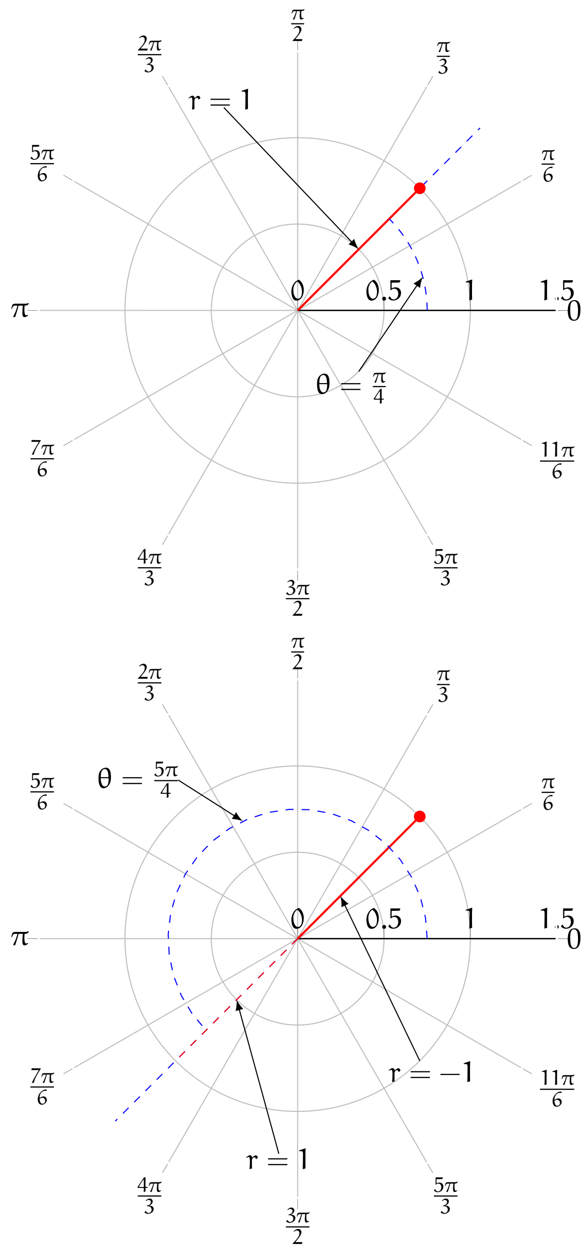



Figure 4.5: The polar coordinates points $(1, \frac{\pi}{4})$ and $(-1, \frac{5\pi}{4})$ are the same location on a polar axis

## 4.3 Changing coordinate systems

### 4.3.1 Cartesian to Polar

From the example above, you should see that a given Cartesian coordinate, $(x, y)$, can also be expressed as a polar coordinate, $(r, \theta)$, where $r$ is the distance from the origin and $\theta$ is the angle of rotation from the horizontal. (Note: Polar functions are generally given as $r$ defined in terms of $\theta$, which means the *dependent* variable is listed first in the coordinate pair, unlike Cartesian coodinates.) Additionally,

$$r = \sqrt{x^2 + y^2}$$

$$\theta = \arctan \frac{y}{x}$$

**Example**: Express the Cartesian point $(-3, 4)$ in polar coordinates.

**Solution**: Taking $x = -3$ and $y = 4$, we find that:

$$r = \sqrt{(-3)^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5$$

We follow the convention of only taking the positive solution to the square root. Finding $\theta$:

$$\theta = \arctan \frac{4}{-3}$$

When you evaluate the arctan with a calculator, you are likely to get back $\theta = -0.928$. Recall that $tan\theta = \tan\theta \pm n\pi$, where $n$ is an integer. We know our Cartesian point, $(-3, 4)$, is in the II quadrant, while the angle $-0.928$ radians would fall in the IV quadrant. So, clearly, $-0.928$ radians is not correct. Most calculators restrict the output of arctan to angles between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, because there are actually multiple angles where $\tan\theta = -\frac{4}{3}$. Since $\tan\theta = \tan\theta \pm n\pi$, we also know that:

$$\arctan -\frac{4}{3} = -0.928 \pm n\pi$$

Another possible $\theta$ is $-0.928 + \pi \approx 2.214$, which does fall in the appropriate quadrant. This means the polar coordinates $(5, 2.214)$ are the same as the Cartesian coordinates $(-3, 4)$. *Note:* It is standard practice to express angles in radians, and not degrees, when using polar coordinates.

### 4.3.2 Polar to Cartesian

We can also leverage our knowledge of right triangles to convert polar coordinates to Cartesian coordinates. Take the polar coordinate $(2, \frac{\pi}{4})$ (see figure 4.6). We can draw a right triangle with legs parallel to the $x$ and $y$ axes (not shown in the figure) and a hypotenuse that goes from the origin to the polar coordinate $(2, \frac{\pi}{4})$.

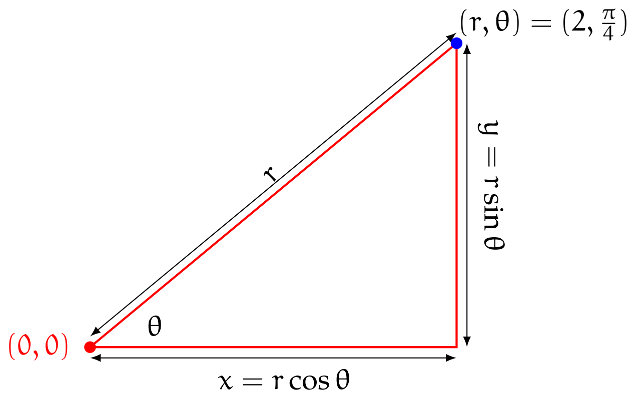



Figure 4.6: To convert from polar to Cartesian coordinates, use the identities $x = r\cos\theta$ and $y = r\sin\theta$

Recall from trigonometry that:

$$\sin\theta = \frac{\text{opposite leg}}{\text{hypotenuse}}$$

We know that the hypotenuse of this triangle has a length of $r$. The opposite leg is vertical and is the same length as the distance of the polar coordinate from the $x$-axis. Therefore, the length of the vertical leg represents the $y$ value of that same polar coordinate if it were expressed in Cartesian coordinates. So, we can say that:

$$\sin\theta = \frac{y}{r}$$

And therefore:

$$y = r\sin\theta$$

By a similar process, we also see that:

$$x = r\cos\theta$$

This is easy to visualize and understand for $0 \leq \theta \leq \frac{\pi}{2}$, but it also holds for other values of $\theta$.

**Example**: Express the polar coordinate $(\frac{3}{2}, \frac{2\pi}{3})$ in Cartesian coordinates.

**Solution**: From the polar coordinate, we see that $\theta = \frac{2\pi}{3}$ and $r = \frac{3}{2}$. Therefore:

$$x = r\cos\theta = \frac{3}{2} \cdot \cos\frac{2\pi}{3} = \frac{3}{2} \cdot -\frac{1}{2} = -\frac{3}{4}$$

$$y = r\sin\theta = \frac{3}{2} \cdot \sin\frac{2\pi}{3} = \frac{3}{2} \cdot \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{4}$$

The Cartesian coordinate $(-\frac{3}{4}, \frac{3\sqrt{3}}{4})$ has the same location as the given polar coordinate.

## Exercise 4

Convert the following polar coordinates to Cartesian coordinates:

1. $(2, \frac{3\pi}{2})$
2. $(\sqrt{2}, \frac{3\pi}{4})$
3. $(3, -\frac{\pi}{4})$
4. $(-3, -\frac{\pi}{3})$
5. $(2, -\frac{\pi}{2})$

*Working Space*

*Answer on Page 52*

### Exercise 5

Convert the following Cartesian coordinates to polar coordinates. Restrict $\theta$ to $0 \leq \theta < 2\pi$.

1. $(-4, 4)$
2. $(3, 3\sqrt{3})$
3. $(\sqrt{3}, -1)$
4. $(-6, 0)$
5. $(-2, -2)$

**Working Space**

*Answer on Page 52*

## 4.4 Circles in Polar Coordinates

Many conic sections, including circles, are simpler to express as polar functions than as Cartesian functions. Consider a circle with a radius of 2 centered about the origin. The polar function for this is $r = 2$ for all $\theta$. Let's write a Cartesian function for the same circle.

We know that for every point on the circle, the distance to the origin is 2. This means that, by the Pythagorean theorem,

$$r^2 = x^2 + y^2$$

.

(see figure 4.7)

We can solve this equation for $y$, given that $r = 2$ (in this case):

$$y = \pm\sqrt{2^2 - x^2}$$

Notice that this is really two equations: $y = \sqrt{2^2 - x^2}$ and $y = -\sqrt{2^2 - x^2}$. This is more complex than the polar equation, $r = 2$.

As seen above, the equation of a circle with radius $R$ centered on the origin is simply $r = R$

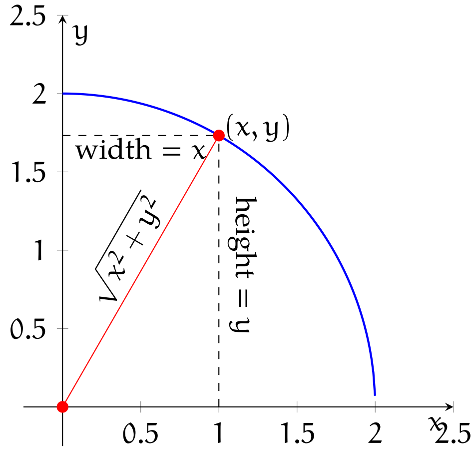



Figure 4.7: All $(x, y)$ pairs on the circle are the same distance from the origin

in polar coordinates. What if we want a circle centered somewhere else? Polar coordinates are best when a circle is bisected by the $x$ or $y$ axis. Consider the polar equation $r = 3\sin\theta$. Let's use a table to find some points and plot the function:

| $\theta$ | $r = 3\sin\theta$ |
|---|---|
| $0$ | $0$ |
| $\frac{\pi}{6}$ | $\frac{3}{2}$ |
| $\frac{\pi}{4}$ | $\frac{3\sqrt{2}}{2}$ |
| $\frac{\pi}{3}$ | $\frac{3\sqrt{3}}{2}$ |
| $\frac{\pi}{2}$ | $3$ |
| $\frac{2\pi}{3}$ | $\frac{3\sqrt{3}}{2}$ |
| $\frac{3\pi}{4}$ | $\frac{3\sqrt{2}}{2}$ |
| $\frac{5\pi}{6}$ | $\frac{3}{2}$ |
| $\pi$ | $0$ |

Here is how those points look plotted (see figures 4.8 and 4.9):

So, the polar equation $r = 3\sin\theta$ gives a circle with radius $\frac{3}{2}$ centered at $(0, \frac{3}{2})$.

**Example**: Describe the graph of $r = \cos\theta$. Feel free to make a rough plot on the blank polar axis below:

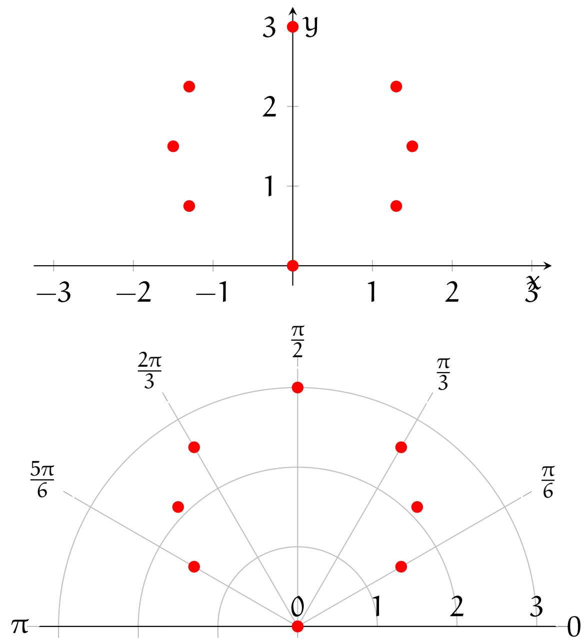



Figure 4.8: Several points for $r = 3 \sin \theta$ plotted on Cartesian and polar coordinate systems

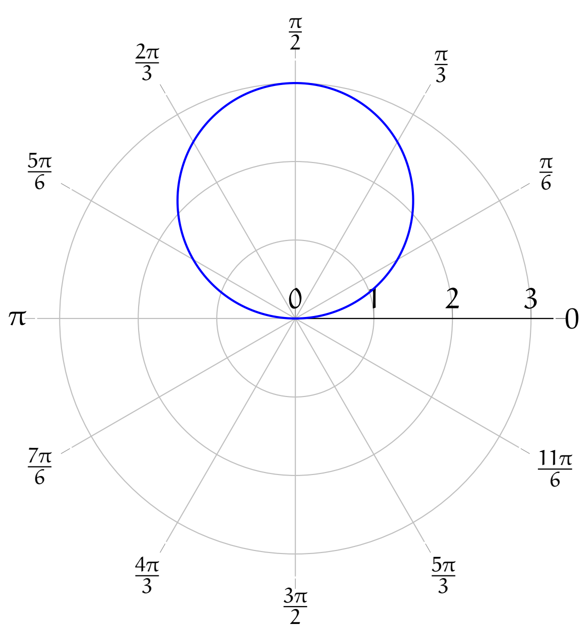
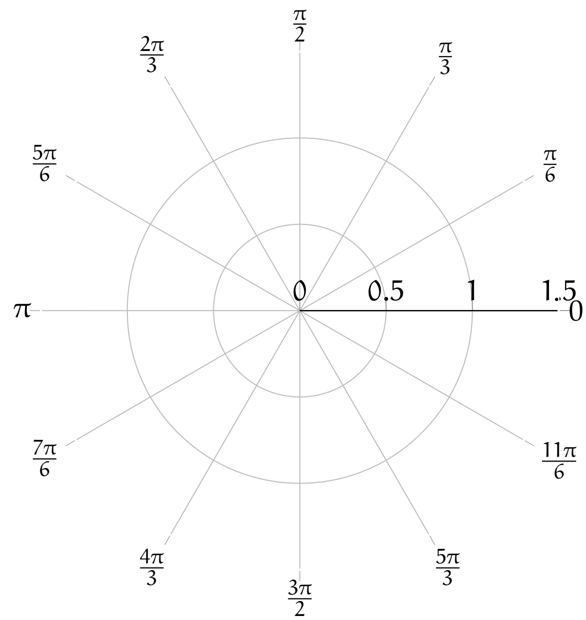
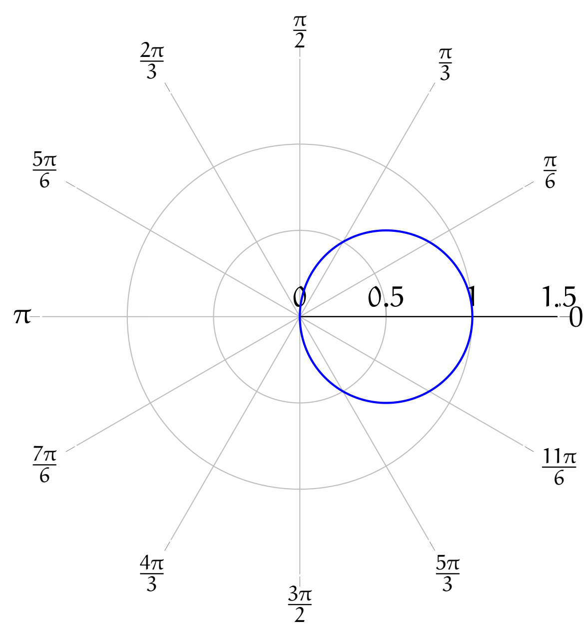



Figure 4.9: $r = 3 \sin \theta$ plotted on a polar coordinate system



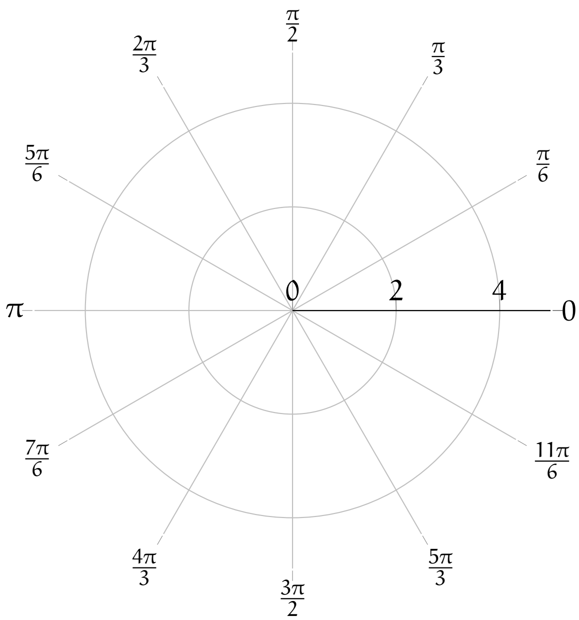

**Solution**: This plot will look like a circle of radius 0.5 centered at $(0.5, 0)$ (in polar coordinates).



### *Exercise 6*

Sketch the following polar functions on the provided polar axis for $0 \leq \theta < 2\pi$:

1. $r = 3$
2. $\theta = \pi$
3. $r = 2\cos\frac{\theta}{2}$
4. $r = -4\sin\theta$
5. $r = \theta$




*Working Space*

*Answer on Page 53*

# Chapter 5

# Sound

When you set off a firecracker, it makes a sound.

Let's break that down a little more. Inside the cardboard wrapper of the firecracker, there is potassium nitrate ($KNO_3$), sulfur ($S$), and carbon($C$). These are all solids. When you trigger the chemical reactions with a little heat, these atoms rearrange themselves to be potassium carbonate ($K_2CO_3$), potassium sulfate ($K_2SO_4$), carbon dioxide ($CO_2$), and nitrogen ($N_2$). Note that the last two are gasses.

The molecules of a solid are much more tightly packed than the molecules of a gas. So after the chemical reaction, the molecules expand to fill a much bigger volume. The air molecules nearby get pushed away from the firecracker. They compress the molecules beyond them, and those compress the molecules beyond them.

This compression wave radiates out as a sphere; its radius growing at about 343 meters per second ("The speed of sound").

The energy of the explosion is distributed around the surface of this sphere. As the radius increases, the energy is spread more and more thinly around. This is why the firecracker seems louder when you are closer to it. (If you set off a firecracker in a sewer pipe, the sound will travel much, much farther.)

This compression wave will bounce off of hard surfaces. If you set off a firecracker 50 meters from a big wall, you will hear the explosion twice. We call the second one an "echo".

The compression wave will be absorbed by soft surfaces. If you covered that wall with pillows, there would be almost no echo.

The study of how these compression waves move and bounce is called *acoustics*. Before you build a concert hall, you hire an acoustician to look at your plans and tell you how to make it sound better.

## 5.1 Pitch and frequency

The string on a guitar is very similar to the weighted spring example. The farther the string is displaced, the more force it feels pushing it back to equilibrium (remember the tension force?). Thus, it moves back and forth in a sine wave. (OK, it isn't a pure sine wave, but we will get to that later.)

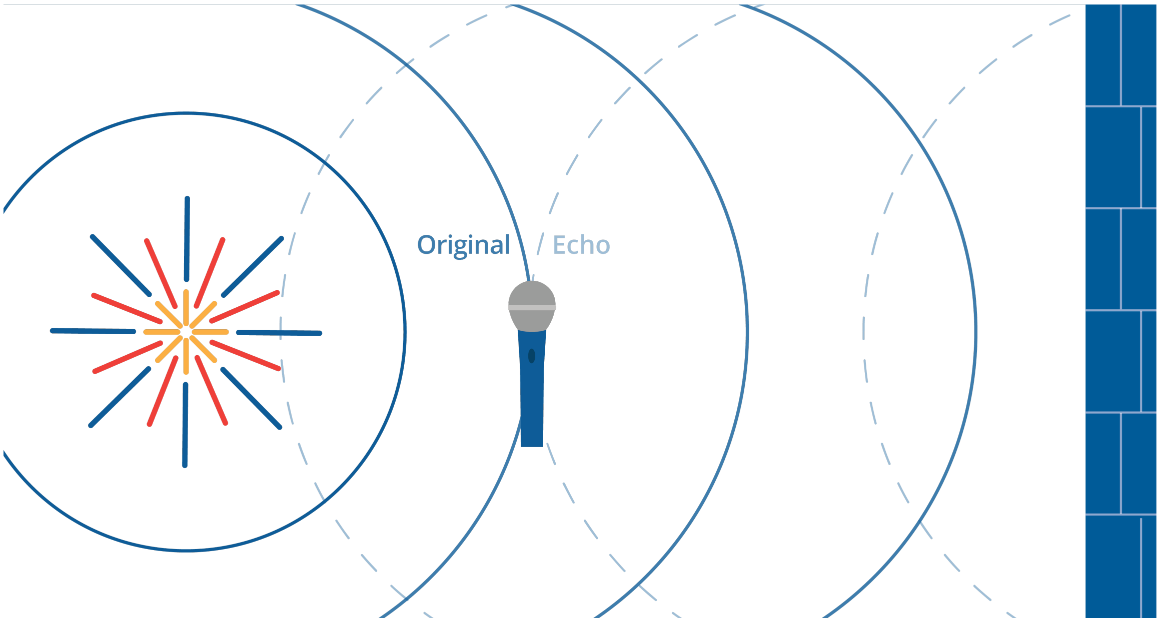



Figure 5.1: A firecracker exploding causes an initial sound wave and an echo.

The string is connected to the center of the boxy part of the guitar, which is pushed and pulled by the string. That creates compression waves in the air around it.

If you are in the room with the guitar, those compression waves enter your ear and push and pull your eardrum, which is attached to bones that move a fluid that tickles tiny hairs, called *cilia*, in your inner ear. This is how you hear.

We sometimes see plots of sound waveforms. The $x$-axis represents time. The $y$-axis represents the amount the air is compressed at the microphone that converted the air pressure into an electrical signal.

If the guitar string is made tighter (by the tuning pegs) or shorter (by the guitarist's fingers on the strings), the string vibrates more times per second. We measure the number of waves per second and we call it the *frequency* of the tone. The unit for frequency is *Hertz*: cycles per second. The *period* is opposite of frequency: it is the time it takes for one cycle to complete. The unit for period is seconds per cycle.

Musicians have given the different frequencies names. If the guitarist plucks the lowest note on his guitar, it will vibrate at 82.4 Hertz. The guitarist will say "That pitch is low E." If the string is made half as long (by a finger on the 12th fret), the frequency will be twice as fast (164.8 Hertz), and the guitarist will say "That is E an octave up."

For any note, the note that has twice the frequency is one octave up. The note that has half the frequency is one octave down.

The octave is a very big jump in pitch, so musicians break it up into 12 smaller steps. If

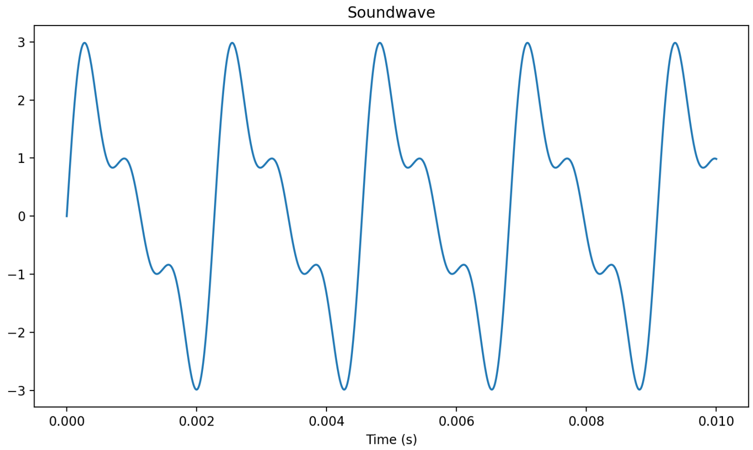



Figure 5.2: A soundwave graph over time.

the guitarist shortens the E string by one fret, the frequency will be $82.4 \times 1.059463 \approx 87.3$ Hertz.

Shortening the string one fret always increases the frequency by a factor of 1.059463. Why?

Because $1.059463^12 = 2$. That is, if you take 12 of these hops, you end up an octave higher. This, the smallest hop in western music, is referred to as a *half step*.

## *Exercise 7* Notes and frequencies

*Working Space*

The note A near the middle of the piano, is 440Hz. The note E is 7 half steps above A. What is its frequency?

*Answer on Page 55*

## 5.2 Chords and harmonics

Of course, a guitarist seldom plays only one string at a time. Instead, they use the frets to pick a pitch for each string and strums all six strings.

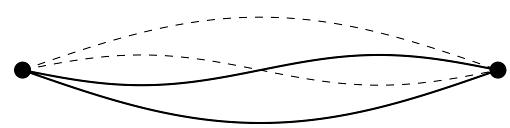
Some combinations of frequencies sound better than others. We have already talked about the octave: If one string vibrates twice for each vibration of another, they sound sweet together.

Musicians speak of "the fifth". If one string vibrates three times and the other vibrates twice in the same amount of time, they sound sweet together.

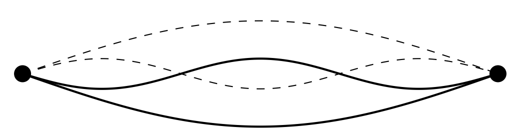
Likewise, if one string vibrates 4 times while the other vibrates 3 times, they sound sweet together. Musicians call this "the third."

Each of these different frequencies tickle different cilia in the inner ear, so you can hear all six notes at the same time when the guitarist strums their guitar.

When a string vibrates, it doesn't create a single sine wave. Yes, the string vibrates from end-to-end, and this generates a sine wave at what we call *the fundamental frequency*. However, there are also "standing waves" on the string. One of these standing waves is still at the centerpoint of the string, but everything to the left of the centerpoint is going up, while everything to the right is going down. This creates *an overtone* that is twice the frequency of the fundamental.



The next overtone has two still points — it divides the string into three parts. The outer parts are up, while the inner part is down. Its frequency is three times the fundamental frequency.



And so on. 4 times the fundamental, 5 times the fundamental, etc.

In general, tones with many overtones tend to sound bright. Tones with just the fundamental sound thin.

Humans can generally hear frequencies from 20Hz to 20,000Hz (or 20kHz). Young people tend to be able to hear very high sounds better than older people.

Dogs can generally hear sounds in the 65Hz to 45kHz range.

## 5.3 Making waves in Python

Let's make a sine wave and add some overtones to it. Create a file named `harmonics.py`.

```
import matplotlib.pyplot as plt
import math

# Constants: frequency and amplitude
fundamental_freq = 440.0 # A = 440 Hz
fundamental_amp = 2.0

# Up an octave
first_freq = fundamental_freq * 2.0 # Hz
first_amp = fundamental_amp * 0.5

# Up a fifth more
second_freq = fundamental_freq * 3.0 # Hz
second_amp = fundamental_amp * 0.4

# How much time to show
max_time = 0.0092 # seconds

# Calculate the values 10,000 times per second
time_step = 0.00001 # seconds

# Initialize
time = 0.0
times = []
totals = []
fundamentals = []
firsts = []
seconds = []

while time <= max_time:
    # Store the time
    times.append(time)

    # Compute value each harmonic
    fundamental = fundamental_amp * math.sin(2.0 * math.pi * fundamental_freq * time)
    first = first_amp * math.sin(2.0 * math.pi * first_freq * time)
    second = second_amp * math.sin(2.0 * math.pi * second_freq * time)

    # Sum them up
    total = fundamental + first + second
```

```
    # Store the values
    fundamentals.append(fundamental)
    firsts.append(first)
    seconds.append(second)
    totals.append(total)

    # Increment time
    time += time_step

# Plot the data
fig, ax = plt.subplots(2, 1)

# Show each component
ax[0].plot(times, fundamentals)
ax[0].plot(times, firsts)
ax[0].plot(times, seconds)
ax[0].legend()

# Show the totals
ax[1].plot(times, totals)
ax[1].set_xlabel("Time (s)")

plt.show()
```

When you run it, you should see a plot of all three sine waves and another plot of their sum:

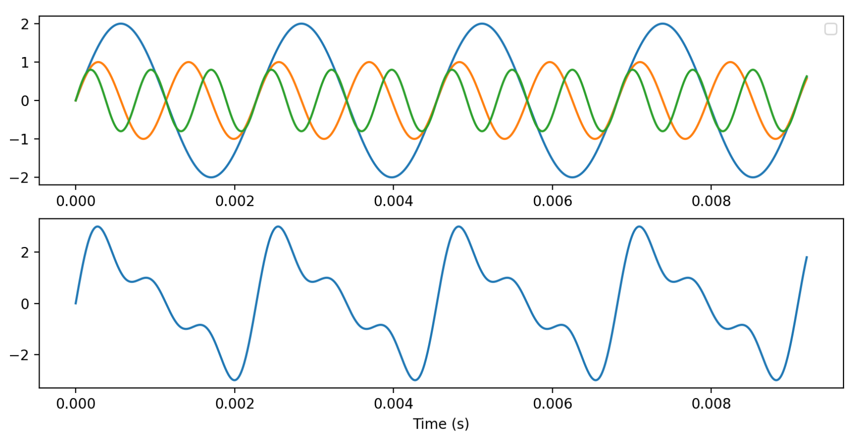



Figure 5.3: The output of haronics.py

### 5.3.1 Making a sound file

The graph is pretty to look at, but make let's a file that we can listen to.

The WAV audio file format is supported on pretty much any device, and a library for writing WAV files comes with Python. Let's write some sine waves and some noise into a WAV file.

Create a file called `soundmaker.py`

```
import wave
import math
import random

# Constants
frame_rate = 16000 # samples per second
duration_per = 0.3 # seconds per sound
frequencies = [220, 440, 880, 392] # Hz
amplitudes = [20, 125]
baseline = 127 # Values will be between 0 and 255, so 127 is the baseline
samples_per = int(frame_rate * duration_per) # number of samples per sound

# Open a file
wave_writer = wave.open('sound.wav', 'wb')

# Not stereo, just one channel
wave_writer.setnchannels(1)

# 1 byte audio means everything is in the range 0 to 255
wave_writer.setsampwidth(1)

# Set the frame rate
wave_writer.setframerate(frame_rate)

# Loop over the amplitudes and frequencies
for amplitude in amplitudes:
    for frequency in frequencies:
        time = 0.0
        # Write a sine wave
        for sample in range(samples_per):
            s = baseline + int(amplitude * math.sin(2.0 * math.pi * frequency * time))
            wave_writer.writeframes(bytes([s]))
            time += 1.0 / frame_rate

        # Write some noise after each sine wave
```

```
        for sample in range(samples_per):
            s = baseline + random.randint(0, 15)
            wave_writer.writeframes(bytes([s]))

# Close the file
wave_writer.close()
```

When you run it, it should create a sound file with several tones of different frequencies and volumes. Each tone should be followed by some noise.

Appendix A

# Answers to Exercises

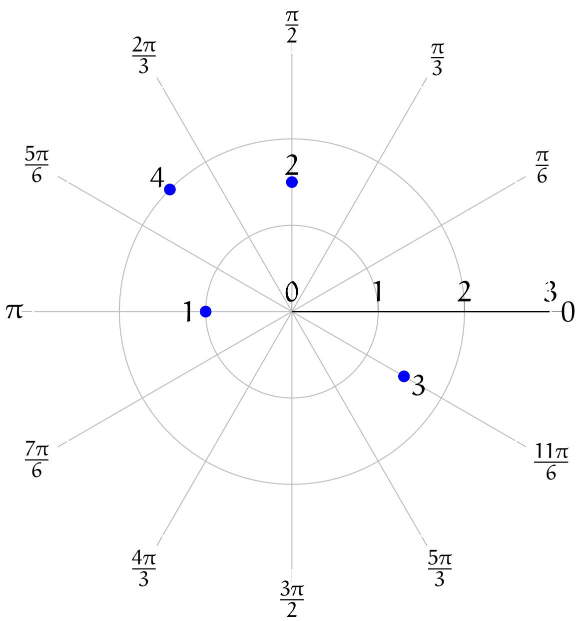
## Answer to Exercise 1 (on page 11)

- $[1, 2, 3] \cdot [4, 5, -6] = 4 + 10 - 18 = -4$
- $[\pi, 2\pi] \cdot [2, -1] = 2\pi - 2\pi = 0$
- $[0, 0, 0, 0] \cdot [10, 10, 10, 10] = 0 + 0 + 0 + 0 = 0$

## Answer to Exercise 2 (on page 16)

- $[1, 0] \cdot [0, 1] = 0$. The angle must be $\pi/2$.
- $[3, 4] \cdot [4, 3] = 24$. $|[3, 4]||[4, 3]| \cos(\theta) = 24$. $\cos(\theta) = \frac{24}{(5)(5)}$. $\theta = \arccos(\frac{24}{25}) \approx 0.284$ radians.
- $[2, -1, 2] \cdot [-1, 2, -2] = 4-2-4 = -2$. $|[2, -1, 2]| = \sqrt{4+1+4} = \sqrt{9} = 3$. $|[-1, 2, -2]| = \sqrt{1+4+4} = \sqrt{9} = 3$. $3(3) \cos\theta = -2$. $\theta = \arccos(-2/9) \approx 1.795$ radians.
- $[-5, 0, -1] \cdot [2, 3, -4] = -10+0+4 = -6$. $|[-5, 0, 1]| = \sqrt{25+0+1} = \sqrt{26}$. $|[2, 3, -4]| = \sqrt{4+9+16} = \sqrt{29}$. $\sqrt{26}(\sqrt{29}) \cos\theta = -6$. $\theta = \arccos(\frac{-6}{\sqrt{26}\sqrt{29}}) \approx 1.791$ radians.

## Answer to Exercise 3 (on page 32)




## Answer to Exercise 4 (on page 37)

1. $(0,-2)$. $x = 2 \cdot \cos \frac{3\pi}{2} = 2 \cdot 0 = 0$ and $y = 2 \cdot \sin \frac{3\pi}{2} = 2 \cdot -1 = -2$.
2. $(-1,1)$. $x = \sqrt{2} \cdot \cos \frac{3\pi}{4} = \sqrt{2} \cdot -\frac{\sqrt{2}}{2} = \frac{2}{2} = -1$ and $y = \sqrt{2} \cdot \sin \frac{3\pi}{4} = \sqrt{2} \cdot \frac{\sqrt{2}}{2} = \frac{2}{2} = 1$.
3. $(\frac{3\sqrt{2}}{2}, -\frac{3\sqrt{2}}{2})$. $x = 3 \cdot \cos -\frac{\pi}{4} = 3 \cdot \frac{\sqrt{2}}{2} = \frac{3\sqrt{2}}{2}$ and $y = 3 \cdot \sin -\frac{\pi}{4} = 3 \cdot -\frac{\sqrt{2}}{2} = -\frac{3\sqrt{2}}{2}$.
4. $(-\frac{3}{2}, -\frac{3\sqrt{3}}{2})$. $x = (-3) \cdot \cos \frac{\pi}{3} = (-3) \cdot \frac{1}{2} = -\frac{3}{2}$ and $y = (-3) \cdot \sin \frac{\pi}{3} = (-3) \cdot \frac{\sqrt{3}}{2} = -\frac{3\sqrt{3}}{2}$.
5. $(0,-2)$. $x = 2 \cdot \cos -\frac{pi}{2} = 2 \cdot 0 = 0$ and $y = 2 \cdot \sin -\frac{\pi}{2} = 2 \cdot -1 = -2$.

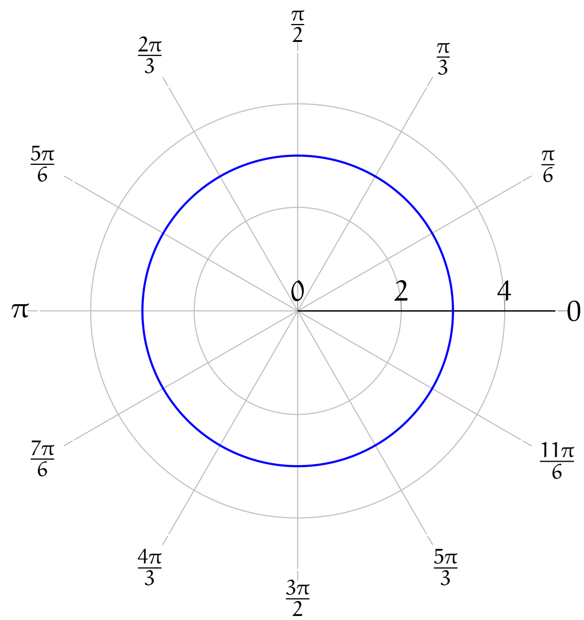
## Answer to Exercise 5 (on page 38)

1. $(4\sqrt{2}, \frac{3\pi}{4})$. $r = \sqrt{x^2 + y^2} = \sqrt{32} = 4\sqrt{2}$. $\arctan \frac{y}{x} = \arctan \frac{4}{-4} = \arctan -1 = -\frac{\pi}{4} + n\pi$. We take $\theta = \frac{3\pi}{4}$ to satisfy the domain restriction and be in the correct quadrant.
2. $(6, \frac{\pi}{3})$. $r = \sqrt{3^2 + \left(3\sqrt{3}\right)^2} = \sqrt{9 + 27} = \sqrt{36} = 6$. $\arctan \frac{3\sqrt{3}}{3} = \arctan \sqrt{3} = \frac{\pi}{3} + n\pi$. We take $\theta = \frac{\pi}{3}$ to satisfy the domain restriction and be in the correct quadrant.

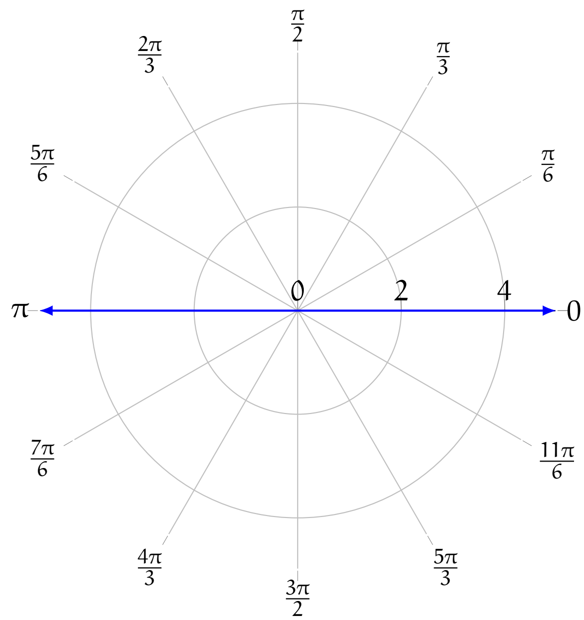
3. $(2, \frac{11\pi}{6})$. $r = \sqrt{\sqrt{3}^2 + (-1)^2} = \sqrt{3+1} = 2$. $\arctan \frac{-1}{\sqrt{3}} = -\frac{\pi}{6} + n\pi$. We take $\theta = \frac{11\pi}{6}$ to satisfy the domain restriction and have the point in the correct quadrant.
4. $(6, \pi)$. $r = \sqrt{(-6)^2 + 0^2} = 6$. $\arctan \frac{0}{-6} = \pi + n\pi$. We take $\theta = \pi$ to satisfy the domain restriction.
5. $(2\sqrt{2}, \frac{5\pi}{4})$. $r = \sqrt{(-2)^2 + (-2)^2} = \sqrt{8} = 2\sqrt{2}$. $\arctan \frac{-2}{-2} = \arctan 1 = \frac{\pi}{4} + n\pi$. We take $\theta = \frac{5\pi}{4}$ to satisfy the domain restriction and be in the correct quadrant.

## Answer to Exercise ?? (on page 42)

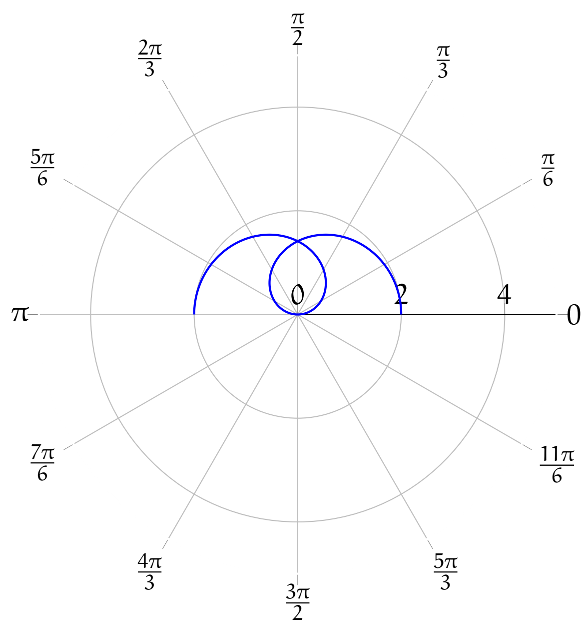
1. $r = 3$



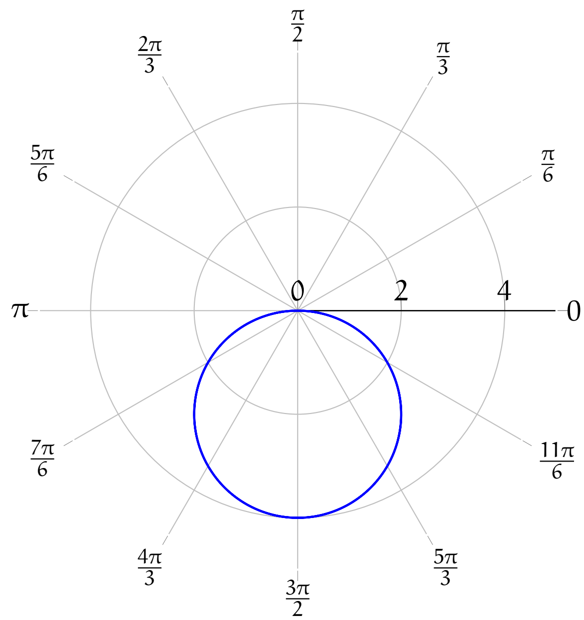

2. $\theta = \pi$ Because $r$ includes all real numbers, negative $r$ is possible and the line $\theta = \pi$ extends in both directions



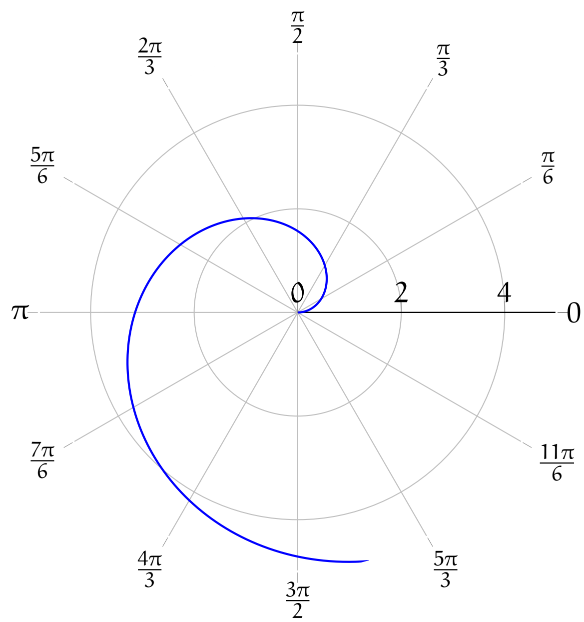

3. $r = 2\cos\frac{\theta}{2}$




4. $r = -4\sin\theta$




5. $r = \theta$ (The spiral continues, but is beyond the boundary of the graph)




## Answer to Exercise 7 (on page 45)

A is 440 Hz. Each half-step is a multiplication by $\sqrt[12]{2} = 1.059463094359295$ So the frequency of E is $(440)(2^{7/12}) = 659.255113825739859$

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