Contents

Chapter 1

Inverse Trigonometric Functions

Recall from the chapter on functions that an inverse of a function is a machine that turns y back into x. The inverses of trigonometric functions are essential to solving certain integrals (you will learn in a future chapter why integrals are useful — for now, trust us that they are!). Let's begin by discussing the sin function and its inverse, $\sin^{-1}$, also called arcsin.

Examine the graph of $\sin x$ in figure 1.1. See how the dashed horizontal line crosses the function at many points? This means the function $\sin x$ is not one-to-one. In other words, there is not a unique x-value for every y-value. This means that if we do not restrict the domain of $\arcsin x$, the result will not be a function (see figure 1.2). In figure 1.2, you can see that just reflecting the graph across $y = x$ fails the vertical line test: an x value has more than one y value.

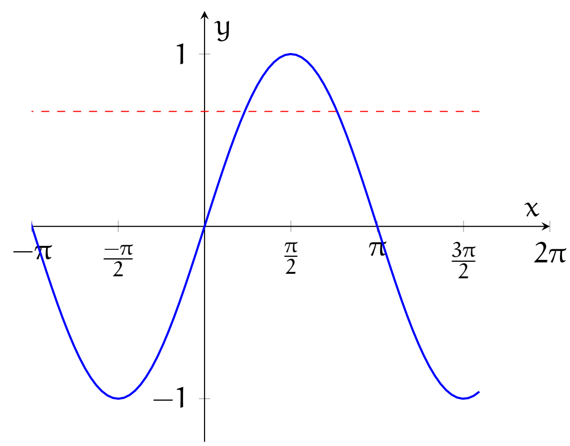

Figure 1.1: The horizontal line $y = \frac{2}{3}$ crosses $y = \sin x$ more than once

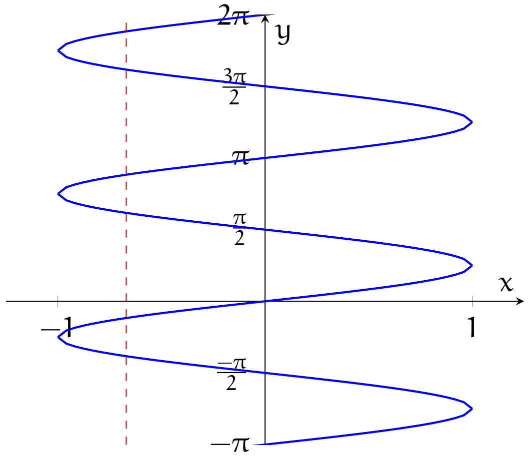

Figure 1.2: The inverse of an unrestricted sin function fails the vertical line test

1.1 Derivatives of Inverse Trigonometric Functions

f	f'
$\arcsin x$	$\frac{1}{\sqrt{1-x^2}}$
$\arccos x$	$-\frac{1}{\sqrt{1-x^2}}$
$\arctan x$	$\frac{1}{1+x^2}$
$\text{arccsc } x$	$-\frac{1}{x\sqrt{x^2-1}}$
$\text{arcsec } x$	$\frac{1}{x\sqrt{x^2-1}}$
$\text{arccot } x$	$-\frac{1}{1+x^2}$

1.2 Practice

Exercise 1

Find the f'. Give your answer in a simplified form.

1. $f(x) = \arctan x^2$
2. $f(x) = x\text{arcsec}\,(x^3)$
3. $f(x) = \arcsin \frac{1}{x}$

Working Space

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Chapter 2

Trigonometric Identities

2.1 The Unit Circle

There are some values of $\sin\theta$ and $\cos\theta$ that will be useful to know off the top of your head. The Unit Circle will help you in this memorization process (see figure 2.1). When a circle of radius 1 is centered at the origin, the Cartesian coordinates of any point on the circle correspond to the values of cosine and sine of the angle above the horizontal (how far you've rotated from the positive x-axis).

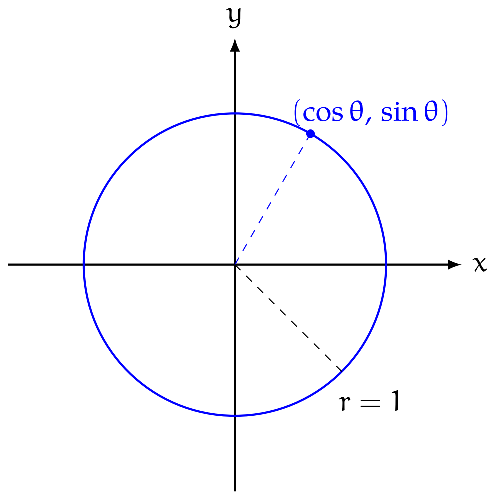

Figure 2.1: The Unit Circle is a circle with radius 1 centered at the origin

Let's take a closer look at a triangle in the first quadrant to see why this is true. Imagine some point on the circle, (x_o, y_o). Drawing a line from that point back to the origin creates an angle θ between the imaginary line and the positive x-axis (see figure 2.2). Extending an imaginary vertical down to $(x_o, 0)$, then an imaginary horizontal from $(x_o, 0)$ to the origin, creates a right triangle. What can we say about the legs of the triangle?

Recall SOH-CAH-TOA from a previous chapter. This acronym tells us that, for a right triangle, the sine of an angle is given by the ratio of the length of the leg opposite the angle to the hypotenuse. In our case, then, $\sin\theta = \frac{y_o}{1} = y_o$. [Remember: We are dealing with the Unit Circle, which has a radius of one. Examining figure 2.2 shows you that the hypotenuse of the imaginary triangle is the same as the circle's radius.] This means that the y-coordinate of any point on the Unit Circle is the sine of the angle of rotation from the horizontal.

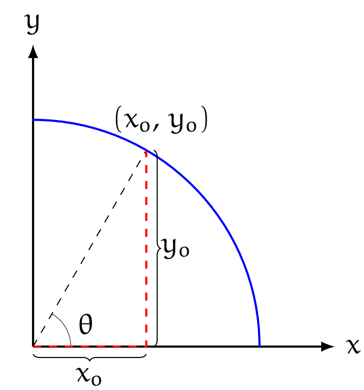

Figure 2.2: Drawing a line from any point on the circle to the origin creates an angle with the horizontal

Exercise 2

In a similar manner as we did with $\sin\theta$ above, prove the x-coordinate of any point on the unit circle is equal to $\cos\theta$, where θ is the angle of rotation from the horizontal.

Working Space

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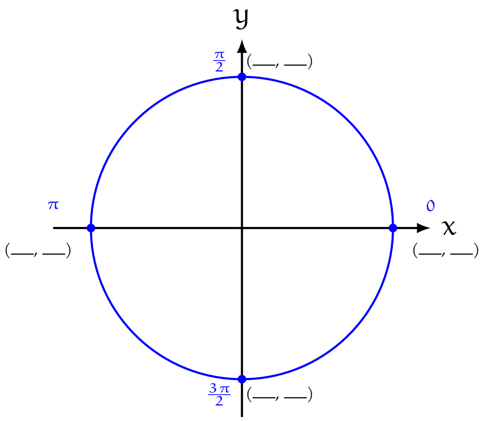
Exercise 3

Fill in the unit circle with the coordinates for $\theta = 0$, $\pi/2$, π, and $3\pi/2$. Use this to determine:

1. $\sin \frac{\pi}{2}$
2. $\cos \frac{3\pi}{2}$
3. $\sin \pi$
4. $\cos -\pi$



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2.1.1 Exact Values of Key Angles

We will examine two triangles. First, a 30-60-90 triangle, then a 45-45-90 triangle. As shown in figure 2.3, you can get a 30-60-90 triangle with hypotenuse 1 by dividing an equilateral triangle in half. We will label the horizontal leg of the 30-60-90 triangle A and the vertical leg B.

From the figure, we see that the length of A is half that of the hypotenuse, which in this case is $\frac{1}{2}$. This means the $\cos 60° = \cos \frac{\pi}{3} = \frac{1}{2}$. To find the length of side B, we can use the Pythagorean theorem:

$$B^2 = C^2 - A^2, \text{ where C is the hypotenuse}$$

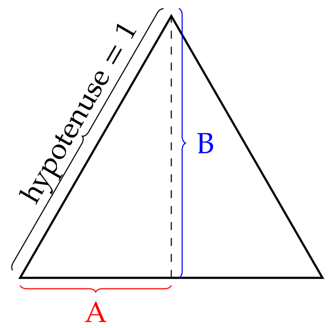


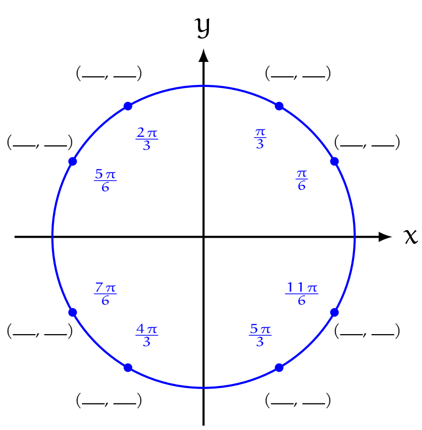
Figure 2.3: A 30-60-90 triangle is made by vertically bisecting an equilateral triangle

$$B^2 = 1^2 - \left(\frac{1}{2}\right)^2$$
$$B^2 = \frac{3}{4}$$
$$B = \frac{\sqrt{3}}{2}$$

Therefore, $\sin 60^\circ = \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$.

Exercise 4

Use symmetry to complete the blank unit circle below. (Hint: We just showed that the (x, y) coordinate for $\frac{\pi}{3}$ is $(1/2, \sqrt{3}/2)$).



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Now we will look at a 45-45-90 triangle (see figure 2.4), which will allow us to complete our Unit Circle. Recall that a 45-45-90 triangle is an isosceles triangle in addition to being a right triangle. This means both the legs are the same length. Using the Pythagorean theorem, we would say $A = B$. We also know that $C = 1$, since our triangle is inscribed in the unit circle. Let's find A:

$$A^2 + B^2 = C^2$$
$$A^2 + A^2 = 1^2$$
$$2A^2 = 1$$
$$A^2 = \frac{1}{2}$$
$$A = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

Therefore, each leg has a length of $\sqrt{2}/2$, and the (x, y) coordinates for $\theta = 45° = \pi/4$ are $(\sqrt{2}/2, \sqrt{2}/2)$.

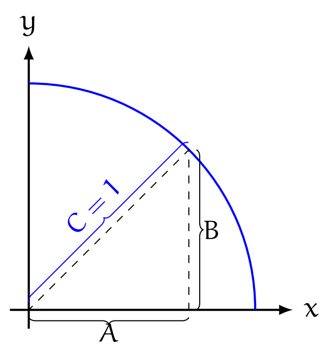
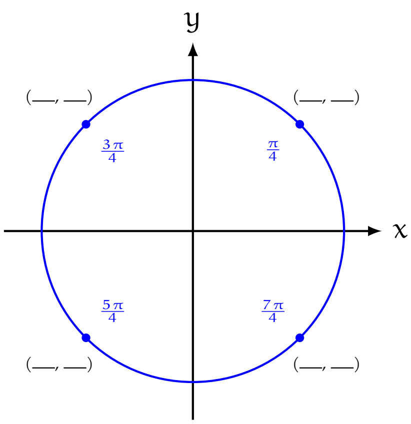


Figure 2.4: The two legs of a 45-45-90 triangle are the same length

Exercise 5

Use symmetry to complete the blank unit circle below.



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Exercise 6

Without a calculator and using only your completed unit circles, determine the value requested (angles are given in radians unless otherwise indicated).

1. $\cos \frac{3\pi}{2}$
2. $\sin \frac{\pi}{4}$
3. $\sin -\frac{\pi}{6}$
4. $\cos \frac{4\pi}{3}$
5. $\sin \frac{3\pi}{4}$
6. $\cos -\frac{\pi}{3}$
7. $\sin 45°$
8. $\sin 270°$
9. $\sin -60°$
10. $\sin 150°$

Working Space

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2.2 Sum and Difference Formulas

Consider 4 points on the unit circle: B at $(1, 0)$, Q at some angle β, P at some angle α, and A at angle $\alpha - \beta$ (see figure 2.5).

The distance from P to Q is the same as the distance from A to B, since ΔPOQ is a rotation of ΔAOB. Because this is a Unit Circle, $P = (\cos\alpha, \sin\alpha)$, $Q = (\cos\beta, \sin\beta)$, and $A = (\cos\alpha - \beta, \sin\alpha - \beta)$. Let's use the distance formula to find the length of $\overline{PQ}$:

$$\overline{PQ} = \sqrt{(\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2} =$$

$$\sqrt{\cos^2\alpha - 2\cos\alpha\cos\beta + \cos^2\beta + \sin^2\alpha - 2\sin\alpha\sin\beta + \sin^2\beta} =$$

$$\sqrt{\left(\cos^2\alpha + \sin^2\alpha\right) + \left(\cos^2\beta + \sin^2\beta\right) - 2\cos\alpha\cos\beta - 2\sin\alpha\sin\beta}$$

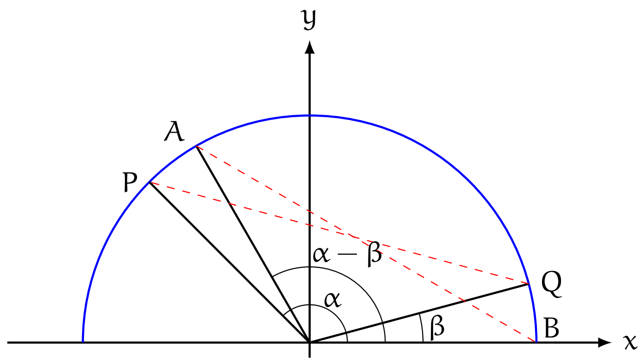

Figure 2.5: $\overline{AB} = \overline{PQ}$

Recall that for any angle, θ, $\sin^2\theta + \cos^2\theta = 1$. Substituting this identity, we see that:

$$\overline{PQ} = \sqrt{1 + 1 - 2\sin\alpha\sin\beta - 2\cos\alpha\cos\beta} = \sqrt{2 - 2\sin\alpha\sin\beta - 2\cos\alpha\cos\beta}$$

Let's leave this simplified equation for $\overline{PQ}$ alone for the moment and similarly find $\overline{AB}$:

$$\overline{AB} = \sqrt{[\cos(\alpha-\beta) - 1]^2 + [\sin(\alpha-\beta) - 0]^2} =$$

$$\sqrt{\cos^2(\alpha-\beta) - 2\cos(\alpha-\beta) + 1 + \sin^2(\alpha-\beta)} =$$

$$\sqrt{\cos^2(\alpha-\beta) + \sin^2(\alpha-\beta) + 1 - 2\cos(\alpha-\beta)}$$

$$= \sqrt{2 - 2\cos(\alpha-\beta)} = \overline{AB}$$

Recall that we've established $\overline{AB} = \overline{PQ}$. We can set the statements equal to each other:

$$\sqrt{2 - 2\sin\alpha\sin\beta - 2\cos\alpha\cos\beta} = \sqrt{2 - 2\cos(\alpha-\beta)}$$

Squaring both sides and subtracting 2, we find:

$$-2\sin\alpha\sin\beta - 2\cos\alpha\cos\beta = -2\cos(\alpha-\beta)$$

Finally, we can divide both sides by negative 2 to get the difference of angles formula for cosine:

$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

There are similar formulas for the sine and cosine of the sum of two angles, and for the sine of the difference of two angles, which we won't derive here.

Sum and Difference Formulas

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$
$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$
$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$
$$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

Exercise 7

Without a calculator, find the exact value requested:

1. $\sin\frac{\pi}{12}$
2. $\cos\frac{7\pi}{12}$
3. $\tan\frac{13\pi}{12}$ (hint: $\tan\theta = \sin\theta / \cos\theta$)

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2.3 Double and Half Angle Formulas

We can easily derive a formula for twice an angle by letting $\alpha = \beta$ for a sum formula.

Example: Derive a formula for $\cos 2\theta$ in terms of trigonometric functions of θ.

Solution: Using the sum formula for cosine, we see that:

$$\cos 2\theta = \cos(\theta + \theta)$$

$$= \cos\theta\cos\theta - \sin\theta\sin\theta = \cos^2\theta - \sin^2\theta$$

Noting that $\sin^2\theta = 1 - \cos^2\theta$:

$$\cos 2\theta = 2\cos^2\theta - 1$$

Alternatively, we could note that $\cos^2\theta = 1 - \sin^2\theta$:

$$\cos 2\theta = 1 - 2\sin^2\theta$$

Exercise 8

Derive a formula for $\sin 2\theta$ in terms of trigonometric functions of θ.

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We can use these double-angle formulas for find half-angle formulas. Consider the double-angle formula for cosine:

$$\cos 2\theta = 2\cos^2\theta - 1$$

Let $\theta = \alpha/2$, then:

$$\cos\alpha = 2\cos^2(\alpha/2) - 1$$

Rearranging to solve for $\cos(\alpha/2)$:

$$2\cos^2(\alpha/2) = \cos\alpha + 1$$

$$\cos^2(\alpha/2) = \frac{\cos\alpha + 1}{2}$$

$$\cos(\alpha/2) = \pm\sqrt{\frac{1 + \cos\alpha}{2}}$$

Exercise 9

Derive a formula for $\sin(\alpha/2)$.

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There are two identities that will be very useful for integrals in a future chapter:

Squared Trigonometric Identities

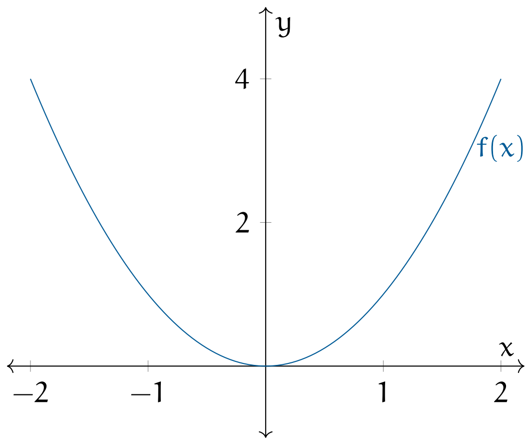
$$\sin^2\theta = \frac{1-\cos 2\theta}{2}$$
$$\cos^2\theta = \frac{1+\cos 2\theta}{2}$$

These are just specific re-writings of the half-angle identities.

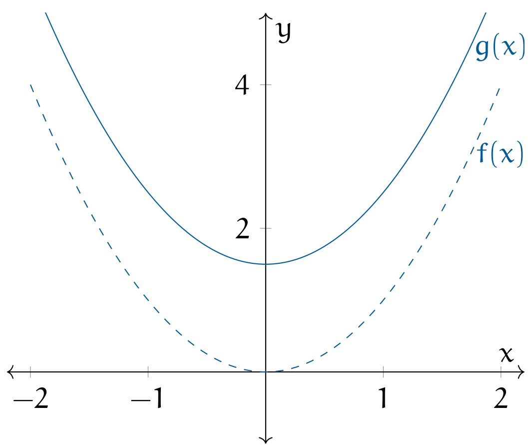
Chapter 3

Transforming Functions

Let's say we gave you the graph of a function f, like this:



We then tell you that the function is $g(x) = f(x) + 1.5$. Can you guess what the graph of g would look like? It is the same graph, just translated up 1.5:



There are four kinds of transformations that we do all the time:

- Translation up and down in the direction of y axis (the one you just saw)
- Translation left and right in the direction of the x axis
- Scaling up and down along the y axis

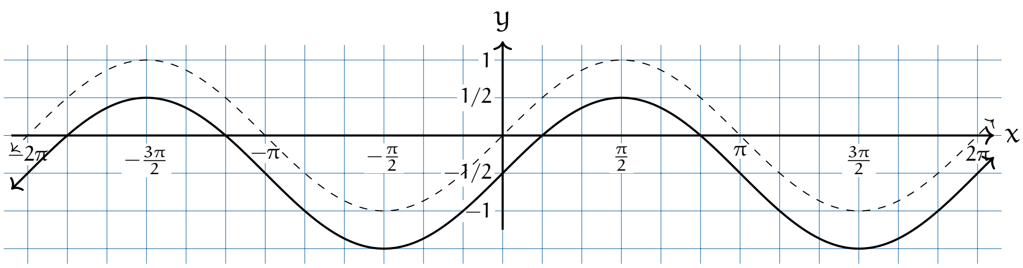
- Scaling up and down along the x axis

Next, we will demonstrate each of the four using the graph of $\sin(x)$.

3.1 Translation up and down

When you add a positive constant to a function, you translate the whole graph up that much. A negative constant translates it down.

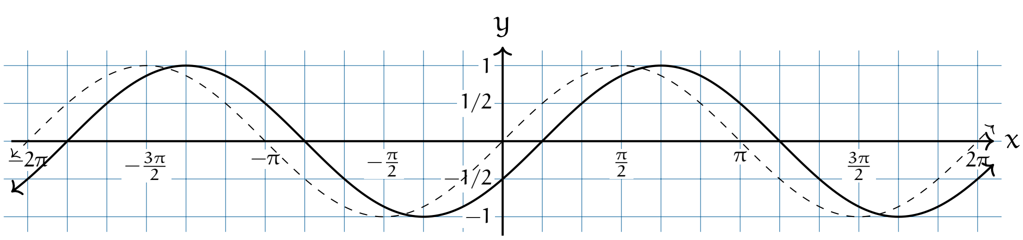
Here is the graph of $\sin(x) - 0.5$:



3.2 Translation left and right

When you add a positive number to x before running it through f, you translate the graph to the left by that amount. Adding a negative number translates the graph to the right.

Here is the graph of $\sin(x - \pi/6)$:



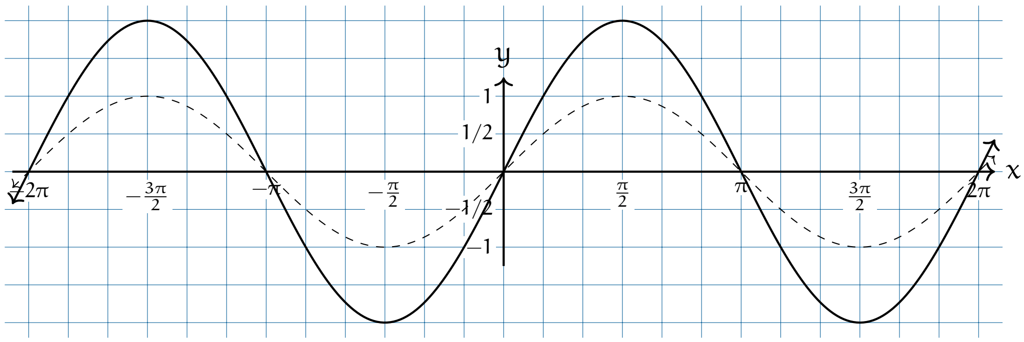
Notice the sign:

- Adding to x before processing with the function translates the graph to the *left*.
- Subtracting from x before processing with the function translates the graph to the *right*

3.3 Scaling up and down in the y direction

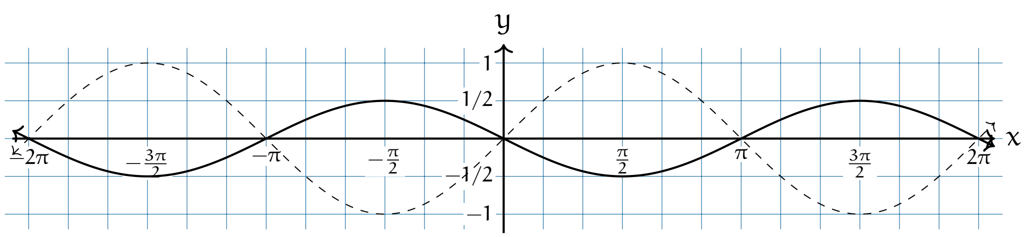
To scale the function up and down, you multiply the result of the function by a constant. If the constant is larger than 1, it stretches the function up and down.

Here is $y = 2\sin(x)$:



With a wave like this, we speak of its *Amplitude*, which you can think of as its height. The baseline that this wave oscillates around is zero. The maximum distance that it gets from that baseline is its amplitude. Thus, the amplitude here has been increased from 1 to 2.

If you multiply by a negative number, the function gets flipped. Here is $y = -0.5\sin(x)$:

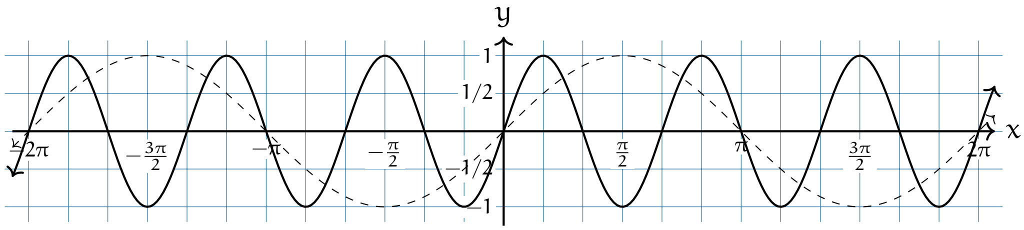


Amplitude is never negative. Thus, the amplitude of this wave is 0.5.

3.4 Scaling up and down in the x direction

If you multiply x by a number larger than 1 before running it through the function, the graph gets compressed toward zero.

Here is $y = \sin(3x)$:

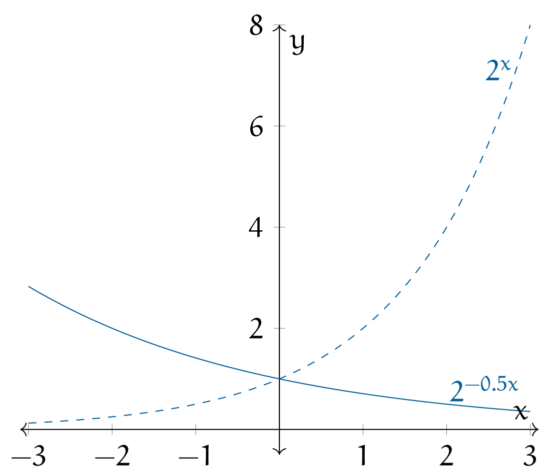


The distance between two peaks of a wave is known as its *wavelength*. The original wave had a wavelength of 2π. The compressed wave has a wavelength of $2\pi/3$.

If you multiply x by a number smaller than 1, it will stretch the function out, away from the y axis.

If you multiply x by a negative number, it will flip the function around the y axis.

Here is $y = 2^{(-0.5x)}$. Notice that it has flipped around the y axis and is stretched out along the x axis.

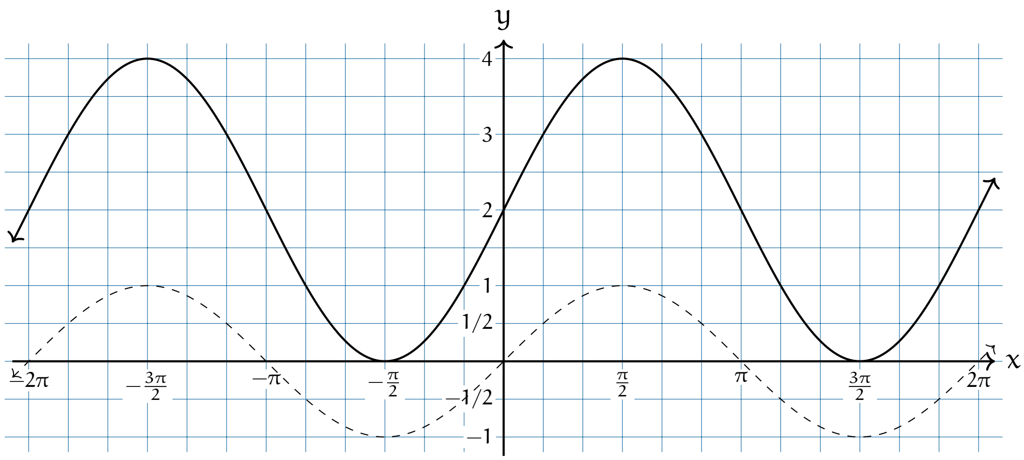


Reflection over *x*-axis	$(x, y) \to (x, -y)$
Reflection over *y*-axis	$(x, y) \to (-x, y)$
Translation	$(x, y) \to (x + a, y + b)$
Dilation	$(x, y) \to (kx, ky)$
Rotation 90° counterclockwise	$(x, y) \to (-y, x)$
Rotation 180°	$(x, y) \to (-x, -y)$

3.5 Order is important!

We can combine these transformations. This allows us, for example, to translate a function up 2, then scale along the y axis by 3.

Here is $y = 2.0(\sin(x) + 1)$:

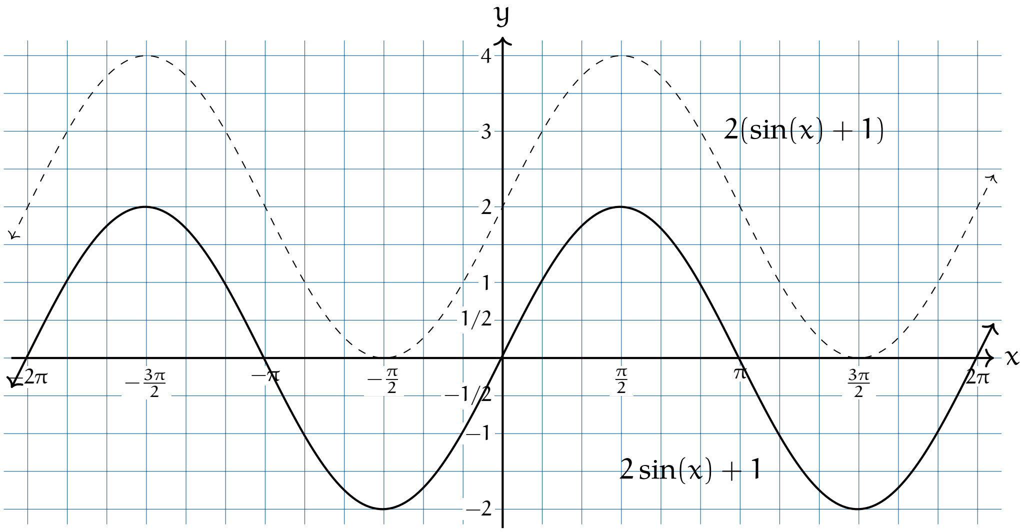


A function is often a series of steps. Here are the steps in $f(x) = 2(\sin(x) + 1)$:

1. Take the sine of x
2. Add 1 to that
3. Multiply that by 2

What if we change the order? Here are the steps in $g(x) = 2\sin(x) + 1$:

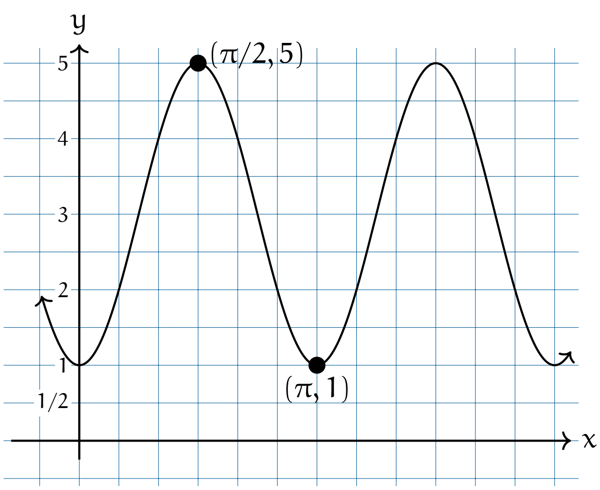
1. Take the sine of x
2. Multiply that by 2
3. Add 1 to that



The moral: You can do multiple transformations of your function, but the order in which you do them is important.

Exercise 10 Transforms

Find a function that creates a sine wave such that the top of the first crest is at the point $(\frac{\pi}{2}, 5)$ and the bottom of the trough that follows is at $(\pi, 1)$.



Working Space

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Chapter 4

Polar Coordinates

We have already seen how to plot a function with (x, y) coordinates. For every x that we put into a function, it returns a y. These pairs of coordinates tell us where on the xy-plane to graph the function. This coordinate system, where x and y are oriented horizontally and vertically, is called the *Cartesian* coordinate system. It can be used to describe 2D space, but it is not the only way.

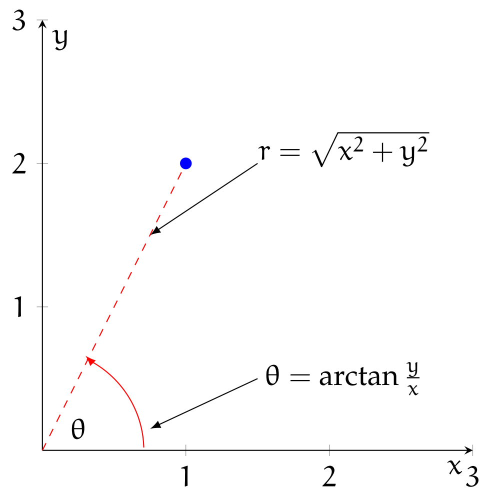


Figure 4.1: The point $(1, 2)$ is $\sqrt{5}$ units from the origin and approximately 1.107 radians counterclockwise from horizontal

Instead of thinking about the horizontal and vertical position, we could think about distance from the origin and rotation about the origin. Take the Cartesian coordinate point $(1, 2)$ (see figure 4.1). How far is $(1, 2)$ from the origin, $(0, 0)$? We can create a right triangle, where the legs are parallel to the x and y axes. This means the leg lengths are 1 and 2, and we can use the Pythagorean theorem to find the length of the hypotenuse (which is the distance from the origin to the point):

$$c^2 = a^2 + b^2$$

$$c^2 = 1^2 + 2^2 = 1 + 4 = 5$$

$$c = \sqrt{5}$$

Therefore, the Cartesian point $(1, 2)$ is $\sqrt{5}$ units from the origin. This is not enough to find our point: there are infinite points that are $\sqrt{5}$ from the origin (see 4.2). To identify a particular point that is a distance of $\sqrt{5}$ from the origin, we also need an *angle of rotation*. By convention, angles are measured from the positive x- axis. This means points on the positive x-axis have an angle of $\theta = 0$, points on the positive y-axis have an angle of $\theta = \frac{\pi}{2}$, and so on.

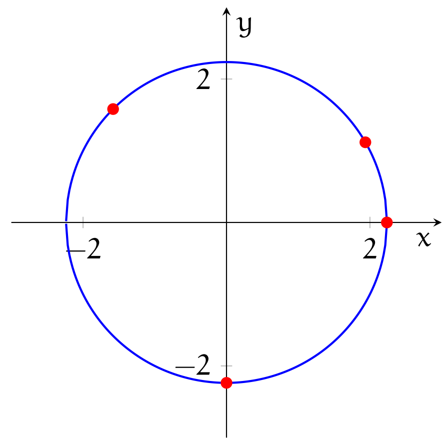


Figure 4.2: There are infinite points $\sqrt{5}$ from the origin, represented by the circle with a radius of $\sqrt{5}$ centered about the origin

We can use trigonometry to find the appropriate angle of rotation for our Cartesian point. There are many ways to do this, but using arctan is the most straightforward. Recall that:

$$\tan\theta = \frac{opposite}{adjacent}$$

That is, for a given angle in a right triangle, the tangent of that angle is given by the length of the opposite leg divided by the adjacent leg. In our case, the opposite leg is the vertical distance (y-value of the Cartesian point) and the adjacent leg is the horizontal distance (x-value of the Cartesian point), which means:

$$\tan\theta = \frac{2}{1}$$

$$\theta = \arctan 2 \approx 1.107 \text{ radians}$$

4.1 Plotting Polar Coordinate Points

How do we plot polar coordinate points? Begin by locating the angle given by the second coordinate (remember, the angle is measured counterclockwise from the horizontal). Your point will lie somewhere on this line. Next, move outwards along the angle by the radius given by the first coordinate.

Example: Plot the polar coordinate point $(2, \frac{\pi}{3})$.

Solution: Begin by locating $\theta = \frac{\pi}{3}$ (see figure 4.3)

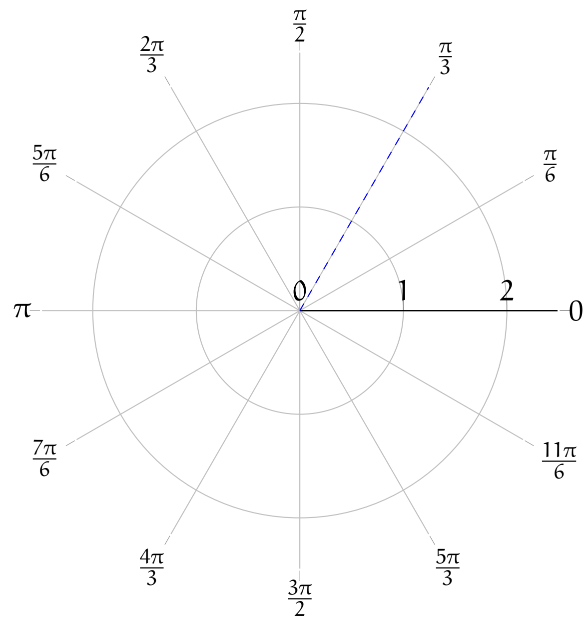


Figure 4.3: $\theta = \frac{\pi}{3}$

ThNexten, move your finger or pencil along the line $\theta = \frac{\pi}{3}$ until you reach $r = 2$ (see figure 4.4).

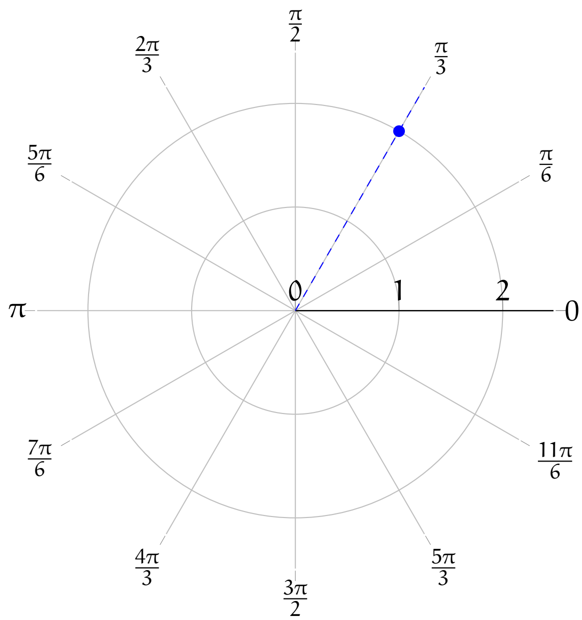
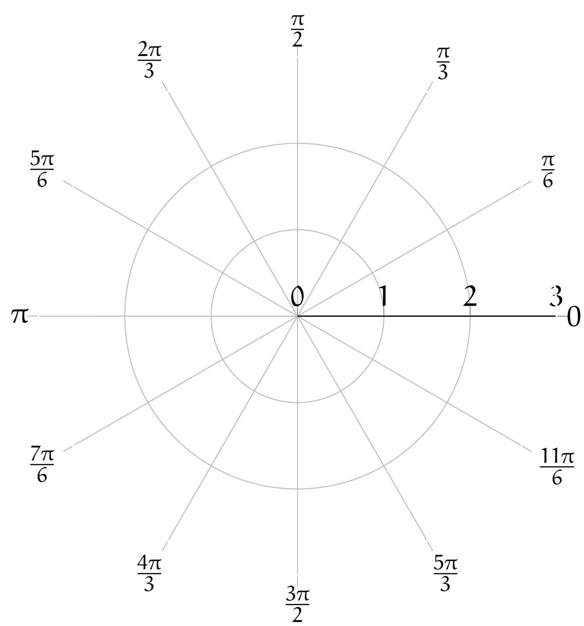


Figure 4.4: $(2, \frac{\pi}{3})$

Exercise 11

Plot the following polar coordinate points on the provided polar axis (hint: negative angles are taken counterclockwise):

1. $(1, \pi)$
2. $(1.5, \frac{\pi}{2})$
3. $(1.5, -\frac{\pi}{6})$
4. $(2, \frac{3\pi}{4})$



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4.2 Equivalent Points

Unlike the Cartesian coordinate system, two different coordinates may lie at the same location. Consider the points $(1, \frac{\pi}{4})$ and $(-1, \frac{5\pi}{4})$ (see figure 4.5). When a radius is negative, you move *backwards* back over the origin, like a mirror image.

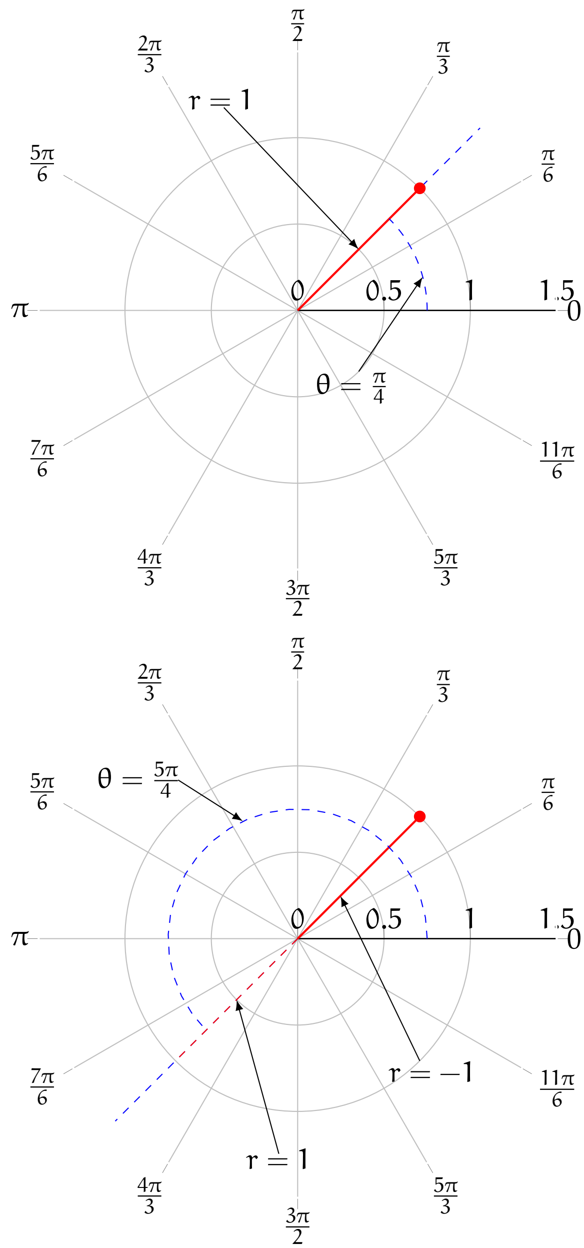


Figure 4.5: The polar coordinates points $(1, \frac{\pi}{4})$ and $(-1, \frac{5\pi}{4})$ are the same location on a polar axis

4.3 Changing coordinate systems

4.3.1 Cartesian to Polar

From the example above, you should see that a given Cartesian coordinate, (x, y), can also be expressed as a polar coordinate, (r, θ), where r is the distance from the origin and θ is the angle of rotation from the horizontal. (Note: Polar functions are generally given as r defined in terms of θ, which means the *dependent* variable is listed first in the coordinate pair, unlike Cartesian coodinates.) Additionally,

$$r = \sqrt{x^2 + y^2}$$

$$\theta = \arctan \frac{y}{x}$$

Example: Express the Cartesian point $(-3, 4)$ in polar coordinates.

Solution: Taking $x = -3$ and $y = 4$, we find that:

$$r = \sqrt{(-3)^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5$$

We follow the convention of only taking the positive solution to the square root. Finding θ:

$$\theta = \arctan \frac{4}{-3}$$

When you evaluate the arctan with a calculator, you are likely to get back $\theta = -0.928$. Recall that $tan\theta = \tan\theta \pm n\pi$, where n is an integer. We know our Cartesian point, $(-3, 4)$, is in the II quadrant, while the angle -0.928 radians would fall in the IV quadrant. So, clearly, -0.928 radians is not correct. Most calculators restrict the output of arctan to angles between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, because there are actually multiple angles where $\tan\theta = -\frac{4}{3}$. Since $\tan\theta = \tan\theta \pm n\pi$, we also know that:

$$\arctan -\frac{4}{3} = -0.928 \pm n\pi$$

Another possible θ is $-0.928 + \pi \approx 2.214$, which does fall in the appropriate quadrant. This means the polar coordinates $(5, 2.214)$ are the same as the Cartesian coordinates $(-3, 4)$. *Note:* It is standard practice to express angles in radians, and not degrees, when using polar coordinates.

4.3.2 Polar to Cartesian

We can also leverage our knowledge of right triangles to convert polar coordinates to Cartesian coordinates. Take the polar coordinate $(2, \frac{\pi}{4})$ (see figure 4.6). We can draw a right triangle with legs parallel to the x and y axes (not shown in the figure) and a hypotenuse that goes from the origin to the polar coordinate $(2, \frac{\pi}{4})$.

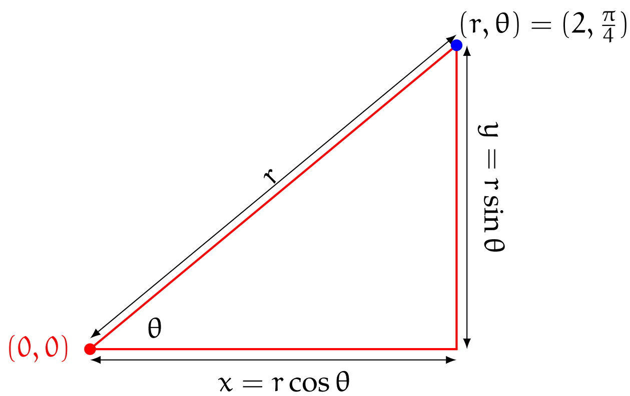


Figure 4.6: To convert from polar to Cartesian coordinates, use the identities $x = r\cos\theta$ and $y = r\sin\theta$

Recall from trigonometry that:

$$\sin\theta = \frac{\text{opposite leg}}{\text{hypotenuse}}$$

We know that the hypotenuse of this triangle has a length of r. The opposite leg is vertical and is the same length as the distance of the polar coordinate from the x-axis. Therefore, the length of the vertical leg represents the y value of that same polar coordinate if it were expressed in Cartesian coordinates. So, we can say that:

$$\sin\theta = \frac{y}{r}$$

And therefore:

$$y = r\sin\theta$$

By a similar process, we also see that:

$$x = r\cos\theta$$

This is easy to visualize and understand for $0 \le \theta \le \frac{\pi}{2}$, but it also holds for other values of θ.

Example: Express the polar coordinate $(\frac{3}{2}, \frac{2\pi}{3})$ in Cartesian coordinates.

Solution: From the polar coordinate, we see that $\theta = \frac{2\pi}{3}$ and $r = \frac{3}{2}$. Therefore:

$$x = r\cos\theta = \frac{3}{2} \cdot \cos\frac{2\pi}{3} = \frac{3}{2} \cdot -\frac{1}{2} = -\frac{3}{4}$$

$$y = r\sin\theta = \frac{3}{2} \cdot \sin\frac{2\pi}{3} = \frac{3}{2} \cdot \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{4}$$

The Cartesian coordinate $(-\frac{3}{4}, \frac{3\sqrt{3}}{4})$ has the same location as the given polar coordinate.

Exercise 12

Convert the following polar coordinates to Cartesian coordinates:

1. $(2, \frac{3\pi}{2})$
2. $(\sqrt{2}, \frac{3\pi}{4})$
3. $(3, -\frac{\pi}{4})$
4. $(-3, -\frac{\pi}{3})$
5. $(2, -\frac{\pi}{2})$

Working Space

Answer on Page 49

Exercise 13

Convert the following Cartesian coordinates to polar coordinates. Restrict θ to $0 \leq \theta < 2\pi$.

1. $(-4, 4)$
2. $(3, 3\sqrt{3})$
3. $(\sqrt{3}, -1)$
4. $(-6, 0)$
5. $(-2, -2)$

Working Space

Answer on Page 49

4.4 Circles in Polar Coordinates

Many conic sections, including circles, are simpler to express as polar functions than as Cartesian functions. Consider a circle with a radius of 2 centered about the origin. The polar function for this is $r = 2$ for all θ. Let's write a Cartesian function for the same circle.

We know that for every point on the circle, the distance to the origin is 2. This means that, by the Pythagorean theorem,

$$r^2 = x^2 + y^2$$

.

(see figure 4.7)

We can solve this equation for y, given that $r = 2$ (in this case):

$$y = \pm\sqrt{2^2 - x^2}$$

Notice that this is really two equations: $y = \sqrt{2^2 - x^2}$ and $y = -\sqrt{2^2 - x^2}$. This is more complex than the polar equation, $r = 2$.

As seen above, the equation of a circle with radius R centered on the origin is simply $r = R$

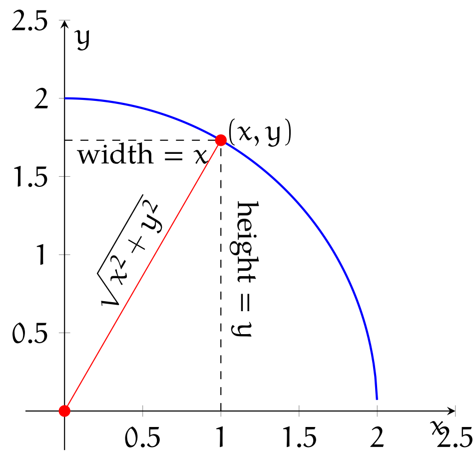


Figure 4.7: All (x, y) pairs on the circle are the same distance from the origin

in polar coordinates. What if we want a circle centered somewhere else? Polar coordinates are best when a circle is bisected by the x or y axis. Consider the polar equation $r = 3\sin\theta$. Let's use a table to find some points and plot the function:

θ	$r = 3\sin\theta$
0	0
$\frac{\pi}{6}$	$\frac{3}{2}$
$\frac{\pi}{4}$	$\frac{3\sqrt{2}}{2}$
$\frac{\pi}{3}$	$\frac{3\sqrt{3}}{2}$
$\frac{\pi}{2}$	3
$\frac{2\pi}{3}$	$\frac{3\sqrt{3}}{2}$
$\frac{3\pi}{4}$	$\frac{3\sqrt{2}}{2}$
$\frac{5\pi}{6}$	$\frac{3}{2}$
π	0

Here is how those points look plotted (see figures 4.8 and 4.9):

So, the polar equation $r = 3\sin\theta$ gives a circle with radius $\frac{3}{2}$ centered at $(0, \frac{3}{2})$.

Example: Describe the graph of $r = \cos\theta$. Feel free to make a rough plot on the blank polar axis below:

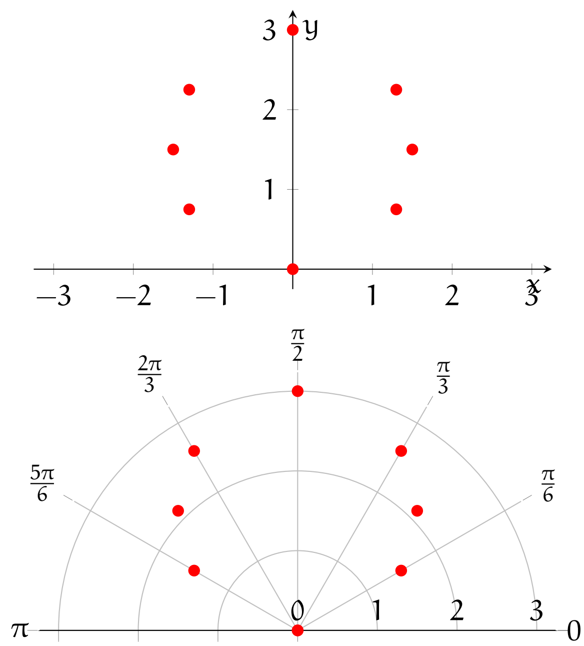


Figure 4.8: Several points for $r = 3\sin\theta$ plotted on Cartesian and polar coordinate systems

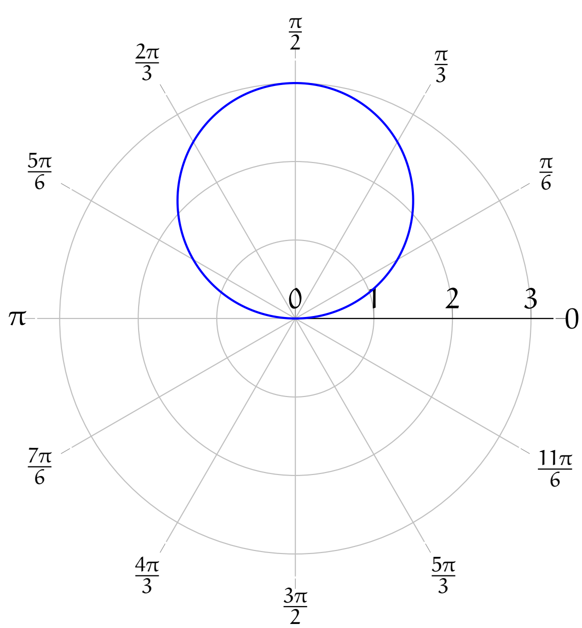
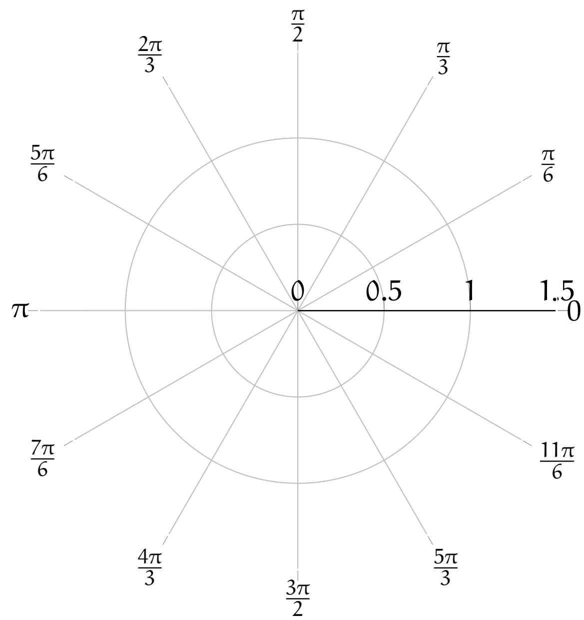
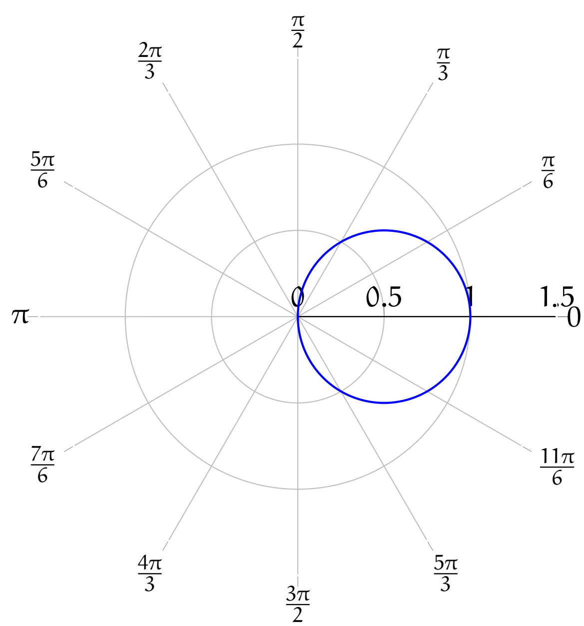


Figure 4.9: $r = 3\sin\theta$ plotted on a polar coordinate system



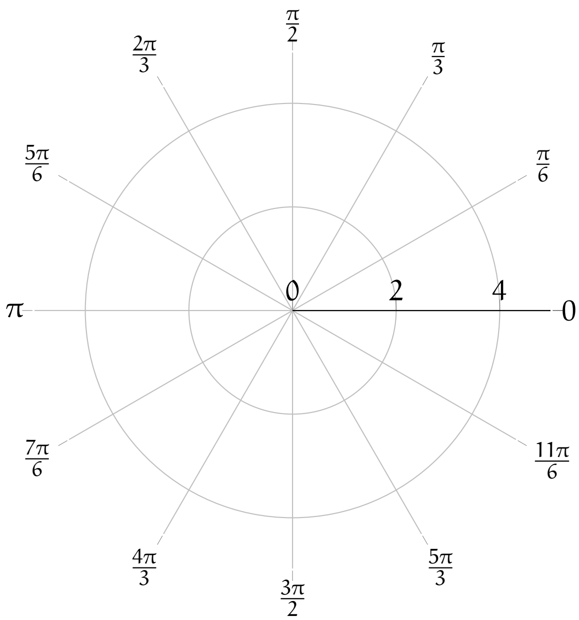
Solution: This plot will look like a circle of radius 0.5 centered at $(0.5, 0)$ (in polar coordinates).



Exercise 14

Sketch the following polar functions on the provided polar axis for $0 \leq \theta < 2\pi$:

1. $r = 3$
2. $\theta = \pi$
3. $r = 2\cos\frac{\theta}{2}$
4. $r = -4\sin\theta$
5. $r = \theta$



Working Space

Answer on Page 50

Chapter 5

Sound

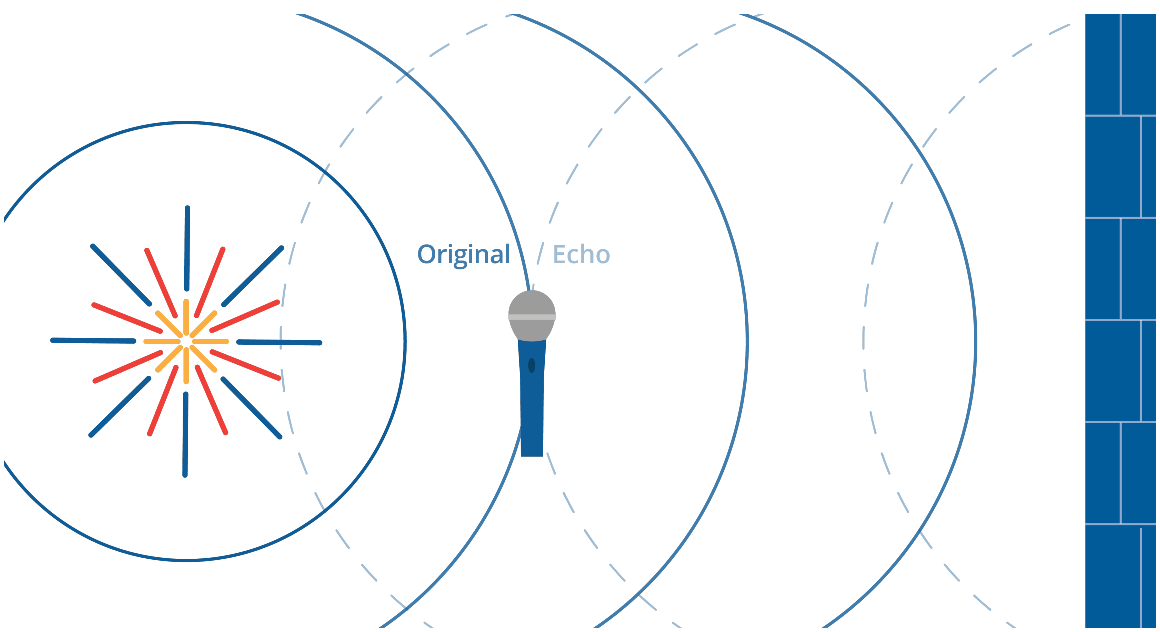
When you set off a firecracker, it makes a sound.

Let's break that down a little more. Inside the cardboard wrapper of the firecracker, there is potassium nitrate (KNO_3), sulfur (S), and carbon(C). These are all solids. When you trigger the chemical reactions with a little heat, these atoms rearrange themselves to be potassium carbonate (K_2CO_3), potassium sulfate (K_2SO_4), carbon dioxide (CO_2), and nitrogen (N_2). Note that the last two are gasses.

The molecules of a solid are much more tightly packed than the molecules of a gas. So after the chemical reaction, the molecules expand to fill a much bigger volume. The air molecules nearby get pushed away from the firecracker. They compress the molecules beyond them, and those compress the molecules beyond them.

This compression wave radiates out as a sphere; its radius growing at about 343 meters per second ("The speed of sound").

The energy of the explosion is distributed around the surface of this sphere. As the radius increases, the energy is spread more and more thinly around. This is why the firecracker seems louder when you are closer to it. (If you set off a firecracker in a sewer pipe, the sound will travel much, much farther.)



This compression wave will bounce off of hard surfaces. If you set off a firecracker 50 meters from a big wall, you will hear the explosion twice. We call the second one an "echo".

The compression wave will be absorbed by soft surfaces. If you covered that wall with pillows, there would be almost no echo.

The study of how these compression waves move and bounce is called *acoustics*. Before you build a concert hall, you hire an acoustician to look at your plans and tell you how to make it sound better.

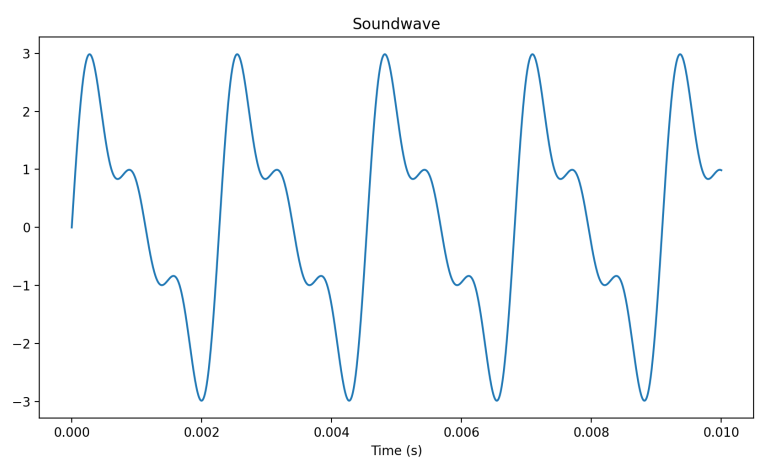
5.1 Pitch and frequency

The string on a guitar is very similar to the weighted spring example. The farther the string is displaced, the more force it feels pushing it back to equilibrium. Thus, it moves back and forth in a sine wave. (OK, it isn't a pure sine wave, but we will get to that later.)

The string is connected to the center of the boxy part of the guitar, which is pushed and pulled by the string. That creates compression waves in the air around it.

If you are in the room with the guitar, those compression waves enter your ear and push and pull your eardrum, which is attached to bones that move a fluid that tickles tiny hairs, called *cilia*, in your inner ear. This is how you hear.

We sometimes see plots of sound waveforms. The x-axis represents time. The y-axis represents the amount the air is compressed at the microphone that converted the air pressure into an electrical signal.



If the guitar string is made tighter (by the tuning pegs) or shorter (by the guitarist's fingers on the strings), the string vibrates more times per second. We measure the number of waves per second and we call it the *frequency* of the tone. The unit for frequency is *Hertz*: cycles per second.

Musicians have given the different frequencies names. If the guitarist plucks the lowest note on his guitar, it will vibrate at 82.4 Hertz. The guitarist will say "That pitch is low E." If the string is made half as long (by a finger on the 12th fret), the frequency will be twice as fast (164.8 Hertz), and the guitarist will say "That is E an octave up."

For any note, the note that has twice the frequency is one octave up. The note that has half the frequency is one octave down.

The octave is a very big jump in pitch, so musicians break it up into 12 smaller steps. If the guitarist shortens the E string by one fret, the frequency will be $82.4 \times 1.059463 \approx 87.3$ Hertz.

Shortening the string one fret always increases the frequency by a factor of 1.059463. Why?

Because $1.059463^{1}2 = 2$. That is, if you take 12 of these hops, you end up an octave higher.

This, the smallest hop in western music, is referred to as a *half step*.

Exercise 15 Notes and frequencies

Working Space

The note A near the middle of the piano, is 440Hz. The note E is 7 half steps above A. What is its frequency?

Answer on Page 52

5.2 Chords and harmonics

Of course, a guitarist seldom plays only one string at a time. Instead, they use the frets to pick a pitch for each string and strums all six strings.

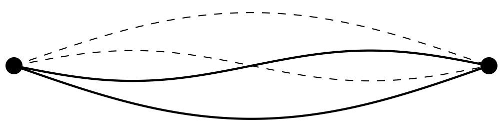
Some combinations of frequencies sound better than others. We have already talked about the octave: If one string vibrates twice for each vibration of another, they sound sweet together.

Musicians speak of "the fifth". If one string vibrates three times and the other vibrates twice in the same amount of time, they sound sweet together.

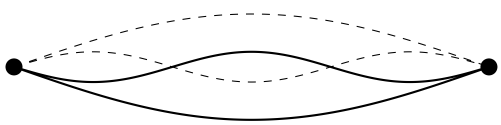
Likewise, if one string vibrates 4 times while the other vibrates 3 times, they sound sweet together. Musicians call this "the third."

Each of these different frequencies tickle different cilia in the inner ear, so you can hear all six notes at the same time when the guitarist strums their guitar.

When a string vibrates, it doesn't create a single sine wave. Yes, the string vibrates from end-to-end, and this generates a sine wave at what we call *the fundamental frequency*. However, there are also "standing waves" on the string. One of these standing waves is still at the centerpoint of the string, but everything to the left of the centerpoint is going up, while everything to the right is going down. This creates *an overtone* that is twice the frequency of the fundamental.



The next overtone has two still points — it divides the string into three parts. The outer parts are up, while the inner part is down. Its frequency is three times the fundamental frequency.



And so on. 4 times the fundamental, 5 times the fundamental, etc.

In general, tones with many overtones tend to sound bright. Tones with just the fundamental sound thin.

Humans can generally hear frequencies from 20Hz to 20,000Hz (or 20kHz). Young people tend to be able to hear very high sounds better than older people.

Dogs can generally hear sounds in the 65Hz to 45kHz range.

5.3 Making waves in Python

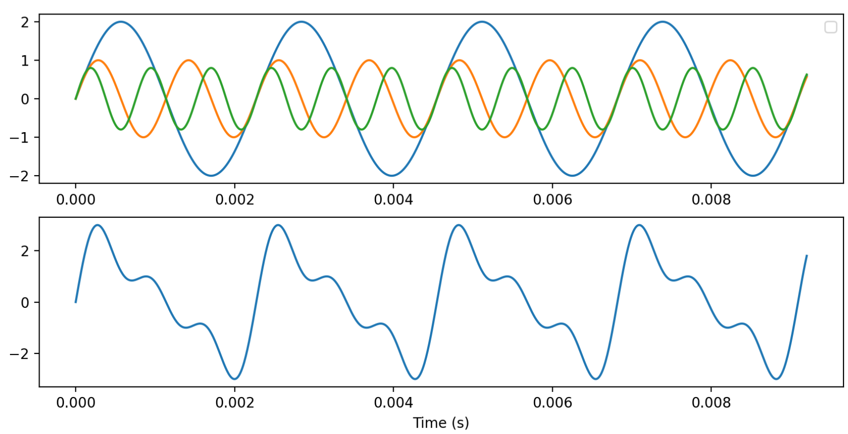
Let's make a sine wave and add some overtones to it. Create a file named `harmonics.py`.

```
import matplotlib.pyplot as plt
import math
```

```
# Constants: frequency and amplitude
fundamental_freq = 440.0 # A = 440 Hz
fundamental_amp = 2.0

# Up an octave
first_freq = fundamental_freq * 2.0 # Hz
first_amp = fundamental_amp * 0.5

# Up a fifth more
second_freq = fundamental_freq * 3.0 # Hz
second_amp = fundamental_amp * 0.4

# How much time to show
max_time = 0.0092 # seconds

# Calculate the values 10,000 times per second
time_step = 0.00001 # seconds

# Initialize
time = 0.0
times = []
totals = []
fundamentals = []
firsts = []
seconds = []

while time <= max_time:
    # Store the time
    times.append(time)

    # Compute value each harmonic
    fundamental = fundamental_amp * math.sin(2.0 * math.pi * fundamental_freq * time)
    first = first_amp * math.sin(2.0 * math.pi * first_freq * time)
    second = second_amp * math.sin(2.0 * math.pi * second_freq * time)

    # Sum them up
    total = fundamental + first + second

    # Store the values
    fundamentals.append(fundamental)
    firsts.append(first)
    seconds.append(second)
    totals.append(total)

    # Increment time
```

```
    time += time_step

# Plot the data
fig, ax = plt.subplots(2, 1)

# Show each component
ax[0].plot(times, fundamentals)
ax[0].plot(times, firsts)
ax[0].plot(times, seconds)
ax[0].legend()

# Show the totals
ax[1].plot(times, totals)
ax[1].set_xlabel("Time (s)")

plt.show()
```

When you run it, you should see a plot of all three sine waves and another plot of their sum:



5.3.1 Making a sound file

The graph is pretty to look at, but make let's a file that we can listen to.

The WAV audio file format is supported on pretty much any device, and a library for writing WAV files comes with Python. Let's write some sine waves and some noise into a WAV file.

Create a file called `soundmaker.py`

```
import wave
import math
import random

# Constants
frame_rate = 16000 # samples per second
duration_per = 0.3 # seconds per sound
frequencies = [220, 440, 880, 392] # Hz
amplitudes = [20, 125]
baseline = 127 # Values will be between 0 and 255, so 127 is the baseline
samples_per = int(frame_rate * duration_per) # number of samples per sound

# Open a file
wave_writer = wave.open('sound.wav', 'wb')

# Not stereo, just one channel
wave_writer.setnchannels(1)

# 1 byte audio means everything is in the range 0 to 255
wave_writer.setsampwidth(1)

# Set the frame rate
wave_writer.setframerate(frame_rate)

# Loop over the amplitudes and frequencies
for amplitude in amplitudes:
    for frequency in frequencies:
        time = 0.0
        # Write a sine wave
        for sample in range(samples_per):
            s = baseline + int(amplitude * math.sin(2.0 * math.pi * frequency * time))
            wave_writer.writeframes(bytes([s]))
            time += 1.0 / frame_rate

        # Write some noise after each sine wave
        for sample in range(samples_per):
            s = baseline + random.randint(0, 15)
            wave_writer.writeframes(bytes([s]))

# Close the file
wave_writer.close()
```

When you run it, it should create a sound file with several tones of different frequencies and volumes. Each tone should be followed by some noise.

Appendix A

Answers to Exercises

Answer to Exercise 1 (on page 4)

1. By the chain rule, $f'(x) = 2\arctan x \times \frac{d}{dx}\arctan x = 2\arctan x \frac{1}{1+x^2}$

2. By the Product rule, $f'(x) = x\frac{d}{dx}\text{arcsec}\,(x^3) + \text{arcsec}\,(x^3)$. Further, by the chain rule, $\frac{d}{dx}\text{arcsec}\,(x^3) = \frac{1}{(x^3)\sqrt{(x^3)^2-1}} \times \frac{d}{dx}(x^3) = \frac{3x^2}{x^3\sqrt{x^6-1}}$. Therefore, $f'(x) = \frac{3}{\sqrt{x^6-1}} + \text{arcsec}\,(x^3)$

3. By the chain rule, $f'(x) = \frac{1/x}{\sqrt{1-(1/x)^2}} \times -\frac{1}{x^2} = -\frac{1}{x^3\sqrt{1-\frac{1}{x^2}}}$

Answer to Exercise 2 (on page 6)

We know that for a right triangle, $\cos\theta = \frac{\text{adjacent leg}}{\text{hypotenuse}}$. For a right triangle inscribed in the Unit Circle, the adjacent leg is parallel to the x-axis and has the same length as the x-value of the coordinate point on the circle. Additionally, the length of the hypotenuse is 1. Therefore, $\cos\theta = \frac{x_o}{1} = x_o$.

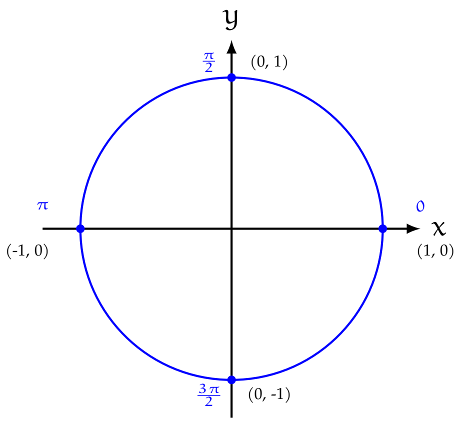
Answer to Exercise 3 (on page 7)

1. $\sin\frac{\pi}{2} = 1$

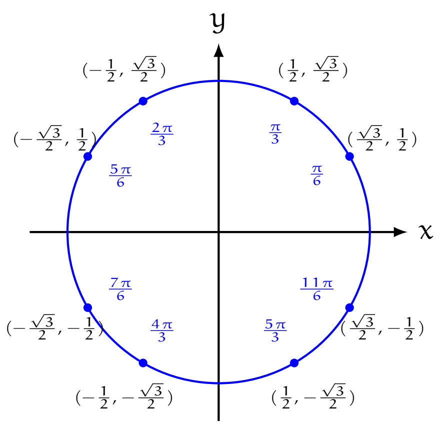
2. $\cos\frac{3\pi}{2} = 0$

3. $\sin\pi = 0$

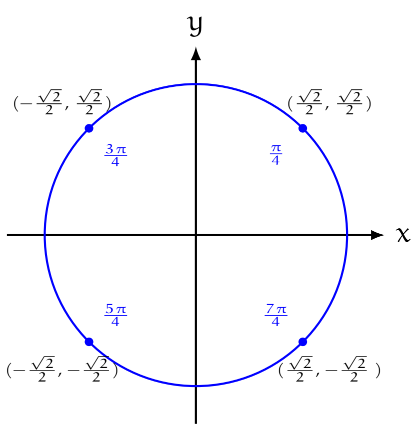
4. $\cos -\pi = -1$ (Negative angles are measured clockwise from the x-axis, so $\theta = -\pi$ is at the same angle as $\theta = \pi$.)



Answer to Exercise 4 (on page 8)



Answer to Exercise 5 (on page 10)



Answer to Exercise 6 (on page 11)

1. 0
2. $\sqrt{2}/2$
3. $-1/2$
4. $-1/2$
5. $\sqrt{2}/2$
6. $1/2$
7. $\sqrt{2}/2$
8. -1
9. $-\sqrt{3}/2$
10. $1/2$

Answer to Exercise 7 (on page 13)

1. $\sin(\pi/12) = \sin(\pi/3 - \pi/4) = \sin\frac{\pi}{3}\cos\frac{\pi}{4} - \cos\frac{\pi}{3}\sin\frac{\pi}{4} = \left(\frac{\sqrt{3}}{3}\right)\cdot\left(\frac{\sqrt{2}}{2}\right) - \left(\frac{1}{2}\right)\cdot\left(\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4} = \frac{\sqrt{6}-\sqrt{2}}{4}$
2. $\cos(7\pi/12) = \cos(4\pi/12 + 3\pi/12) = \cos(\pi/3 + \pi/4) = \cos\frac{\pi}{3}\cos\frac{\pi}{4} - \sin\frac{\pi}{3}\sin\frac{\pi}{4} = \left(\frac{1}{2}\right)\cdot\left(\frac{\sqrt{2}}{2}\right) - \left(\frac{\sqrt{3}}{2}\right)\cdot\left(\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{4} - \frac{\sqrt{6}}{4} = \frac{\sqrt{2}-\sqrt{6}}{4}$
3. $\tan(13\pi/12) = \frac{\sin(13\pi/12)}{\cos(13\pi/12)}$ First, we will find $\sin(13\pi/12)$: $\sin(13\pi/12) = \sin(3\pi/12 + 10\pi/12) = \sin(\pi/4 + 5\pi/6) = \sin\frac{\pi}{4}\cos\frac{5\pi}{6} + \cos\frac{\pi}{4}\sin\frac{5\pi}{6} = \left(\frac{\sqrt{2}}{2}\right)\cdot\left(\frac{-\sqrt{3}}{2}\right) + \left(\frac{\sqrt{2}}{2}\right)\cdot\left(\frac{1}{2}\right) = \frac{-\sqrt{6}}{4} = \frac{\sqrt{2}}{4} = \frac{\sqrt{2}-\sqrt{6}}{4}$. Next we find $\cos(13\pi/12) = \cos(\pi/4 + 5\pi/6) = \cos\frac{\pi}{4}\cos\frac{5\pi}{6} - \sin\frac{\pi}{4}\sin\frac{5\pi}{6} = \left(\frac{\sqrt{2}}{2}\right)\cdot\left(\frac{-\sqrt{3}}{2}\right) - \left(\frac{\sqrt{2}}{2}\right)\cdot\left(\frac{1}{2}\right) = \frac{-\sqrt{6}}{4} - \frac{\sqrt{2}}{4} = \frac{-\sqrt{6}-\sqrt{2}}{4}$. And therefore $\tan(13\pi/12) = \frac{\sin 13\pi/12}{\cos 13\pi/12} = \frac{\sqrt{2}-\sqrt{6}}{4}\cdot\frac{4}{-\sqrt{6}-\sqrt{2}} = \frac{\sqrt{6}-\sqrt{2}}{\sqrt{6}+\sqrt{2}} = \frac{\sqrt{6}-\sqrt{2}}{\sqrt{6}+\sqrt{2}}\cdot\frac{\sqrt{6}-\sqrt{2}}{\sqrt{6}-\sqrt{2}} = \frac{6-2\sqrt{12}+2}{6-2} = \frac{8-4\sqrt{3}}{4} = 2-\sqrt{3}$

Answer to Exercise 8 (on page 14)

$\sin 2\theta = \sin\theta + \theta = \sin\theta\cos\theta + \cos\theta\sin\theta = 2\sin\theta\cos\theta$

Answer to Exercise 9 (on page 15)

Similar to $\cos(\alpha/2)$, we begin with the double angle formula for cosine, but another version:

$$\cos 2\theta = 1 - 2\sin^2\theta$$

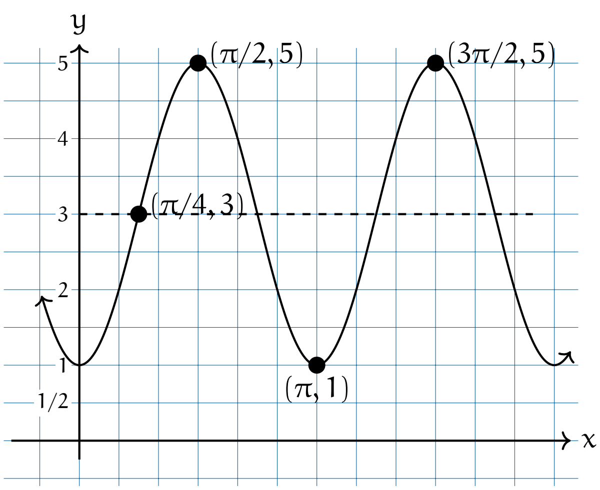
Substituting $\theta = \alpha/2$:

$$\cos\alpha = 1 - 2\sin^2(\alpha/2)$$

And rearranging to solve for $\sin(\alpha/2)$:

$$\sin(\alpha/2) = \pm\sqrt{\frac{1-\cos\alpha}{2}}$$

Answer to Exercise 10 (on page 22)



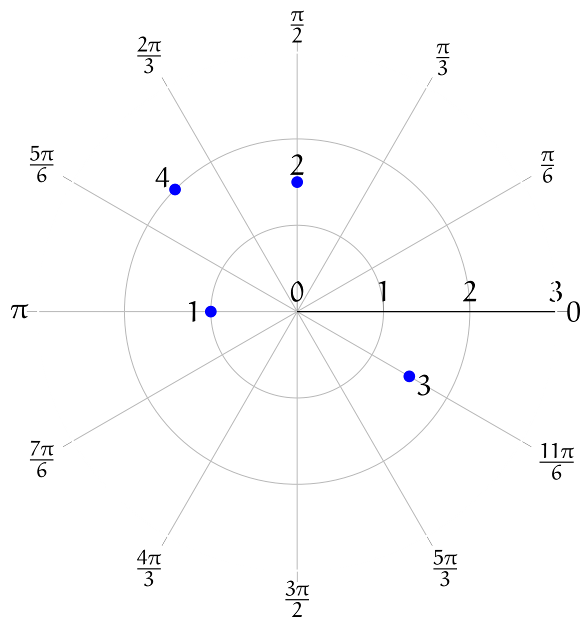
This wave has an amplitude of 2; its baseline has been translated up to 3.

This wave has wavelength of π. A sine wave usually has a wavelength of 2π, so we need to compress the x axis by a factor of 2.

The wave first crosses its baseline at $pi/4$. The sine wave starts by crossing its baseline, so we need to translate the curve right by $\pi/4$.

$$f(x) = 2\sin(2x - \frac{\pi}{4}) + 3$$

Answer to Exercise 11 (on page 26)



Answer to Exercise 12 (on page 31)

1. $(0, -2)$. $x = 2 \cdot \cos \frac{3\pi}{2} = 2 \cdot 0 = 0$ and $y = 2 \cdot \sin \frac{3\pi}{2} = 2 \cdot -1 = -2$.
2. $(-1, 1)$. $x = \sqrt{2} \cdot \cos \frac{3\pi}{4} = \sqrt{2} \cdot -\frac{\sqrt{2}}{2} = \frac{2}{2} = -1$ and $y = \sqrt{2} \cdot \sin \frac{3\pi}{4} = \sqrt{2} \cdot \frac{\sqrt{2}}{2} = \frac{2}{2} = 1$.
3. $(\frac{3\sqrt{2}}{2}, -\frac{3\sqrt{2}}{2})$. $x = 3 \cdot \cos -\frac{\pi}{4} = 3 \cdot \frac{\sqrt{2}}{2} = \frac{3\sqrt{2}}{2}$ and $y = 3 \cdot \sin -\frac{\pi}{4} = 3 \cdot -\frac{\sqrt{2}}{2} = -\frac{3\sqrt{2}}{2}$.
4. $(-\frac{3}{2}, -\frac{3\sqrt{3}}{2})$. $x = (-3) \cdot \cos \frac{\pi}{3} = (-3) \cdot \frac{1}{2} = -\frac{3}{2}$ and $y = (-3) \cdot \sin \frac{\pi}{3} = (-3) \cdot \frac{\sqrt{3}}{2} = -\frac{3\sqrt{3}}{2}$.
5. $(0, -2)$. $x = 2 \cdot \cos -\frac{pi}{2} = 2 \cdot 0 = 0$ and $y = 2 \cdot \sin -\frac{\pi}{2} = 2 \cdot -1 = -2$.

Answer to Exercise 13 (on page 32)

1. $(4\sqrt{2}, \frac{3\pi}{4})$. $r = \sqrt{x^2 + y^2} = \sqrt{32} = 4\sqrt{2}$. $\arctan \frac{y}{x} = \arctan \frac{4}{-4} = \arctan -1 = -\frac{\pi}{4} + n\pi$. We take $\theta = \frac{3\pi}{4}$ to satisfy the domain restriction and be in the correct quadrant.
2. $(6, \frac{\pi}{3})$. $r = \sqrt{3^2 + \left(3\sqrt{3}\right)^2} = \sqrt{9 + 27} = \sqrt{36} = 6$. $arctan \frac{3\sqrt{3}}{3} = \arctan \sqrt{3} = \frac{\pi}{3} + n\pi$. We take $\theta = \frac{\pi}{3}$ to satisfy the domain restriction and be in the correct quadrant.

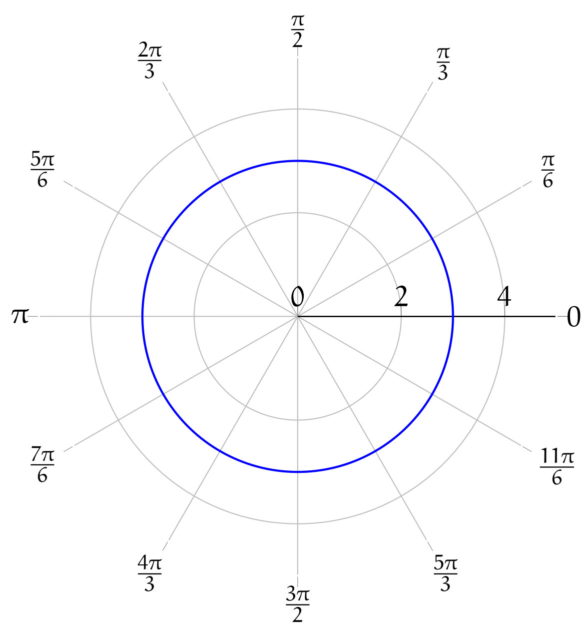
3. $(2, \frac{11\pi}{6})$. $r = \sqrt{\sqrt{3}^2 + (-1)^2} = \sqrt{3+1} = 2$. $\arctan \frac{-1}{\sqrt{3}} = -\frac{\pi}{6} + n\pi$. We take $\theta = \frac{11\pi}{6}$ to satisfy the domain restriction and have the point in the correct quadrant.

4. $(6, \pi)$. $r = \sqrt{(-6)^2 + 0^2} = 6$. $\arctan \frac{0}{-6} = \pi + n\pi$. We take $\theta = \pi$ to satisfy the domain restriction.

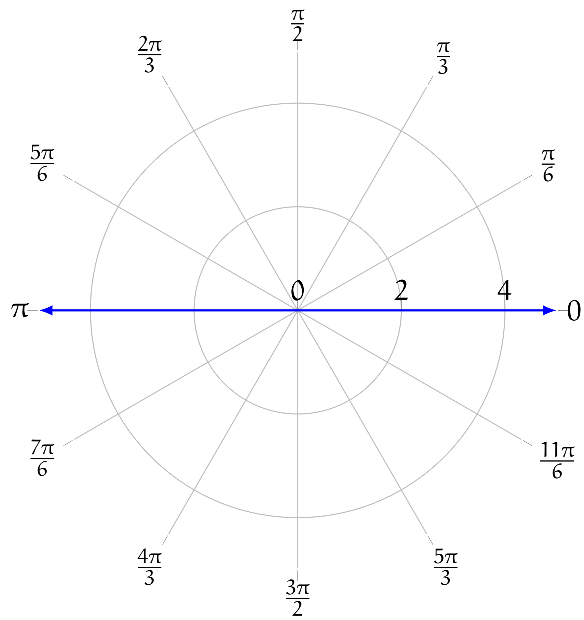
5. $(2\sqrt{2}, \frac{5\pi}{4})$. $r = \sqrt{(-2)^2 + (-2)^2} = \sqrt{8} = 2\sqrt{2}$. $\arctan \frac{-2}{-2} = \arctan 1 = \frac{\pi}{4} + n\pi$. We take $\theta = \frac{5\pi}{4}$ to satisfy the domain restriction and be in the correct quadrant.

Answer to Exercise ?? (on page 36)

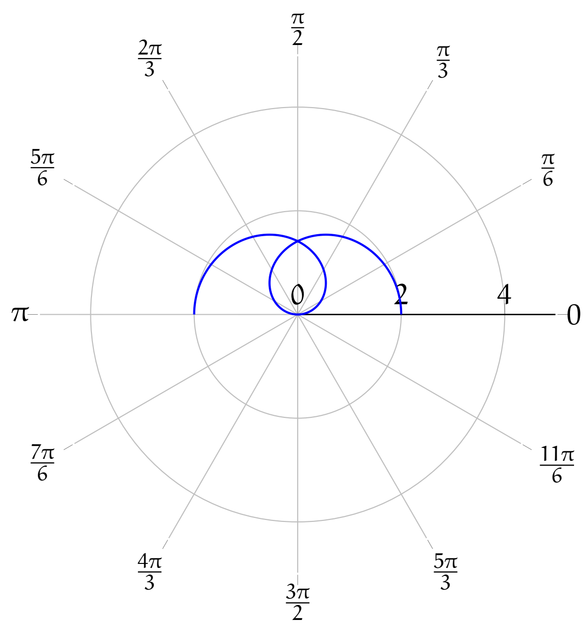
1. $r = 3$



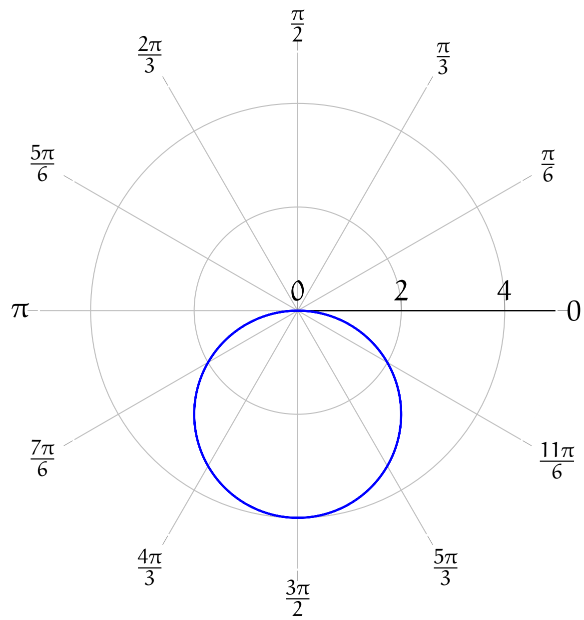
2. $\theta = \pi$ Because r includes all real numbers, negative r is possible and the line $\theta = \pi$ extends in both directions



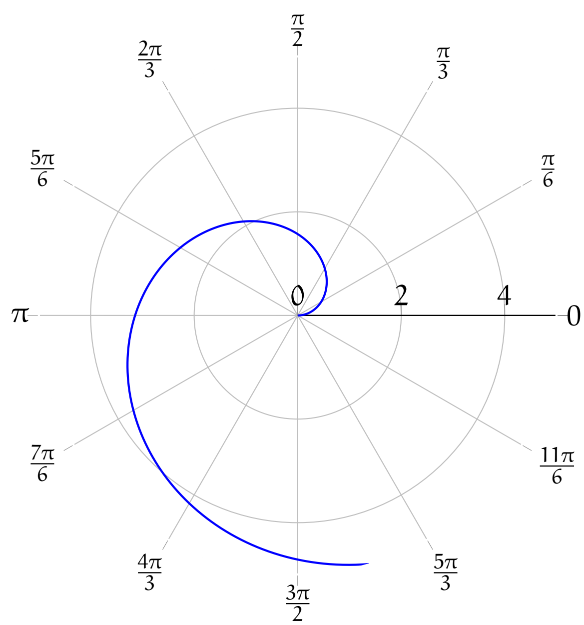
3. $r = 2\cos\frac{\theta}{2}$



4. $r = -4\sin\theta$



5. $r = \theta$ (The spiral continues, but is beyond the boundary of the graph)



Answer to Exercise 15 (on page 39)

A is 440 Hz. Each half-step is a multiplication by $\sqrt[12]{2} = 1.059463094359295$ So the frequency of E is $(440)(2^{7/12}) = 659.255113825739859$