Contents

Chapter 1

Exponents

Let's take quick look at exponents. Ancient scientists started coming up with a lot of formulas that involved multiplying the same number several times. For example, if they knew that a sphere was r centimeters in radius, its volume in milliliters was

$$V = \frac{4}{3} \times \pi \times r \times r \times r$$

They did two things to make the notation less messy. First, they decided that if two numbers were written next to each other, the reader would assume that meant "multiply them". Second, they came up with the exponent, a little number that was lifted off the baseline of the text, that meant "multiply it by itself". For example 5^3 was the same as $5 \times 5 \times 5$.

Now, the formula for the volume of a sphere is written

$$V = \frac{4}{3}\pi r^3$$

Tidy, right? In an exponent expression like this, we say that r is *the base* and 3 is *the exponent*.

1.1 Identities for Exponents

What about exponents of exponents? What is $\left(5^3\right)^2$?

$$\left(5^3\right)^2 = (5 \times 5 \times 5)^2 = (5 \times 5 \times 5)(5 \times 5 \times 5) = 5^6$$

In general, for any a, b, and c:

$$\left(a^b\right)^c = a^{(bc)}$$

If you have $\left(5^3\right)\left(5^4\right)$, that is just $5 \times 5 \times 5 \times 5 \times 5 \times 5 \times 5$ or 5^7

The general rule is that for any a, b, and c

$$\left(a^b\right)\left(a^c\right) = a^{(b+c)}$$

Mathematicians *love* this rule, so we keep extending the idea of exponents to keep this rule true. For example, at some point, someone asked "What about 5^0?" According to the rule, 5^2 must equal $5^{(2+0)}$ which must equal $\left(5^2\right)\left(5^0\right)$. Thus, 5^2 must be 1. So mathematicians declared "Anything to the power of 0 is 1".

We don't typically assume that $0^0 = 1$. It is just too weird. So, we say that for any a not equal to zero,

$$a^0 = 1$$

What about $5^{(-2)}$? By our beloved rule, we know that $\left(5^{-2}\right)\left(5^5\right)$ must be equal to 5^3, right? So 5^{-2} must be equal to $\frac{1}{5^2}$.

We say that for any a not equal to zero and any b,

$$a^{-b} = \frac{1}{a^b}$$

This makes dividing one exponential expression by another (with the same base) easy:

$$\frac{a^b}{a^c} = a^{(b-c)}$$

We often say "cancel out" for this. Here, we can "cancel out" x^2:

$$\frac{x^5}{x^2} = x^3$$

What about $5^{\frac{1}{3}}$? By the beloved rule, we know that $5^{\frac{1}{3}}5^{\frac{1}{3}}5^{\frac{1}{3}}$ must equal 5^1. Thus, $5^{\frac{1}{3}} = \sqrt[3]{5}$.

We say that for any a and b not equal to zero and any c greater than zero,

$$a^{\frac{b}{c}} = \left(\sqrt[c]{a}\right)^b = \left(\sqrt[c]{a^b}\right)$$

Before you go on to the exercises, note that the beloved rule demands a common base.

- We can combine these: $\left(5^2\right)\left(5^4\right) = 5^6$
- We cannot combine: $\left(5^2\right)\left(3^5\right)$

With that said, we note for any a,b, and c:

$$(ab)^c = (a^c)(b^c)$$

So, for example, if we were asked to simplify $\left(3^4\right)\left(6^2\right)$, we would note that $6 = 2 \times 3$, so

$$\left(3^4\right)\left(6^2\right) = \left(3^4\right)\left(3^2\right)\left(2^2\right) = \left(3^6\right)\left(2^2\right)$$

If these ideas are new to you (or maybe you are just having trouble remembering them), watch the Khan Academy's **Intro to rational exponents** video at https://youtu.be/lZfXc4nHooo.

Chapter 2

Compound Interest

When you loan money to someone, you typically charge them some sort of interest. The most common loan of this sort is what the bank calls a "savings account". Any money you put in the account is loaned to the bank. The bank then lends it to someone else, who pays interest to the bank. The bank gives some of that interest to you. However, what if you leave the interest in your account? And you start making *interest on the interest*? This is known as *compound interest*.

2.1 An example with annual interest payments

Let's say that you put $1000 in a savings account that pays 6% interest every year. How much money would you have after 12 years? Let's make a spreadsheet.

	A	B	C
1	Interest Rate	6.00%	
2			
3	After year:	Interest	Balance
4	0	$0.00	1000
5	1	$60.00	$1,060.00

Create a new spreadsheet and edit the cells to look like this. All the cells in rows 1 - 4 are just values: just type in what you see.

The fifth row is all formulas:

After year	Interest	Balance
= A4 + 1	= B$1 * C4	= C4 + B5

The interest rate field should be formatted as a percentage. One thing to know when dealing with percentages in the spreadsheet: If the field says "600%", its value is 6.

The cells in the Interest and Balance column should be formatted as currency.

You are about to make several copies of the cells in the fifth row, so make sure they look right.

Click on A5 and shift-click on C5 to select all three cells. Drag the lower-right corner down to fill the rows 6 - 15.

A5:C16 fx = A4 + 1

	A	B	C
1	**Interest Rate**	6.00%	
2			
3	**After year**	**Interest**	**Balance**
4	0	$0.00	1000
5	1	$60.00	$1,060.00
6	2	$63.60	$1,123.60
7	3	$67.42	$1,191.02
8	4	$71.46	$1,262.48
9	5	$75.75	$1,338.23
10	6	$80.29	$1,418.52
11	7	$85.11	$1,503.63
12	8	$90.22	$1,593.85
13	9	$95.63	$1,689.48
14	10	$101.37	$1,790.85
15	11	$107.45	$1,898.30
16	12	$113.90	$2,012.20
17			

Look at the numbers. The first interest payment is $60, but the last is $113.90. Your balance has more than doubled!

2.2 Exponential Growth

We figured this out numerically by repeatedly multiplying the balance by the interest rate. What if you wanted to know what the balance would be n years after investing P_0 dollars with an annual interest rate of r? (Note that r in our example would be 0.06, not 6.0.)

Each year, the balance is multiplied by $1+r$, so after one year, P_0 would become $P_0 \times (1+r)$. The next year you would multiply this number by $(1+r)$ again: $P_0 \times (1+r) \times (1+r)$. The next year? $P_0 \times (1+r) \times (1+r) \times (1+r)$ See the pattern? P_n is this balance after n years, then

$$P_n = P_0(1+r)^n$$

Because n is an exponent, we call this *exponential growth*. There are few things as terrifying to a scientist as the phrase "The population is undergoing exponential growth".

If the principal is not compounded annually, we can modify the equation to account for that bu changing n.

$$n = mt$$

Where t is the frequency of the compounding, called the period: (1: annually, 12: monthly, 52: weekly, 365: daily), and m is the number of periods.

2.3 Sensitivity to interest rate

For most people, the first surprising thing about compound interest is how quickly your money grows after a few years. The second thing that is surprising is how much difference a small change in the percentage rate makes.

Let's add another set of columns that shows what happens to your money if you convince the bank to pay you 8% instead of 6%.

Copy everything from columns B and C:

	A	B	C	D	E
1	**Interest Rate**	6.00%		6.00%	
2					
3	**After year**	**Interest**	**Balance**	**Interest**	**Balance**
4	0	$0.00	1000	$0.00	1000
5	1	$60.00	$1,060.00	$60.00	$1,060.00
6	2	$63.60	$1,123.60	$63.60	$1,123.60
7	3	$67.42	$1,191.02	$67.42	$1,191.02
8	4	$71.46	$1,262.48	$71.46	$1,262.48
9	5	$75.75	$1,338.23	$75.75	$1,338.23
10	6	$80.29	$1,418.52	$80.29	$1,418.52
11	7	$85.11	$1,503.63	$85.11	$1,503.63
12	8	$90.22	$1,593.85	$90.22	$1,593.85
13	9	$95.63	$1,689.48	$95.63	$1,689.48
14	10	$101.37	$1,790.85	$101.37	$1,790.85
15	11	$107.45	$1,898.30	$107.45	$1,898.30
16	12	$113.90	$2,012.20	$113.90	$2,012.20
17					

Now edit the second interest rate to be 8%:

	A	B	C	D	E
1	**Interest Rate**	6.00%		8.00%	
2					
3	**After year**	**Interest**	**Balance**	**Interest**	**Balance**
4	0	$0.00	1000	$0.00	1000
5	1	$60.00	$1,060.00	$80.00	$1,080.00
6	2	$63.60	$1,123.60	$86.40	$1,166.40
7	3	$67.42	$1,191.02	$93.31	$1,259.71
8	4	$71.46	$1,262.48	$100.78	$1,360.49
9	5	$75.75	$1,338.23	$108.84	$1,469.33
10	6	$80.29	$1,418.52	$117.55	$1,586.87
11	7	$85.11	$1,503.63	$126.95	$1,713.82
12	8	$90.22	$1,593.85	$137.11	$1,850.93
13	9	$95.63	$1,689.48	$148.07	$1,999.00
14	10	$101.37	$1,790.85	$159.92	$2,158.92
15	11	$107.45	$1,898.30	$172.71	$2,331.64
16	12	$113.90	$2,012.20	$186.53	$2,518.17
17					

Chapter 3

Logarithms

After the world created exponents, it needed the opposite. We could talk about the quantity $? = 2^3$, that is, “What is the product of 2 multiplied by itself three times?” We needed some way to talk about $2^? = 8$, that is, “2 to the what is 8?” This is why we developed the logarithm.

Here is an example:

$$\log_2 8 = 3$$

In English, you would say, “The logarithm base 2 of 8 is 3.”

The base (2, in this case) can be any positive number. The argument (8, in this case) can also be any positive number.

Try this one: What is the logarithm base 2 of 1/16?

You know that $2^{-4} = \frac{1}{16}$, so $\log_2 \frac{1}{16} = -4$.

3.1 Logarithms in Python

Most calculators have pretty limited logarithm capabilities, but python has a nice `log` function that lets you specify both the argument and the base. Start python, import the math module, and try taking a few logarithms:

```
>>> import math
>>> math.log(8,2)
3.0
>>> math.log(1/16, 2)
-4.0
```

Let’s say that a friend offers you 5% interest per year on your investment for as long as you want. You wonder, “How many years before my investment is 100 times as large?” You can solve this problem with logarithms:

```
>>> math.log(100, 1.05)
```

```
94.3872656381287
```

If you leave your investment with your friend for 94.4 years, the investment will be worth 100 times what you put in.

3.2 Logarithm Identities

The logarithm is defined this way:

$$\log_b a = c \iff b^c = a$$

Notice that the logarithm of 1 is always zero, and $\log_b b = 1$.

The logarithm of a product:

$$\log_b ac = \log_b a + \log_b c$$

This follows from the fact that $b^{a+c} = b^a b^c$. What about a quotient?

$$\log_b \frac{a}{c} = \log_b a - \log_b c$$

Exponents?

$$\log_b (a^c) = c \log_b a$$

Notice that because logs and exponents are the opposite of each other, they can cancel each other out:

$$b^{\log_b a} = a$$

and

$$\log_b (b^a) = a$$

3.3 Changing Bases

We mentioned that most calculators have pretty limited logarithm capabilities. Most calculators don't allow you to specify what base you want to work with. All scientific calculators have a button for "log base 10". So, you need to know how to use that button to get logarithms for other bases. Here is the change-of-base identity:

$$\log_b a = \frac{\log_c a}{\log_c b}$$

For example, if you wanted to find $\log_2 8$, you would ask the calculator for $\log_{10} 8$, then divide that by $\log_{10} 2$. You should get 3.

3.4 Natural Logarithm

When you learn about circles, you are told that the circumference of a circle is about 3.141592653589793 times its diameter. Because we use this unwieldy number a lot, we give it a name: We say "The circumference of a circle is π times its diameter."

There is a second unwieldy number that we will eventually use a great deal in solving problems. It is about 2.718281828459045 (but the digits actually go on forever, just like π). We call this number e. (We are not going to talk about why e is special quite yet, but we will soon.)

Most calculators have a button labeled "ln". That is the *natural logarithm* button. It takes the log in base e.

Similarly, in python, if you don't specify a base, the logarithm is done in base e:

```
>>> math.log(10)
2.302585092994046
>>> math.log(math.e)
1.0
```

3.5 Logarithms in Spreadsheets

Spreadsheets have three log functions:

- `LOG` takes both the argument and the base. `LOG(8,2)` returns 3.
- `LOG10` takes just the argument and uses 10 as the base.

- `LN` takes just the argument and uses e as the base.

Here is a plot from a spreadsheet of a graph of $y = \text{LOG}(x, 2)$.

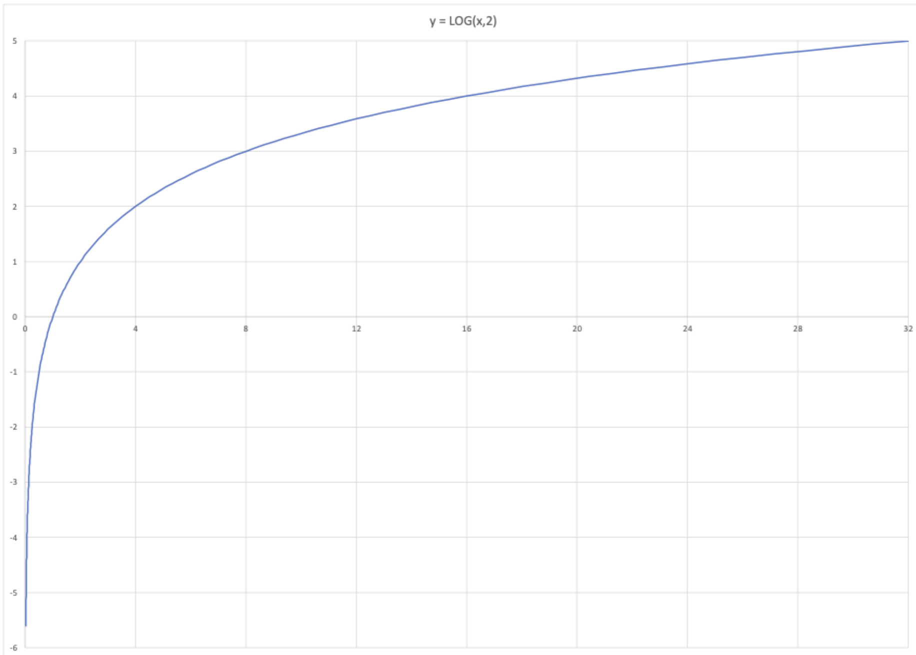

Figure 3.1: A graph of log base 2.

Spreadsheets also have the function `EXP(x)`, which returns e^x. For example, `EXP(2)` returns 7.38905609893065.

Chapter 4

Exponential Decay

In a previous chapter, we saw that an investment of P getting compound interest with an annual interest rate of r, grows exponentially. At the end of year t, your balance would be

$$P(1 + r)^t$$

Because r is positive, this number grows as time passes. You get a nice exponential growth curve that looks something like this:

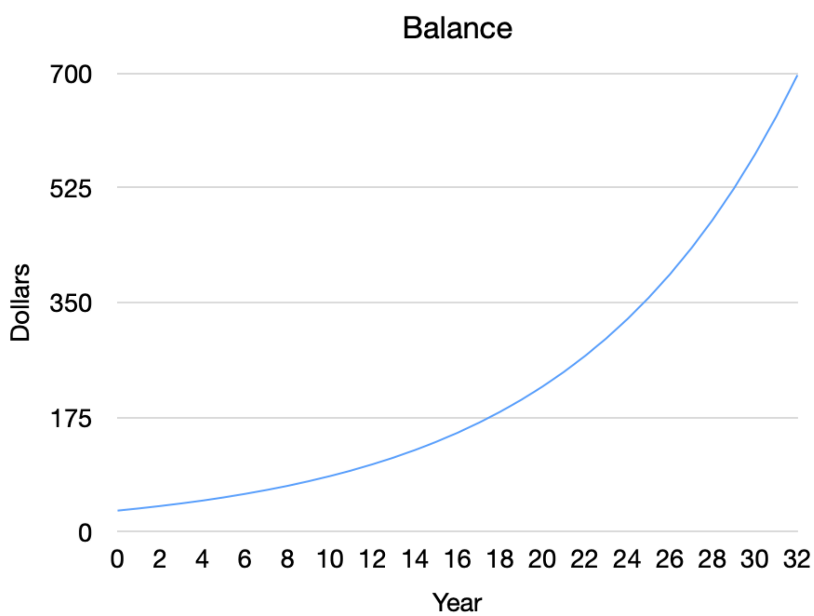

Figure 4.1: A diagram showing (exponential) compound interest.

This is \$30 invested with a 10% annual interest rate. So, the formula for the balance after t years would be

$$(30)(1.1)^t$$

What if r were negative? This would be *exponential decay*.

4.1 Radioactive Decay

Until around 1970, there were companies making watches whose faces and hands were coated with radioactive paint. The paint usually contained radium. When a radium atom decays, it gives off some energy, loses two protons and two neutrons, and becomes becomes a different element (radon). Some of the energy given off is visible light. Thus, these watches glow in the dark.

How many of the radium atoms in the paint decay each century? About 4.24%.

Notice the quantity of atoms lost is proportional to the number of atoms you have. This is exponential decay. If we assume that we start with a million radium atoms, the number of atoms decreases over time like this:

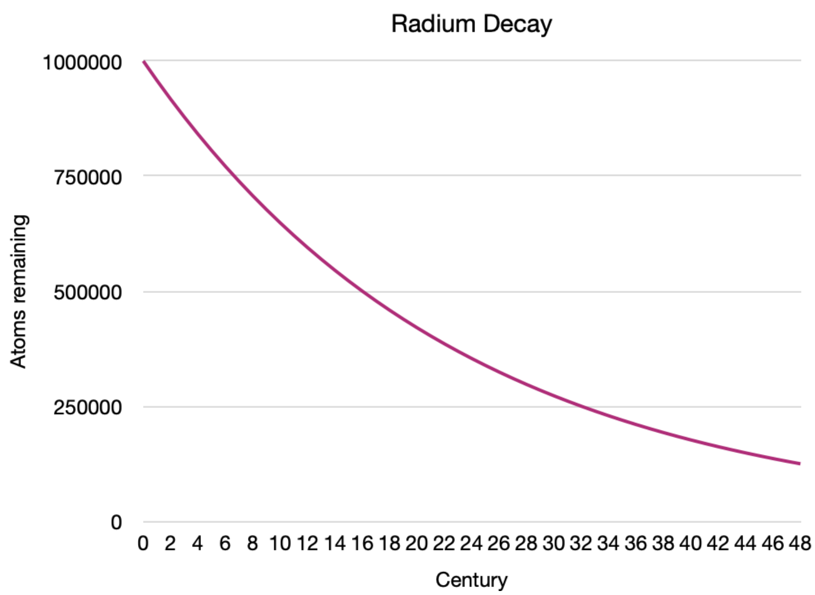

Figure 4.2: A diagram showing the decay and half life of radium.

- We start with 1,000,000 atoms.
- At 16 centuries, we have only 500,000 (half as many) left.
- 16 centuries after that, we have only 250,000 (half again) left.
- 16 centuries after that, we have only 125,000 (half again) left.

A nuclear chemist would say that radium has a *half-life* of 1,600 years; its lifespan decreases by half its original amount every 1,600 years. Note that this means that if you bought a watch with glowing hands in 1960, it will be glowing half as brightly in the year 3560.

How do we calculate the amount of radium left at the end of century t? If you start with

P atoms, at the end of the t-th century, you will have

$$P(1 - 0.0424)^t$$

This is exponential decay.

4.2 Model Exponential Decay

Let's say you get hired to run a company with 480,000 employees. Each year, 1/8 of your employees leave the company for one reason or another (retirement, quitting, etc.). For some reason, you never hire any new employees.

Make a spreadsheet that indicates how many of the original 480,000 employees will still be around at the end of each year for the next 12. Next, make a bar graph from that data.

Chapter 5

Inverse Trigonometric Functions

Recall from the chapter on functions that an inverse of a function is a machine that turns y back into x. The inverses of trigonometric functions are essential to solving certain integrals (you will learn in a future chapter why integrals are useful — for now, trust us that they are!). Let's begin by discussing the sin function and its inverse, $\sin^{-1}$, also called arcsin. (Note: this is not saying sin to the power of -1, it is a method of stating the inverse of the function).

Examine the graph of $\sin x$ in figure 5.1. See how the dashed horizontal line crosses the function at many points? This means the function $\sin x$ is not one-to-one. In other words, there is not a unique x-value for every y-value. This means that if we do not restrict the domain of $\arcsin x$, the result will not be a function (see figure 5.2). In figure 5.2, you can see that just reflecting the graph across $y = x$ fails the vertical line test: an x value has more than one y value.

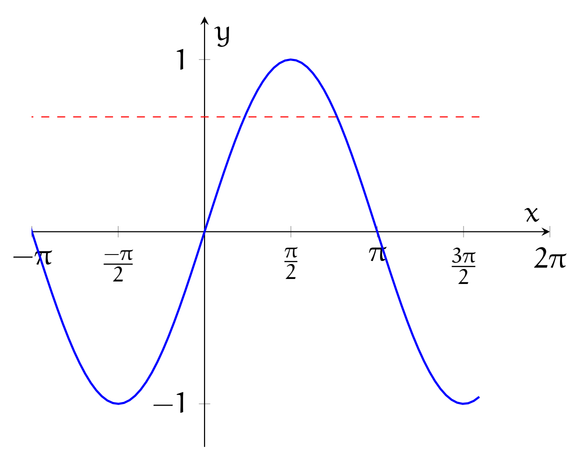

Figure 5.1: The horizontal line $y = \frac{2}{3}$ crosses $y = \sin x$ more than once

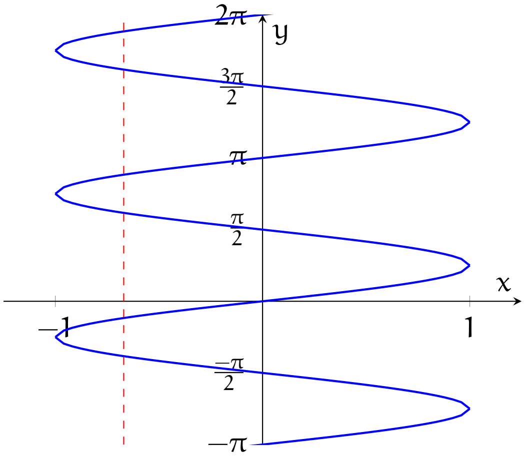

Figure 5.2: The inverse of an unrestricted sin function fails the vertical line test

5.1 Derivatives of Inverse Trigonometric Functions

f	f'
$\arcsin x$	$\frac{1}{\sqrt{1-x^2}}$
$\arccos x$	$-\frac{1}{\sqrt{1-x^2}}$
$\arctan x$	$\frac{1}{1+x^2}$
$\text{arccsc}\ x$	$-\frac{1}{x\sqrt{x^2-1}}$
$\text{arcsec}\ x$	$\frac{1}{x\sqrt{x^2-1}}$
$\text{arccot}\ x$	$-\frac{1}{1+x^2}$

5.2 Practice

Exercise 1

Find the f'. Give your answer in a simplified form.

1. $f(x) = \arctan x^2$
2. $f(x) = x\text{arcsec}\ (x^3)$
3. $f(x) = \arcsin \frac{1}{x}$

Working Space

Answer on Page 21

Appendix A

Answers to Exercises

Answer to Exercise 1 (on page 20)

1. By the chain rule, $f'(x) = 2\arctan x \times \frac{d}{dx}\arctan x = 2\arctan x \frac{1}{1+x^2}$
2. By the Product rule, $f'(x) = x\frac{d}{dx}\text{arcsec}\,(x^3) + \text{arcsec}\,(x^3)$. Further, by the chain rule, $\frac{d}{dx}\text{arcsec}\,(x^3) = \frac{1}{(x^3)\sqrt{(x^3)^2-1}} \times \frac{d}{dx}(x^3) = \frac{3x^2}{x^3\sqrt{x^6-1}}$. Therefore, $f'(x) = \frac{3}{\sqrt{x^6-1}} + \text{arcsec}\,(x^3)$
3. By the chain rule, $f'(x) = \frac{1/x}{\sqrt{1-(1/x)^2}} \times -\frac{1}{x^2} = -\frac{1}{x^3\sqrt{1-\frac{1}{x^2}}}$

Index