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Chapter 1

Atmospheric Pressure

The air you breathe is a blend of gases:

1. 78% nitrogen in the form of N_2
2. 21% oxygen in the form O_2
3. 1% other gases (mostly argon)

If you fill a balloon with helium (He), the helium will push against the interior of the balloon with a certain amount of pressure. The pressure is the same at every point in the interior of the balloon. Pressure, then, is force spread over some area. Force is commonly measured in newtons. Pressure is measured in *pascals*. A pascal is 1 newton per square meter.

We don't usually think about it, but the air outside the balloon is also pushing against the exterior of the balloon. We call this *barometric pressure* or *atmospheric pressure*, and it is caused by gravity pulling on the gas molecules above the balloon.

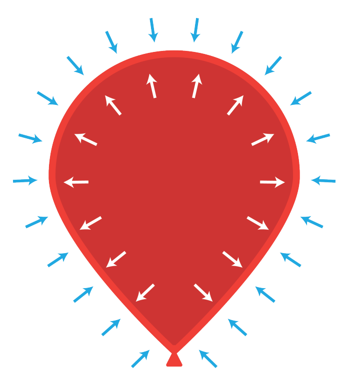


Figure 1.1: Air pushes both ways on a balloon.

Imagine a square meter on the ground at sea level. Now imagine the column of air above it — reaching all the way to the top of the atmosphere.

The air inside that column has a mass of about 10,340 kg. One kilogram on the earth experiences a gravitational force of 9.8 N. So the atmospheric pressure all around you is about 101,332 pascals. When dealing with such large numbers, we often use kilopascals.

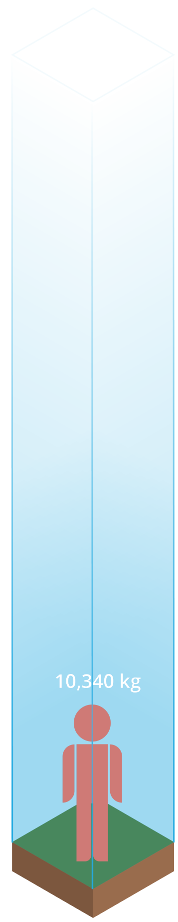


Figure 1.2: An air column surronding a person experiences a large amount of pressure.

We'd say the barometric pressure at sea level is about 101.3 kPa.

That's a lot of pressure! Why doesn't your ribcage collapse crushing your lungs? The air *inside* your lungs is the same pressure as the air push on the outside of your rib cage.

This is why we go about our lives relatively oblivious to this huge force that is all around us — but you can see it sometimes. For example, if you suck the air out of a plastic bottle, the bottle will be crushed by the barometric pressure.

1.1 Altitude and Atmospheric Pressure

If you let go of the balloon, as it rises through this column there will be less and less air mass above it, and thus less and less atmospheric pressure on the outside of the balloon.

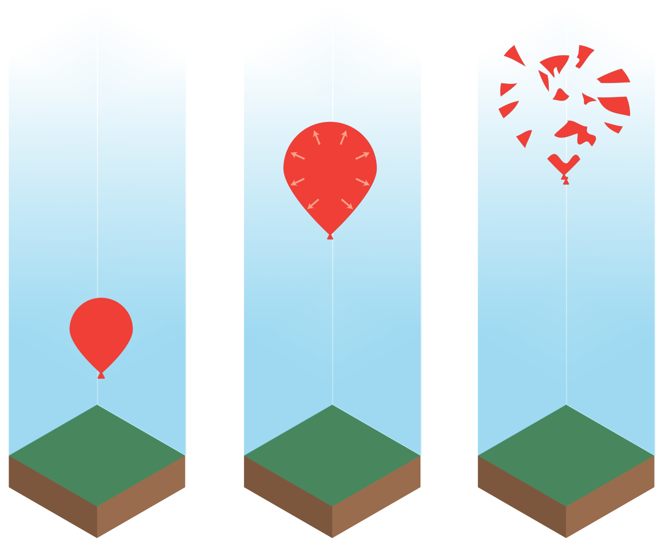


Figure 1.3: As a balloon rises, it reaches a point of less and less outer pressure.

What would be the atmospheric pressure at h meters above sea level? Here is a handy formula for that:

$$p = 101,332 \times \left(1 - \left(2.25577 \times 10^{-5} \times h\right)\right)^{5.25588}$$

where p is the atmospheric pressure in pascals.

Exercise 1 Atmospheric Pressure

Working Space

You are thinking about riding your bicycle to the top of Mount Everest. You are worried when the atmospheric pressure outside the tire drops, the tire will fail. (I have had a tire fail before; It is very, very loud.)

Calculate the atmospheric pressure at the top of Mount Everest (9,144 meters above sea level).

Answer on Page 41

1.2 How a Drinking Straw Works

When you suck on a drinking straw, why does the beverage rise? It is actually pushed by atmospheric pressure.

Before you put your mouth on the straw, the atmospheric pressure is pressing on the entire surface of the liquid (even inside the straw) evenly. Gravity pulls on the liquid making the surface level.

When you suck some air out of the straw, the pressure on the surface inside the straw drops. The atmospheric pressure on the surface outside the straw pushes into the straw and the beverage rises.

Of course, gravity is still trying to pull the liquid inside the straw back down. And for every inch that you lift the liquid in the straw, the force of gravity gets greater, demanding more suction.

1.2.1 The Longest Usable Straw

Assuming you are drinking water in a place with 100 kPa of atmospheric pressure, how high could you suck water with a perfect vacuum? That is, given a very, very long and

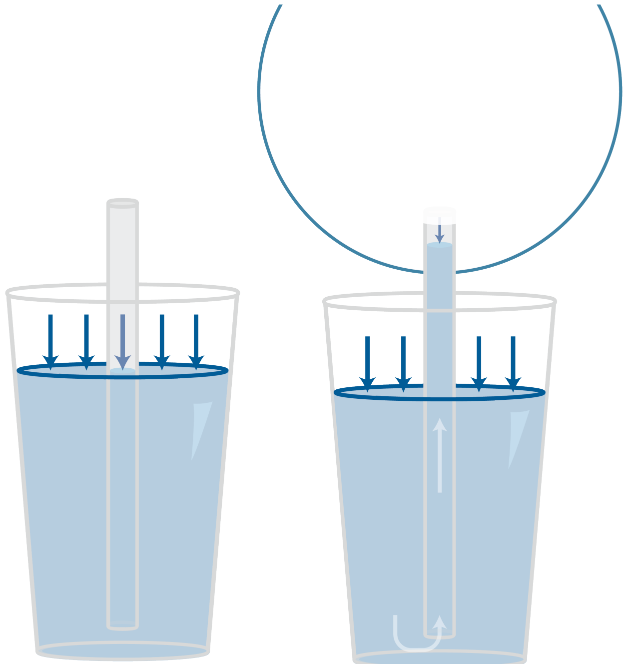
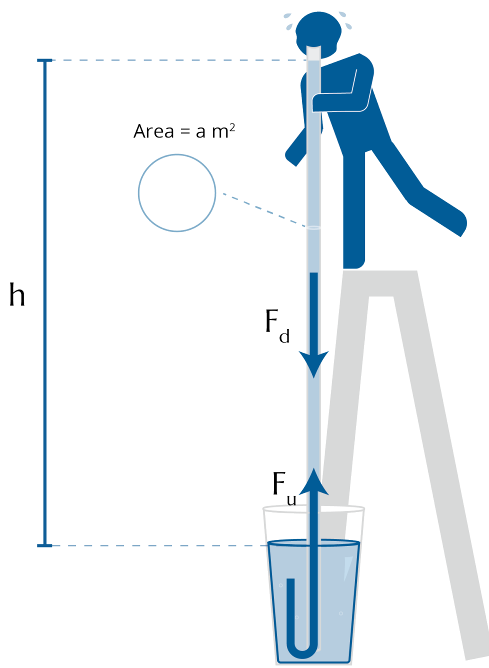


Figure 1.4: You use pressure to counteract gravity in a straw.

very, very stiff drinking straw, if you created a pressure of 0 Pa inside, how far above the surface of the glass could you get the water?

Let's say a cross-section of the straw has an area of a square meters, and the very top of the column of water is h meters above the surface in the glass.



With how many newtons of force is the atmosphere pushing the water upward? 100 kPa = 100,000 newtons per square meter. So:

$$F_u = (100,000)a$$

With how many newtons of force is gravity pulling the water in the straw downward? The volume of the water is ah. A cubic meter of liquid water weights 1000 kg. The force of gravity is 9.8 Newtons per kg.

$$F_d = (ah)(1000)(9.8)$$

The water will stop rising when $F_u = F_d$. So, to find h, we substitute in:

$$(100,000)a = (ah)(1000)(9.8)$$

Notice that we can divide both sides by a getting:

$$h = \frac{100,000}{9,800} = 10.2 \text{ meters}$$

A perfect vacuum would only be able to drag the water up 10.2 meters.

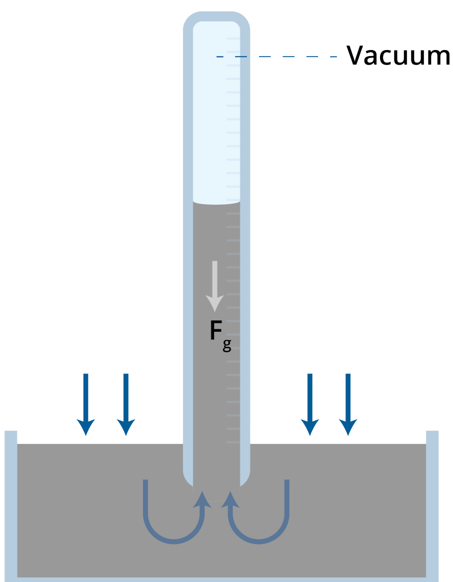
1.2.2 Millimeters Mercury

The density of mercury is 13,500 kg per cubic meter. How far up a straw would a perfect vacuum pull mercury?

$$h = \frac{100,000}{(9.8)(13,500)} \approx 0.756 \text{ meters}$$

When the atmospheric pressure is 100kPa, the mercury will rise 756 mm into a vacuum.

This is actually how scientists measure atmospheric pressure. They have a long glass tube filled with mercury. One end is closed off and pointed into the sky (exactly opposite the direction of gravity). The other end is placed into a dish of mercury. There are millimeter marks on the glass tube.



We use fluctuations in the atmospheric pressure to help us predict the weather. You might hear a weather nerd with a barometer in their house say, ”Wow, the barometer has gone from 752 to 761 millimeters mercury in the last hour. A high-pressure system is moving in.”

1.3 How Siphon Works

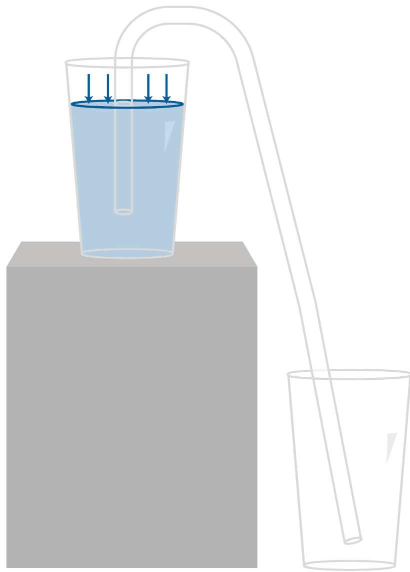
Let’s say you had two cups on a table; one is filled with water, one is empty. And you connected them with an empty U -shaped straw. Water will not crawl up the empty straw — the pressure on each end of the straw is the same, and crawling the straw (against gravity) would require energy.

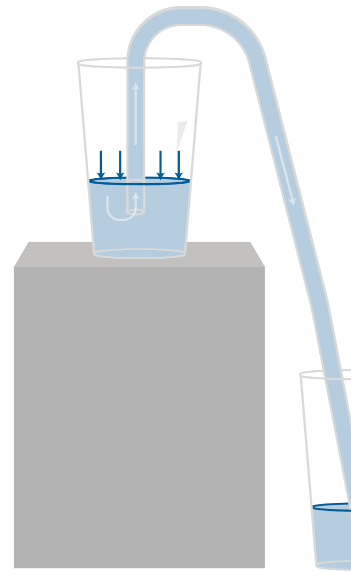
However, what if the straw were filled with water? Then the force of gravity is pulling the water on each side of the hump in different directions. However, the side going into the empty glass pulls a little harder. That is sufficient to create enough suction to pull the water in the other side up and over the hump.

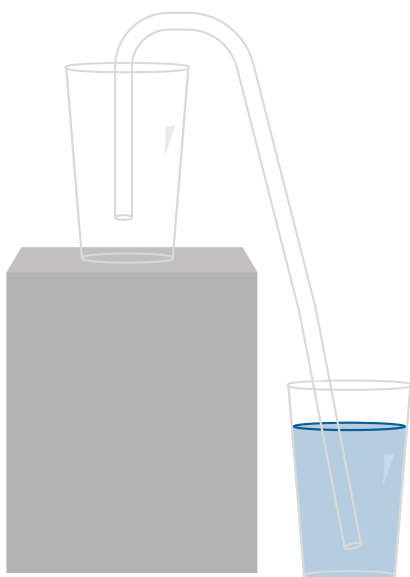
In turn, this will pull more water. Water will continue to flow from the full glass to the empty one until their surfaces are at the same level. At this point, the pull of gravity is the same on each side of the tube.

This is known as a *siphon*. Notice that atmospheric pressure makes the siphon possible. When the water on one side is pulled down by gravity, the atmospheric pressure pushes the other side up. If you were on a planet with plenty of gravity, but no atmosphere, a siphon wouldn’t work.

A siphon is especially useful when you want to get liquid out of a container that is too big to pour. For example, if you wanted to take the gasoline out of a car, you could use any flexible tubing to make a siphon. You would put the hose in the gas tank, suck enough gas up into the hose to get the siphon going, then put the hose into your jug. (If you ever do this, be extra careful not to suck any of the gasoline into your mouth; ingesting even a little bit of gasoline can make you incredibly sick, or even kill you.)







There are two rules to siphons:

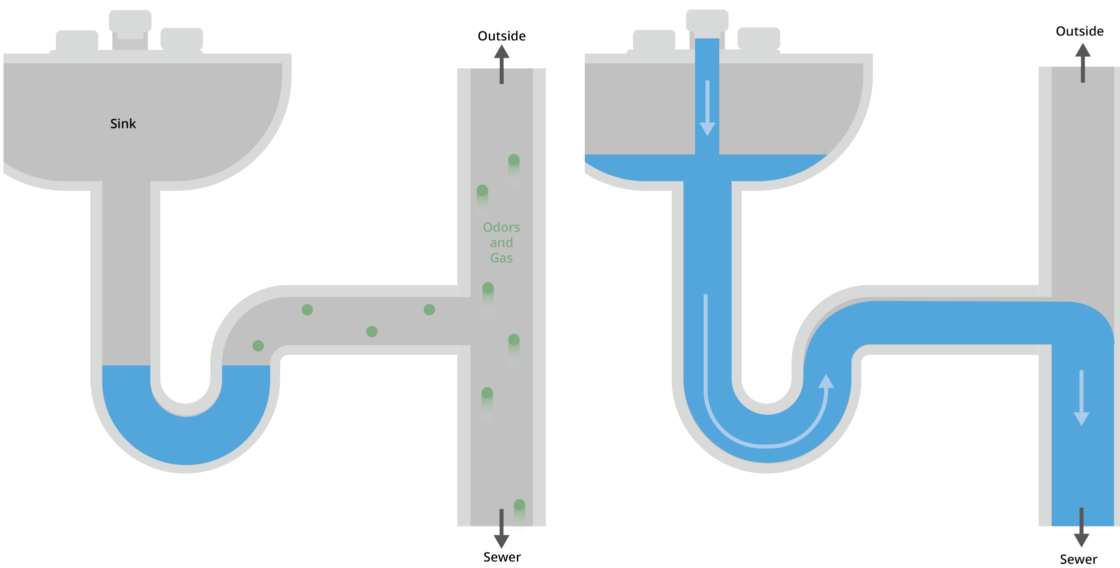
- The peak of the siphon needs to be low enough for atmospheric pressure to push the liquid that high. For water at sea-level, for example, the peak of the siphon can't be more than 10.2 meters above the surface of the source liquid.
- The tube has to carry the liquid to a lower level than the surface of the source liquid. If the destination end of the siphon is submerged, the surface of the liquid it is submerged in must be lower than the surface of the source liquid. If the destination

end of the siphon is not submerged, its opening must be lower than the surface of the source liquid.

As long as you follow these two rules, your siphons can be very creative. For example, every toilet has a siphon in it.

1.4 How a Toilet Works

Before we talk about toilets, you should know about P-traps. The drain from every sink, shower, and toilet in your house curves up and then down. This is known as a *P-trap*. The P-trap should always have some water in it. That keeps stinky (and flammable!) gases in the sewer from coming up and into your house.



If one of your fixtures (especially one that hasn't been used in a while) smells like raw sewage, run some water to ensure that the P-trap is full.

Now, back to the toilet! The drain in the bottom of the toilet is connected to a siphon into the sewer. The siphon is filled with air most of the time. However, when you flush the toilet, water rushes from the tank into the bowl, which also fills the siphon. Once the siphon is filled, it pulls water out of the toilet until air starts to enter the siphon. At that point, the water stops flowing.

The toilet tank is pretty simple: it has a float and valve that opens with the float is too low. So, anytime the water-level is too low, the value is open and slowly filling the tank. This means the tank is nearly always filled with a precise amount of water.

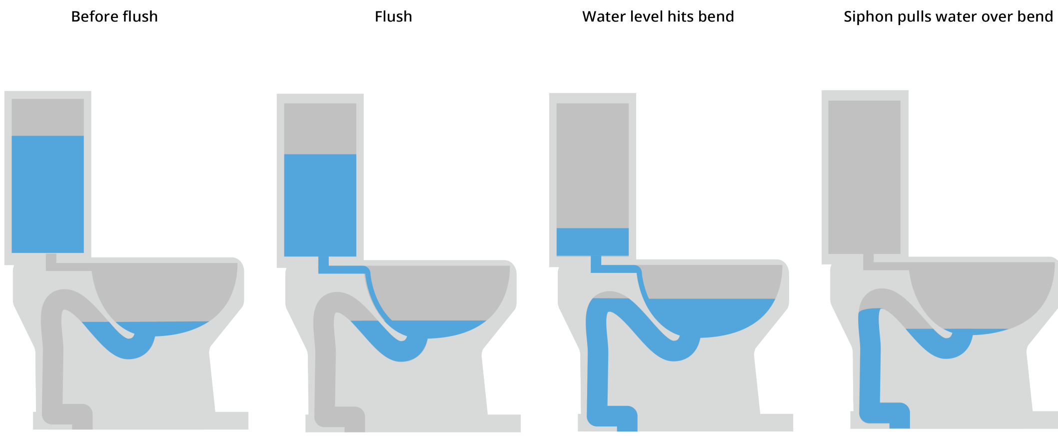


Figure 1.5: The steps a toilet takes when siphoning out waste.

When you flush, a small door in the bottom of the tank is opened and the water rushes into the bowl. When the water is out, the door closes again so the tank can refill.

Chapter 2

Cognitive Biases 1

In this section, we are going to take a look at research findings about *cognitive biases*. These are universal quirks found in the human thought process. Cognitive biases are a bit different from other kinds of biases, such as racial biases. Everyone, regardless of nationality, race, or gender is subject to these cognitive traps. You might be wondering, why do I need to learn about cognitive science in order to be an engineer? The most important tool we have as problem solvers is our own minds. We are going to be looking at ways that our minds can trip us up.

Our brains were designed over millions of years by the evolutionary process. The resulting mind is an amazing and powerful tool, but it is not flawless. The human brain has tendencies (or biases) that nudge us toward bad judgment and poor decisions.

When someone first gave you a hammer they handed it over with a warning: ”Don't hit your thumb!” No matter how careful you are with the hammer, at some point you will still hit your thumb. It's the same with cognitive biases. In the course of life, all of us will fall prey to these cognitive biases. Knowing about them is the first step in protecting ourselves.

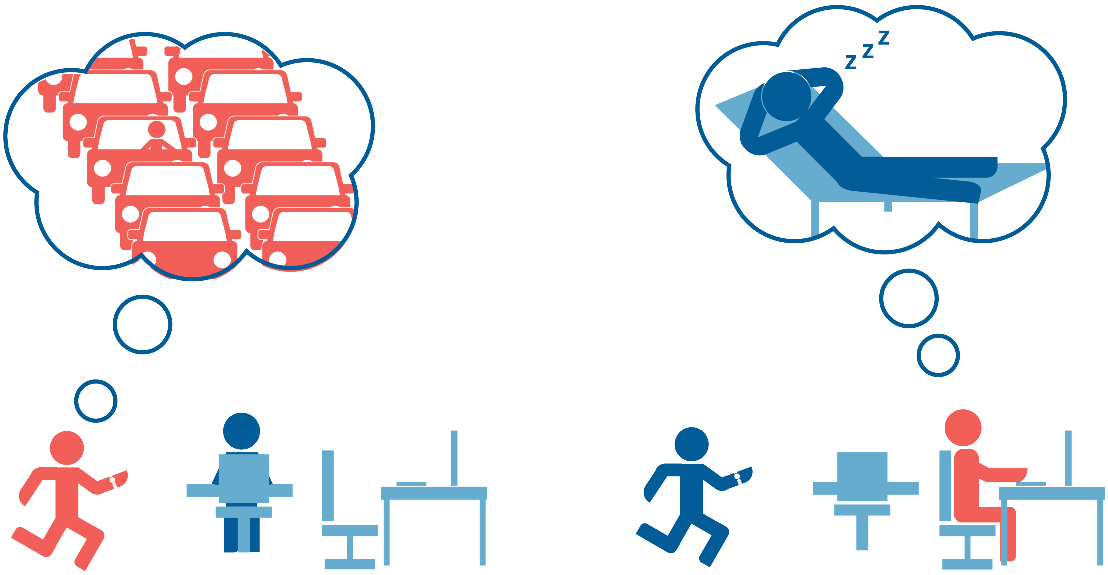
It would be irresponsible to teach you powerful ideas without also teaching you about the cognitive biases that follow them. There are about 50 that you should know about, but let's start with only a few.

2.1 Fundamental Attribution Error

You tend to attribute the mistakes of another person to their character, but attribute your own mistakes to the situation.

If someone asks you “Why were you late for work today?” you are likely to have an excuse, such as “I got stuck in a crazy traffic jam.”

However, if you notice your coworker is late for work, you are likely to think “My coworker is lazy.”



The solution? Cut people some slack. You probably don't know the whole story, so assume that their character is as strong as yours.

Or maybe you also need to hold yourself to a higher standard? Do you find yourself frequently rationalizing your bad judgment, lateness, or rudeness? This could be an opportunity for you to become a better person whose character is stronger regardless of the situation.

2.2 Self-Serving Bias

Self-serving bias is when you blame the situation for your failures, but attribute your successes to your strengths.

For example, when asked "Why did you lose the match?" you are likely to answer "The referee wasn't fair." When you are asked "Why did you win the match?" you are likely to answer "Because I have been training for weeks, and I was very focused."

This bias tends to make us feel better about ourselves, but it makes it difficult for us to be objective about our strengths and weaknesses.

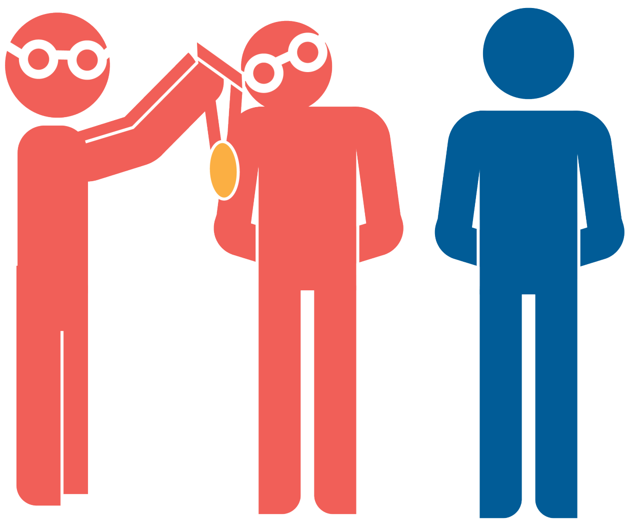
2.3 In-group favoritism

In-group favoritism: We tend to favor people who are in a group with us over people who are not in groups with us.

When asked "Who is the better goalie, Ted or John?"

If Ted is a Star Trek fan like you, you are likely to think he is also a good goalie.

As you might imagine, this unconscious tendency is the source of a lot of subtle discrimination based on race, gender, age, and religion. As we mentioned earlier, racial bias isn't a cognitive bias, but one can still feed into the other.



2.4 The Bandwagon Effect and Groupthink

The bandwagon effect is our tendency to believe the same things that the people around us believe. This is how fads spread so quickly: one person buys in, and then the people they know have a strong tencdency to buy in as well.

Groupthink is similar: To create harmony with the people around us, we go along with things we disagree with.

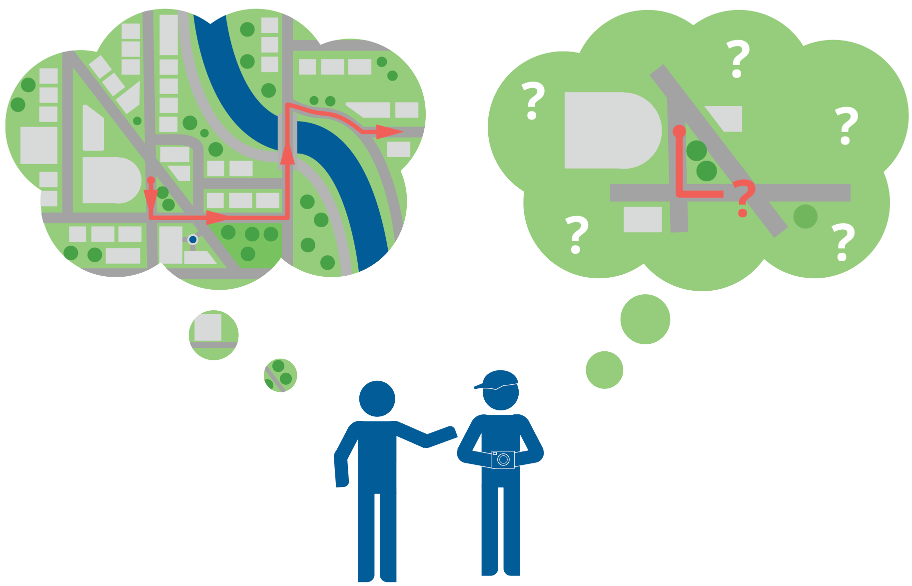
It takes a lot of perspective to recognize when those around us are wrong. And it takes even more courage to openly disagree with them.

2.5 The Curse of Knowledge

Once you know something, you tend to assume everyone else knows it too.

This is why teaching is sometimes difficult; a teacher will assume that everyone in the audience already knows the same things the teacher knows.

For example, imagine a local who has lived in a city for years giving directions to a tourist. The local has an in-depth understanding of the city, and gives overly quick and detailed instructions. The tourist politely smiles and nods, but stopped following after the local began listing unfamiliar street names and landmarks.



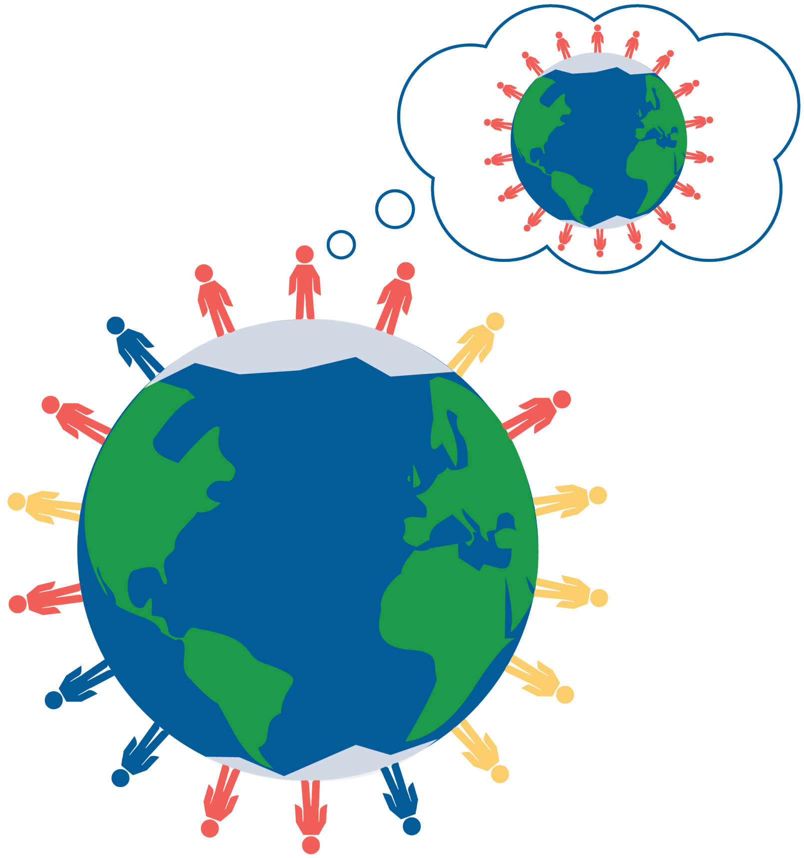
When we learn that a friend doesn't know something that we know, we are often very surprised. This surprise can sometimes manifest as hurtful behavior.

When we find a gap in our friend's knowledge, we try to remind ourselves that the friend certainly knows many things that we don't. We also try to imagine how it would feel if they teased us for our ignorance.

2.6 False Consensus

We tend to believe that more people agree with us than is actually the case. For example, if you are a member of a particular religion, you tend to overestimate the percentage of people in the world who are members of that religion.

When people vote in elections, they are often surprised when their preferred candidate loses. "Everyone, and I mean EVERYONE, voted for Smith!" they yell. "There must have been a mistake in counting the votes."

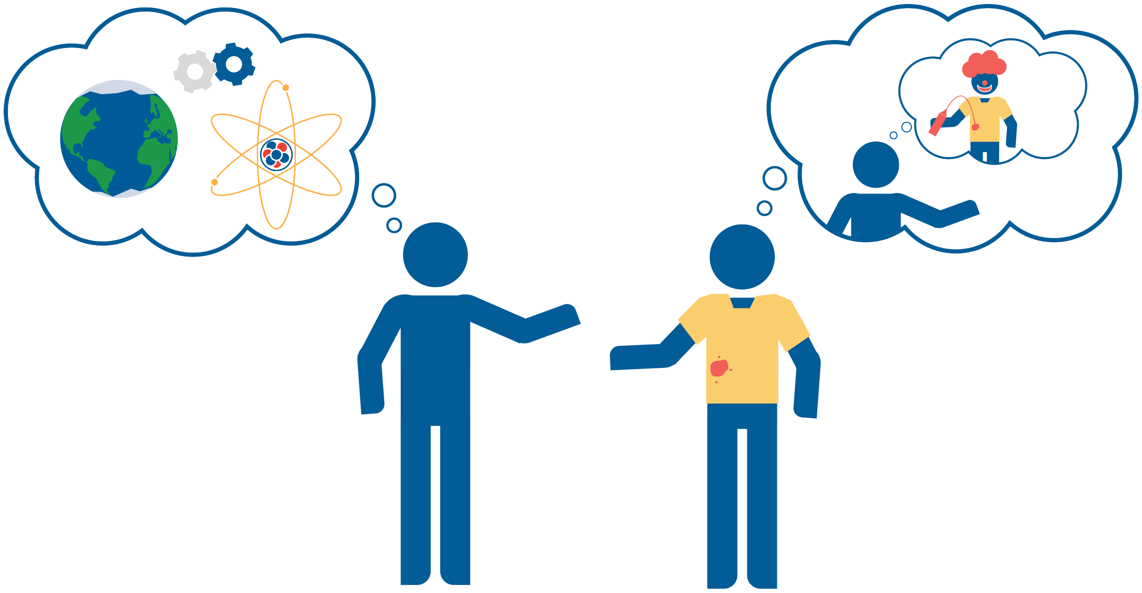


2.7 The Spotlight Effect

You tend to overestimate how much other people are paying attention to your behavior and appearance.

Think of six people that you talked to today. Can you even remember what shoes most of them were wearing? Do you care? Do you think any of them remember which shoes you wore today?

There is an old saying, "You would worry a lot less about how people think of you, if you realized how rarely they do."



2.8 The Dunning-Kruger Effect

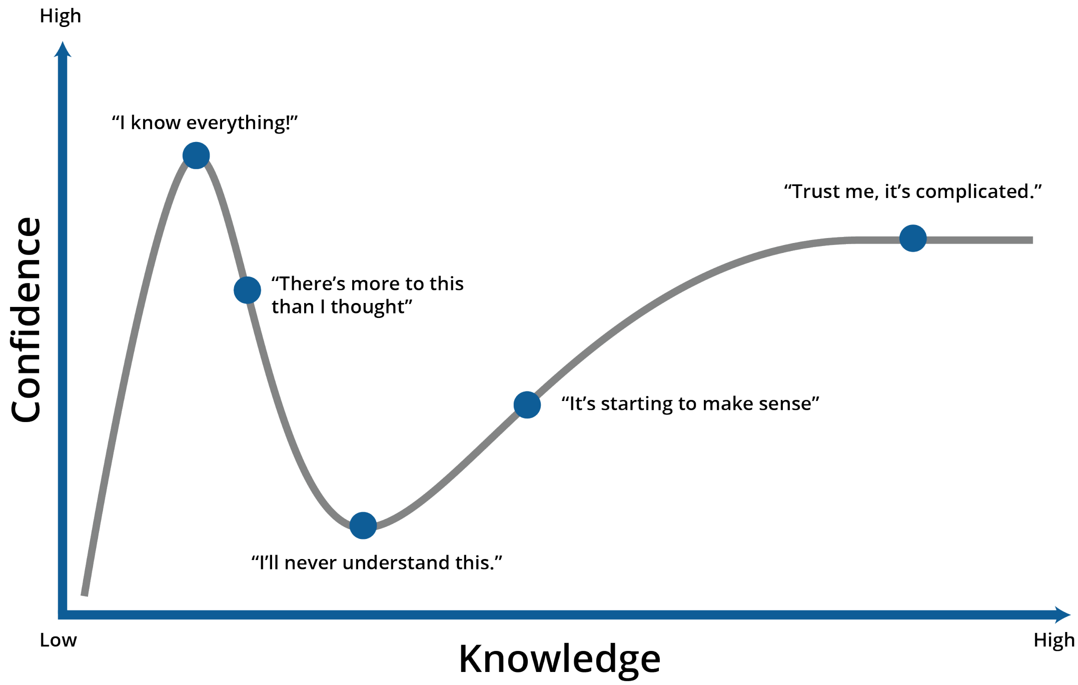
The less you know, the more confident you are.

When a person doesn't know all the nuance and context in which a question is asked, the question seems simple. Thus, person tends to be confident in their answer. As they learn more about the complexity of the space in which the question lives, they often realize the answer is not nearly so obvious.

For example, a lot of people will confidently proclaim "Taxes are too high! We need to lower taxes." An economist who has studied government budgets, deficits, history, and monetary policy, might say something like "Maybe taxes *are* too high. Or maybe they are too low. Or maybe we are taxing the wrong things. It is a complex question."

When we are talking with people about a particular topic, we do our best to defer to the person in the conversation who we think has the most knowledge in the area. If we disagree with the person, we try to figure out why our opinions are different.

Similarly, you should assume that any opinion that is voiced in an internet discussion is, at best, wildly over-simplified. If you really care about the subject, read a book by a respected expert. Yes, a whole book — there are few interesting topics that can be legitimately explained in under 100 pages.

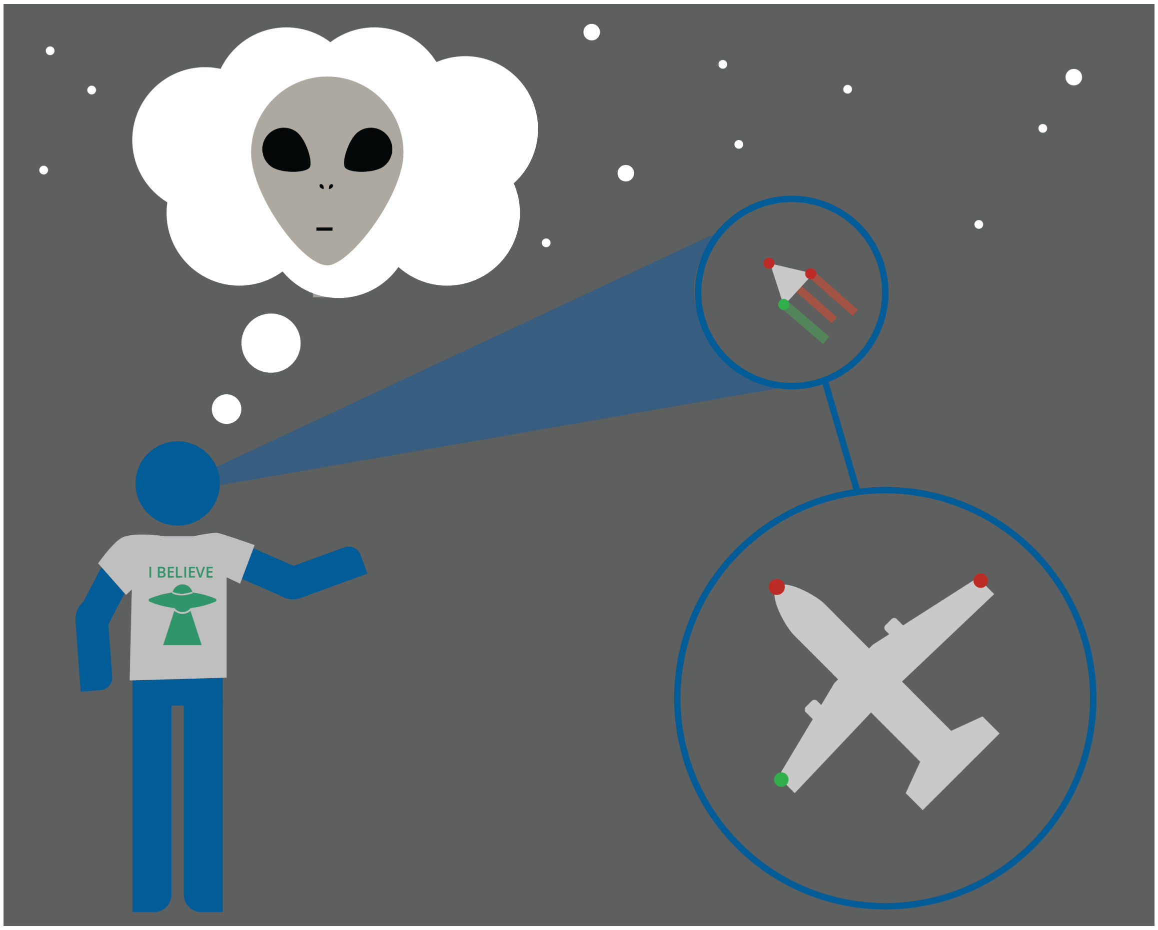


2.9 Confirmation Bias

You tend to find and remember information that supports beliefs you already have. You also tend to avoid and dismiss information that contradicts your beliefs.

If you believe that intelligent creatures have visited from other planets, you will tend to look for data to support your beliefs. When you find data that shows that it is just too far for any creature to travel, you will try to find a reason why the data is incorrect.

Confirmation bias is one reason why people don't change their beliefs more often.



Confirmation bias wrecks many, many studies. The person doing the study often has a hypothesis that they believe and very much want to prove true. It is very tempting to discard data that doesn't support the hypothesis. A person may even throw all the data away and experiments again and again until they get the result they want.

When you design an experiment, you must describe it explicitly before you start. You must tell someone: "If the hypothesis I love is incorrect, the results will look like this. If the hypothesis I love is correct, the results will look like that. And if the results look any other way, I have neither proved nor disproved the hypothesis."

Once the experiment is underway, you must not change the plan and you must not discard any data.

This is scientific integrity. You should demand it from yourself, and you should expect it from others.

Watch a TED Talk and Learn More About Confirmation Bias: What shapes our perceptions (and misperceptions) about science? In an eye-opening talk, meteorologist J. Marshall Shepherd explains how confirmation bias, the Dunning-Kruger effect and cognitive dissonance impact what we think we know – and shares ideas for how we can replace them with something much more powerful: knowledge.

https://www.ted.com/talks/j_marshall_shepherd_3_kinds_of_bias_that_shape_your_worldview

2.10 Survivorship bias

You will pay more attention to those that survived a process than those who failed.

After looking at several old houses, you might say "In the 1880s, they built great houses." However, you haven't seen the houses that were built in the 1880s and didn't survive. Which houses tended to survive for a long time? Only the great houses – you are basing your opinion on a very skewed sample.

Chapter 3

Solving Quadratics

A quadratic function has three terms: ax^2+bx+c. a, b, and c are known as the *coeffcients*. The coefficients can be any constant, except that a can never be zero. (If a is zero, it is a linear function, not a quadratic.)

When you have an equation with a quadratic function on one side and a zero on the other, you have a quadratic equation. For example:

$$72x^2 - 12x + 1.2 = 0$$

How can you find the values of x that will make this equation true?

You can always reduce a quadratic equation so that the first coefficient is 1, so that your equation looks like this:

$$x^2 + bx + c = 0$$

For example, if you are asked to solve $4x^2 + 8x - 19 = -2x^2 - 7$

$4x^2 + 8x - 19 = -2x^2 - 7$

$$6x^2 + 8x - 12 = 0$$

$$x^2 + \frac{4}{3}x - 2 = 0$$

Here, $b = \frac{4}{3}$ and $c = -2$.

$x^2 + bx + c = 0$ when

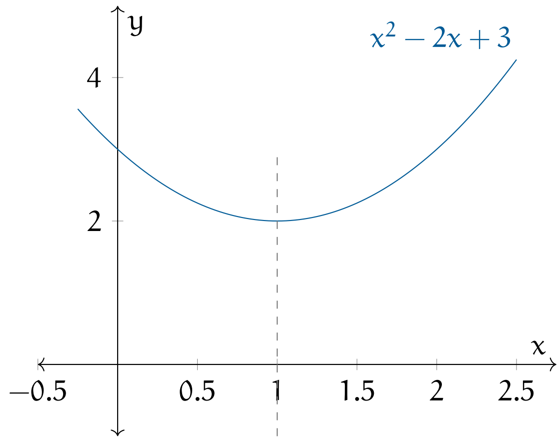
$$x = -\frac{b}{2} \pm \frac{\sqrt{b^2 - 4c}}{2}$$

Only when $a = 1$

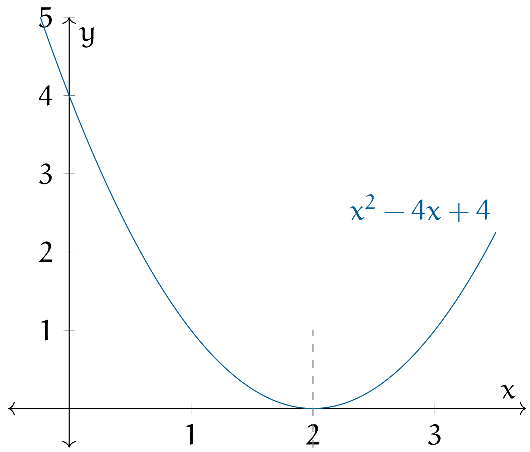
What does this mean?

For any b and c, the graph of $x^2 + bx + c$ is a parabola that goes up on each end. Its low point is at $x = -\frac{b}{2a}$. This is referred to as the vertex formula.

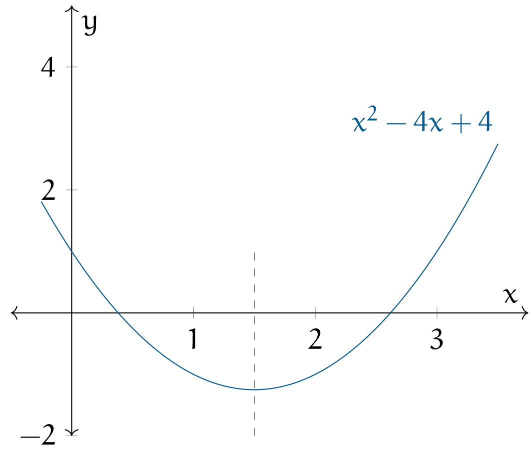
If there are no real roots ($b^2 - 4c < 0$), which means the parabola never gets low enough to cross the x-axis:



If there is one real root ($b^2 - 4c = 0$), it means that the parabola only touches the x-axis.



If there are two real roots ($b^2 - 4c > 0$), it means that the parabola crosses the x-axis twice as it dips below and then returns:



Exercise 2 Roots of a Quadratic

Working Space

In the last chapter, you found that the function for the height of your flying hammer is:

$$p = -\frac{1}{2}9.8t^2 + 12t + 2$$

At what time will the hammer hit the ground?

Answer on Page 41

3.1 The Traditional Quadratic Formula

If the last explanation was a little tricky to understand, the quadratic formula is a nifty tool.

The Quadratic Formula

$ax^2 + bx + c = 0$ when

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

3.2 Factoring Quadratics

Sometimes, the quadratic gets a bit clunky and hard so solve, especially without a calculator. That's when factoring quadratics becomes a good tool to use as well.

Factoring quadratics is a way to seperate a quadratic, usually in the form x^2+bx+c where $(a = 1)$, into two multiplied equations in the form $(px + q)(rx + s) = 0$. Solving these for x also gives the roots of the equation.

Types of Factoring Techniques

1. Factoring out the GCF (Greatest Common Factor)
   Always check for a common factor first:
   $$2x^2 + 4x = 2x(x + 2)$$
2. Factoring Trinomials: $a = 1$
   For expressions like $x^2 + bx + c$, find two numbers that:
   - Multiply to c
   - Add to b
   $$x^2 + 5x + 6 = (x + 2)(x + 3)$$
3. Factoring Trinomials: $a \neq 1$
   Use the "AC method":
   (a) Multiply $a \times c$
   (b) Find two numbers that multiply to ac and add to b
   (c) Split the middle term and factor by grouping
   $$6x^2 + 11x + 4 = 6x^2 + 3x + 8x + 4 = (3x + 2)(2x + 2)$$
4. Special Cases:
   - Perfect Square Trinomials:
     $a^2 + 2ab + b^2 = (a + b)^2$
     $x^2 + 6x + 9 = (x + 3)^2$
   - Difference of Squares:
     $a^2 - b^2 = (a - b)(a + b)$
     $x^2 - 16 = (x - 4)(x + 4)$

Tips

- Always factor out the GCF first.

- Check your result by expanding (FOIL). The inverse of factoring is expanding using the FOIL method.
- Not all quadratics are factorable over real integers.

Chapter 4

Complex Numbers

Complex numbers are an extension of real numbers, which in turn are an extension of rational numbers. In mathematics, the set of complex numbers is a number system that extends the real number line to a full two dimensions, using the imaginary unit, which is denoted by i, with the property that $i^2 = -1$.

An imaginary number is a number that is a multiple of the imaginary unit i. For example, $5i$ and $-3i$ are both imaginary numbers. While imaginary numbers are a subset of complex numbers (with real part equal to zero), not all complex numbers are imaginary. Complex numbers can have both a real and an imaginary part, while imaginary numbers specifically refer to those with no real part.

4.1 Definition

A complex number is a number of the form $a + bi$, where a and b are real numbers, and i is the imaginary unit, with the property that $i^2 = -1$. The real part of the complex number is a, and the imaginary part is bi.

4.2 Why Are Complex Numbers Necessary?

Complex numbers are essential to many fields of science and engineering. Here are a few reasons why:

4.2.1 Roots of Negative Numbers

In the real number system, the square root of a negative number does not exist, because there is no real number that you can square to get a negative number. The introduction of the imaginary unit i, which has the property that $i^2 = -1$, allows us to take square roots of negative numbers and gives rise to complex numbers.

4.2.2 Polynomial Equations

The fundamental theorem of algebra states that every non-constant polynomial equation with complex coefficients has a complex root. This theorem guarantees that polynomial equations of degree n always have n roots in the complex plane.

4.2.3 Physics and Engineering

In physics and engineering, complex numbers are used to represent waveforms in control systems, in quantum mechanics, and many other areas. Their properties make many mathematical manipulations more convenient.

4.3 Adding Complex Numbers

The addition of complex numbers is straightforward. If we have two complex numbers $z_1 = a + bi$ and $z_2 = c + di$, their sum is defined as:

$$z_1 + z_2 = (a + c) + (b + d)i \tag{4.1}$$

In other words, you add the real parts to get the real part of the sum, and add the imaginary parts to get the imaginary part of the sum.

4.4 Multiplying Complex Numbers

The multiplication of complex numbers is a bit more involved. If we have two complex numbers $z_1 = a + bi$ and $z_2 = c + di$, their product is defined as:

$$z_1 \cdot z_2 = (a + bi) \cdot (c + di) = ac + adi + bci - bd = (ac - bd) + (ad + bc)i \tag{4.2}$$

Note the last term comes from $i^2 = -1$. You multiply the real parts and the imaginary parts just as you would in a binomial multiplication, and remember to replace i^2 with -1. See the Khan academy video in the digital resources for a more in-depth explanation.

Chapter 5

Introduction to Sequences

A sequence is a list of numbers in a particular order. {1, 3, 5, 7, 9} is a sequence. So is $\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \cdots\}$. There are many types of sequences. We will present two of the most common types in this chapter: arithmetic and geometric sequences.

Sequences are generally represented like this:

$$a_1, a_2, a_3, a_4, ..., a_n, ...$$

The first number, a_1, is called the *first term*, a_2 is the *second term*, and a_n is the *nth term*. A sequence can be finite or infinite. If the sequence is infinite, we represent that with ellipses $(\cdots)$ at the end of the list, to indicate that there are more numbers.

We can also write formulas to represent a sequence. Take the first example, the finite sequence {1, 3, 5, 7, 9}. Notice that each term is two more than the previous term. We can define the sequence *recursively* by defining the n^{th} term as a function of the $(n-1)^{th}$ term. In our example, we see that $a_n = a_{n-1} + 2$ with $a_1 = 1$ for $1 \leq n \leq 5$. This is called a recursive formula, because you have to already know the $(n-1)^{th}$ term to find the n^{th} term.

Another way to write a formula for a sequence is to find a rule for the n^{th} term. In our example sequence, the first term is 1 plus 0 times 2, the second term is 1 plus 1 times 2, the third term is 1 plus 2 times 2, and so one. Did you notice the pattern? The n^{th} term is 1 plus (n-1) times 2. We can write this mathematically:

$$a_n = 1 + 2(n-1) \text{ for } 1 \leq n \leq 5$$

This is called the *explicit* formula because each term is explicitly defined. Notice that for the second way of writing a formula, we don't have to state what the first term is — the formula tells us.

5.1 Arithemtic sequences

Our first example sequence, {1, 3, 5, 7, 9} is a *finite*, *arithmetic* sequence. We know it is finite because there is a limited number of terms in the sequence (in this case, 5). How do we know it is arithmetic?

An arithmetic sequence is one where you add the same number every time to get the next term. Our example is an arithmetic sequence because you add 2 to get the next term every time. That number that you add is called the *common difference*, so we can say the sequence {1, 3, 5, 7, 9} has a common difference of 2. The common difference can be positive (in the case of an increasing arithmetic sequence) or negative (in the case of a decreasing arithmetic sequence). Formally, we can find the common difference of an arithmetic sequence by subtracting the $(n-1)^{th}$ term from the n^{th} term:

$$d = a_n - a_{n-1}$$

Exercise 3

Which of the following are arithmetic sequences? For the arithmetic sequences, find the common difference.

1. $\{\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, ...\}$
2. {5, 8, 11, 14, 17, ...}
3. {3, -1, -5, -9, ...}
4. {-1, 2, -3, 4, -5, 6, ...}

Working Space

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5.1.1 Formulas for arithmetic sequences

If you are given an arithmetic sequence, you can write an explicit or recursive formula. You can think of the formula as a function where the domain (input) is restricted to integers greater than or equal to one. Let's write explicit and recursive formulas for the sequence {3, -1, -5, -9, ...}.

For either type of formula, we need to identify the common difference. Since each term is 4 less than the previous term, the common difference is -4 (see figure 5.1). This means the n^{th} term is the $(n-1)^{th}$ term minus 4. The general form of a recursive formula is $a_n = a_{n-1} + d$, where d is the common difference. For our example, the common difference is -4, so we can write a recursive formula:

$$a_n = a_{n-1} - 4$$

However, this formula doesn't tell us what a_1 is! For recursive formulas, you have to specify the first term in the sequence. So, the *complete* recursive formula for the sequence is:

$$a_n = a_{n-1} - 4$$

$$a_1 = 3$$

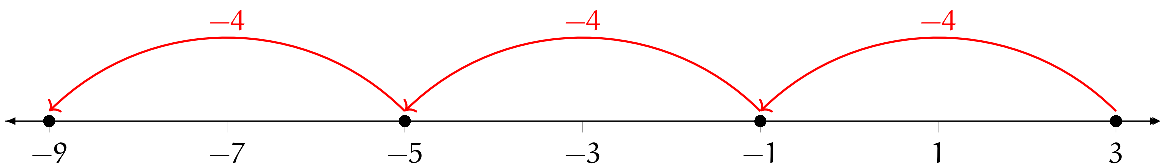

Figure 5.1: The common difference in the sequence {3, -1, -5, -9,...} is -4

Recursive formulas make it easy to see how each term is related to the next term. However, it is difficult to use recursive formulas to find a specific term. Say we wanted to know the 7^{th} term in the sequence. Well, from the formula, we know that:

$$a_7 = a_6 - 4$$

What is a_6? Again, we see that

$$a_6 = a_5 - 4$$

Now we have to find a_5! If we keep going, we see that:

$$a_5 = a_4 - 4$$

$$a_4 = a_3 - 4$$

$$a_3 = a_2 - 4$$

$$a_2 = a_1 - 4$$

Since we were told a_1, we can find a_2 and propagate our terms back up the chain to find a_7:

$$a_2 = 3 - 4 = -1$$

$$a_3 = a_2 - 4 = -1 - 4 = -5$$

$$a_4 = a_3 - 4 = -5 - 4 = -9$$

$$a_5 = a_4 - 4 = -9 - 4 = -13$$

$$a_6 = a_5 - 4 = -13 - 4 = -17$$

$$a_7 = a_6 - 4 = -17 - 4 = -21$$

Ultimately, we see that $a_7 = -21$. That was a lot of work! You can imagine that for higher

n terms, such as the 100^{th} or 1000^{th} term, this method becomes cumbersome. This is where the explicit formula is more useful.

The general form of an explicit formula for an arithmetic sequence is

$$a_n = a_1 + d \times (n - 1)$$

where d is the common difference. For our example sequence, {3, -1, -5, -9, ...}, the common difference is -4. So the explicit formula is

$$a_n = 3 + (-4)(n - 1) = 3 - 4(n - 1)$$

You may be tempted to distribute and simplify, which is fine and yields an equivalent formula:

$$a_n = 7 - 4n$$

Now, to find the 7^{th} term, all we have to do is substitute $n = 7$:

$$a_7 = 3 - 4(7 - 1) = 3 - 4(6) = 3 - 24 = -21$$

We get the same answer with much less effort!

Another example of a recursive sequence is the Fibonacci sequence which we will cover in-depth later

Exercise 4

An arithmetic sequence is defined by the recursive formula $a_n = a_{n-1} + 5$ with $a_1 = -4$. Write the first 5 terms of the sequence and determine an explicit formula for the same sequence.

Working Space

Answer on Page 42

Exercise 5

The first four terms of an arithmetic sequence are $\{\pi, \frac{3\pi}{2}, 2\pi, \frac{5\pi}{2}\}$. What is the common difference? Write explicit and recursive formulas for the infinite sequence.

Working Space

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5.2 Geometric sequences

Let's look at the other sequence given as an example at the beginning of the chapter: $\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \cdots\}$. How is each term related to the previous term? Well, $\frac{1}{4}$ is half of $\frac{1}{2}$, and $\frac{1}{8}$ is half of $\frac{1}{4}$, so each term is the previous term multiplied by $\frac{1}{2}$. When each term in a sequence is a multiple of the previous term, this is a *geometric* sequence. The number we multiply by each time (in our example, this is $\frac{1}{2}$), which is called the *common ratio*. The common ratio can be positive or negative, but not zero.

An easy way to determine the common ratio (r) is to divide the n^{th} term by the $(n-1)^{th}$ term. In our example sequence, the first term is $\frac{1}{2}$ and the second is $\frac{1}{4}$.

$$r = \frac{a_2}{a_1} = \frac{1/4}{1/2} = \frac{1}{2}$$

which returns the common ratio we already identified, $r = \frac{1}{2}$.

If the common ratio is negative, then the sequence will "flip" back and forth from positive to negative. For example, suppose there is a geometric sequence such that $a_1 = 1$ and $r = -2$. Then the first 5 terms are {1, -2, 4, -8, 16}. Whenever you see a sequence going back and forth from positive to negative, that means the common ratio is negative.

For positive ratios ($r > 1$), the sequence is increasing. And for negative ratios, ($r < 1$), the sequence is decreasing.

5.2.1 Formulas for geometric sequences

Like arithmetic sequences, we can write recursive and explicit formulas. For geometric sequences, the recursive formula has the general form:

$$a_n = r(a_{n-1})$$

where r is the common ratio and a_1 is specified. For our example sequence, $\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \cdots\}$, the recursive formula is:

$$a_n = \frac{1}{2}a_{n-1}$$
$$a_1 = \frac{1}{2}$$

In a geometric sequence, each term is the first term, a_1, multiplied by the common ratio, r, $n-1$ times. Therefore, the general form of an explicit formula for a geometric function is:

$$a_n = (a_1)r^{n-1}$$

Again, for our example sequence, $a_1 = \frac{1}{2}$ and $r = \frac{1}{2}$, so the explicit formula is:

$$a_n = (\frac{1}{2})(\frac{1}{2})^{(n-1)}$$

Exercise 6

Which of the following are geometric sequences? For each geometric sequence, determine the common ratio.

1. {2, -4, 6, -8,...}
2. {4, 2, 1, $\frac{1}{2}$, ...}
3. {-5, 25, -125, 525, ...}
4. {2, 0, -2, -4, ...}

Working Space

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Exercise 7

A geometric sequence is defined by the recursive formula $a_n = a_{n-1} \times \frac{3}{2}$ with $a_1 = 1$. Write the first five terms of the sequence and determine an explicit formula for the same sequence.

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Exercise 8

The first four terms of a geometric sequence are {-4, 2, -1, $\frac{1}{2}$}. What is the common ratio? Write recursive and explicit formulas for the infinite sequence.

Working Space

Answer on Page 42

Appendix A

Answers to Exercises

Answer to Exercise 1 (on page 6)

$$p = 101,332 \times \left(1 - 2.25577 \times 10^{-5} \times h\right)^{5.25588}$$

and $h = 9,144$. Thus,

$$p \approx 30.1\text{kPa}$$

Answer to Exercise 2 (on page 27)

For what t is $-4.9t^2 + 12t + 2 = 0$? Start by dividing both sides of the equation by -4.9.

$$t^2 - 2.45t - 0.408 = 0$$

The roots of this are at

$$x = -\frac{b}{2} \pm \frac{\sqrt{b^2 - 4c}}{2} = -\frac{-2.45}{2} \pm \frac{\sqrt{(-2.45)^2 - 4(-0.408)}}{2} = 1.22 \pm 1.36$$

We only care about the root after we release the hammer $(t > 0)$.

$1.22 + 1.36 = 2.58$ seconds after releasing the hammer, it will hit the ground.

Answer to Exercise 3 (on page 34)

1. not arithmetic
2. arithmetic, common difference is 3

3. arithmetic, common difference is -4
4. not arithmetic

Answer to Exercise 4 (on page 36)

The first five terms are {-4, 1, 6, 11, 16} and an explicit formula is $a_n = -4 + 5(n-1)$.

Answer to Exercise 5 (on page 37)

The common difference is $\frac{3\pi}{2} - \pi = \frac{\pi}{2}$. The recursive formula is $a_n = a_{n-1} + \frac{\pi}{2}$ with $a_1 = \pi$. The explicit formula is $a_n = \pi + \frac{\pi}{2}(n-1)$.

Answer to Exercise 6 (on page 38)

1. not geometric
2. geometric sequence with common ratio $r = \frac{1}{2}$
3. geometric sequence with common ration $r = -5$
4. not geometric

Answer to Exercise 7 (on page 39)

The first five terms are $\{1, \frac{3}{2}, \frac{9}{4}, \frac{27}{8}, \frac{81}{16}\}$. An explicit formula for this sequence is $a_n = 1(\frac{3}{2})^{(n-1)}$.

Answer to Exercise 8 (on page 39)

The common ratio is $\frac{a_n}{a_{n-1}} = \frac{2}{-4} = -\frac{1}{2}$. A recursive formula would be $a_n = a_{n-1} \times -\frac{1}{2}$ with $a_1 = -4$. An explicit formula would be $a_n = (-4)(-\frac{1}{2})^{(n-1)}$.

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