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Chapter 1

The Dot Product

If you have two vectors $\mathbf{u} = [u_1, u_2, \ldots, u_n]$ and $\mathbf{v} = [v_1, v_2, \ldots, v_n]$, we define the *dot product* $\mathbf{u} \cdot \mathbf{v}$ as

$$\mathbf{u} \cdot \mathbf{v} = (u_1 \times v_1) + (u_2 \times v_2) + \cdots + (u_n \times v_n)$$

For example,

$$[2, 4, -3] \cdot [5, -1, 1] = 2 \times 5 + 4 \times -1 + -3 \times 1 = 3$$

This may not seem like a very powerful idea, but dot products are *incredibly* useful. The enormous GPUs (Graphics Processing Units) that let video games render scenes so quickly? They primarily function by computing huge numbers of dot products at mind-boggling speeds.

Exercise 1 Basic dot products

Compute the dot product of each pair of vectors:

- $[1, 2, 3]$, $[4, 5, -6]$
- $[\pi, 2\pi]$, $[2, -1]$
- $[0, 0, 0, 0]$, $[10, 10, 10, 10]$

Working Space

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1.1 Properties of the dot product

Sometimes we need an easy way to say "The vector of appropriate length is filled with zeros." We use the notation $\vec{0}$ to represent this. Then, for any vector $\mathbf{v}$, this is true:

$$\mathbf{v} \cdot \vec{0} = 0$$

The dot product is commutative:

$$\mathbf{v} \cdot \mathbf{u} = \mathbf{u} \cdot \mathbf{v}$$

The dot product of a vector with itself is its magnitude squared:

$$\mathbf{v} \cdot \mathbf{v} = |\mathbf{v}|^2$$

If you have a scalar a, then:

$$(\mathbf{v}) \cdot (a\mathbf{u}) = a(\mathbf{v} \cdot \mathbf{u})$$

So, if **v** and **w** are vectors that go in the same direction,

$$\mathbf{v} \cdot \mathbf{w} = |\mathbf{v}||\mathbf{w}|$$

If **v** and **w** are vectors that go in opposite directions,

$$\mathbf{v} \cdot \mathbf{w} = -|\mathbf{v}||\mathbf{w}|$$

If **v** and **w** are vectors that are perpendicular to each other, their dot product is zero:

$$\mathbf{v} \cdot \mathbf{w} = 0$$

1.2 Cosines and dot products

Furthermore, dot products' interaction with cosine makes them even more useful is what makes them so useful: If you have two vectors **v** and **u**,

$$\mathbf{v} \cdot \mathbf{u} = |\mathbf{v}||\mathbf{u}| \cos \theta$$

where θ is the angle between them.

So, for example, if two vectors **v** and **u** are perpendicular, the angle between them is $\pi/2$. The cosine of $\pi/2$ is 0. The dot product of any two perpendicular vectors is always 0. In fact, if the dot product of two non-zero vectors is 0, the vectors *must be* perpendicular (see figure 1.1 for an example of perpendicular 2-dimensional vectors).

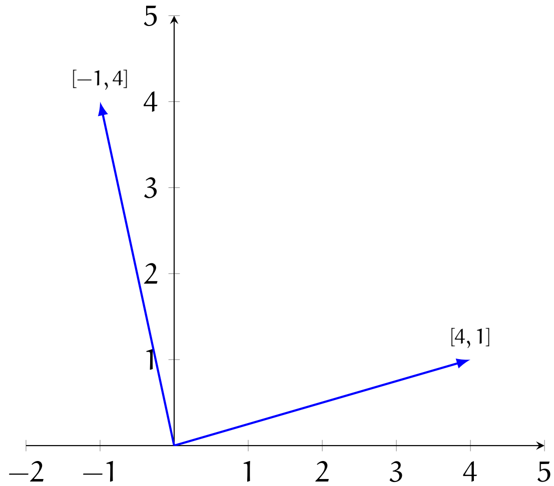

Figure 1.1: The dot product of any two perpendicular vectors is zero.

If you have two non-zero vectors **v** and **u**, you can always compute the angle between them:

$$\theta = \arccos\left(\frac{\mathbf{v} \cdot \mathbf{u}}{|\mathbf{v}||\mathbf{u}|}\right)$$

Arccos is short for arccosine, or $\cos^{-}1$, and it is a function that is the inverse of cosine. Cosine takes an angle and gives back the scaled x-component of the angle. Arccosine takes the x-component of an angle and returns an angle with that x-component. However, there is a limit to what arccos can return. Let's look at cosine and its inverse, arccos (see figures 1.2 and 1.3).

When you use a calculator to evaluate arccos, the calculator automatically restricts the results to between 0 and π. Let's look at an example of using the dot product to determine the angle between two vectors:

Example: What is the angle between $\mathbf{u} = \left[\sqrt{3}, 1\right]$ and $\mathbf{v} = [0, -1]$?

Solution: We know that $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}|\,|\mathbf{v}| \cos\theta$. Therefore, we also know that:

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|}$$

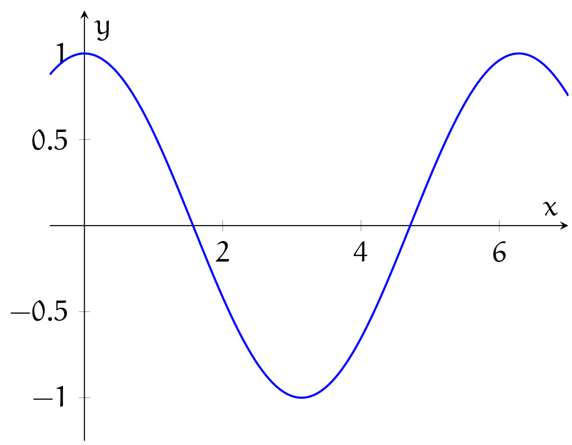

Figure 1.2: Cosine is a function: there is exactly one output for every input.

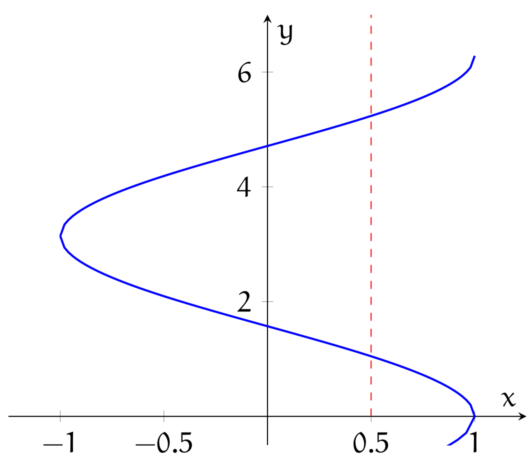

Figure 1.3: Arccos is not a function: there are many angles with the same x-component. Notice that one input value has many output values (see the red dashed line).

First, let's compute the dot product:

$$\mathbf{u} \cdot \mathbf{v} = \sqrt{3} \cdot 0 + 1 \cdot -1 = -1$$

And therefore:

$$\cos\theta = \frac{-1}{|\mathbf{u}|\,|\mathbf{v}|}$$

Now, let's find the magnitudes of both vectors:

$$|\mathbf{u}| = \sqrt{\left(\sqrt{3}\right)^2 + (1)^2} = 2$$

$$|\mathbf{v}| = \sqrt{(0)^2 + (-1)^2} = 1$$

Substituting for the magnitudes, we find that:

$$\cos\theta = \frac{-1}{2 \cdot 1} = \frac{-1}{2}$$

To solve for θ, we take the arccos of both sides:

$$\arccos\left(\cos\theta\right) = \theta = \arccos\frac{-1}{2}$$

What angles have a cosine of $-1/2$? We know that $2\pi/3$, $4\pi/3$, $8\pi/3$, etc., all have a cosine of $-1/2$. Because the range of arccos is restricted to between 0 and π, our result is:

$$\theta = \arccos\frac{-1}{2} = \frac{2\pi}{3}$$

.

Therefore, the angle between $\mathbf{u}$ and $\mathbf{v}$ is $2\pi/3$ (or $120°$).

Exercise 2 Using dot products

What is the angle between these each pair of vectors:

- $[1, 0]$, $[0, 1]$
- $[3, 4]$, $[4, 3]$
- $[2, -1, 2]$, $[-1, 2, -2]$
- $[-5, 0, -1]$, $[2, 3, -4]$

Working Space

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1.3 Dot products in Python

NumPy will let you do dot products using the the symbol @. Open `first_vectors.py` and add the following to the end of the script:

```
# Take the dot product
d = v @ u
print("v @ u =", d)

# Get the angle between the vectors
a = np.arccos(d / (mv * mu))
print(f"The angle between u and v is {a * 180 / np.pi:.2f} degrees")
```

When you run it you should get:

```
v @ u = 4
The angle between u and v is 78.55 degrees
```

1.4 Work and Power

Earlier, we mentioned that mechanical work is the product of the force you apply to something and the amount it moves. For example, if you push a train with a force of 10 newtons as it moves 5 meters, you have done 50 joules of work.

What if you try to push the train sideways? It moves down the track 5 meters, but you push it as if you were trying to derail it — perpendicular to its motion. You have done no work, because the train didn't move at all in the direction you were pushing.

Now that you know about dot products: The work you do is the dot product of the force vector you apply and the displacement vector of the train. (The displacement vector is the vector that tells how the train moved while you pushed it.)

Similarly, we mentioned that power is the product of the force you apply and the velocity of the mass you are applying it to. It is actually the dot product of the force vector and the velocity vector.

For example, if you are pushing a sled with a force of 10 newtons and it is moving 2 meters per second, but your push is 20 degrees off, you aren't transferring 20 watts of power to the sled. You are transferring $10 \times 2 \times \cos(20 \text{ degrees}) \approx 18.8$ watts of power.

Chapter 2

Boats

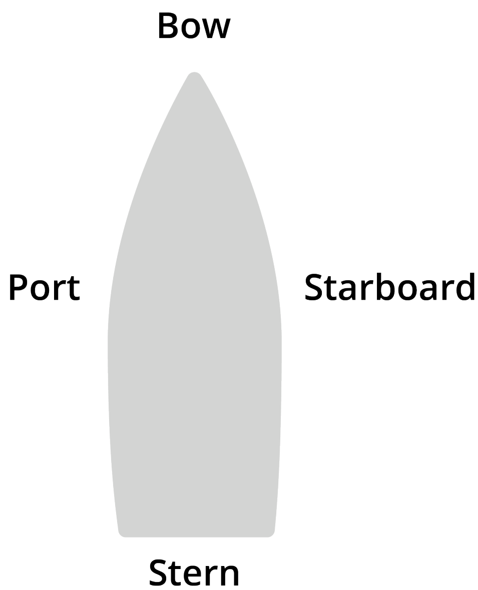
Engineers have been building boats for centuries. Through boat design, humanity learned the lessons that made airplanes and rockets possible. Learning about how boats work will give you a foundation for understanding more advanced concepts we will be covering later.

2.1 Basic Terminology

The front of a boat is called *the bow* (pronounced exactly the same as ”bough”). The back of the boat is called *the stern*.

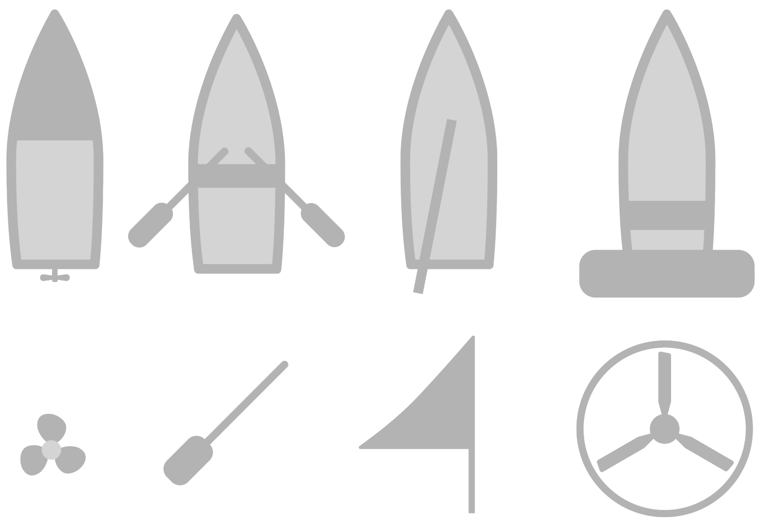
The underside of the boat is called *the hull*. The top of the boat is called *the deck*.

If you are standing at the stern and looking toward the bow, everything on your left is the *port* side. Everything on your right is the *starboard* side.



There are several different ways that boats are propelled:

- A motor turns a screw in the water, as in a motorboat. The screw is known as a *propeller*.
- A human pushes the water with a stick. If the stick is attached to the boat with a pivot (as in a rowboat) it is an *oar*. If the blade is not attached to the boat (as in a canoe), it is a *paddle*.
- A human pushes the ground beneath the water with a long pole. These boats are called *punts*, and are mainly used in more shallow waters where an oar or a paddle would be less convenient.
- The wind pushes the boat, as in a sailboat. The sails are held up by a *mast*.
- Some boats have a big fan that pushes the boat. These are called *airboats*. Airboats are not the most efficient boats, but they can travel on waterways with water just a couple of inches deep.



In the terms of physics, each of these method provide a *thrust vector* which is applied to the boat at a particular place and in a particular direction.

The speed of a boat is usually measured in *knots*. 1 knot is 1 nautical mile per hour, or 1.852 km per hour.

2.2 Why Boats Float Upright

Early in this sequence, we discussed buoyancy as a quantity. The magnitude of the buoyant force is equivalent to the weight of the liquid displaced.

We can also talk about the direction of the buoyant force: buoyancy pushes in the opposite direction as gravity.

How do we design boats so that they don't flip over?

2.2.1 Center of Buoyancy

Let's say you have a rowboat. If you push down on point on the floor near the front, the front of the boat will go down and the back of the boat will rise. In other words, the boat will rotate in the direction you are pushing. Likewise, if you push on the floor near the back of the boat, the back will sink a little lower and the front will rise. However, there is a place, near the center of the boat, where if you push down, the boat will not rotate at all; it will simply sink a little lower in the water. That point is known as the *center of buoyancy*.

How can we calculate the center of buoyancy? Imagine the shape of the water that was displaced by the boat. Now imagine that shape filled with water. The center of mass of that water is the center of buoyancy of the boat.

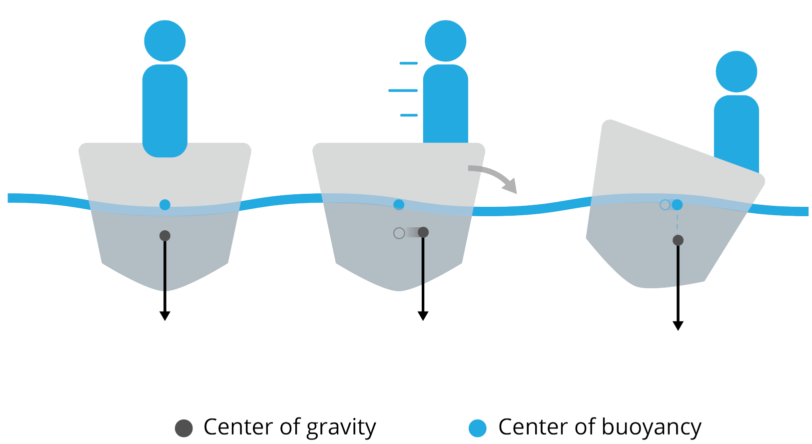
2.2.2 Center of Mass

Your boat and everything in it can be thought of as one object. That object has a center of mass. If you found the center of mass, you could balance the whole boat on it.

In a boat, if you move your body from the center of the boat to once side, you will have moved the center of mass. The boat will lean in that direction, which will change the center of buoyancy.

If you imagine a line is parallel to the force of gravity that passes through the center of mass of your boat, the boat will continue to increase its lean until the center of buoyancy is on that line.

If water comes over the sides of the boat before the center of gravity and center of buoyancy align, your boat will sink.



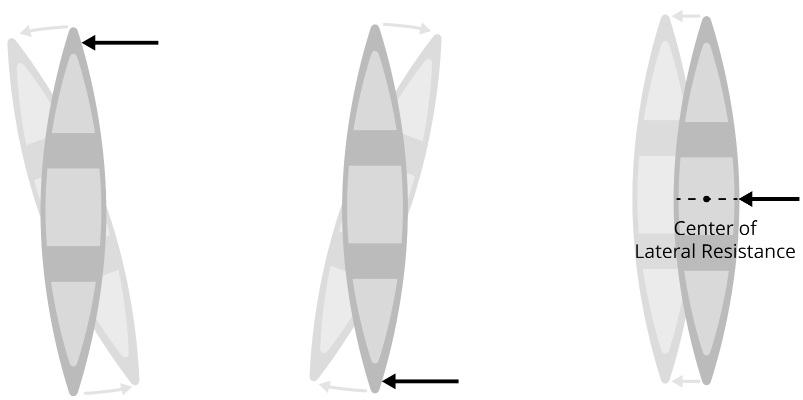
2.3 Center of Lateral Resistance

It isn't enough for a boat to float — for a boat to be useful, it must also be able to travel in a straight line.

Imagine that you are standing knee-deep in a lake next to a canoe. If you push the front of the canoe away from you, it will rotate — the back end will actually swing toward you. There is a point near the middle of the canoe where if you push it will not rotate in either direction — the boat will just slide sideways. This point is known as the *center of lateral resistance*.

The trick to making a boat travel in a straight line is to make sure that the line that contains the thrust vector passes through the center of lateral resistance.

An outboard motor allows you to direct the thrust vector. When the line of thrust passes the center of lateral resistance on the starboard side of the center of lateral resistance, the boat turns toward the port side.

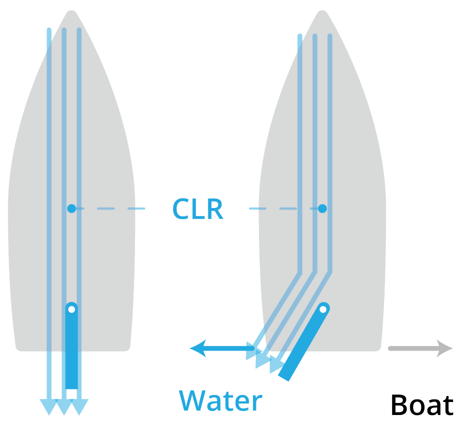


2.4 Steering with a Rudder

While outboard motors and airboats let you direct the thrust vector, most boats have a *rudder*. The rudder is a blade on a pivot near the back of the boat. The angle of the rudder can be adjusted so that water rushing past it gets pushed to one side or the other.

According to Newton's third law, when the water gets pushed to the left, the back of the boat gets pushed (with the same force) to the right. This causes the boat to rotate around its center of lateral resistance.

Note that a rudder only works when the boat is passing through the water.



2.5 Boat Length and Resistance

FIXME: Write about wave length, boat length, and Froude number.

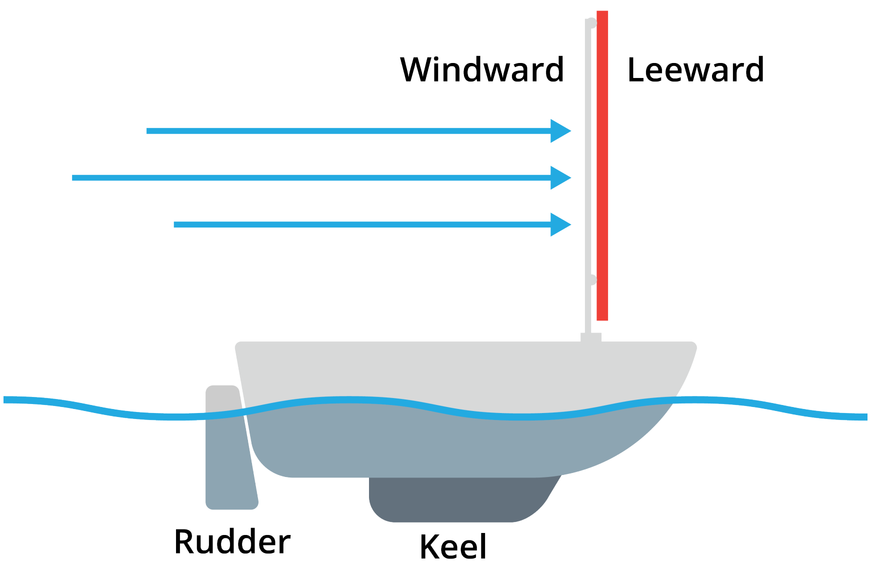
Chapter 3

Sailboats

Imagine that you have a canoe, and you are about to paddle from one island to another that is directly east of where you are standing. There is also a steady wind coming from the west, and you have a big piece of plywood. You might be inspired to use it as a sail.

This situation is the most simple form of sailing: Wind comes from behind the boat and hits the sail which generates a force that pushes the boat in the direction of the wind.

The sail has two sides: The *windward* side is the one that is getting hit with the wind. The *leeward* side is the side away from the wind.



3.1 Magnitude of the Wind Force

The first natural question is: How much force will I have pushing my canoe through the water?

Wind Force

When the sail is perpendicular to the wind, the force of the wind on the sail in newtons (F_w) will be given by:

$$F_w = A\frac{dv^2}{2}$$

where A is the area of the sail in square meters d is the density of the gas in kg per cubic meter, and v is the wind speed in meters per second.

For air at STP, d is about 1.225 kg per cubic meter.

We call $\frac{dv^2}{2}$ the *wind pressure*. This is the amount of pressure that the windward side of plywood is experiencing that is above the pressure that the leeward side of the plywood is experiencing. (The leeward side might experience some turbulence, but the pressure it is experiencing is approximately 1 atmosphere.)

Let's say your canoe is standing still and the wind is 0.5 m/s. This means the wind pressure is

$$P = \frac{1.225(0.5^2)}{2} = 0.153125 \text{ newtons per square meter}$$

Let's say your plywood sail is 2 meters tall and 1.5 meters wide. What will be the force of the wind?

$$F_w = AP = (3)(0.153125) \approx 0.46 \text{ newtons}$$

This is a very intuitive idea: There is a difference between the pressure on the windward side and the pressure on the leeward side, and the plywood experiences a force that pushes the boat through the water.

3.2 Direction and Location of the Wind Force

If there is low pressure on one side of the sail and high pressure on the other, the force vector will be perpendicular to the sail.

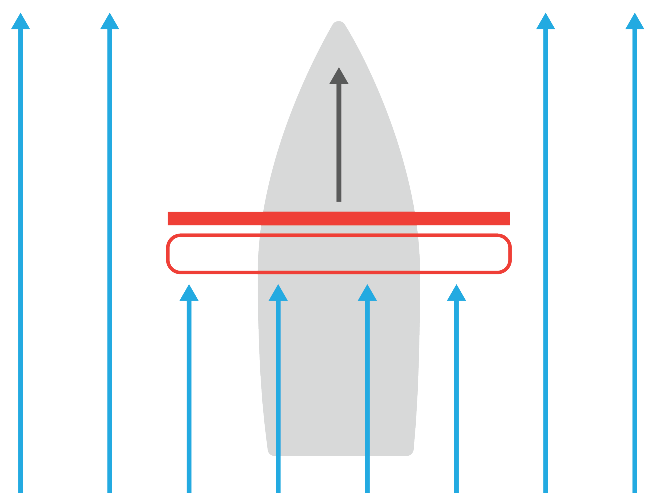
Where is this force vector applied? We can think of the force as being applied at the geometric center of the sail. This is called *the center of effort*. In this case, the center of effort is the exact center of the rectangular plywood.

The mast on a windsurfing board can be tilted from side to side. When the center of effort is over the center of lateral resistance, the board goes straight. To steer, the sailor moves the mast to one side of the center of lateral resistance, which rotates the board.

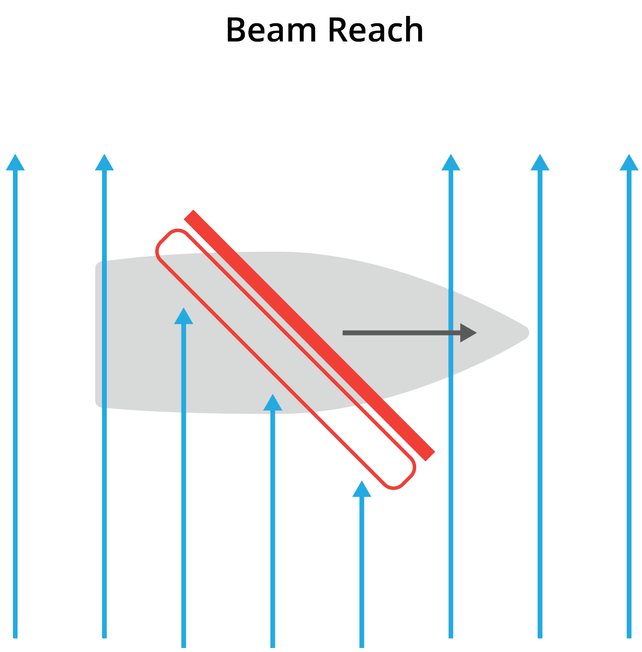
3.3 Beam Reach

When you are sailing in the same direction as the wind, sailors say you are *running*.

Running



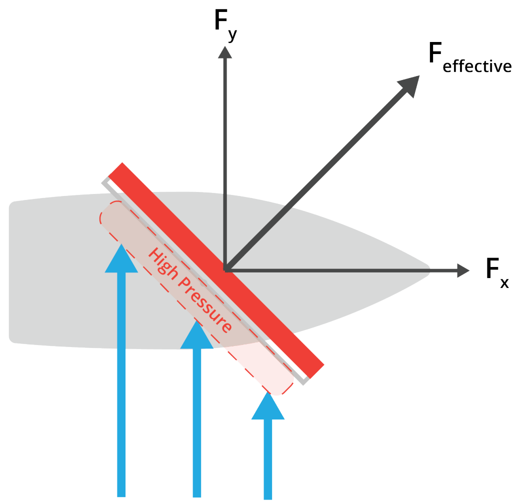
What if you want to go east and the wind (still 0.5 m per second) is now coming from the south? Sailing perpendicular to the wind is known as a *beam reach*.



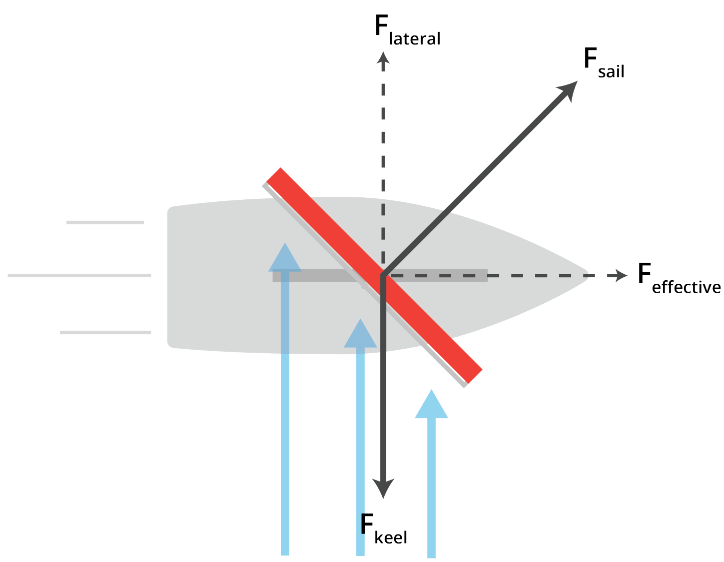
To do a beam reach, instead of mounting the plywood perpendicular to the boats direction of travel, you would mount it at a 45 degree angle. The wind pressure will build on the windward side of the plywood, and the plywood will experience a force pushing it at a 45° angle to the boat.

We can think of this force as having two components:

- One component pushes the boat forward (Yay!)
- One component pushes the boat sideways (Ugh!)



To minimize the effect of the sideways force, sailboats typically have a keel — a long fin on its underside that slows its sideways sliding.



Notice also that the ”wind shadow” of the plywood is smaller when it is at a 45° angle to the wind. How much smaller? The effective area of your plywood has gone from 3 square meters to $\frac{3}{\sqrt{2}} \approx 2.12$ square meters. This means the force generated will be smaller, and some of it will be wasted pushing the boat sideways.

If we assume that the wind pressure is still 0.153125 newtons per square meter. The force on the plywood will be about

$$F_w = AP = \frac{3}{\sqrt{2}}(0.154125) \approx 0.325 \text{ newtons}$$

However, the direction of that force is not all in the direction you want to go, so the effective force is

$$F = F_w \frac{1}{\sqrt{2}} = \frac{3}{2}(0.154125) = 0.2311875 \text{ newtons}$$

Notice that we got twice as much effective force when we were running with the wind as when we are on a beam reach. However, any sailor will tell you that you can go much faster on a beam reach than you can running. Why?

3.4 Apparent Wind

When you are running, you can never go faster than the wind. As you go faster and faster, the wind that the boat experience decreases. For example, if you are going 0.2 m/s in a wind of 0.5 m/s, you (and your sail) will only experience wind at 0.3 m/s.We call the wind as experienced by the boat the *apparent wind*. The wind as observed by a stationary observer is called the *actual wind*.

If you are running with the wind, as you approach the speed of the wind, the force of that wind will decrease towards zero.

On a beam reach, as you go faster, the direction of the wind seems to change. If you are going 0.2 m/s east and the actual wind is 0.5 m/s from the south, the direction of the apparent wind will seem to come from about 22 degrees east of true south. The speed of the apparent wind will be about 0.54 m/s.

3.5 Close Reach

What if you want to go east, and the wind is coming from 40 degrees east of south? This would mean that you were sailing just 50 degrees away from straight into the wind. Is this possible?

If you put your sail at a 25 degree angle, you will still catch some wind and create some pressure on one side of the sail. Most of the resulting force would be trying to push your boat sideways, but some of it would be in the direction you were trying to travel.

Picking an angle for your sail that creates high pressure that makes a desirable force is known as the ”angle of attack”.

This is a result does not feel intuitive. A boat can sail into the wind!? The boat can’t sail directly into the wind – with each degree that the boat gets closer to straight into the wind, the force pushing it forward decreases and the force pushing it back increases. However, most boats can get within 45% if they have a well-shaped sail.

3.6 Shaping the Sail

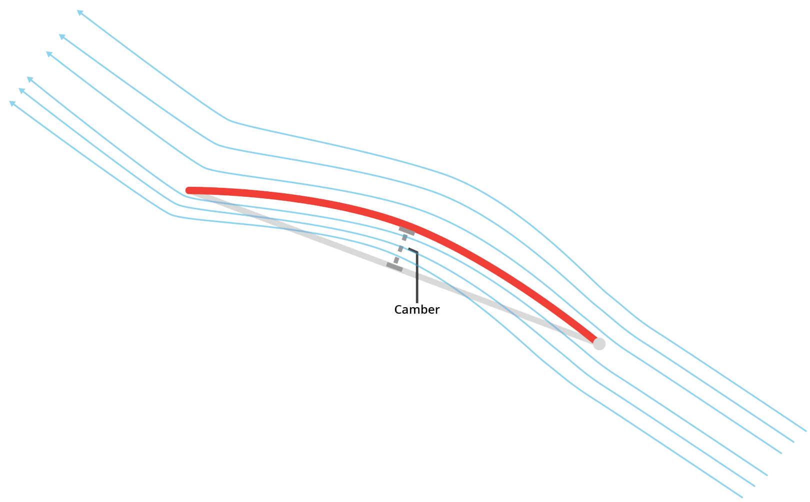
Most of the power of the wind can be captured with a piece of flat plywood. The wind hits it and creates a high pressure on the windward side. What about the other side of the plywood?

It turns out that if we can get the wind to travel smoothly over the back side of the plywood, the pressure on that side will be a little lower than if we had turbulence there. (We are not going to go too deeply into why. If you want to learn more, look up the Coanda effect.)

For example, if we were on a close reach, the very best sail we could have would gently pull the wind along its backside. It would look like this: FIXME: Like what? - Tony

Of course, for the sail to work on either side of the boat, this asymmetrical design would not work. (Although, we should note that this design works great for airplane wings.)

When we make a sail out of cloth, we give it some curve known as *camber*. Slow winds require just a little camber; fast winds require more.



Some newer sailboats have wing sails that have two pieces that can be arranged to redirect the most air possible.

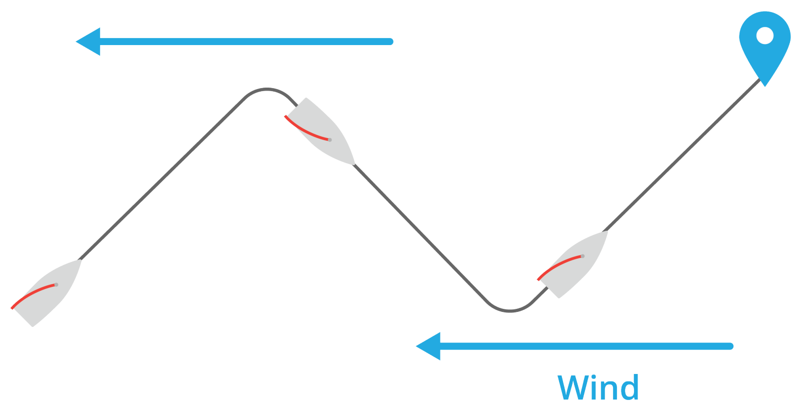
Note: when running with the wind, the turbulence on the leeward side of the sail is unavoidable. But when traveling perpendicular to the wind or on a close reach, the air should move smoothly over the leeward side of the sail. Many sailors have a piece of yarn taped on each side of the sail so they can see if the air is moving smoothly.

Many sailboats also have multiple sails. Besides the increase in the sail area, each sail also redirects the wind to pass smoothly over the leeward surface of the sail behind it.

3.7 Tacking into the Wind

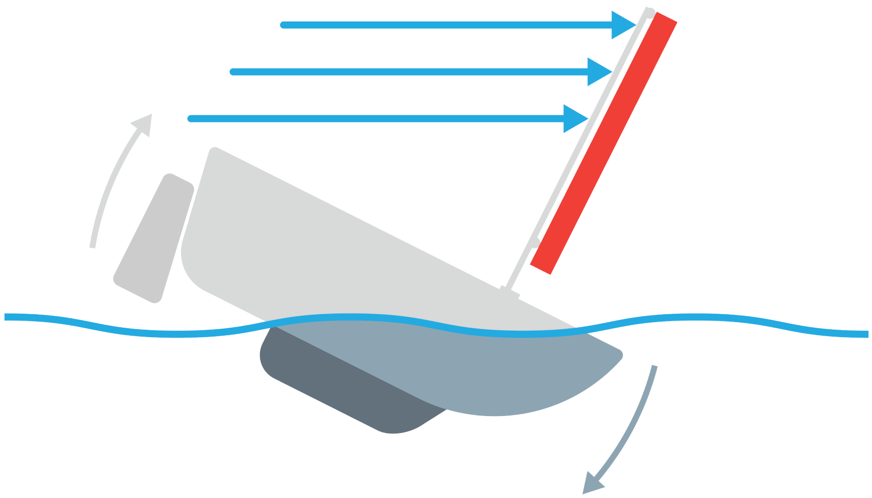
What if the wind is coming from the east and you really need to go directly into the wind?

The boat will not travel straight into the wind, instead you will travel on a close reach with the wind coming from one side of the boat. You will then turn into the wind and continue turning until you are on a close reach with the wind coming from the other side. This is known as *tacking*.

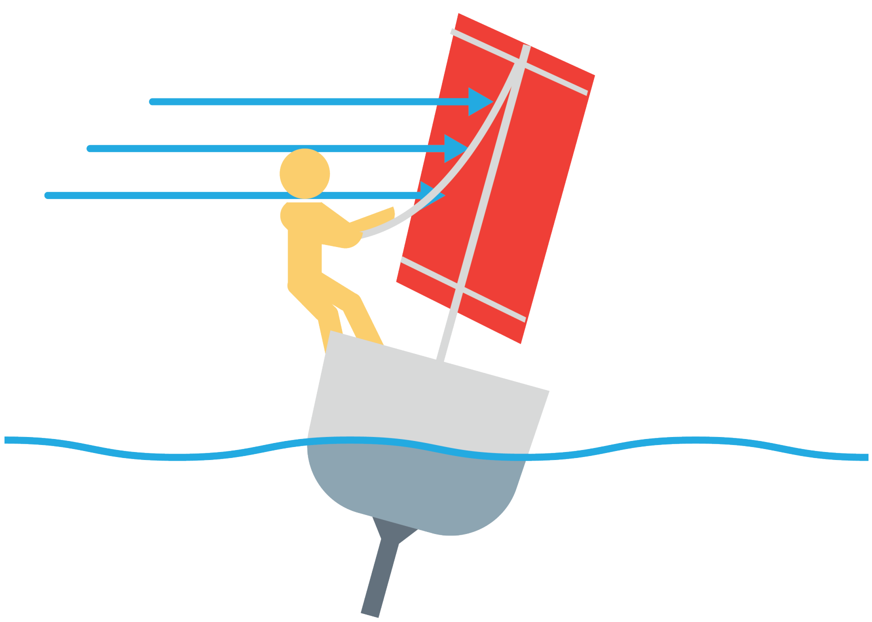


3.8 Heeling

The center of effort is near the center of your sail, which is usually pretty high in the air. Yes, the force generated will push your boat, but it will also rotate your boat. Sailors call the resulting lean *heeling*. Heeling too far is problematic — the rudder gets pulled out of the water, which makes it hard to steer the boat.



If a boat is heeling too far, sailors will move their weight to the windward side of the boat — some even wear harnesses that push their weight out beyond the edge of the hull. If that doesn't work, they will reduce the sideways force on the sail.



Can the boat flip over? (The word sailors use is *capsize*.) Most larger boats should not be able to capsize; as the sail gets pushed down toward the water, it loses power. So, putting some weight in the keel will ensure that the boat doesn't turn upside-down. Small boats (think 3 or 4 meters long) can capsize, but they are small enough that the sailor can usually get them upright again without assistance.

There are boats with multiple hulls. A catamaran has two identical hulls that are side-by-side. As a result, the catamaran will not heel much at all. However, if a big gust of wind comes up, one hull can be pulled completely out of the water. Catamarans can be capsized.

If a boat is running when a big gust comes up, that rotating force will try to push the bow underwater. If the boat is going very fast, this can result in a somersault as the front of the boat slows and dips suddenly and the back of the boat is pitched up over it.

Chapter 4

Airplanes

Now that you understand how a sail works, you are ready to understand how an airplane works.

4.1 Drag and Thrust

As an airplane flies, the air around it creates a large force that is trying to slow the plane down. Pilots call this *drag*.

To overcome drag, an airplane must have a source of *thrust*. This is usually done by pushing air toward the back of plane.

There are three basic types of propulsion on airplanes:

- Propeller planes have fan blades that are spun by an internal combustion engine and push the air toward the back of the plane.
- Jet planes suck air into a tube, mix it with fuel, and ignite the mixture. There are fan blades inside the tube that ensure the power from the burn is efficiently converted into thrust.
- Turboprop planes use both propeller and jet technology — their propellers are turned by jet engines.

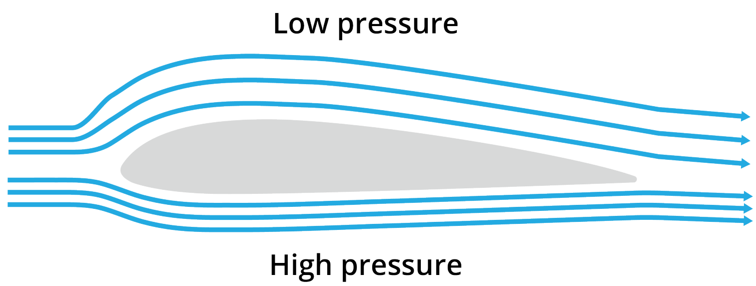
Drag is proportional to the square of the velocity. Tor example, if you double how fast you are flrying, the drag force will increase by a factor of 4.

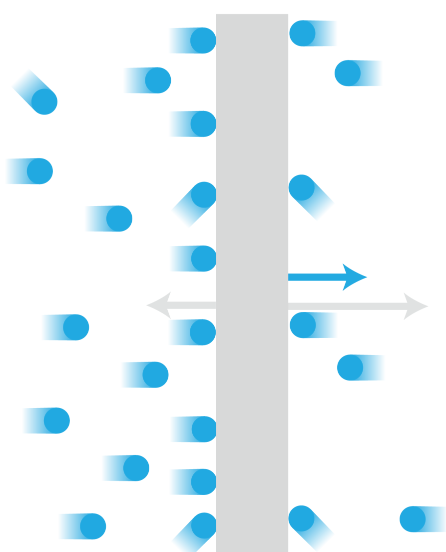
Passenger airplane fly fast — they have no trouble reaching 1000 km per hour. A Boeing 747 burns about 4 liters of fuel per second; the flight from London to New York requires about 70,000 kg of fuel. When the plane takes off, the fuel for the journey often weighs twice as much as the passengers.

4.2 Lift

Unlike a hot air balloon, an airplane is heavier than the air it displaces, so the force of gravity will pull it from the sky unless there is a counteracting force. We call the counteracting force *lift*.

Lift works just like sailing into the wind. By picking an angle of attack, we create high pressure under the wing. By creating a nice smooth path for the air to travel over the top of the wing, we create low pressure over the wing. This difference creates the lift on the wings.





(There are some very bad explanations of this idea that overstate the importance of the rounded top of the wing. Most airplanes can fly upside down – if the most important part were the shape of the top, this would be impossible. Instead, the most important part is the angle of attack.)

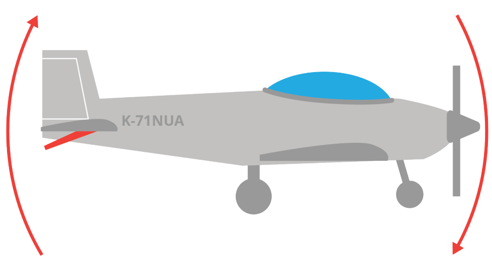
At high altitudes, the air is less dense. If two identical planes are flying at the same speed, but at different altitudes:

- There is less drag on the higher airplane.
- The wings provide less left on the higher airplane.
- The air around the higher plane has less oxygen per liter, which can affect how the fuel burns.

4.3 Control

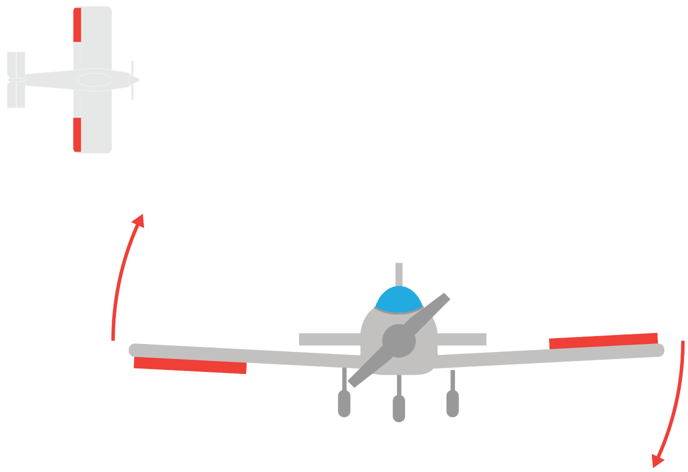
The first control that a pilot has is the throttle. By increasing the throttle, the pilot increases the plane's thrust.

The pilot has a stick. Pulling back on the stick causes the *elevator* on the tail of the plane to go up. Air hitting the elevator pushes the tail down and the nose up.

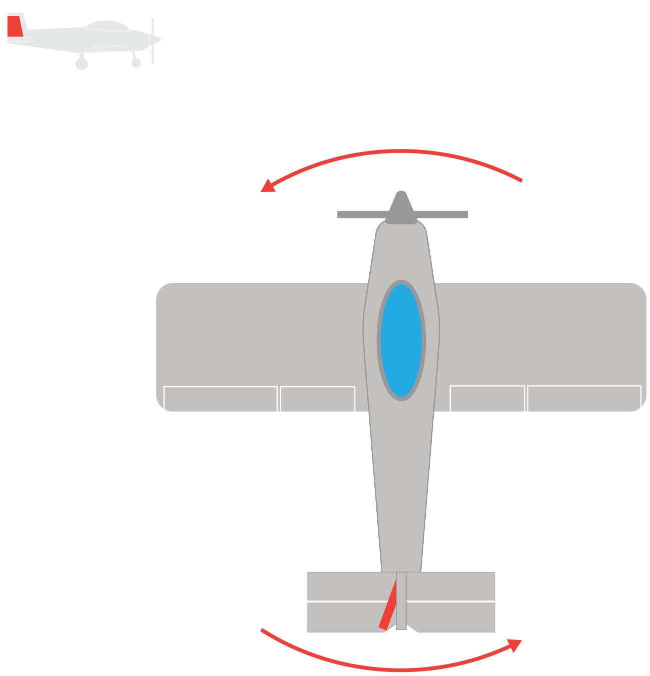


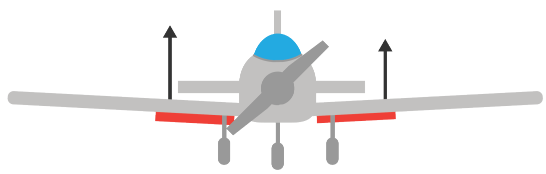
Pushing the stick forward will push the tail up and the nose down. We say ”The elevator controls the *pitch* of the airplane.”

The stick also goes left and right. Pushing the stick to the left, lifts the *aileron* on the left wing and lowers the aileron on the right wing. This pushes the left wing down and the right wing up. We say ”The ailerons control the *roll* of the airplane.”



The pilot controls the *rudder* on the tail with his feet. Pushing the right side down will push the rudder to the right. This will push the tail of the plane to the left and the nose to the right. We say "The rudder controls the *yaw* of the airplane."

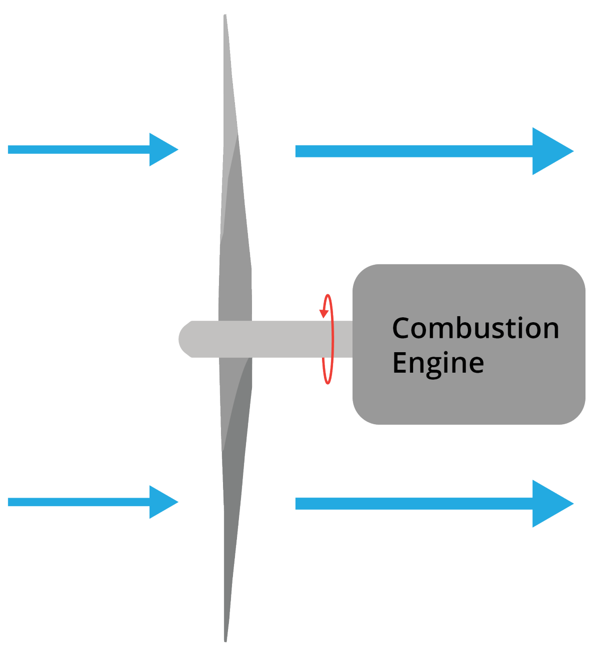




4.4 Thrust

There are several common ways that airplanes produce thrust.

The first is a propeller powered by a standard piston engine.

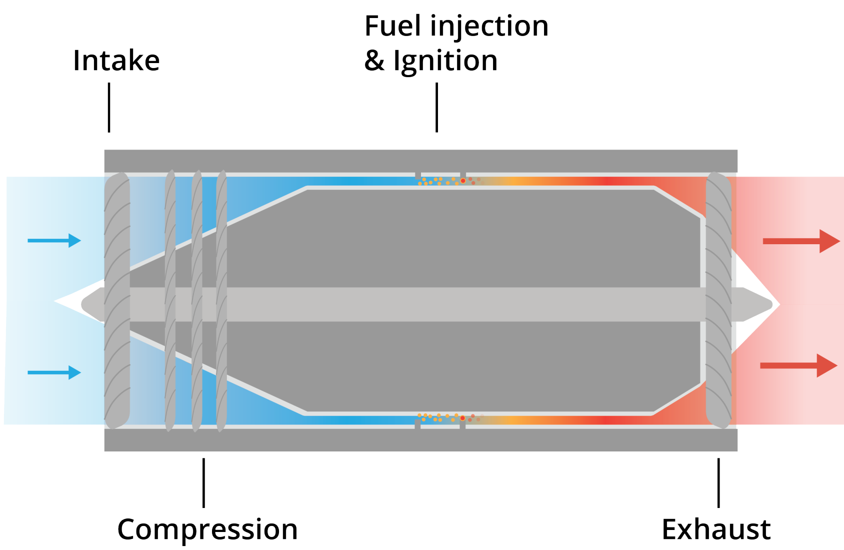


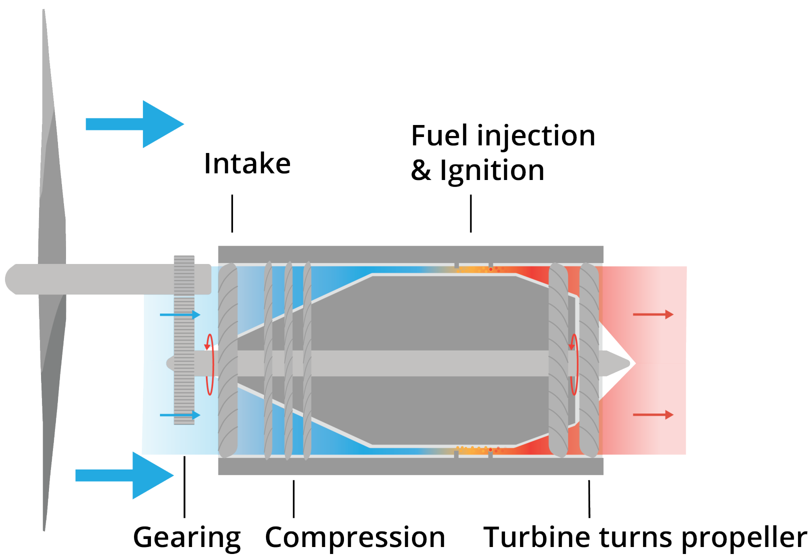
The propeller [HOW PROPELLER USES PRESSURE]

Another way to produce thrust is by using a *jet engine*. Jet engines take in air, increase its pressure and temperature, and shoots it out of the back of the engine. The combination of making air move much faster while also

and XXX can make a jet engine more powerful than standard pistol-driven propellers.

Jet engines suck in air, compress it, add a fuel mixture, ignite the fuel, and shoot it out the back. Once the air exits the engine, it expands at a high speed, which further helps create thrust.



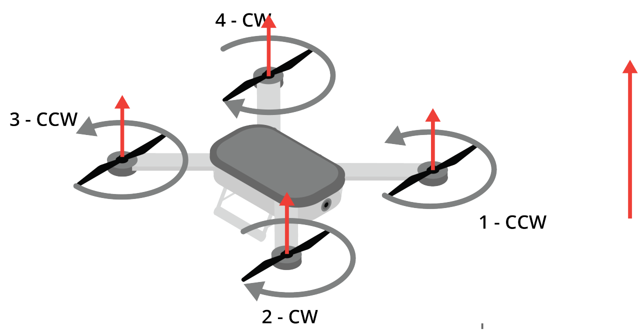


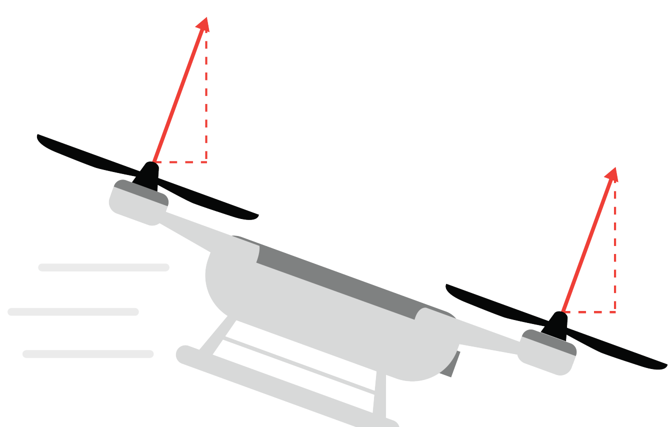
4.5 Gliders

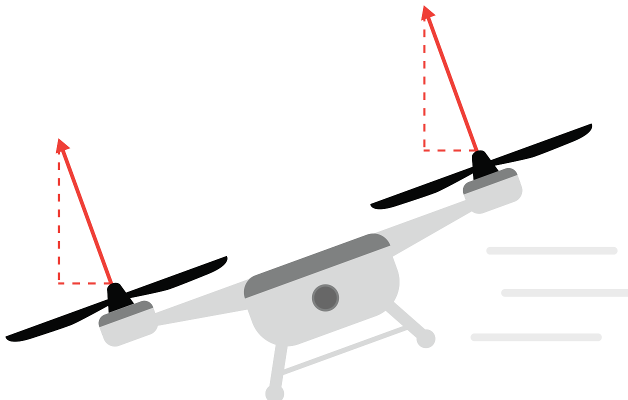
It is interesting to note that gliders fly without an engine. Gliding usually starts high in the air, where the glider has lots of potential energy. To stay aloft for a long time, glider pilots will look for places where air is rising, so they can ride those updrafts and regain that potential energy.

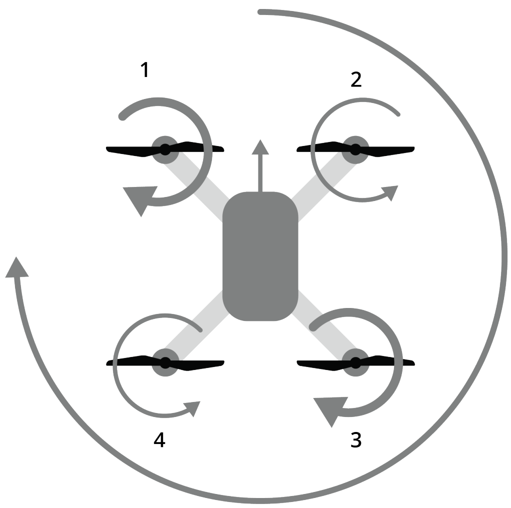
Chapter 5

Quadcopters








1
2
4
3

Appendix A

Answers to Exercises

Answer to Exercise 1 (on page 3)

- $[1, 2, 3] \cdot [4, 5, -6] = 4 + 10 - 18 = -4$
- $[\pi, 2\pi] \cdot [2, -1] = 2\pi - 2\pi = 0$
- $[0, 0, 0, 0] \cdot [10, 10, 10, 10] = 0 + 0 + 0 + 0 = 0$

Answer to Exercise 2 (on page 8)

- $[1, 0] \cdot [0, 1] = 0$. The angle must be $\pi/2$.
- $[3, 4] \cdot [4, 3] = 24$. $|[3, 4]||[4, 3]| \cos(\theta) = 24$. $\cos(\theta) = \frac{24}{(5)(5)}$. $\theta = \arccos(\frac{24}{25}) \approx 0.284$ radians.
- $[2, -1, 2]\cdot[-1, 2, -2] = 4-2-4 = -2$. $|[2, -1, 2]| = \sqrt{4 + 1 + 4} = \sqrt{9} = 3$. $|[-1, 2, -2]| = \sqrt{1 + 4 + 4} = \sqrt{9} = 3$. $3(3) \cos\theta = -2$. $\theta = \arccos(-2/9) \approx 1.795$ radians.
- $[-5, 0, -1]\cdot[2, 3, -4] = -10+0+4 = -6$. $|[-5, 0, 1]| = \sqrt{25 + 0 + 1} = \sqrt{26}$. $|[2, 3, -4]| = \sqrt{4 + 9 + 16} = \sqrt{29}$. $\sqrt{26}(\sqrt{29}) \cos\theta = -6$. $\theta = \arccos(\frac{-6}{\sqrt{26}\sqrt{29}}) \approx 1.791$ radians.

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