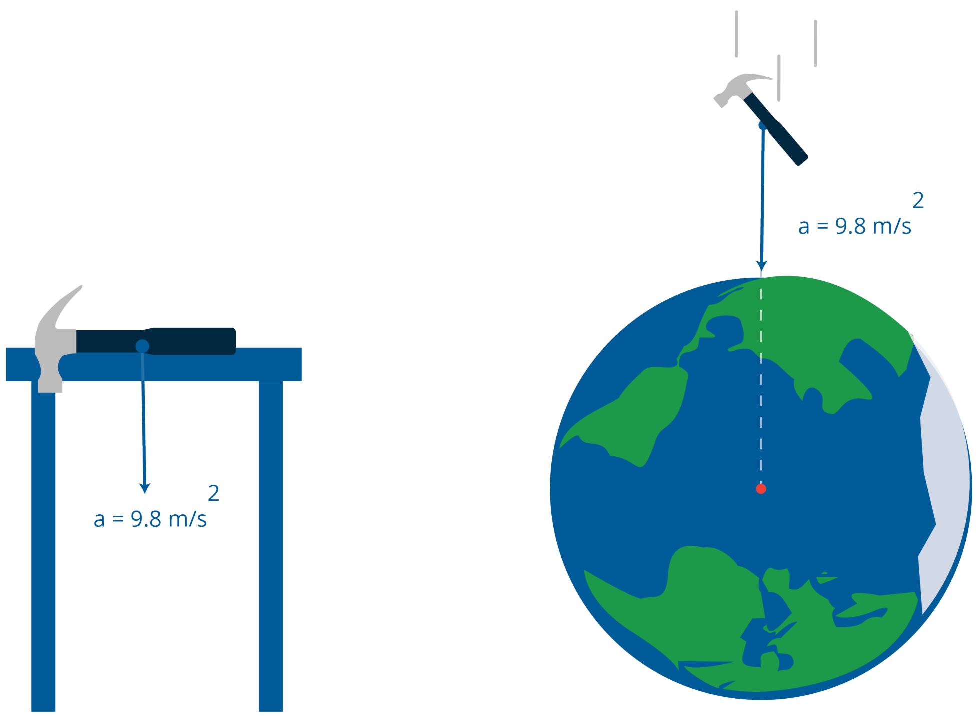
Contents

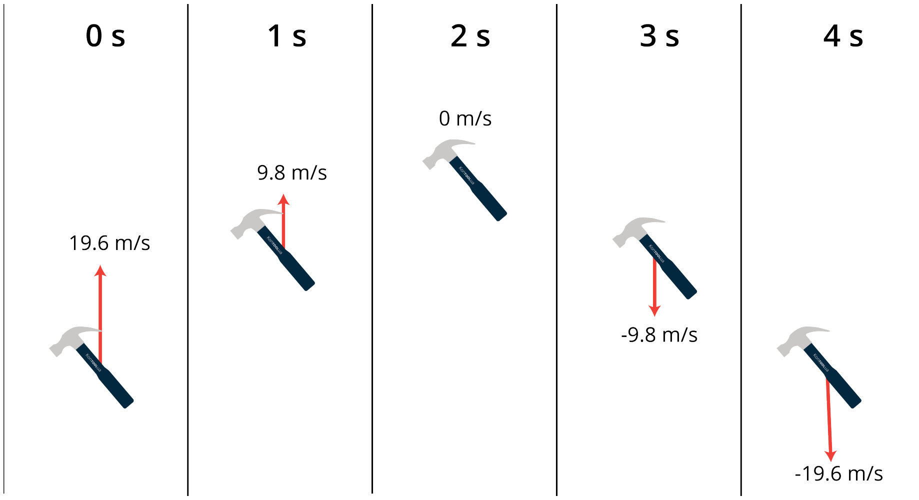
Chapter 1

Falling Bodies

Gravity exists all around us. If you throw a hammer straight up in the air, from the moment it leaves your hand until it hits the ground, it is accelerating toward the center of the earth at a constant rate.



Acceleration can be defined as change in velocity. If the hammer leaves your hand with a velocity of 12 meters per second upward, one second later, it will be rising, and its velocity will have slowed to 2.2 meters per second. One second after that, the hammer will be falling at a rate of 7.6 meters per second. Every second the hammer's velocity is changing by 9.8 meters per second, and that change is always toward the center of the earth. When the hammer is going up, gravity is slowing it down by 9.8 meters per second, each second it is in the air. When the hammer is coming down, gravity is increasing the speed of its descent by 9.8 meters per second.



Acceleration due to gravity on earth is a constant negative 9.8 meters per second per second:

$$a = -9.8\frac{m}{s^2}$$

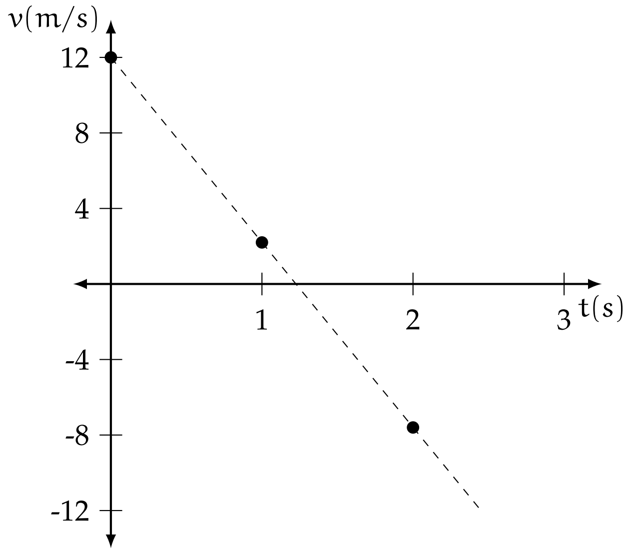
(Why is it negative? We are talking about height, which is generally considered to be increasing as you go away from the center of the earth. Since "up" is the positive direction, we take "down" as a negative direction. Therefore, since gravity pulls things "down", the acceleration due to gravity is considered negative.)

1.1 Calculating the Velocity

Given that the acceleration is constant, it makes sense that the velocity is a straight line. Assuming once again that the hammer leaves your hand at 12 meters per second, let's create a quick data table and graph of the hammer's velocity. Every second, we will subtract 9.8 m/s from the previous velocity.

Time elapsed (s)	Velocity (m/s)
0	12
1	2.2
2	-7.6

Here is a plot of these data with a dashed fit line:



We can find an equation for the line: $v = 12 - 9.8t$. What is this saying? It says that at any time, t, after $t = 0s$, the velocity of the hammer is its starting velocity (12 m/s) plus the acceleration (-9.8 m/s^2) multiplied by the time elapsed (t). (Why did we say "plus" instead of "minus"? Well, since the acceleration *is negative*, $v_0 - 9.8t$ is the same as $v_0 + (-9.8t)$, and we are really adding the negative acceleration. When the acceleration is positive, we'll see a plus sign!)

Exercise 1 When is the apex of flight?

Given the hammer's velocity is calculated as $12 - 9.8t$, at what time (in seconds) does it stop rising and begin to fall?

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We can generalize this to other falling bodies. For any falling object with initial velocity v_0 and constant acceleration a, its velocity is given by:

$$v(t) = v_0 + a \cdot t$$

Example: The acceleration due to gravity on the Moon is approximately 1.63 m/s^2. If a hammer tossed upwards on the Moon takes 3.4 seconds to reach its apex, what was the hammer's initial velocity?

Solution: We know that $a = -1.63\text{m/s}^2$ and $t = 3.4\text{s}$. Substituting and solving the velocity equation:

$$0\frac{m}{s} = v_0 + \left(-1.63\frac{m}{s^2}\right)(3.4s)$$
$$v - 0 = (1.63 \cdot 3.4)\,\frac{m}{s} \approx 5.5\frac{m}{s}$$

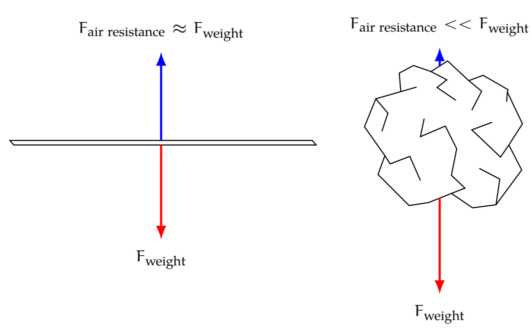
This hammer must have been tossed upwards with an initial velocity of approximately 5.5 m/s.

1.1.1 Air Resistance

At this point, we need to acknowledge air resistance. Gravity is not the only force on the hammer; as it travels through the air, friction with the air slows it down. This force is called *air resistance*, and for a large, fast-moving object (like an airplane) it is GIGANTIC force. For a dense object (like a hammer) moving at a slow speed (what you generate with your hand), air resistance doesn't significantly affect acceleration.

Consider two otherwise identical pieces of paper: one completely flat and unfolded, the other balled up as tightly as possible. If you dropped them both from the same height, which would hit the ground first? Intuitively, you know the balled-up paper would: but *why*? Since the papers have the same mass, you know that the force of gravity is the same on both papers! However, if you give this a try (go ahead, all you need to two sheets of printer paper and a safe place to drop them from), the acceleration of the two papers is different.

We can apply Newton's Second Law to explain our observations. First, let's draw diagrams to represent the forces acting on the papers:



For the crumpled paper, the decrease in surface area results in very little air resistance. As a result, the net force on the crumpled paper is approximately equal to the paper's weight.

$$F_{\text{net}}^{\text{crumpled}} = F_{\text{air resistance}} + F_{\text{weight}} \approx F_{\text{weight}}$$

$$F_{net}^{crumpled} = ma_{crumpled} \approx F_{weight} = mg$$
$$a_{crumpled} \approx g$$

Thus, you observe the crumpled paper to fall at nearly -9.8 m/s^2. The flat paper, on the other hand, has a much larger surface area, and therefore experiences significantly more air resistance. Thus, the observed acceleration of the flat paper is much slower:

$$F_{net}^{flat} = F_{air\ resistance} + F_{weight}$$
$$ma_{flat} = F_{air\ resistance} + mg$$
$$a_{flat} = \frac{F_{air\ resistance}}{m} + g$$

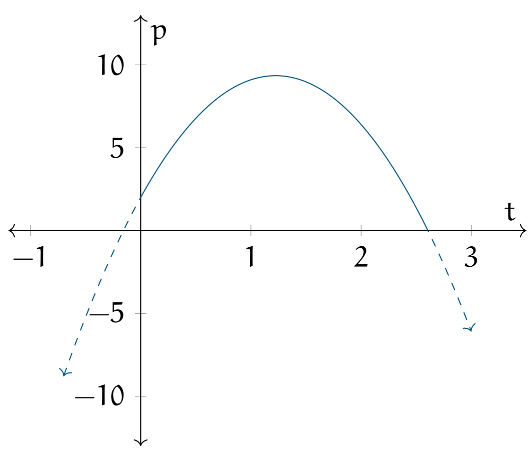
Since $g < 0$ and $F_{air\ resistance}$ is positive, the greater the air resistance the slower the paper falls.

1.2 Calculating Position

If you let go of the hammer when it is 2 meters above the ground, the height of the hammer is given by:

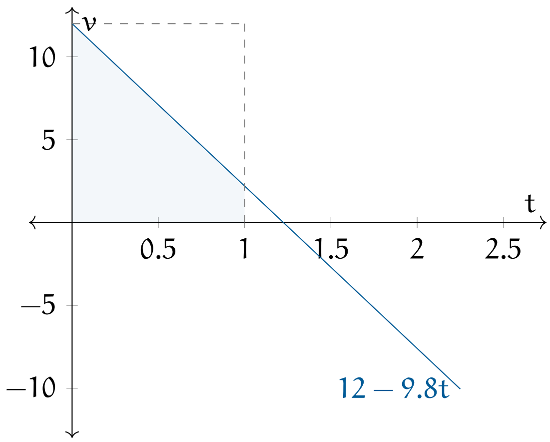
$$p = -\frac{9.8}{2}t^2 + 12t + 2$$

Here is a graph of this function:



How do we know? **The change in position between time 0 and any time t is equal to the area under the velocity graph between $x = 0$ and $x = t$.**

Let's use the velocity graph to figure out how much the position has changed in the first second of the hammer's flight. Here is the velocity graph with the area under the graph for the first second filled in:



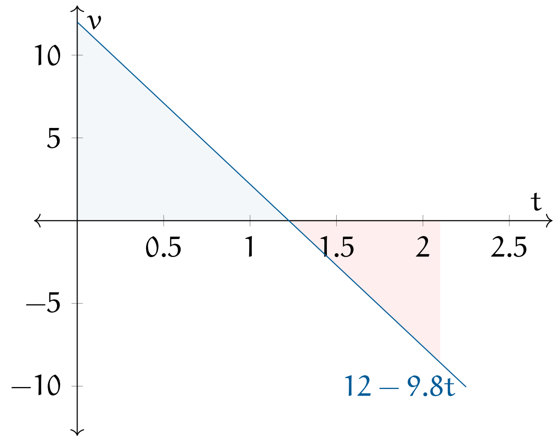
The blue filled region is the area of the dashed rectangle minus that empty triangle in its upper left. The height of the rectangle is twelve and its width is the amount of time the hammer has been in flight (t). The triangle is t wide and $9.8t$ tall. Thus, the area of the blue region is given by $12t - \frac{1}{2}9.8t^2$.

That's the change in position. Where was it originally? 2 meters off the ground. This means the height is given by $p = 2 + 12t - \frac{1}{2}9.8t^2$. We usually write terms so that the exponent decreases, so:

$$p = -\frac{1}{2}9.8t^2 + 12t + 2$$

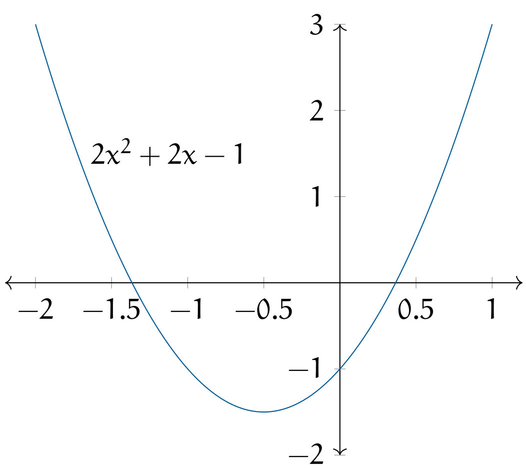
Finding the area under the curve like this is called *integration*. We say "To find a function that gives the change in position, we just integrate the velocity function." Much of the study of calculus is learning to integrate different sorts of functions.

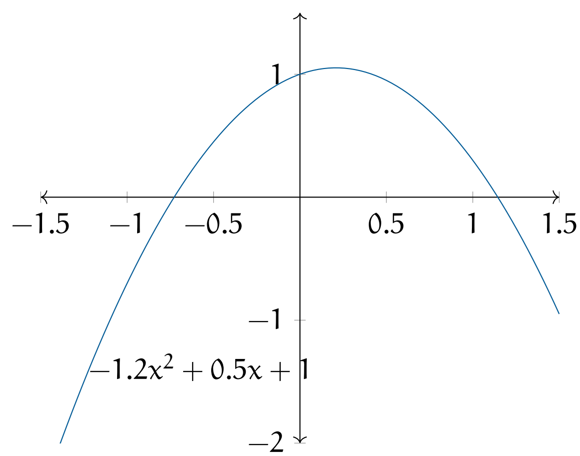
One important note about integration: Any time the curve drops under the x-axis, the area is considered negative. (Which makes sense, right? If the velocity is negative, the hammer's position is decreasing.)



1.3 Quadratic functions

Functions of the form $f(x) = ax^2 + bx + c$ are called *quadratic functions*. If $a > 0$, the ends go up. If $a < 0$, the ends go down.





The graph of a quadratic function is a *parabola*.

1.4 Simulating a falling body in Python

Now you are going to write some Python code that simulates the flying hammer. First, we are just going to print out the position, speed, and acceleration of the hammer for every 1/100th of a second after it leaves your hand. (Later, we will make a graph.)

Create a file called `falling.py` and type this into it:

```
# Acceleration on earth
acceleration = -9.8 # m/s/s

# Size of time step
time_step = 0.01 # seconds

# Initial values
speed = 12  # m/s upward
height = 2  # m above the ground
current_time = 0.0  # seconds after release

# Is the hammer still aloft?
while height > 0.0:

    # Show the values
    print(f"{current_time:.2f} s:")
    print(f"\tacceleration: {acceleration:.2f} m/s/s")
    print(f"\tspeed: {speed:.2f} m/s")
    print(f"\theight: {height:.2f} m")

    # Update height
    height = height + time_step * speed

    # Update speed
    speed = speed + time_step * acceleration

    # Update time
    current_time = current_time + time_step


print(f"Hit the ground: Complete")
```

When you run it, you will see something like this:

```
0.00 s:
        acceleration: -9.80 m/s/s
        speed: 12.00 m/s
        height: 2.00 m
0.01 s:
        acceleration: -9.80 m/s/s
        speed: 11.90 m/s
        height: 2.12 m
0.02 s:
        acceleration: -9.80 m/s/s
        speed: 11.80 m/s
        height: 2.24 m
```

```
0.03 s:
        acceleration: -9.80 m/s/s
        speed: 11.71 m/s
        height: 2.36 m
...
2.60 s:
        acceleration: -9.80 m/s/s
        speed: -13.48 m/s
        height: 0.20 m
2.61 s:
        acceleration: -9.80 m/s/s
        speed: -13.58 m/s
        height: 0.07 m
Hit the ground: Complete
```

Note that the acceleration isn't changing at all, but it is changing the speed, and the speed is changing the height.

We can see that the hammer in our simulation hits the ground just after 2.61 seconds.

1.4.1 Graphs and Lists

Now, we are going to graph the acceleration, speed, and height using a library called matplotlib. However, to make the graphs, we need to gather all the data into lists.

For example, we will have a list of speeds, and the first three entries will be 12.0, 11.9, and 11.8.

We create an empty list and assign it to a variable like this:

```
x = []
```

Next, we can add items like this:

```
x.append(3.14)
```

To get the first time back, we can ask for the object at index 0.

```
y = x[0]
```

Note that the list starts at 0. If you have 32 items in the list, the first item is at index 0; the last item is at index 31.

Duplicate the file `falling.py` and name the new copy `falling_graph.py`

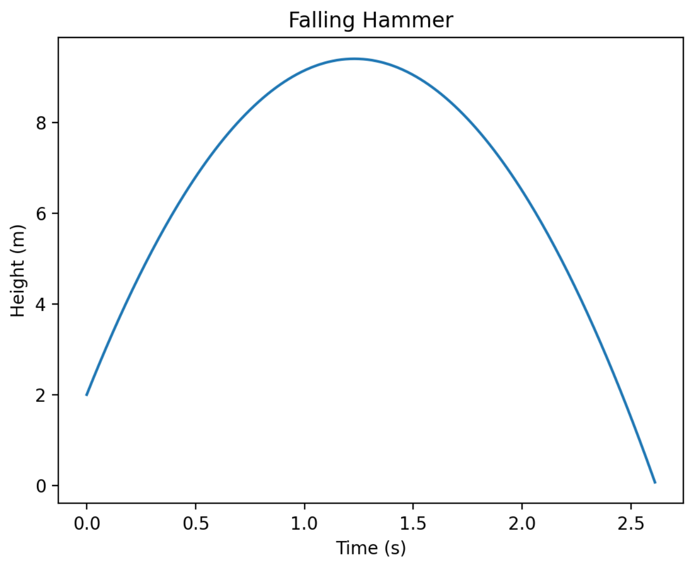
We are going to make a plot of the height over time. At the start of the program, you will import the matplotlib library. At the end of the program, you will create a plot and show it to the user.

In `falling_graph.py`, add the bold code:

```
import matplotlib.pyplot as plt

# Acceleration on earth
acceleration = -9.8 # m/s/s

# Size of time step
time_step = 0.01 # seconds

# Initial values
speed = 12  # m/s upward
height = 2  # m above the ground
current_time = 0.0  # seconds after release

# Create empty lists
accelerations = []
speeds = []
heights = []
times = []

# Is the hammer still aloft?
while height > 0.0:

    # Add the data to the lists
    times.append(current_time)
    accelerations.append(acceleration)
    speeds.append(speed)
    heights.append(height)

    # Update height
    height = height + time_step * speed

    # Update speed
    speed = speed + time_step * acceleration

    # Update time
    current_time = current_time + time_step

# Make a plot
```

```
fig, ax = plt.subplots()
fig.suptitle("Falling Hammer")
ax.set_xlabel("Time (s)")
ax.set_ylabel("Height (m)")
ax.plot(times, heights)
plt.show()
```

When you run the program, you should see a graph of the height over time.

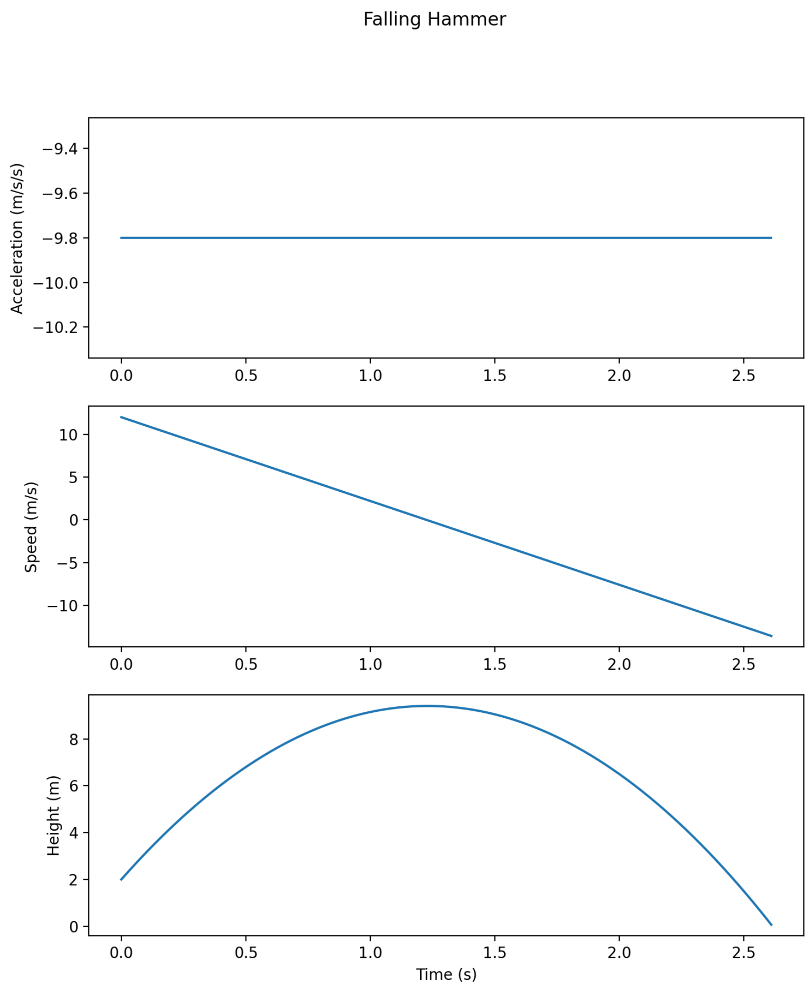


It is more interesting if we can see all three: acceleration, speed, and height. So, let's make three stacked plots. Change the plotting code in `falling_graph.py` to:

```
# Make a plot with three subplots
fig, ax = plt.subplots(3,1)
fig.suptitle("Falling Hammer")

# The first subplot is acceleration
ax[0].set_ylabel("Acceleration (m/s/s)")
ax[0].plot(times, accelerations)

# Second subplot is speed
ax[1].set_ylabel("Speed (m/s)")
ax[1].plot(times, speeds)

# Third subplot is height
ax[2].set_xlabel("Time (s)")
ax[2].set_ylabel("Height (m)")
ax[2].plot(times, heights)
```

```
plt.show()
```

You will now get plots of all three variables:



This is what we expected, right? The acceleration is a constant negative number. The speed is a straight line with a negative slope. The height is a parabola. **The slope of the height graph is the speed, and the slope of the speed graph is acceleration**

A natural question at this point is "When exactly will the hammer hit the ground?" In other words, when does $\texttt{height} = 0$? The values of t where a function is zero are known as its *roots*. Height is given by a quadratic function. In the next chapter, you will get the trick for finding the roots of any quadratic function.

Chapter 2

Kinematics

How can we describe the motion of objects other than falling bodies? Kinematics is the description of motion.

Kinematic Equations

$$x_f = x_0 + v_0 t + \frac{1}{2} a t^2 \tag{2.1}$$

$$v_f = v_0 + a t \tag{2.2}$$

$$x_f = x_0 + \frac{1}{2} \left(v_f + v_0\right) t \tag{2.3}$$

$$v_f^2 = v_0^2 + 2a \left(x_f - x_0\right) \tag{2.4}$$

$$\Delta x - x_0 = v_{avg} t \tag{2.5}$$

Note that v_0 and v_i are synonomous, some professors will use v_0 to mean "velocity not" or inital velocity.

Note that each equation is missing a certain variable (or two)

- (5.1) is missing v_f.
- (5.2) is missing x_i or x_f
- (5.3) is missing a
- (5.4) is missing t
- (5.5) is missing a

It is important to note that these equations only work when the acceleration is held constant, which is referred to as *uniformly accelerated motion*.

Example: Terri and Jerry are running a race. Terri has a maximum acceleration of 3.4 m/s^2 and a top speed of 9.2 m/s. Jerry has a maximum acceleration of 3.7 m/s^2 and a top speed of 8.7 m/s. If the race is 200 m long, who will win? Assume that each runner has maximum acceleration until they reach their top speed and that they will maintain that top speed once they reach it.

Solution: We want to know how long it takes each runner to complete the 200 m race. For each runner, we will divide their run into two sections:

1. The time they are accelerating to their top speed
2. The time they are maintaining their top speed

We can use a table to track the results of our calculations:

Runner	Terri	Jerry
Leg 1 (s)		
Leg 2 (s)		
Total (s)		

We'll begin with Terri's first leg. Taking the starting line as $x = 0$, we know that:

$$x_0 = 0 \text{ m}$$
$$v_0 = 0 \ \frac{\text{m}}{\text{s}}$$
$$a = 3.4 \ \frac{\text{m}}{\text{s}^2}$$

And since we want to know how long it takes Terri to reach her top speed, we also know that $v_f = 9.2$ m/s. Since we don't know how far Terri will run before she reaches her top speed (and we're not looking for that quantity), we need to select an equation that does not include x:

$$v_f = v_0 + at$$
$$9.2 \frac{\text{m}}{\text{s}} = 0 \frac{\text{m}}{\text{s}} + \left(3.4 \frac{\text{m}}{\text{s}^2}\right) t$$
$$t = \frac{9.2 \frac{\text{m}}{\text{s}}}{3.4 \frac{\text{m}}{\text{s}^2}} \approx 2.7 \text{ s}$$

With a similar method, we can find how long it takes Jerry to reach his top speed:

$$t = \frac{v_f}{a} = \frac{8.7 \frac{\text{m}}{\text{s}}}{3.7 \frac{\text{m}}{\text{s}^2}} \approx 2.4 \text{ s}$$

Let's go ahead and record this in our table:

Runner	Terri	Jerry
Leg 1 (s)	2.7	2.4
Leg 2 (s)		
Total (s)		

Now that we know how much time it takes each runner to reach their top speed, we need to figure out how much time it takes them to complete the race from the point at which each reaches their top speed. To do this, we will first have to find *where* each runner hits their top speed. (This is because we can't use $v_f = v_0 + at$ to find a time anymore, since from now on the runners' accelerations are zero, and all the other equations involve x.) For each runner, we know a, v_0, v_f, x_0, and t. You could choose any equation, but we will use this one:

$$x_f = x_0 + \frac{1}{2}(v_f + v_0)\,t$$

Since each runner begins on the starting line, $x_0 = 0$ m and $v_0 = 0$ m/s:

$$x_f = \frac{v_f \cdot t}{2}$$

For Terri:

$$x_f = \frac{\left(9.2\frac{m}{s}\right)(2.7\ s)}{2} \approx 12.4\ m$$

For Jerry:

$$x_f = \frac{\left(8.7\frac{m}{s}\right)(2.4\ s)}{2} \approx 10.2\ m$$

Now that we know where they reach their top speed, we can take that position as x_0 and find how long it takes each runner to reach the finish line at $x_f = 200$ m. Since $a = 0$ m/s^2, we can use:

$$x_f = x_0 + v_0 t$$

Rearranging to solve for t:

$$t = \frac{x_f - x_0}{v_0}$$

For Terri:

$$t = \frac{200\ m - 12.4\ m}{9.2\ \frac{m}{s}} \approx 20.4\ s$$

And for Jerry:

$$t = \frac{200\ m - 10.2\ m}{8.7\ \frac{m}{s}} \approx 21.8\ s$$

Completing our table, we see that Terri will win the race by finishing in the least amount of time:

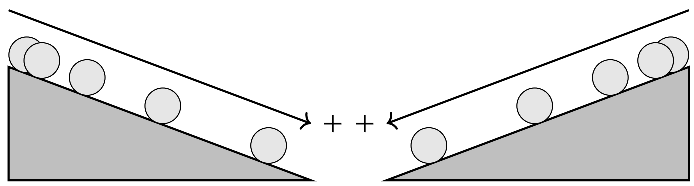
Runner	Terri	Jerry
Leg 1 (s)	2.7	2.4
Leg 2 (s)	20.4	21.8
Total (s)	23.1	24.2

2.1 Graphing Motion

2.1.1 Motion Diagrams

Example: Create a motion diagram of a ball rolling with a constant acceleration down an incline (the acceleration is in the same direction as motion).

Solution: The ball is accelerating down the ramp, so it will cover more distance each second. You could draw your ramp going left or right, as long as the distance covered each time interval increases as time passes.



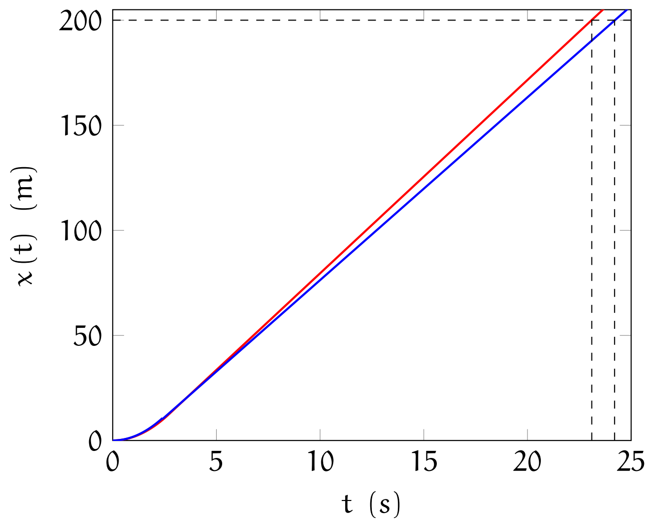
2.1.2 Position-Time and Velocity-Time Graphs

In the example problem above, graphs of each runner's motion would allow us to immediately see who would win. We can describe Terri's and Jerry's runs with piecewise functions:

$$x_{\text{Terri}}(t) = \begin{cases} \left(1.7\ \frac{\text{m}}{\text{s}^2}\right) t^2 & \text{if } 0 \leq t < 2.7 \text{ s} \\ 12.4 \text{ m} + \left(9.2\ \frac{\text{m}}{\text{s}}\right)(t - 2.7 \text{ s}) & \text{if } t \geq 2.7 \text{ s} \end{cases}$$

$$x_{\text{Jerry}}(t) = \begin{cases} \left(1.85\ \frac{\text{m}}{\text{s}^2}\right) t^2 & \text{if } 0 \leq t < 2.4 \text{ s} \\ 10.2 \text{ m} + \left(8.7\ \frac{\text{m}}{\text{s}}\right)(t - 2.4 \text{ s}) & \text{if } t \geq 2.4 \text{ s} \end{cases}$$

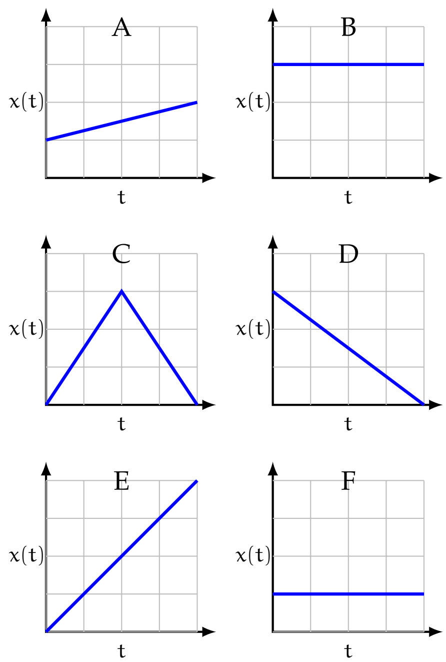
Graphing Terri in red and Jerry in blue:



Since the red function (Terri) crosses $x(t) = 200$ m first, we see that Terri will win.

Exercise 2

The following graphs show the position of an object from $t = 0$ s to $t = 10$ s. The scales on the y-axes are all the same.



Rank the objects from least to greatest (all negative values are lower than all positive values) in terms of:

1. displacement from $t = 0$ s to $t = 10$ s.
2. instantaneous velocity at $t = 7.5$ s.
3. distance traveled from $t = 0$ s to $t = 10$ s

Working Space

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2.2 Separation of Components

In the current chapter, we have learned how to use kinematics to describe one-dimensional motion. In the next chapter, you will learn to describe two-dimensional motion. It turns out that you can treat the different dimensions (horizontal and vertical motion) separately! Consider this scenario: A cannonball is shot from a cliff and the same instant an identical cannonball is dropped from the same cliff (see...). If the cannon is aimed horizontally, which cannonball will hit the ground first?

Go ahead and jot down what you think would happen: would the dropped ball hit first, the launched ball hit first, or would they both reach the ground below at the same time? Then, take a look at this video: https://www.youtube.com/watch?v=zMF4CD7i3hg. In it, two balls are released from the same height at the same time. One is dropped from rest while the other is launched horizontally (that is, its initial velocity is entirely in the x-direction). Based on the video, was your cannonball prediction correct? We'll learn to explain this phenomenon in the next chapter.

Chapter 3

Projectile Motion

A projectile is an object that, once thrown or dropped, continues to move only under the influence of gravity. Throwing a baseball, shooting a cannon, and diving off a high diving board are all examples. NASA flight planners use projectile motion to plan flight paths for space vehicles, such as sending rovers to Mars. You've already learned how to describe and model one-dimensional projectile motion in the Falling Bodies chapter. Now, we consider projectiles that also have horizontal motion, and therefore are moving in two dimensions.

First, we will compare the motion of projectiles that are dropped versus horizontally launched from the same height. This will frame our discussion of the important concept of independence of motion: the vertical and horizontal motions of a projectile can be considered and described independent from each other. This will allow you to predict how far horizontally launched objects will travel before hitting the ground. Next, you'll learn to describe the motion of projectiles launched at an angle (like some heavy ground artillery). Finally, you'll use what you've learned to create a model of any projectile motion.

3.1 Comparing Projectiles

This video was mentioned at the end of the kinematics chapter: `https://www.youtube.com/watch?v=zMF4CD7i3hg`. From the video, we can see that the addition of horizontal motion does not effect how fast an object is acted upon by gravity. Both objects hit the ground at the same time, regardless of whether horizontal motion was added or not. This is because of a concept called *independence of motion*.

3.2 Independence of Motion

In projectile motion, such as a ball being thrown off of a cliff, the horizontal and vertical components of motion are independent of each other. This means that horizontal motion (and forces in the horizontal direction) do not affect vertical motion (and forces in the vertical direction), and vice versa. This is because gravity only acts in the vertical direction.[1]

[1] This holds true for objects on sloped surfaces. Gravitational motion is just separated into components parallel and perpendicular to the slope. This will be covered in a future chapter.

Horizontal Motion	Vertical Motion
$\Delta x = v_{0x} t$ $v_x = v_{0x}$ (constant!) $a_x = 0$	$\Delta y = v_{0y} t + \frac{1}{2}(-g)t^2$ $v_y = v_{0y} + (-g)t$ $a_y = -g$

Figure 3.1: Projectile motion equations from kinematics equations.

If you recall the falling bodies chapter, you know that the vertical motion of an object in free fall is described by the equations of unformly accelerated motion, with a constant acceleration of $9.8\frac{m}{s^2}$ downward (this may simplified to $10\frac{m}{s^2}$ for simplicity in many calculations).

In the horizontal direction, if we ignore air resistance (a common practice for elementary physics), there are no forces acting on the object. This means that the object will continue to move at a constant velocity in the horizontal direction. Note that from launch to landing, the time must be the same for both components. Take a minute to think about why this is if you are unsure.

Because the horizontal and vertical motions are independent of each other, we can use different equations to describe the motion in each direction (See Figure 3.1). Note that the velocity in the horizontal direction is ***constant***, while the velocity in the vertical direction is ***not constant*** due to the acceleration of gravity ($9.8\frac{m}{s^2}$ downward).

Let's take a look at some simple graphs comparing x motions and y motions, shown in Figures 3.2 and 3.3. Note that these are a general representation of projectile motion, and the scales or values may not be accurate.

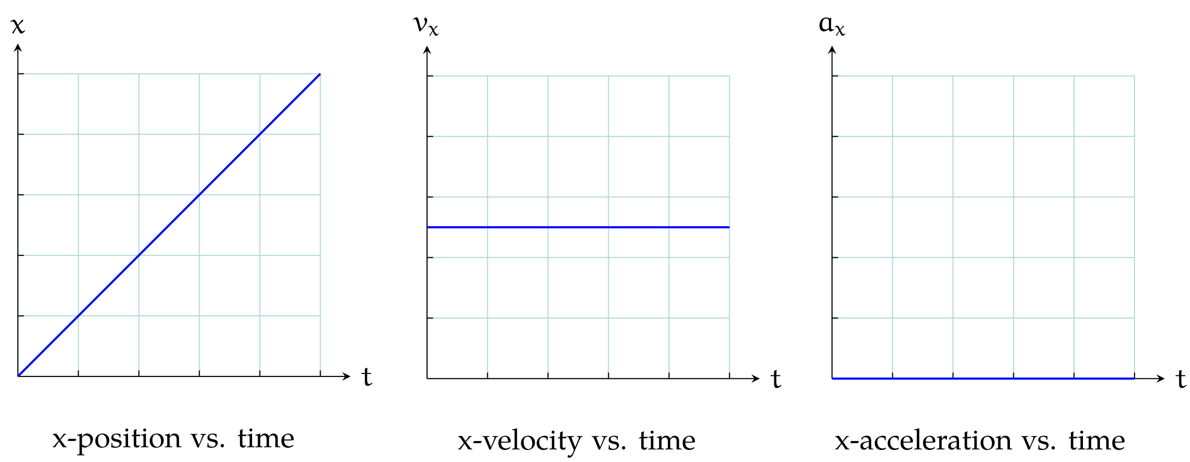


Figure 3.2: Vertical motion graphs

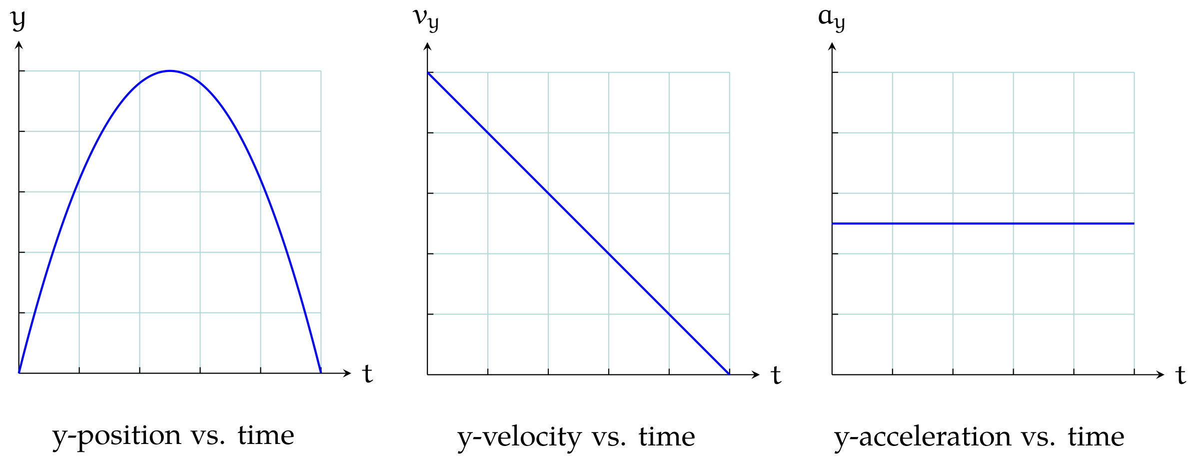


Figure 3.3: Vertical motion graphs

3.3 Horizontally-launched Projectiles

Imagine a ball is thrown horizontally off of a cliff. The ball has an initial horizontal velocity of 20 m/s, but no initial vertical velocity. The ball will continue to move horizontally at a constant velocity, while simultaneously accelerating downward due to gravity. The ball falls for 5 seconds before hitting the ground. We will use $-10\frac{m}{s^2}$ for the acceleration due to gravity to make calculations easier. See figure 3.4.

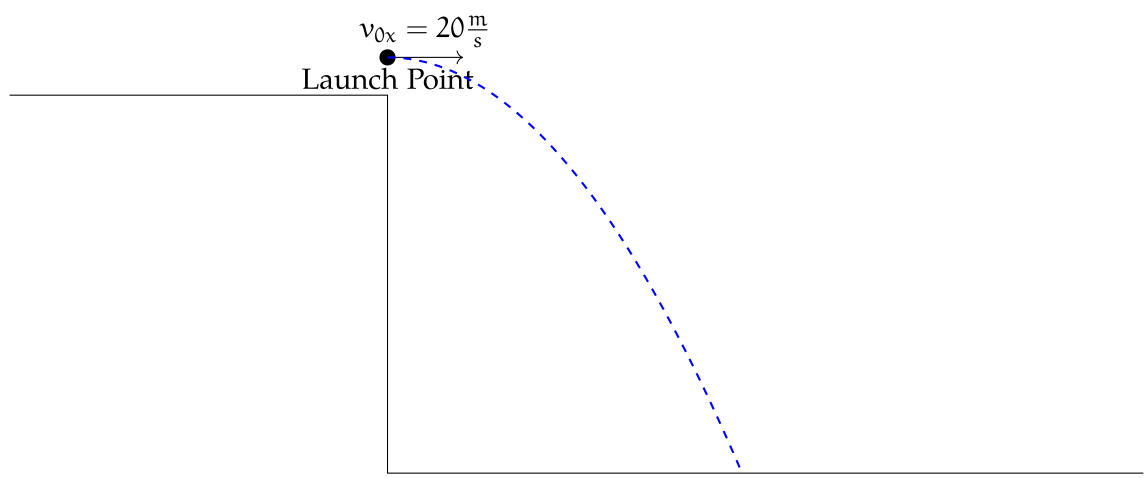


Figure 3.4: Diagram of a ball being thrown horizontally off of a cliff. Note: figure is not to scale.

Let's calculate the change in y.

The vertical motion can be described by the equation:

$$\Delta y = v_{0y}t + \frac{1}{2}(-g)t^2$$

Since the initial vertical velocity $v_{0y} = 0$, this simplifies to:

$$\Delta y = \frac{1}{2}(-g)t^2$$

Substituting in the values for g and t:

$$\Delta y = \frac{1}{2}(-10\frac{m}{s^2})(5s)^2$$

Calculating this gives:

$$\Delta y = -125m$$

So the ball falls a vertical distance of 122.5 m from its origin before hitting the ground.

Now let's calculate the change in x. The horizontal motion can be described by the equation:

$$\Delta x = v_{0x}t$$

Substituting in the values for v_{0x} and t:

$$\Delta x = (20\frac{m}{s})(5s) = 100m$$

Exercise 3 Horizontally-launched Projectile

A rock is thrown horizontally off of the edge of a cliff with an initial velocity of 8 m/s. If the rock falls for 3 seconds before hitting the ground, (a) how far from the base of the cliff does the rock land? (b) How high is the cliff?

Use $-10\frac{m}{s^2}$ for the acceleration due to gravity.

Working Space

Answer on Page 56

3.3.1 Newton's Cannon and Escape Velocity

A real world application of horizontally launched projectiles is a thought experiment known as Newton's cannon (sometimes called Newton's cannonball). Newton theorized that a cannon on the top of a mountain above Earth's atmosphere could launch a cannonball at some high velocity could do one of three things:

- The cannonball would not have enough horizontal velocity, and fall back to the Earth.
- The cannonball would get shot at enough velocity to be in continuous orbit around the Earth. (This is how satellites stay in orbit around the Earth!).
- The cannonball would get shot at such a high velocity that it would escape the Earth's gravitational pull, and fly off into space. This is known as *escape velocity*.

Take a look at this simulation of Newton's cannon: `https://physics.weber.edu/schroeder/software/NewtonsCannon.html`. Then, watch this video: `https://www.youtube.com/watch?v=ALRdYPMpqQs`.

At anywhere below 7000 m/s, the cannonball falls back to Earth. In a range of $7,000$ m/s to $8,000$ m/s (the simulation doesn't go above $8,000$ m/s), the cannonball enters an orbit around the Earth. For Earth, the escape velocity is $\approx 11,200$ m/s.

For any given planet, the escape velocity can be calculated using the formula:

$$v_e = \sqrt{\frac{2GM}{d}}$$

where G is the gravitational constant, M is the mass of the planet, and d is the distance from the center of the planet to the object. We will cover this in more detail in the orbits chapter.

3.4 Projectiles launched at an Angle

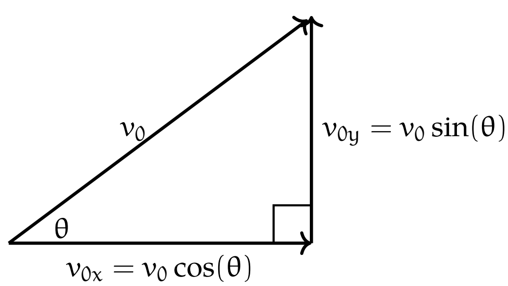
3.4.1 From the Ground

Now, let's imagine a ball is launched from the ground at at angle. The angle of launch influences both horizontal and vertical components of the initial velocity. We can use trigonometry to separate the initial velocity into its horizontal and vertical components.

$$v_{0x} = v_0 \cos(\theta) \qquad v_{0y} = v_0 \sin(\theta)$$

In the previous section, when an object is thrown directly horizontally, $\theta = 0°$, so $v_{0x} = v_0$. However, we now have v_0 being separated into two components, so the v_{0x} will be less than v_0, as $\cos(\theta)$ and $\sin(\theta)$ are always less than or equal to 1.

The vertical component of v_0 is the initial velocity, v_{0y}. This component, unlike v_{0x}, is constantly changing due to gravity, at a rate of $-9.8\frac{m}{s^2}$. The peak of an object being launched at some angle is when v_{0y} is equal to 0.

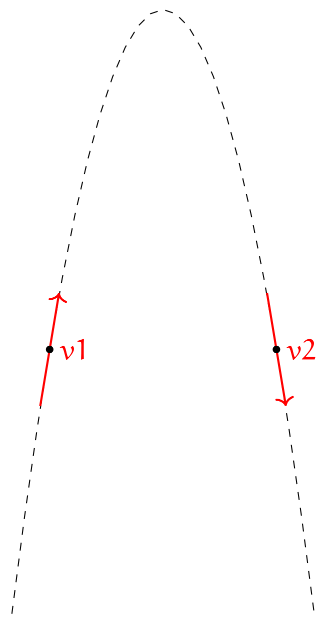


For example, take a watermelon launched at $20\frac{m}{s}$ at an angle of 30° above the horizontal. What would the horizontal and vertical components of the initial velocity be?

Well, we can use the equations above to find out:

$$v_{0x} = (20)\cos(30°) \approx 17.32 \text{ m/s}, \qquad v_{0y} = (20)\sin(30°) = 10 \text{ m/s}$$

It is important to note that the upwards velocity and downwards velocity are equal in magnitude at any given height (given the same vertical initial and final height), but *opposite in direction*, due to the symmetric nature of projectile motion.



$$\mathbf{v_1} = -\mathbf{v_2}$$

One last formula of note is the range formula, describing the horizontal range of any object launched at an angle on *level ground*:

Range Formula

$$R = \frac{v_0^2 \sin(2\theta)}{g} \tag{3.1}$$

where the object starts and lands at the same vertical height.

Exercise 4 Projectile Motion at an angle

A pumpkin is launched at 25 m/s at an angle of 40° above the horizontal. Find (a) the peak height of the pumpkin, and (b) the horizontal distance the pumpkin travels before hitting the ground. Use $-10\frac{m}{s^2}$ for the acceleration due to gravity. You may assume the ground is level the entire time.

Working Space

Answer on Page 56

Exercise 5 Launch Angle for Maximum Range

(a) At what angle should you launch for an object to go the furthest given a maximum launch velocity? (b) How can you get the same horizontal distance with two different launch angles?

Working Space

Answer on Page 57

Exercise 6 Projectile Motion at an angle

There is a target 100 meters away. I must shoot a bow at 14 meters per second. At what angle will I be able to hit my target? If I cannot hit the target, calculate the velocity neeeded to reach the target. Use $-10\frac{m}{s^2}$ for the acceleration due to gravity. You may assume the ground is level the entire time.

Working Space

Answer on Page 58

3.4.2 From a Height Above the Ground

Now we can imagine a ball being launched at some angle from a height above the ground. This is similar to the horizontally launched projectile, except now we have an initial vertical velocity component.

Let's say a ball is launched from a height of 50 m at an angle of 30° above the horizontal with an initial velocity of 20 m/s. We can find the horizontal and vertical components of the initial velocity as follows:

$$v_{0x} = (20)\cos(30^\circ) \approx 17.32 \text{ m/s}, \qquad v_{0y} = (20)\sin(30^\circ) = 10 \text{ m/s}.$$

The vertical motion can be described by the equation:

$$\Delta y = y_0 + v_{0y}t + \frac{1}{2}(-g)t^2 = 50 + 10t - 5t^2$$

Likewise, the horizontal motion of the projectile can be described by the equation:

$$\Delta x = v_{0x}t = 17.32t$$

To find the peak of the projectile, we can set $v_y = 0$ and solve for t:

$$v_y = v_{0y} + (-g)t$$
$$0 = 10 - 5t^2$$
$$t_{peak} = \sqrt{2} \text{ s} \approx 1.414 \text{ s}$$

Substituting this time into the vertical and horizontal equations, we can find the peak height and horizontal distance at the peak:

$$\Delta y = 50 + 10(\sqrt{2}) - 5(\sqrt{2})^2 = 50 + 10\sqrt{2} - 10 = 40 + 10\sqrt{2} \approx 54.14 \text{ m}$$

$$\Delta x = 17.32(\sqrt{2}) \approx 24.49 \text{ m}$$

And to find the total time of flight, we can set $\Delta y = 0$ and solve for t:

$$0 = 50 + 10t - 5t^2$$
$$0 = -5t^2 + 10t + 50$$
$$0 = t^2 - 2t - 10$$

Using the quadratic formula, we find:

$$t = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-10)}}{2(1)}$$
$$= \frac{2 \pm \sqrt{4 + 40}}{2}$$
$$= \frac{2 \pm \sqrt{44}}{2}$$
$$= 1 \pm \sqrt{11} \approx 4.32 \text{ s (taking the positive root)}$$

3.5 Simulating Projectile Motion

Chapter 4

Free Body Diagrams

Now that you've mastered modeling *how things move*, you can learn to model *why things move*. Recall Newton's Second Law:

$$F = ma$$

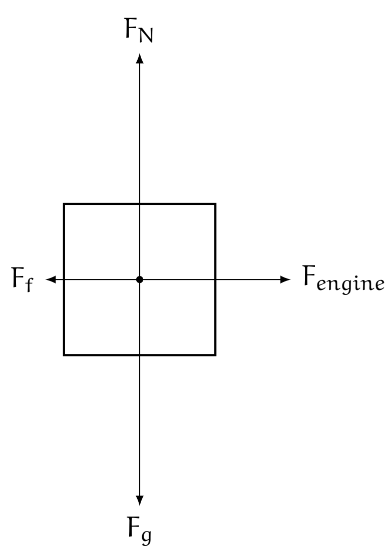
This is a simplification: what Newton's Second Law says in total is "the acceleration of a body is directly proportional to the *net force* acting on the body and inversely proportional to the mass of the body." Mathematically, that would be:

$$a = \frac{\Sigma F}{m}$$

Where ΣF is the vector sum of all the individual forces acting on the body. This sum is also called the *net force*. To visualize the magnitude and direction of the net force, it can be very helpful to draw a free body diagram.

4.1 Interpreting Free Body Diagrams

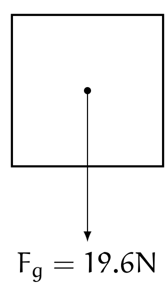
Before you learn to draw your own free body diagrams (FBDs), let's examine and analyze a few. Here is a FBD for an accelerating car:



We can consider the x and y axes separately. On the y-axis, we see that the weight (F_g) is the same magnitude as the normal force (F_N) because the arrows are the same length. This means in the y direction, the forces are balanced, and thus we do not expect to see acceleration in the y direction. This makes intuitive sense: if you are accelerating your car on a flat road, your car does not go up into the air. How about the x direction? Here we see friction (F_f) pulling the car backwards while the engine is pushing the car forwards. In this case, the forces are *not* balanced. See how the F_{engine} arrow is longer than the F_f arrow? This means the forward force from the engine is *greater in magnitude* than the backward force of F_f. Therefore, the *net force* in the x direction is to the right, and the car will accelerate to the right.

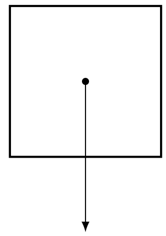
4.2 Drawing Free Body Diagrams

A free body diagram consists of the object, usually represented as a simple square, with various arrows representing the forces acting on the object. The direction and relative lengths of the arrows show the direction and relative strength of the forces. Here is a free body diagram for a 2-kg hammer in free fall through the air on Earth:



4.2.1 Objects At Rest

If an object is at rest, then its acceleration must be $0\frac{m}{s^2}$. By Newton's Second Law, the *net force* acting on that object must also be zero. Does this mean there are no forces acting on the object? NO! It means all the forces are *balanced*: for every upwards force, there is an equal downwards force, and so on. Consider the same 2-kg hammer, but this time it is sitting on a table. We know gravity must be acting on the hammer, so let's begin with that:

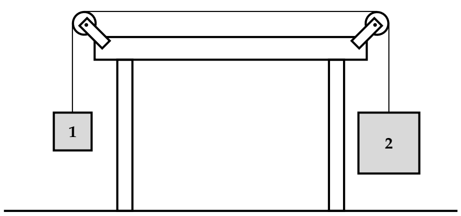


$F_g = 19.6N$

Exercise 7 Blocks on a Table

Working Space

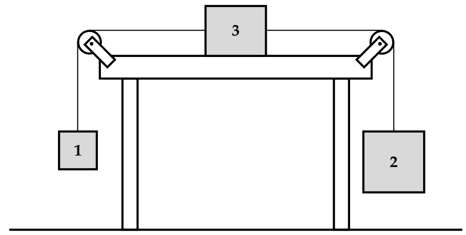
[This exercise was originally presented as a free-response question on the 2015 AP Physics 1 exam.] Two blocks are connected with a string and two pulleys over a table, as shown in the figure. Block 1 has a mass of 3 kg and Block 2 has a mass of 9 kg. The blocks are released from rest. Treat the string as massless and the pulleys as massless and frictionless when answering the questions below.



1. Draw free body diagrams for both blocks. Correctly show the relative magnitude of the forces using the relative lengths of the vectors.
2. Before doing any calculations, describe each block's acceleration as (a) positive or negative, (b) having a magnitude greater or less than g. Take the upward direction as positive.
3. Calculate the acceleration of each block. Take the upward direction as positive and express your answer in terms of g.

Answer on Page 59

Now here's an interesting question: what happens if we add another mass to the system? Imagine that a third block with a mass of 15 kg is laid on the table from the previous exercise and the string is attached to each side, as shown below.



We could redraw free body diagrams for all three blocks, noting that the tensions on either side of Block 3 will be different. Then, we would have another system of equations to solve and we could find the acceleration. Or, we could think about the entire system. Originally, the net force on the two-block system was 58.8 Newtons (the weight of the larger block minus the weight of the smaller block). And the mass of the entire two-block system was 12 kilograms. We can apply Newton's Second Law:

$$a = \frac{58.8\text{N}}{12\text{kg}} = 4.9\frac{\text{m}}{\text{s}^2} = \frac{1}{2}g$$

(Notice, this is the same answer we found in the previous exercise). What changes when we add a third, 15-kg block? The net force on the system doesn't change: the table pushes back up on the third block with a force equal to the block's weight. However, the total mass of the system has changed: it's increased. Now the total mass is 27 kg, and we can again apply Newton's Second Law:

$$a = \frac{58.8\text{N}}{27\text{kg}} \approx 2.18\frac{\text{m}}{\text{s}^2}$$

This answer makes sense: the mass of the system has increased while the net force acting on the system has stayed the same, resulting in a slower acceleration.

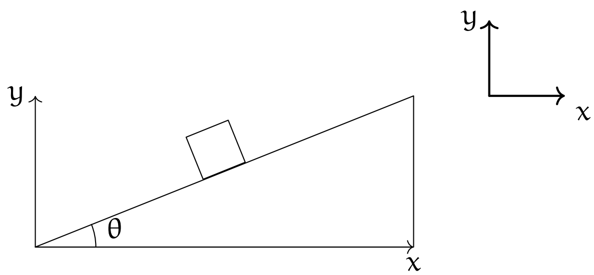
Now, what if instead of blocks resting on level tables, the blocks were on an incline? Let's take a look at this and more FBD's in the next chapter.

Chapter 5

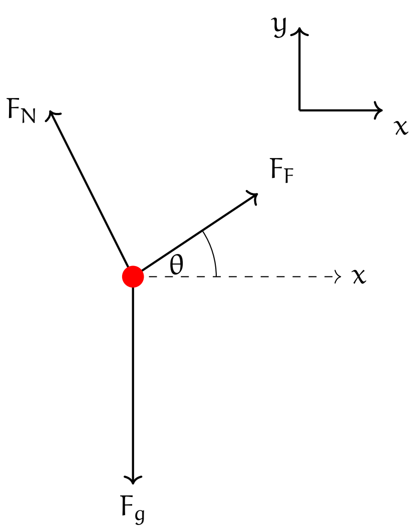
Inclined planes

Take a look at this box resting on an incline plane due to static friction and gravity.

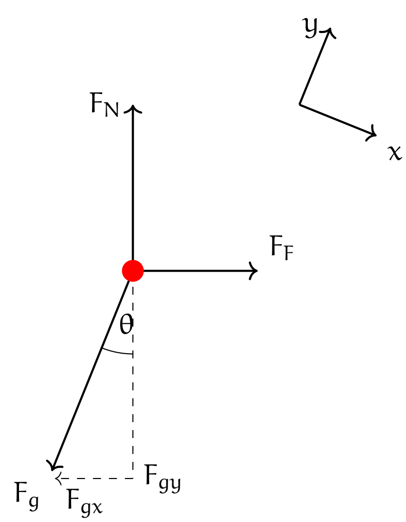
Figure 5.1: A box on an inclined plane with $a = 0$ due to static friction and gravity.



Now for objects on an incline, such as a box on a ramp, we will want to tip our coordinate system. Why you ask? Because we are not restricted to having our y-axis 90° vertically and our x-axis flat horizontally. In this case we can tip our axes so that the x-axis is parallel to the incline and the y-axis is perpendicular to the incline, aligning with the normal force. This will make our calculations much easier.



(a) Free body diagram in standard orientation.



(b) Axes rotated by θ.

Figure 5.2: Free body diagrams on an incline.

From this new perspective, we can see that the gravitational force F_g can be broken down into two components: one parallel to the incline (F_{gx})[1] and one perpendicular to the incline (F_{gy}). The normal force F_N acts perpendicular to the surface of the incline, while the frictional force F_F acts parallel to the surface, opposing motion. The coordinate planes in the corner are included to show the original coordinate system relative to our rotated FBD.[2]

We can set up the following equations based on Newton's second law:

$$\sum F_x = F_{gx} - F_F = ma_x = 0, \qquad \sum F_y = F_N - F_{gy} = ma_y = 0$$

Using the trigonometric components, we can express the forces in terms of the angle θ:

$$\sum F_x = mg\sin(\theta) - F_F = ma_x = 0, \qquad \sum F_y = F_N - mg\cos(\theta) = ma_y = 0$$

Solving for the normal force and frictional force, we find:

$$F_N = mg\cos(\theta), \qquad F_F = mg\sin(\theta)$$

The following relationships are true for acceleration on an incline:

$$F_F = \mu F_N, \qquad a_x = g(\sin\theta - \mu\cos\theta)$$

Where μ is the coefficient of friction between the box and the incline.

Let's apply all of this to an example problem:

[1] Note that some professors will use down the incline as positive. This book will consistently use up the incline as positive as rotating coordinate plane aligns well with this.

[2] Note also that the dashed line components are not actual forces, they are drawn in for illustration of how forces at angles effect the standard forces.

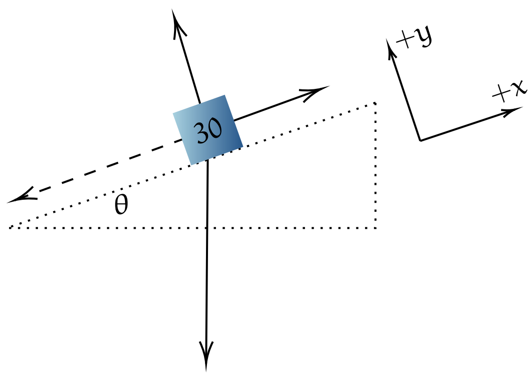
Exercise 8 Box sliding down an incline

A 30 kg block is sliding down an incline plane that makes a 30° with the horizontal. Given that the coefficient of kinetic friction is 0.3, find the following. Note that up the inclined plane is considered the positive x-direction.

(a) The normal force acting on the block.

(b) The frictional force acting on the block.

(c) The acceleration of the block down the incline.



Working Space

Answer on Page 59

Exercise 9 Solving for angle θ

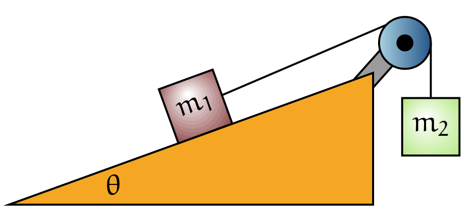
A 40 kg box is sitting on a rough inclined plane, held in place by static friction. The coefficient of static friction μ_s between the box and the incline is found to be 0.464. What is the maximum angle θ that the incline can be tilted to before the box starts to slide down?

Working Space

Answer on Page 60

Exercise 10 Pulleys and Inclines

A block at rest of mass m_1 is resting on a rough inclined plane making an angle θ with the horizontal. The coefficient of kinetic friction between the block and the incline is μ_k. The block is attached by a light string over a frictionless pulley to a hanging block of mass m_2. Once they are connected by a string, the hanging block begins to descend and the block of m_1 moves upwards along the incline. Draw the free body diagrams of each mass. Derive an equation for the acceleration of m_2, only in terms of m_1, m_2, g, θ, and μ_k.



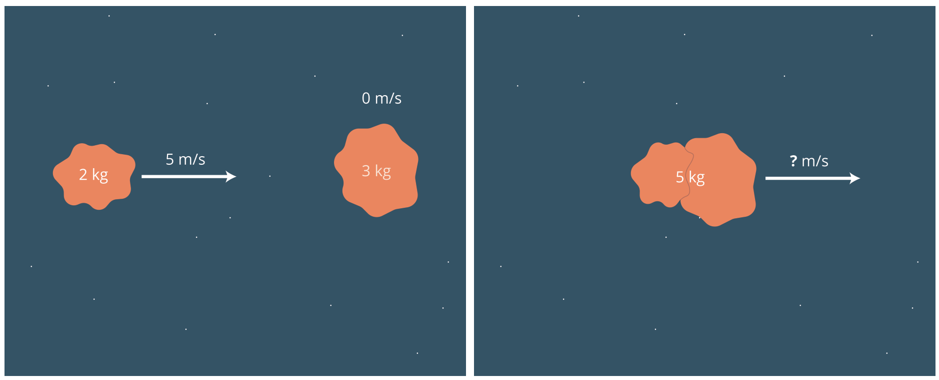
Working Space

Answer on Page 61

Chapter 6

Momentum

Let's say a 2 kg block of putty is flying through space at 5 meters per second, and it collides with a larger 3 kg block of putty that is not moving at all. When the two blocks deform and stick to each other, how fast will the resultant big block be moving?



Formula for Momentum

Every object has *momentum*. The momentum is a vector quantity — it points in the direction that the object is moving and has a magnitude equal to its mass times its speed.

$$p = mv \tag{6.1}$$

Given a set of objects that are interacting, we can sum all their momentum vectors to get the total momenta. In such a set, the total momentum will stay constant.

$$p = p_1 + p_2 + \cdots + p_n = m_1 v_1 + m_2 v_2 + \cdots + m_n v_n \tag{6.2}$$

In our example, one object has a momentum vector of magnitude of 10 kg m/s, the other has a momentum of magnitude 0. Once they have merged, they have a combined mass of 5 kg. This means the velocity vector must have magnitude 2 m/s and pointing in the same direction that the first mass was moving.

Conservation of Momentum

The total momentum of a system remains constant as long as no external forces act on it. In a collision or interaction, the momentum before the event must equal the momentum after the event:

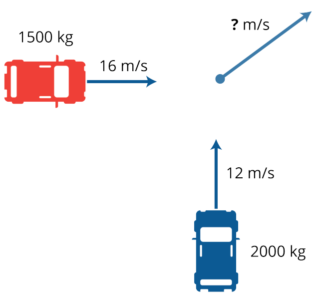
$$p_{\text{initial}} = p_{\text{final}} \tag{6.3}$$

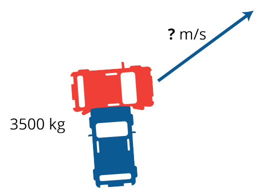
This applies whether the collision is elastic, inelastic, or perfectly inelastic. Even if kinetic energy is lost (as heat, sound, or deformation), momentum is still conserved because it depends only on mass and velocity, not on energy.

Exercise 11 Cars on Ice

A car weighing 1000 kg is going north at 12 m/s. Another car weighing 1500 kg is going east at 16 m/s. They both hit a patch of ice (with zero friction) and collide. Steel is bent, and the two objects become one. How what is the velocity vector (direction and magnitude) of the new object sliding across the ice?

Working Space





Answer on Page 61

Note that kinetic energy ($1/2mv^2$) is *not* conserved here. Before the collision, the moving putty block has $(1/2)(2)(5^2) = 25$ joules of kinetic energy. Afterward, the big block has $(1/2)(5)(2^2) = 10$ joules of kinetic energy. What happened to the energy that was lost? It was used up deforming the putty.

What if the blocks were marble instead of putty? Then there would be very little deforming, so kinetic energy *and* momentum would be conserved. The two blocks would end up having different velocity vectors.

Let's assume for a moment that they strike each other straight on, so there is motion in only one direction, both before and after the collision. Can we solve for the speeds of the first block (v_1) and the second block (v_2)?

We end up with two equations. Conservation of momentum says:

$$2v_1 + 3v_2 = 10$$

Conservation of kinetic energy says:

$$(1/2)(2)(v_1^2) + (1/2)(3)(v_2^2) = 25$$

Using the first equation, we can solve for v_1 in terms of v_2:

$$v_1 = \frac{10 - 3v_2}{2}$$

Substituting this into the second equation, we get:

$$\left(\frac{10 - 3v_2}{2}\right)^2 + \frac{3v_2^2}{2} = 25$$

Simplifying, we get:

$$v_2^2 - 4v_2 + 0 = 0$$

This quadratic has two solutions: $v_2 = 0$ and $v_2 = 4$. $v_2 = 0$ represents the situation before the collision. Substituting in $v_2 = 4$:

$$v_1 = \frac{10 - 3(4)}{2} = -1$$

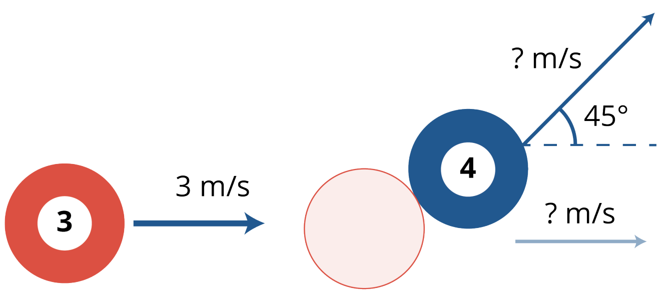
Thus, if the blocks are hard enough that kinetic energy is conserved, after the collision, the smaller block will be heading in the opposite direction at 1 m/s. The larger block will be moving at 4 m/s in the direction of the original motion.

Exercise 12 Billiard Balls

Working Space

A billiard ball weighing 0.4 kg and traveling at 3 m/s hits a billiard ball (same weight) at rest. It strikes obliquely (neither perpendicular nor parallel), so that the ball at rest starts to move at a 45 degree angle from the path of the ball that hit it.

Assuming all kinetic energy is conserved, what is the velocity vector of each ball after the collision?



Answer on Page 62

6.1 Impulse

We can talk about a *change in Momentum* as what we refer to as *Impulse*. When an object has a change in momentum, it is said to have been given an Impulse. Since momentum is a vector quantity, impulse is as well.

Formula for Impulse

Impulse, **J** said to be the change in momentum, is given by:

$$\mathbf{J} = \Delta p \tag{6.4}$$

Equivallently, it can be given as the following equation, in terms of force:

$$\mathbf{J} = F\Delta t \tag{6.5}$$

If the force varies with time, we use integration to find the impulse:

$$\mathbf{J} = \int_{t_0}^{t_1} F(t)dt \tag{6.6}$$

Both Equations (6.5) and (6.6) are referred to as the **Impulse-Momentum Theorem**.

By the impulse momentum theorem, a large force for a short time *or* a small force for a long time can produce the same impulse.

6.1.1 Golf Swings

The best example of impulse in action is a golf swing. Let's analyse this using equations:

The force of your swing will theoretically always be the same, as the maximum force you can apply is limited by human ability. You cannot change the human-provided force, however, you can increase the *contact time* of your club on the ball.

$$\mathbf{J} = F\Delta t$$

Since J is equivalent to Δp, Δt, the contact time of the swing, has a proportional relationship with the momentum of the ball, which starts of as 0. Thus, it can be said:

$$\Delta p \propto \Delta t$$

And, holding F, m, and assuming the golf ball is initially at rest, $p_i = 0$, we can say

$$v_f \propto \Delta t$$

So the longer the contact time of a golf swing (or any swing-based sport, really), the greater the velocity.

6.2 Collisions

When two (or more) objects collide, we can classify their collision as one of three main categories. These classifications tell us when we can apply the conservation of momentum, the consevation of kinetic energy, both, or neither. Momentum is only conserved when the system is isolated, meaning no external forces act on the system.

6.2.1 Elastic Collision

Recall our billiard ball problem from Exercise 12. That problem provides both balls with velocity after the collision. In an ideal and more realistic billiards scenario, one ball transfers all of its velocity onto the other ball (in this case, the red ball stops and the blue continues with close to same velocity that the red ball had initially), as seen in Figure 6.1 This would be a *elastic collision*.

In an elastic collision, *both momentum and kinetic energy are conserved*. There is minimal to zero loss of energy in the collision. Although no collision is ever *truly* elastic (due to existing but miminmal deformation of objects, sound, a transfer of heat through molecular changes, etc), we can think of a collision like this as perfectly elastic. The sound of billiard balls colliding is obsolete (especially compared to a car crash), so very little energy is lost to sound.

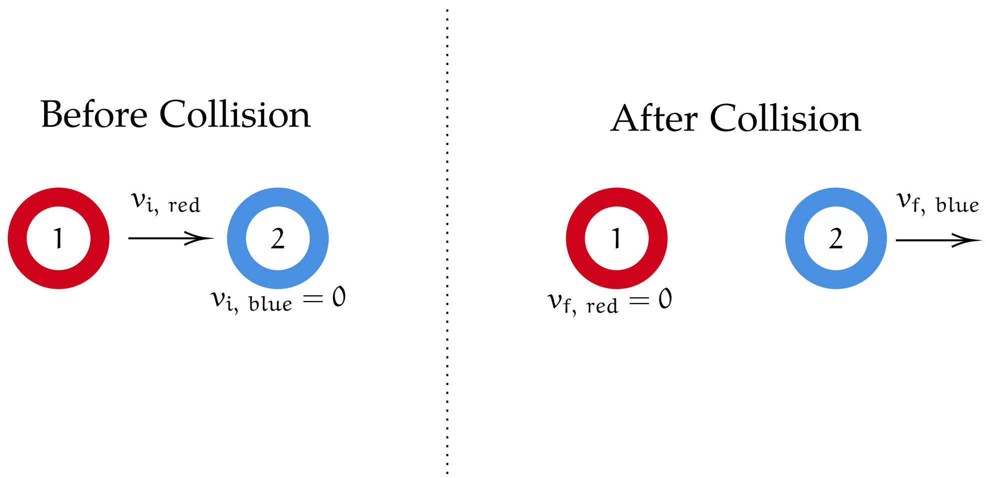


Figure 6.1: Example of an elastic biliard collision.

Another example to consider here is Newton's Cradle. There are two videos demonstrating momentum in Newton's Cradle on your digital resources.

In equation format, Elastic Collisions are represented by the sums of momentum and kinetic energy before and after the collision being equal:

$$\text{total } \mathbf{p}_{\text{before collision}} = \text{total } \mathbf{p}_{\text{after collision}} \implies m_1 v_{i,1} + m_2 v_{i,2} + \cdots = m_1 v_{f,1} + m_2 v_{f,2} + \cdots$$

and

$$\text{total } \mathbf{KE}_{\text{before collision}} = \text{total } \mathbf{KE}_{\text{after collision}}$$

6.2.2 Inelastic Collisions

Inelastic collisions, then, are ones in which the total kinetic energy after the collision is *different* than before the collision, in other words, there is a change in kinetic energy, usually lost due to friction, heat, or deformation.

However, momentum *is* conserved, meaning we can apply conservation of momentum principles.

$$\text{total } \mathbf{p}_{\text{before collision}} = \text{total } \mathbf{p}_{\text{after collision}} \implies m_1 v_{i,1} + m_2 v_{i,2} + \cdots = m_1 v_{f,1} + m_2 v_{f,2} + \cdots$$

A good example of an elastic collision is dropping a basketball on a floor from an initial height. It will never return to its initial height after "colliding" with the floor, as kinetic energy is 'lost' to sound and deformation of the ball. The missing height can be equated to the lost energy. See Figure 6.2

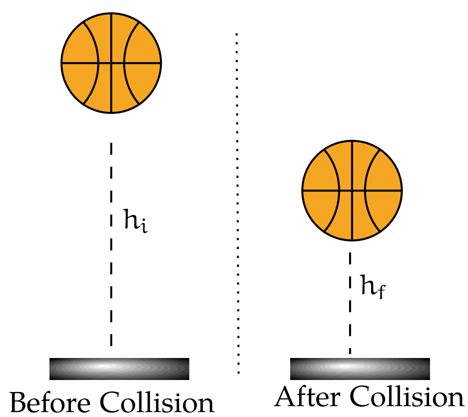


Figure 6.2: An example of a (partially) inelastic collision between the floor and a basketball.

6.2.3 Perfectly Inelastic

A collision is *perfectly inelastic* when the masses stick together after the collision.In this kind of collision, the *maximum* kinetic energy is lost (crumpling, bending, embedding of objects) and share the same final velocity.

The most common example of a perfectly inelastic collision is a car crash, especially high speed T-Bones. In this case, a car at a high speed collides with a car at a lower speed, resulting in a combined final speed as the cars interact on a molecular level and a large portion of energy is lost.

Exercise 13 Two Carts

Two carts on a frictionless track collide elastically.

Cart A has mass 2.0 kg and is moving to the right at 3.0 m/s.

Cart B has mass 1.0 kg and is initially at rest.

They collide head-on elastically.

Find the final velocities of both carts. Hint: You may have to do a few equation substitutions

Working Space

Answer on Page 63

Exercise 14 Car Crash

Sometimes, car companies will intentionally crash cars in order to test the safety of their prototypes.

Two cars collide head-on on a straight, frictionless test track, where east is considered positive.

Car A has mass 1000 kg and is traveling *east* at 20 m/s. Car B has mass 1500 kg and is traveling *west* at 10 m/s.

During the collision, the cars do not stick together. After the collision, Car A is observed to move east at 5 m/s.

Find the final velocity of Car B. You are told kinetic energy is not conserved. Is this an elastic or inelastic collision?

Working Space

Answer on Page 65

Exercise 15 Ready, Aim, Fire!

A $m_{bullet} = 10$ g bullet is fired horizontally at 500 m/s into a wooden block of mass $m_{block} = 2.00$ kg resting on a frictionless horizontal surface. The bullet embeds itself in the block, such that a loud bang was heard and the *block-bullet system* slides together after the collision.

(a) What is the velocity of the *block-bullet system* after the collision?

(b) How much kinetic energy was lost in the collision?

(c) A collision is considered *perfectly inelastic* if more than 95% of the initial energy is lost. Classify the *block-bullet system* collision by first finding the percent of energy lost.

Working Space

Answer on Page 65

6.3 Momentum, Center of Mass, and Explosions

We have previously talked about finding the center of mass of an object (or objects). Now that we know about momentum, we can think about the center of mass in a different way.

Explosions typically involve an object breaking into pieces with internal forces (the forces of the explosion acting between fragments). The key is:

Internal forces cannot change the motion of the center of mass. This means that even during a violent explosion, the center of mass of the system continues moving exactly as it would if no explosion occurred. How does this work? It involves finding the sum of all momenta of the object.

Before and after the explosion: Total external force on the system is the only thing that can change center of mass (COM) motion.

Internal forces (the explosion pushing fragments apart) cancel out due to Newton's Third Law.

Therefore, momentum of the whole system stays the same (as long as there is no external *impulse*).

Let's say a projectile of mass M is moving, then explodes into two pieces with masses m_1 and m_2. Even if the fragments fly off in different directions with different speeds, the COM follows:

- the same trajectory,
- at the same velocity,
- as the original mass would have had if it had not exploded.

What can you do with this information?

- Find missing fragment velocities
- Find angles or directions after explosions
- Track motion of COM even when individual parts are complicated

A classic example of this is a mass following projectile motion. The mass splits into pieces at the peak of its motion. Even if pieces fly backwards, upwards, or along the same path, the COM stays on the original path.

If before the explosion the object moves with velocity v then the COM velocity after explosion is still:

$$\vec{v}_{\text{COM}} = \vec{v}_{\text{before}}$$

After explosion:

$$M\vec{v}_{\text{COM}} = m_1\vec{v}_1 + m_2\vec{v}_2 + \dots$$

You use this to solve unknown fragment velocities or angles.

Exercise 16 Three-Fragment Projectile Explosion

Working Space

A projectile of mass 6.0 kg is launched from the ground and follows a parabolic arc. At the very top of its trajectory, its speed is measured to be 20 m/s, moving horizontally.

At that instant, it explodes into three fragments:

- Fragment A has mass 2.0 kg and flies off at 40° above the horizontal with a speed of 30 m/s.
- Fragment B has mass 1.0 kg and flies off at 60° below the horizontal with a speed of 25 m/s.
- Fragment C has mass 3.0 kg, and its direction is unknown, but it is known to move *somewhere* in the horizontal plane after the explosion.

Gravity acts only after the explosion. During the explosion itself, assume there are no external horizontal forces.

Find the **magnitude and direction** of the velocity of Fragment C.

Answer on Page 66

This chapter has talked about Momentum, Impulse, Types of Collisions, and Explosions, all of which are intertwined topics in Newtonian Physics. Next, we will be talking about physics behind vehicles such as planes, quadcopters, helicopters, and boats.

Appendix A

Answers to Exercises

Answer to Exercise 1 (on page 5)

At it's apex, the hammer's velocity is $0m/s$, so we will find the time where $v = 0m/s$:

$$0\frac{m}{s} = 12\frac{m}{s} - 9.8\frac{m}{s^2} \cdot t$$

$$t = \frac{12\frac{m}{s}}{9.8\frac{m}{s^2}} \approx 1.22s$$

Answer to Exercise 2 (on page 20)

1. D, B/C/F, A, E; displacement is the difference in position between the starting and ending points. Object D moves backwards and therefore has a negative displacement. Objects B, C, and F all end in the same position they started and therefore have zero displacement. Objects A and E both move forward and have positive displacement. Object E moves forward 4 units while Object A only moves forward by 1 unit. Therefore, object E has a greater positive displacement than object A.

2. C, D, B/F, A, E; instantaneous velocity is given by the slope of a position-time graph at the indicated time (in this case, $t = 7.5s$). Since the graphs show 10 seconds of motion and there are 4 tick marks on each x-axis, each unit on the x-axis represents 2.5 seconds of time. Therefore, $t = 7.5s$ is the third tick mark on the x-axis. At $t = 7.5s$, the graphs of objects C and D have negative slopes and therefore negative velocities. Since the slope for object C is steeper than the slope for object D, object C's *speed* is greater than object D's, and object C's *velocity* is more negative than object D's. The graphs of object's B and F are horizontal at $t = 7.5s$ and therefore their velocities are zero. Graphs A and E have positive slopes, and since E is steeper, object E has a greater speed and more positive velocity than object A.

3. B/F, A, D, E, C; distance is a scalar and therefore always positive. B and F do not change position, and therefore travel a distance of 0. A moves forward 1 unit. D moves backwards 3 units (for a *distance* of 3). E moves forward 4 units. C moves forwards 3 units, then backwards 3 additional units, for a total of 6 units of distance.

Answer to Exercise 3 (on page 27)

(a) To find how far from the base of the cliff the rock lands, we can use the horizontal motion equation:

$$\Delta x = v_{0x} t$$
$$\Delta x = (8\tfrac{m}{s})(3s) = 24m$$

(b) To find the height of the cliff, we can use the vertical motion equation. Since $v_{0y} = 0$, and the initial vertical position is at the top of the cliff (we can call this $y_0 = 0$), we have:

$$y_f = v_{0y} t + \frac{1}{2}(-g)t^2$$
$$y_f = 0 + \frac{1}{2}(-10)(3)^2$$
$$y_f = -45m$$

We get a negative value for y_f because the rock is falling downward from its initial position, the top of the cliff which we selected as the origin. Therefore, the height of the cliff is 45 meters.

So the rock lands 24 meters from the base of the cliff, and the cliff is 45 meters high.

Answer to Exercise 4 (on page 30)

Let's first find the horizontal and vertical components of the initial velocity:

$$v_{0x} = (25)\cos(40^\circ) \approx 19.15 \text{ m/s},$$
$$v_{0y} = (25)\sin(40^\circ) \approx 16.07 \text{ m/s}.$$

(a) To find the peak height, we can use the vertical motion equations. Taking it in two steps, let's first find the time to reach the peak height, where the vertical velocity $v_y = 0$:

$$v_y = v_{0y} + (-g)t$$
$$0 = 16.07 - 10t$$
$$t_{peak} = \frac{16.07}{10} \approx 1.607 \text{ s}$$

Now, we can use this time to find the peak height using the vertical displacement equation:

$$\Delta y = v_{0y}t + \frac{1}{2}(-g)t^2$$
$$\Delta y = (16.07)(1.607) + \frac{1}{2}(-10)(1.607)^2$$
$$\Delta y \approx 12.91 \text{ m}$$

(b) To find the total horizontal distance traveled, we first need the total time of flight. Since the motion is symmetric, the total time will be twice the time to reach the peak height:

$$t_{total} = 2t_{peak} \approx 2(1.607) \approx 3.214 \text{ s} = 3.21 \text{ s}$$

Since there is no horizontal acceleration, we can use the horizontal motion equation to find the horizontal distance:

$$\Delta x = v_{0x}t_{total} \approx (19.15)(3.21) \approx 61.5 \text{ m}$$

So, the peak height of the pumpkin is approximately 12.91 m, and the horizontal distance it travels before hitting the ground is approximately 61.5 m.

Answer to Exercise 5 (on page 30)

To maximize the horizontal distance of a projectile being launched, the vertical and horizontal components must be proportionally equal. This is only possible at a launch angle of 45°. Let's prove this mathematically.

Let the initial launch speed be v and the launch angle be θ. The horizontal and vertical velocity components are

$$v_x = v\cos\theta, \qquad v_y = v\sin\theta.$$

The vertical motion kinematics equation can be solved for:

$$0 = t\left(v\sin\theta - \tfrac{1}{2}gt\right)$$

This has two solutions: $t = 0$ (the time at launch) and

$$t = \frac{2v\sin\theta}{g}.$$

Thus, the horizontal range is

$$R_x = v_x \cdot t = v \cos\theta \cdot \frac{2v \sin\theta}{g} = \frac{v^2}{g} \sin(2\theta).$$

The function $\sin(2\theta)$ reaches its maximum value of 1 when $2\theta = 90°$, or $\theta = 45°$. Therefore, the optimal launch angle for maximum horizontal distance is 45°.

The range equation above can result can result in two different values: θ and $90° - \theta$. There are two ways to get a certain horizontal launch distance, one at an angle less than 45° and one at an angle greater than 45°. For example, a projectile launched at 30° will travel the same horizontal distance as one launched at 60°, assuming the same initial velocity. However, the projectile launched at 30° will spend less time in the air and have a lower peak height than the one launched at 60°.

Answer to Exercise 6 (on page 31)

Using the horizontal distance equation, R, with $v_0 = 14$ m/s and $R = 100$ m, we have:

$$\begin{aligned} R &= \frac{v^2 \sin(2\theta)}{g} \\ \sin(2\theta) &= \frac{Rg}{v^2} \\ \sin(2\theta) &= \frac{(100)(10)}{14^2} \approx 5.10 \end{aligned}$$

Since the sine of an angle cannot be greater than 1, it is impossible to hit the target with a launch speed of 14 m/s.

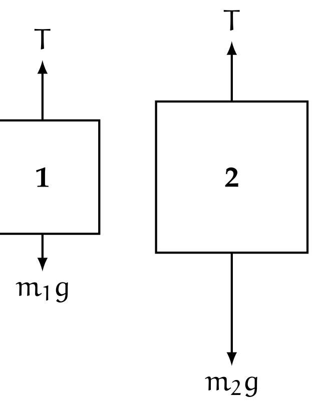
Instead, we can calculate the minimum launch speed needed to hit the target at a 45° angle, while satisfying the 100 m requirement. Assuming the $\sin(2\theta) = 1$ for a 45° launch angle, we can rearrange the range equation to solve for v:

$$\begin{aligned} v &= \sqrt{\frac{Rg}{\sin(2\theta)}} \\ v &= \sqrt{\frac{(100)(10)}{1}} = \sqrt{1000} \approx 31.62 \text{ m/s} \end{aligned}$$

So at an assumed launch angle of 45°, a minimum launch speed of approximately 31.62 m/s is needed to hit the target 100 meters away.

Answer to Exercise 7 (on page 36)

1. Each block is acted on by gravity and the tension of the string. The force of gravity on block 2 is 3 times that of block 1, and the tension vector should be between the lengths of the gravity vectors:



2. Block 1 will have a positive acceleration with a magnitude less than g. Block 2 will have a negative acceleration with a magnitude greater than g.

3. Since the blocks are connected, they will move as a system and therefore have the same magnitude acceleration. From this, the free body diagrams, and Newton's Second Law, we know that:

$$m_1 a = T - m_1 g$$

$$m_2 (-a) = T - m_2 g$$

We know m_1 and m_2, so the two unknowns are T and a. One way to solve this system of equations would be to subtract equation 2 from equation 1:

$$m_1 a + m_2 a = m_2 g - m_1 g$$

$$a = \frac{m_2 - m_1}{m_1 + m_2} g$$

Substituting for the masses, we see that:

$$a = \frac{9 - 3}{9 + 3} g = \frac{6}{12} g = \frac{1}{2} g$$

Therefore, Block 1 is accelerating at $+\frac{1}{2}g$ and Block 2 is accelerating at $-\frac{1}{2}g$.

Answer to Exercise 8 (on page 41)

(a) The normal force must be equal to the perpendicular component of the gravitational force, since there is no acceleration in the y-direction of our rotated y-axis:

$$F_N = mg\cos(\theta) = (30\,\text{kg})(9.8\,\text{m/s}^2)\cos(30°) \approx 254.6\,\text{N}$$

(b) The frictional force can be found using the coefficient of kinetic friction and the normal force:

$$F_F = \mu_k F_N = (0.3)(254.6\,\text{N}) \approx 76.4\,\text{N}$$

(c) The acceleration can be found by setting up the equation for the sum of forces in the x-direction:

$$\sum F_x = F_F - mg\sin(\theta) = ma_x$$

Solving for a_x:

$$\begin{aligned} a_x &= \frac{F_F - mg\sin(\theta)}{m} \\ &= \frac{76.4\,\text{N} - (30\,\text{kg})(9.8\,\text{m/s}^2)\sin(30°) -}{30\,\text{kg}} \\ &= \frac{76.4 - 147}{30} \\ &\approx \frac{-70.6}{30} \\ &\approx -2.35\,\text{m/s}^2 \end{aligned}$$

Answer to Exercise 9 (on page 42)

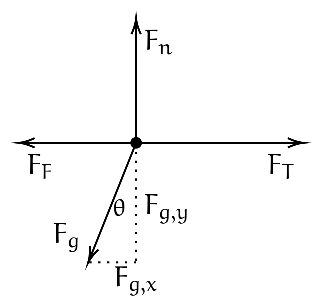
For a box to be on the verge of sliding, the frictional force (static) must equal the component of gravitational force parallel to the incline. Thus, we can set up the following equation:

$$\begin{aligned} \sum F_x &= mg\sin(\theta) - F_F = 0 \\ mg\sin(\theta) &= F_F \\ mg\sin(\theta) &= \mu_s F_N \\ mg\sin(\theta) &= \mu_s mg\cos(\theta) \\ \tan(\theta) &= \mu_s \end{aligned}$$

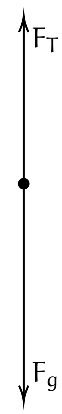
Solving for θ, we get $\theta = \tan^{-1}(\mu_s)$. Plugging in the value of μ_s, we get $\theta = 24.89°$

Answer to Exercise 10 (on page 42)

Let's create free body diagrams for both objects.



FBD Block m_1



FBD Block m_2

Note that the dashed lines are components of forces, not forces themselves. We can see that the common force uniting the two objects is F_T. We can solve for the acceleration in the x-direction of the block of mass m_1 and the acceleration in the y-direction of m_2.

$$\begin{aligned}
F_{net,m_1,x} &= F_T - m_1 g \sin\theta - \mu_k m_1 g \cos\theta \\
&= m_1 a_x \\
F_{net,m_2,y} &= m_2 g - F_T \\
&= m_2 a_x \\
F_{net,m_1,x} + F_{net,m_2,y} &= m_2 g - m_1 g \sin\theta - \mu_k m_1 g \cos\theta = (m_1 + m_2) a \\
a &= \frac{m_2 g - m_1 g \sin\theta - \mu_k m_1 g \cos\theta}{m_1 + m_2}
\end{aligned}$$

Notice that both masses share the same magnitude of acceleration. m_2 accelerates vertically, m_1 accelerates horizontally, but their magnitude is identical. It is *not* the same as the net force on m_2 alone.

Answer to Exercise 11 (on page 44)

The momentum of the first car is 12,000 kg m/s in the north direction.

The momentum of the second car is 24,000 kg m/s in the east direction.

The new object will be moving northeast. What is the angle compared with the east?

$$\theta = \arctan \frac{12,000}{24,000} \approx 0.4636 \text{ radians } \approx 26.565 \text{ degrees north of east}$$

The magnitude of the momentum of the new object is $\sqrt{12,000^2 + 24,000^2} \approx 26,833$ kg m/s

Its new mass is 2,5000 kg. So the speed will be $26,833/2,500 = 10.73$ m/s.

Answer to Exercise 12 (on page 46)

The original forward momentum was 1.2 kg m/s. The original kinetic energy is $(1/2)(0.4)(3^2)$ = 1.8 joules.

Let s be the post-collision speed of the ball that had been at rest. Let x and y be the forward and sideways speeds (post-collision) of the other ball. Conservation of kinetic energy says

$$(1/2)(0.4)(s^2) + (1/2)(0.4)(x^2 + y^2) = 1.8$$

Forward momentum is conserved:

$$0.4\frac{s}{\sqrt{2}} + 0.4x = 1.2$$

Which can be rewritten:

$$x = 3 - \frac{s}{\sqrt{2}}$$

Sideways momentum stays zero:

$$(0.4)\frac{s}{\sqrt{2}} - 0.4y = 0.0$$

Which can be rewritten:

$$y = \frac{s}{\sqrt{2}}$$

Substituting into to the conservation of kinetic energy equation above:

$$(1/2)(0.4)(s^2) + (1/2)(0.4)(\left(3 - \frac{s}{\sqrt{2}}\right)^2 + \left(\frac{s}{\sqrt{2}}\right)^2 = 1.8$$

Which can be rewritten:

$$s^2 - \frac{3}{\sqrt{2}}s + 0 = 0$$

There are two solutions to this quadratic: $s = 0$ (before collision) and $s = \frac{3}{\sqrt{2}}$. Thus,

$$y = \frac{3}{2}$$

and

$$x = 3 - \frac{3}{2} = \frac{3}{2}$$

So, both balls careen off at 45° angles at the exact same speed.

Answer to Exercise 13 (on page 50)

Because both carts collide elastically, the Conservation of Kinetic Energy and Conservation of Momentum both apply.

We can rearrange the Conservation of Kinetic Energy in the following ways:

$$\begin{aligned}
\frac{1}{2}m_1v_{1i}^2 + \frac{1}{2}m_2v_{2i}^2 &= \frac{1}{2}m_1v_{1f}^2 + \frac{1}{2}m_2v_{2f}^2 \\
\frac{1}{2}m_1v_{1i}^2 + \frac{1}{2}m_2v_{2i}^2 - \frac{1}{2}m_1v_{1f}^2 - \frac{1}{2}m_2v_{2f}^2 &= 0 \\
m_1v_{1i}^2 + m_2v_{2i}^2 - m_1v_{1f}^2 - m_2v_{2f}^2 &= 0 \\
m_1v_{1i}^2 - m_1v_{1f}^2 + m_2v_{2i}^2 - m_2v_{2f}^2 &= 0 \\
m_1(v_{1i}^2 - v_{1f}^2) + m_2(v_{2i}^2 - v_{2f}^2) &= 0 \\
m_1(v_{1i} - v_{1f})(v_{1i} + v_{1f}) + m_2(v_{2i} - v_{2f})(v_{2i} + v_{2f}) &= 0 \qquad (1)
\end{aligned}$$

And rearranging Conservation of Momentum:

$$\begin{aligned} m_1 v_{1i} + m_2 v_{2i} &= m_1 v_{1f} + m_2 v_{2f} \\ m_1 v_{1i} + m_2 v_{2i} - m_1 v_{1f} - m_2 v_{2f} &= 0 \\ m_1(v_{1i} - v_{1f}) + m_2(v_{2i} - v_{2f}) &= 0 \\ m_1(v_{1i} - v_{1f}) &= -m_2(v_{2i} - v_{2f}) \end{aligned} \tag{2}$$

Inputting Equation 2 into Equation :

$$\begin{aligned} m_1(v_{1i} - v_{1f})(v_{1i} + v_{1f}) + m_2(v_{2i} - v_{2f})(v_{2i} + v_{2f}) &= 0 \\ -m_2(v_{2i} - v_{2f})(v_{1i} + v_{1f}) + m_2(v_{2i} - v_{2f})(v_{2i} + v_{2f}) &= 0 \\ m_2(v_{2i} - v_{2f})\left[(v_{2i} + v_{2f}) - (v_{1i} + v_{1f})\right] &= 0 \\ (v_{2i} + v_{2f}) - (v_{1i} + v_{1f}) &= 0 \text{ assuming } v_{2i} \neq v_{2f} \\ v_{2f} - v_{1f} &= v_{1i} - v_{2i} \end{aligned} \tag{3}$$

Equation 3 tells us that, in an elastic collision, the relative speed of separation is equal to relative speed of approach. This allows us to do the following plug-ins:

$$\begin{aligned} m_1 v_{1i} + m_2 v_{2i} &= m_1 v_{1f} + m_2(v_{1f} + v_{1i} - v_{2i}) \\ m_1 v_{1i} + m_2 v_{2i} &= (m_1 + m_2)v_{1f} + m_2 v_{1i} - m_2 v_{2i} \\ (m_1 - m_2)v_{1i} + 2m_2 v_{2i} &= (m_1 + m_2)\, v_{1f} \\ v_{1f} &= \frac{m_1 - m_2}{m_1 + m_2} v_{1i} + \frac{2m_2}{m_1 + m_2} v_{2i} \end{aligned}$$

The same process results in v_{2f}:

$$v_{2f} = \frac{2m_1}{m_1 + m_2} v_{1i} + \frac{m_2 - m_1}{m_1 + m_2} v_{2i} \tag{6.7}$$

Equations **??** and 6.7 can be very useful but take a while to derive. Make sure you understand the process to solve for them. Let's plug in the values:

$$v_{1f} = \frac{2-1}{2+1}(3.0) = \frac{1}{3}(3.0) = 1.0 \text{ m/s}$$

$$v_{2f} = \frac{2(2)}{2+1}(3.0) = \frac{4}{3}(3.0) = 4.0 \text{ m/s}$$

Answer to Exercise 14 (on page 51)

Using the Conservation of Momentum theorem:

$$
\begin{aligned}
m_A v_{A,i} + m_B v_{B,i} &= m_A v_{A,f} + m_B v_{B,f} \\
(1000)(20) + (1500)(-10) &= (1000)(5) + (1500) v_{B,f} \\
20000 - 15000 - 5000 &= (1500) v_{B,f} \\
0 &= (1500) v_{B,f} \\
v_{B,f} &= 0 \mathrm{m/s}
\end{aligned}
$$

The final velocity of Car B is 0 m/s. The collision must be inelastic as both velocities change and kinetic energy is not conserved.

Answer to Exercise 15 (on page 52)

(a) Using the conservation of momentum, we can say that:

$$
\begin{aligned}
m_{bullet}(v_{bullet}) + m_{block}(0) &= (m_{bullet} + m_{block}) v_f \\
(0.01)(500) + (2)(0) &= ((0.01) + (2)) v_f \\
v_f &= \frac{(500)(0.01)}{2.01} \\
v_f &\approx 2.49 \mathrm{m/s}
\end{aligned}
$$

The *block-bullet system* moves at a speed of 2.49 m/s, in the same direction the bullet was fired.

(b) To find the initial kinetic energy, we use the initial velocity:

$$KE_i = \frac{1}{2}(0.01)(500)^2 = 1250$$

And, to find the final kinetic energy, we use the summed masses and the calculated final velocity

$$KE_f = \frac{1}{2}(2.01)(2.49)^2 \approx 6.23$$

So the net energy lost is the difference:

$$\Delta KE = KE_i - KE_f \approx 1250 - 6.23 \approx 1243.7\,\mathrm{J}$$

(c)

$$\text{Fraction lost} = \frac{\Delta KE}{KE_i} = \frac{1243.7}{1250} \approx 0.99496$$

So 99.5% percent of the energy is lost. It is impossible for this to be an elastic collision, so this has to be an inelastic collision. Because *more* than 95% of the energy is lost, the collision is perfectly inelastic (by the problem guidelines).

Answer to Exercise 16 (on page 54)

The total momentum before immediately before the explosion can be separated into two components:

$$p_{i,x} = m_i v_{i,x} = (6)(20) = 120 \text{ m/s} \qquad p_{i,y} = m_i v_{i,y} = 0$$

Let's analyze Fragment A:

$$p_{Ax} = m_A v_{Ax} = 2(30\cos 40°), \qquad p_{Ay} = m_A v_{Ay} = 2(30\sin 40°)$$

And Fragment B,

$$p_{Bx} = m_B v_{Bx} = 1(25\cos 60°), \qquad p_{By} = m_B v_{By} = 1(-25\sin 60°)$$

Since the Conservation of Momentum applies to the components:

$$\begin{aligned}
p_{x,\text{final}} &= p_{Ax} + p_{Bx} + p_{Cx} = 120 \text{ kg·m/s} \\
p_{Cx} &= 120 - (p_{Ax} + p_{Bx}) \\
&= 120 - 2(30\cos 40°) - 1(25\cos 60°) \\
v_{Cx} &= \frac{120 - 2(30\cos 40°) - 1(25\cos 60°)}{3.0} \\
&= 20.51 \text{ m/s}
\end{aligned}$$

And same for the Y-components

$$\begin{aligned}
p_{y,\text{final}} &= p_{Ay} + p_{By} + p_{Cy} = 0 \text{ kg·m/s} \\
p_{Cy} &= -(p_{Ay} + p_{By}) \\
&= -(2(30\sin 40°) - 1(-25\sin 60°)) \\
v_{Cy} &= \frac{-(2(30\sin 40°) - 1(-25\sin 60°))}{3} \\
&= -5.64 \text{ m/s}
\end{aligned}$$

So Fragment C moves at $\sqrt{20.51^2 + (-5.64)^2} = 21.3$ m/s with a direction of $\theta = \tan^{-1}\left(\frac{-5.64}{20.51}\right) = -15°$ or 15° below the horizontal.

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