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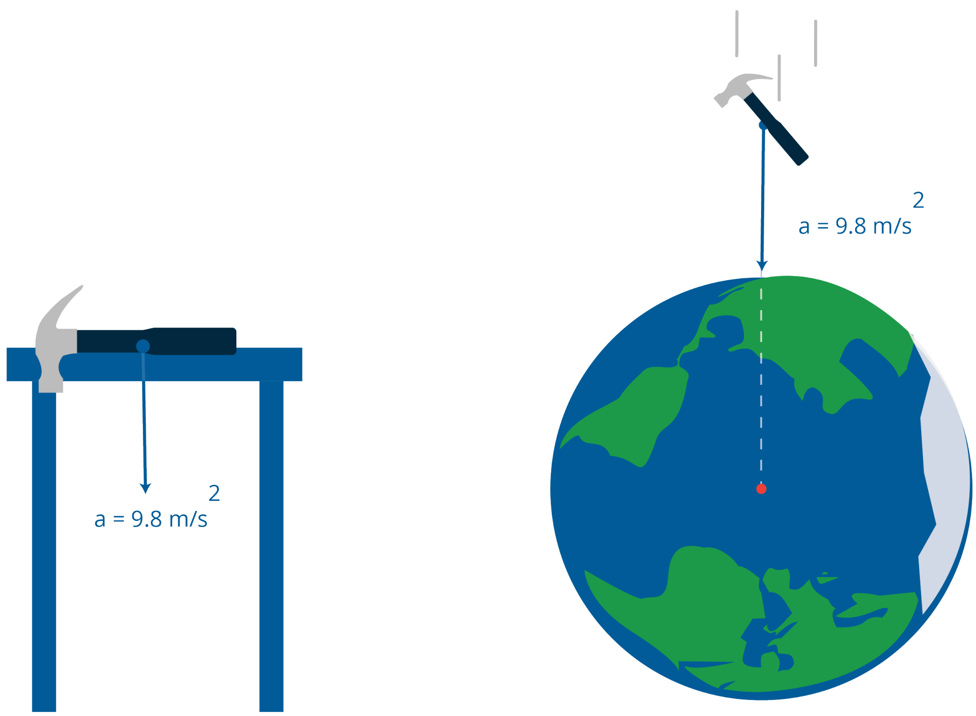
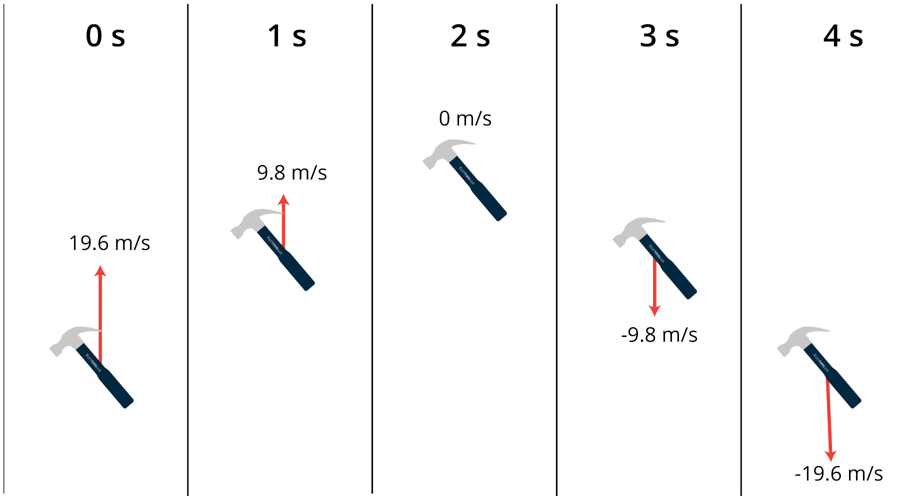
Chapter 1

Falling Bodies

Gravity exists all around us. If you throw a hammer straight up in the air, from the moment it leaves your hand until it hits the ground, it is accelerating toward the center of the earth at a constant rate.



Acceleration can be defined as change in velocity. If the hammer leaves your hand with a velocity of 12 meters per second upward, one second later, it will be rising, and its velocity will have slowed to 2.2 meters per second. One second after that, the hammer will be falling at a rate of 7.6 meters per second. Every second the hammer's velocity is changing by 9.8 meters per second, and that change is always toward the center of the earth. When the hammer is going up, gravity is slowing it down by 9.8 meters per second, each second it is in the air. When the hammer is coming down, gravity is increasing the speed of its descent by 9.8 meters per second.



Acceleration due to gravity on earth is a constant negative 9.8 meters per second per second:

$$a = -9.8\frac{m}{s^2}$$

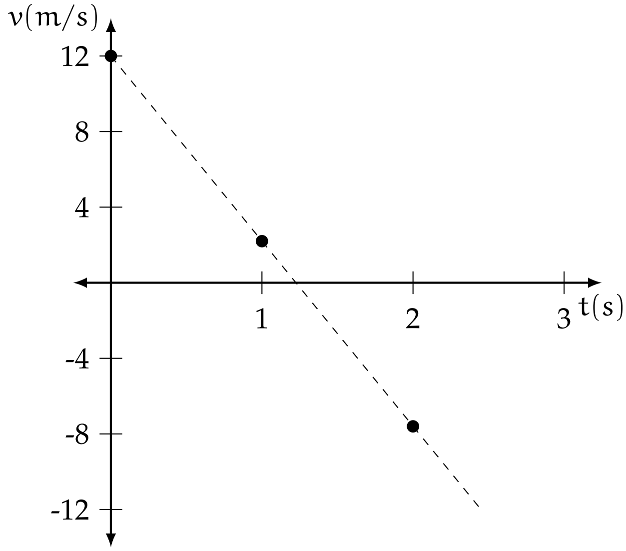
(Why is it negative? We are talking about height, which is generally considered to be increasing as you go away from the center of the earth. Since "up" is the positive direction, we take "down" as a negative direction. Therefore, since gravity pulls things "down", the acceleration due to gravity is considered negative.)

1.1 Calculating the Velocity

Given that the acceleration is constant, it makes sense that the velocity is a straight line. Assuming once again that the hammer leaves your hand at 12 meters per second, let's create a quick data table and graph of the hammer's velocity. Every second, we will subtract 9.8 m/s from the previous velocity.

Time elapsed (s)	Velocity (m/s)
0	12
1	2.2
2	-7.6

Here is a plot of these data with a dashed fit line:



We can find an equation for the line: $v = 12 - 9.8t$. What is this saying? It says that at any time, t, after $t = 0s$, the velocity of the hammer is its starting velocity (12 m/s) plus the acceleration (-9.8 m/s^2) multiplied by the time elapsed (t). (Why did we say "plus" instead of "minus"? Well, since the acceleration *is negative*, $v_0 - 9.8t$ is the same as $v_0 + (-9.8t)$, and we are really adding the negative acceleration. When the acceleration is positive, we'll see a plus sign!)

Exercise 1 When is the apex of flight?

Given the hammer's velocity is calculated as $12 - 9.8t$, at what time (in seconds) does it stop rising and begin to fall?

Working Space

Answer on Page 47

We can generalize this to other falling bodies. For any falling object with initial velocity v_0 and constant acceleration a, its velocity is given by:

$$v(t) = v_0 + a \cdot t$$

Example: The acceleration due to gravity on the Moon is approximately 1.63 m/s^2. If a hammer tossed upwards on the Moon takes 3.4 seconds to reach its apex, what was the hammer's initial velocity?

Solution: We know that $a = -1.63\text{m/s}^2$ and $t = 3.4\text{s}$. Substituting and solving the velocity equation:

$$0\frac{m}{s} = v_0 + \left(-1.63\frac{m}{s^2}\right)(3.4s)$$
$$v - 0 = (1.63 \cdot 3.4)\,\frac{m}{s} \approx 5.5\frac{m}{s}$$

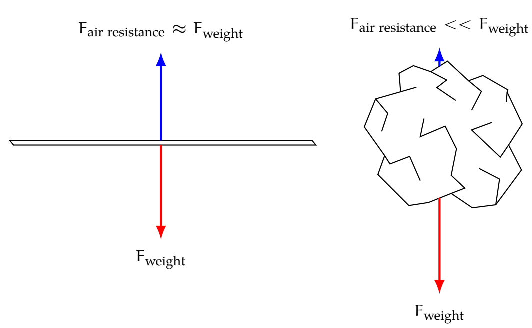
This hammer must have been tossed upwards with an initial velocity of approximately 5.5 m/s.

1.1.1 Air Resistance

At this point, we need to acknowledge air resistance. Gravity is not the only force on the hammer; as it travels through the air, friction with the air slows it down. This force is called *air resistance*, and for a large, fast-moving object (like an airplane) it is GIGANTIC force. For a dense object (like a hammer) moving at a slow speed (what you generate with your hand), air resistance doesn't significantly affect acceleration.

Consider two otherwise identical pieces of paper: one completely flat and unfolded, the other balled up as tightly as possible. If you dropped them both from the same height, which would hit the ground first? Intuitively, you know the balled-up paper would: but *why*? Since the papers have the same mass, you know that the force of gravity is the same on both papers! However, if you give this a try (go ahead, all you need to two sheets of printer paper and a safe place to drop them from), the acceleration of the two papers is different.

We can apply Newton's Second Law to explain our observations. First, let's draw diagrams to represent the forces acting on the papers:



For the crumpled paper, the decrease in surface area results in very little air resistance. As a result, the net force on the crumpled paper is approximately equal to the paper's weight.

$$F_{\text{net}}^{\text{crumpled}} = F_{\text{air resistance}} + F_{\text{weight}} \approx F_{\text{weight}}$$

$$F_{net}^{crumpled} = ma_{crumpled} \approx F_{weight} = mg$$
$$a_{crumpled} \approx g$$

Thus, you observe the crumpled paper to fall at nearly -9.8 m/s^2. The flat paper, on the other hand, has a much larger surface area, and therefore experiences significantly more air resistance. Thus, the observed acceleration of the flat paper is much slower:

$$F_{net}^{flat} = F_{\text{air resistance}} + F_{weight}$$
$$ma_{flat} = F_{\text{air resistance}} + mg$$
$$a_{flat} = \frac{F_{\text{air resistance}}}{m} + g$$

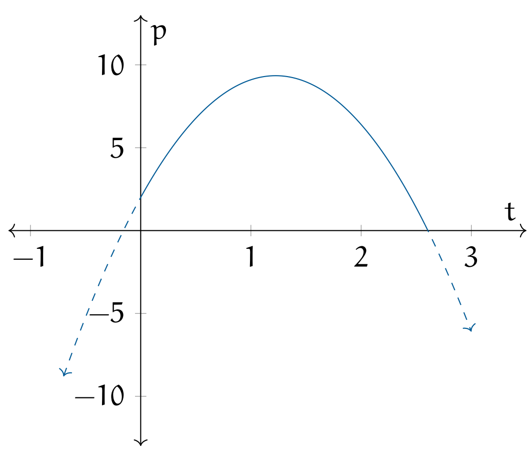
Since $g < 0$ and $F_{\text{air resistance}}$ is positive, the greater the air resistance the slower the paper falls.

1.2 Calculating Position

If you let go of the hammer when it is 2 meters above the ground, the height of the hammer is given by:

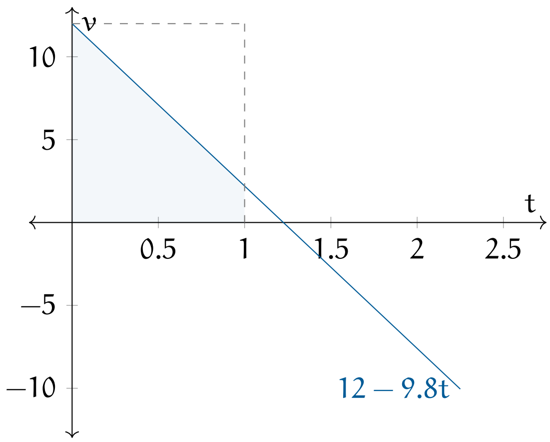
$$p = -\frac{9.8}{2}t^2 + 12t + 2$$

Here is a graph of this function:



How do we know? **The change in position between time 0 and any time t is equal to the area under the velocity graph between $x = 0$ and $x = t$.**

Let's use the velocity graph to figure out how much the position has changed in the first second of the hammer's flight. Here is the velocity graph with the area under the graph for the first second filled in:



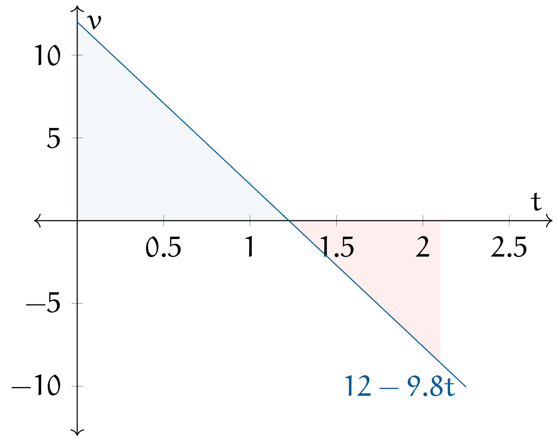
The blue filled region is the area of the dashed rectangle minus that empty triangle in its upper left. The height of the rectangle is twelve and its width is the amount of time the hammer has been in flight (t). The triangle is t wide and $9.8t$ tall. Thus, the area of the blue region is given by $12t - \frac{1}{2}9.8t^2$.

That's the change in position. Where was it originally? 2 meters off the ground. This means the height is given by $p = 2 + 12t - \frac{1}{2}9.8t^2$. We usually write terms so that the exponent decreases, so:

$$p = -\frac{1}{2}9.8t^2 + 12t + 2$$

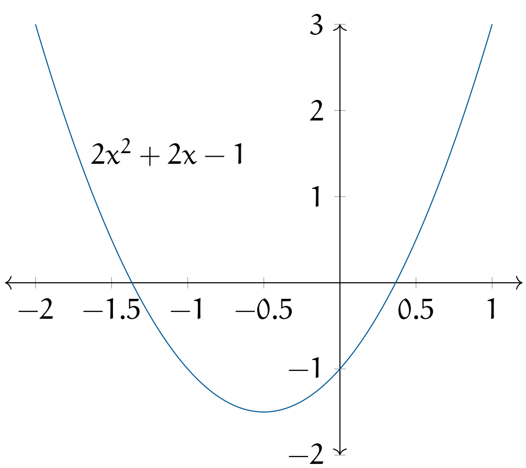
Finding the area under the curve like this is called *integration*. We say "To find a function that gives the change in position, we just integrate the velocity function." Much of the study of calculus is learning to integrate different sorts of functions.

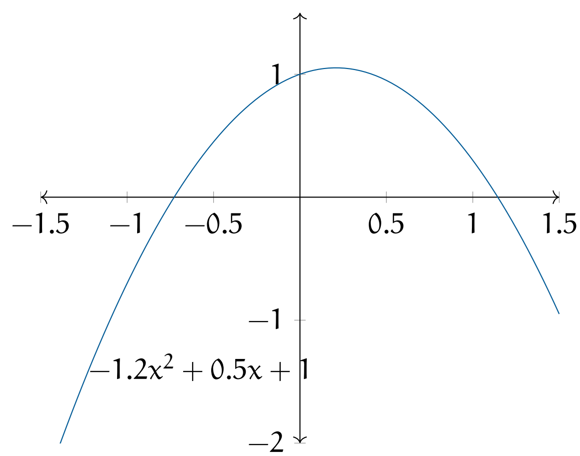
One important note about integration: Any time the curve drops under the x-axis, the area is considered negative. (Which makes sense, right? If the velocity is negative, the hammer's position is decreasing.)



1.3 Quadratic functions

Functions of the form $f(x) = ax^2 + bx + c$ are called *quadratic functions*. If $a > 0$, the ends go up. If $a < 0$, the ends go down.





The graph of a quadratic function is a *parabola*.

1.4 Simulating a falling body in Python

Now you are going to write some Python code that simulates the flying hammer. First, we are just going to print out the position, speed, and acceleration of the hammer for every 1/100th of a second after it leaves your hand. (Later, we will make a graph.)

Create a file called `falling.py` and type this into it:

```
# Acceleration on earth
acceleration = -9.8 # m/s/s

# Size of time step
time_step = 0.01 # seconds

# Initial values
speed = 12  # m/s upward
height = 2  # m above the ground
current_time = 0.0  # seconds after release

# Is the hammer still aloft?
while height > 0.0:

    # Show the values
    print(f"{current_time:.2f} s:")
    print(f"\tacceleration: {acceleration:.2f} m/s/s")
    print(f"\tspeed: {speed:.2f} m/s")
    print(f"\theight: {height:.2f} m")

    # Update height
    height = height + time_step * speed

    # Update speed
    speed = speed + time_step * acceleration

    # Update time
    current_time = current_time + time_step


print(f"Hit the ground: Complete")
```

When you run it, you will see something like this:

```
0.00 s:
        acceleration: -9.80 m/s/s
        speed: 12.00 m/s
        height: 2.00 m
0.01 s:
        acceleration: -9.80 m/s/s
        speed: 11.90 m/s
        height: 2.12 m
0.02 s:
        acceleration: -9.80 m/s/s
        speed: 11.80 m/s
```

```
        height: 2.24 m
0.03 s:
        acceleration: -9.80 m/s/s
        speed: 11.71 m/s
        height: 2.36 m
...
2.60 s:
        acceleration: -9.80 m/s/s
        speed: -13.48 m/s
        height: 0.20 m
2.61 s:
        acceleration: -9.80 m/s/s
        speed: -13.58 m/s
        height: 0.07 m
Hit the ground: Complete
```

Note that the acceleration isn't changing at all, but it is changing the speed, and the speed is changing the height.

We can see that the hammer in our simulation hits the ground just after 2.61 seconds.

1.4.1 Graphs and Lists

Now, we are going to graph the acceleration, speed, and height using a library called matplotlib. However, to make the graphs, we need to gather all the data into lists.

For example, we will have a list of speeds, and the first three entries will be 12.0, 11.9, and 11.8.

We create an empty list and assign it to a variable like this:

```
x = []
```

Next, we can add items like this:

```
x.append(3.14)
```

To get the first time back, we can ask for the object at index 0.

```
y = x[0]
```

Note that the list starts at 0. If you have 32 items in the list, the first item is at index 0; the last item is at index 31.

Duplicate the file `falling.py` and name the new copy `falling_graph.py`

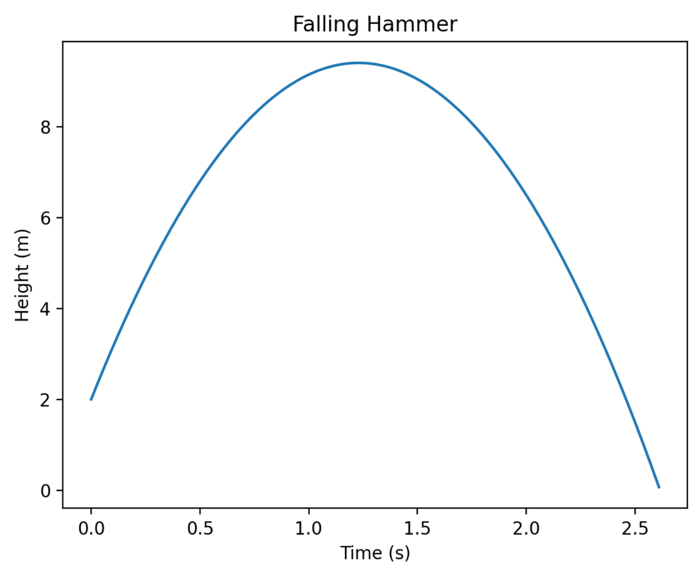
We are going to make a plot of the height over time. At the start of the program, you will import the matplotlib library. At the end of the program, you will create a plot and show it to the user.

In `falling_graph.py`, add the bold code:

```
import matplotlib.pyplot as plt

# Acceleration on earth
acceleration = -9.8 # m/s/s

# Size of time step
time_step = 0.01 # seconds

# Initial values
speed = 12  # m/s upward
height = 2  # m above the ground
current_time = 0.0  # seconds after release

# Create empty lists
accelerations = []
speeds = []
heights = []
times = []

# Is the hammer still aloft?
while height > 0.0:

    # Add the data to the lists
    times.append(current_time)
    accelerations.append(acceleration)
    speeds.append(speed)
    heights.append(height)

    # Update height
    height = height + time_step * speed

    # Update speed
    speed = speed + time_step * acceleration

    # Update time
    current_time = current_time + time_step
```

```
# Make a plot
fig, ax = plt.subplots()
fig.suptitle("Falling Hammer")
ax.set_xlabel("Time (s)")
ax.set_ylabel("Height (m)")
ax.plot(times, heights)
plt.show()
```

When you run the program, you should see a graph of the height over time.

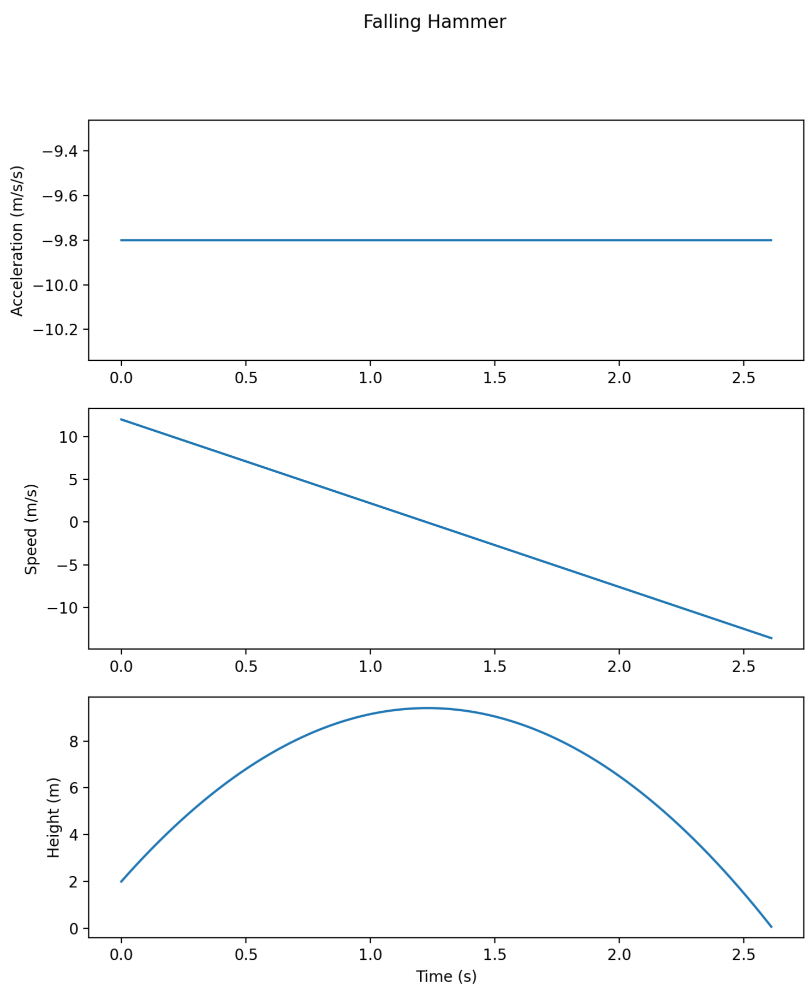


It is more interesting if we can see all three: acceleration, speed, and height. So, let's make three stacked plots. Change the plotting code in `falling_graph.py` to:

```
# Make a plot with three subplots
fig, ax = plt.subplots(3,1)
fig.suptitle("Falling Hammer")

# The first subplot is acceleration
ax[0].set_ylabel("Acceleration (m/s/s)")
ax[0].plot(times, accelerations)

# Second subplot is speed
ax[1].set_ylabel("Speed (m/s)")
ax[1].plot(times, speeds)

# Third subplot is height
```

```
ax[2].set_xlabel("Time (s)")
ax[2].set_ylabel("Height (m)")
ax[2].plot(times, heights)
plt.show()
```

You will now get plots of all three variables:



This is what we expected, right? The acceleration is a constant negative number. The speed is a straight line with a negative slope. The height is a parabola. **The slope of the height graph is the speed, and the slope of the speed graph is acceleration**

A natural question at this point is "When exactly will the hammer hit the ground?" In other words, when does $\texttt{height} = 0$? The values of t where a function is zero are known as its *roots*. Height is given by a quadratic function. In the next chapter, you will get the trick for finding the roots of any quadratic function.

Chapter 2

Kinematics

How can we describe the motion of objects other than falling bodies? Kinematics is the description of motion.

$$x_f = x_0 + v_0 t + \frac{1}{2} a t^2 \tag{2.1}$$

$$v_f = v_0 + at \tag{2.2}$$

$$x_f = x_0 + \frac{1}{2}(v_f + v_0) t \tag{2.3}$$

$$v_f^2 = v_0^2 + 2a(x_f - x_0) \tag{2.4}$$

Note that v_0 and v_i are synonomous, some professors will use v_0 to mean "velocity not" or inital velocity.

Note that each equation is missing a certain variable (or two)

- (5.1) has no v_f.
- (5.2) has no x_i or x_f
- (5.3) has no a
- (5.4) has no t

Example: Terri and Jerry are running a race. Terri has a maximum acceleration of 3.4 m/s^2 and a top speed of 9.2 m/s. Jerry has a maximum acceleration of 3.7 m/s^2 and a top speed of 8.7 m/s. If the race is 200 m long, who will win? Assume that each runner has maximum acceleration until they reach their top speed and that they will maintain that top speed once they reach it.

Solution: We want to know how long it takes each runner to complete the 200 m race. For each runner, we will divide their run into two sections:

1. The time they are accelerating to their top speed
2. The time they are maintaining their top speed

We can use a table to track the results of our calculations:

Runner	Terri	Jerry
Leg 1 (s)		
Leg 2 (s)		
Total (s)		

We'll begin with Terri's first leg. Taking the starting line as $x = 0$, we know that:

$$x_0 = 0 \text{ m}$$
$$v_0 = 0 \ \frac{\text{m}}{\text{s}}$$
$$a = 3.4 \ \frac{\text{m}}{\text{s}^2}$$

And since we want to know how long it takes Terri to reach her top speed, we also know that $v_f = 9.2$ m/s. Since we don't know how far Terri will run before she reaches her top speed (and we're not looking for that quantity), we need to select an equation that does not include x:

$$v_f = v_0 + at$$
$$9.2\frac{\text{m}}{\text{s}} = 0\frac{\text{m}}{\text{s}} + \left(3.4\frac{\text{m}}{\text{s}^2}\right) t$$
$$t = \frac{9.2\frac{\text{m}}{\text{s}}}{3.4\frac{\text{m}}{\text{s}^2}} \approx 2.7 \text{ s}$$

With a similar method, we can find how long it takes Jerry to reach his top speed:

$$t = \frac{v_f}{a} = \frac{8.7\frac{\text{m}}{\text{s}}}{3.7\frac{\text{m}}{\text{s}^2}} \approx 2.4 \text{ s}$$

Let's go ahead and record this in our table:

Runner	Terri	Jerry
Leg 1 (s)	2.7	2.4
Leg 2 (s)		
Total (s)		

Now that we know how much time it takes each runner to reach their top speed, we need to figure out how much time it takes them to complete the race from the point at which each reaches their top speed. To do this, we will first have to find *where* each runner hits their top speed. (This is because we can't use $v_f = v_0 + at$ to find a time anymore, since from now on the runners' accelerations are zero, and all the other equations involve x.) For each runner, we know a, v_0, v_f, x_0, and t. You could choose any equation, but we will use this one:

$$x_f = x_0 + \frac{1}{2}(v_f + v_0)\, t$$

Since each runner begins on the starting line, $x_0 = 0$ m and $v_0 = 0$ m/s:

$$x_f = \frac{v_f \cdot t}{2}$$

For Terri:

$$x_f = \frac{\left(9.2\frac{m}{s}\right)(2.7\ s)}{2} \approx 12.4\ m$$

For Jerry:

$$x_f = \frac{\left(8.7\frac{m}{s}\right)(2.4\ s)}{2} \approx 10.2\ m$$

Now that we know where they reach their top speed, we can take that position as x_0 and find how long it takes each runner to reach the finish line at $x_f = 200$ m. Since $a = 0$ m/s^2, we can use:

$$x_f = x_0 + v_0 t$$

Rearranging to solve for t:

$$t = \frac{x_f - x_0}{v_0}$$

For Terri:

$$t = \frac{200\ m - 12.4\ m}{9.2\ \frac{m}{s}} \approx 20.4\ s$$

And for Jerry:

$$t = \frac{200\ m - 10.2\ m}{8.7\ \frac{m}{s}} \approx 21.8\ s$$

Completing our table, we see that Terri will win the race by finishing in the least amount of time:

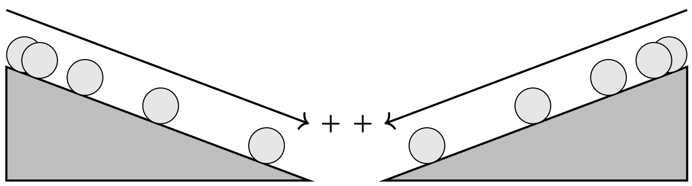
Runner	Terri	Jerry
Leg 1 (s)	2.7	2.4
Leg 2 (s)	20.4	21.8
Total (s)	23.1	24.2

2.1 Graphing Motion

2.1.1 Motion Diagrams

Example: Create a motion diagram of a ball rolling with a constant acceleration down an incline (the acceleration is in the same direction as motion).

Solution: The ball is accelerating down the ramp, so it will cover more distance each second. You could draw your ramp going left or right, as long as the distance covered each time interval increases as time passes.



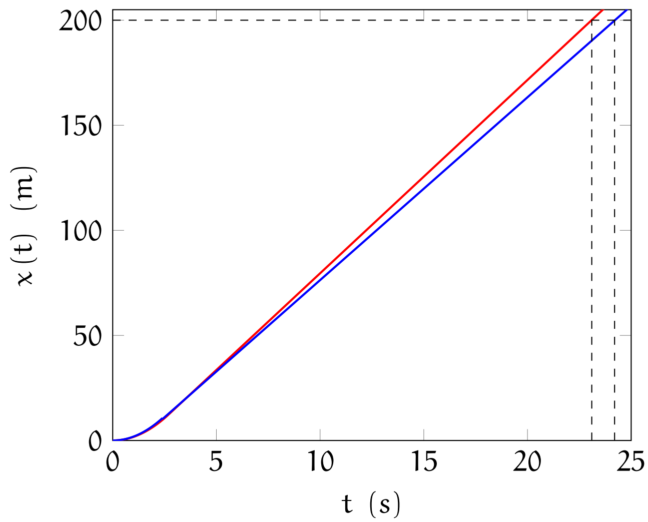
2.1.2 Position-Time and Velocity-Time Graphs

In the example problem above, graphs of each runner's motion would allow us to immediately see who would win. We can describe Terri's and Jerry's runs with piecewise functions:

$$x_{\text{Terri}}(t) = \begin{cases} \left(1.7\ \frac{\text{m}}{\text{s}^2}\right) t^2 & \text{if } 0 \leq t < 2.7\ \text{s} \\ 12.4\ \text{m} + \left(9.2\ \frac{\text{m}}{\text{s}}\right)(t - 2.7\ \text{s}) & \text{if } t \geq 2.7\ \text{s} \end{cases}$$

$$x_{\text{Jerry}}(t) = \begin{cases} \left(1.85\ \frac{\text{m}}{\text{s}^2}\right) t^2 & \text{if } 0 \leq t < 2.4\ \text{s} \\ 10.2\ \text{m} + \left(8.7\ \frac{\text{m}}{\text{s}}\right)(t - 2.4\ \text{s}) & \text{if } t \geq 2.4\ \text{s} \end{cases}$$

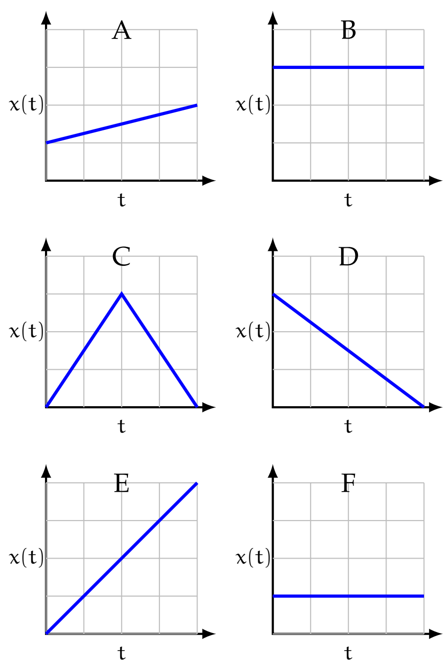
Graphing Terri in red and Jerry in blue:



Since the red function (Terri) crosses $x(t) = 200$ m first, we see that Terri will win.

Exercise 2

The following graphs show the position of an object from $t = 0$ s to $t = 10$ s. The scales on the y-axes are all the same.



Rank the objects from least to greatest (all negative values are lower than all positive values) in terms of:

1. displacement from $t = 0$ s to $t = 10$ s.
2. instantaneous velocity at $t = 7.5$ s.
3. distance traveled from $t = 0$ s to $t = 10$ s

Working Space

Answer on Page 47

2.2 Separation of Components

In the current chapter, we have learned how to use kinematics to describe one-dimensional motion. In the next chapter, you will learn to describe two-dimensional motion. It turns out that you can treat the different dimensions (horizontal and vertical motion) separately! Consider this scenario: A cannonball is shot from a cliff and the same instant an identical cannonball is dropped from the same cliff (see...). If the cannon is aimed horizontally, which cannonball will hit the ground first?

Go ahead and jot down what you think would happen: would the dropped ball hit first, the launched ball hit first, or would they both reach the ground below at the same time? Then, take a look at this video: https://www.youtube.com/watch?v=zMF4CD7i3hg. In it, two balls are released from the same height at the same time. One is dropped from rest while the other is launched horizontally (that is, its initial velocity is entirely in the x-direction). Based on the video, was your cannonball prediction correct? We'll learn to explain this phenomenon in the next chapter.

Chapter 3

Projectile Motion

A projectile is an object that, once thrown or dropped, continues to move only under the influence of gravity. Throwing a baseball, shooting a cannon, and diving off a high diving board are all examples. NASA flight planners use projectile motion to plan flight paths for space vehicles, such as sending rovers to Mars. You've already learned how to describe and model one-dimensional projectile motion in the Falling Bodies chapter. Now, we consider projectiles that also have horizontal motion, and therefore are moving in two dimensions.

First, we will compare the motion of projectiles that are dropped versus horizontally launched from the same height. This will frame our discussion of the important concept of independence of motion: the vertical and horizontal motions of a projectile can be considered and described independent from each other. This will allow you to predict how far horizontally launched objects will travel before hitting the ground. Next, you'll learn to describe the motion of projectiles launched at an angle (like some heavy ground artillery). Finally, you'll use what you've learned to create a model of any projectile motion.

3.1 Comparing Projectiles

This video was mentioned at the end of the kinematics chapter: https://www.youtube.com/watch?v=zMF4CD7i3hg. From the video, we can see that the addition of horizontal motion does not effect how fast an object is acted upon by gravity. Both objects hit the ground at the same time, regardless of whether horizontal motion was added or not.

3.2 Independence of Motion

we can describe the x and y motion separately. you already know how to describe the y motion from falling bodies show 2D kinematics equations graphs comparing x motions and y motions

3.3 Horizontally-launched Projectiles

example

exercise

exercise - Newton's cannon

3.4 Projectiles launched at an Angle

3.4.1 From the Ground

separating vertical and horizontal components of initial motion with trigonometry example

exercise - how far does the object travel?

exercise - at what angle should you launch for an object to go the furthest given a maximum launch velocity?

exercise - I have a target x-meters away, I must launch at v-miles per hour, what angle will allow me to hit my target, if any?

3.5 Simulating Projectile Motion

CHAPTER 4

Free Body Diagrams

Now that you've mastered modeling *how things move*, you can learn to model *why things move*. Recall Newton's Second Law:

$$F = ma$$

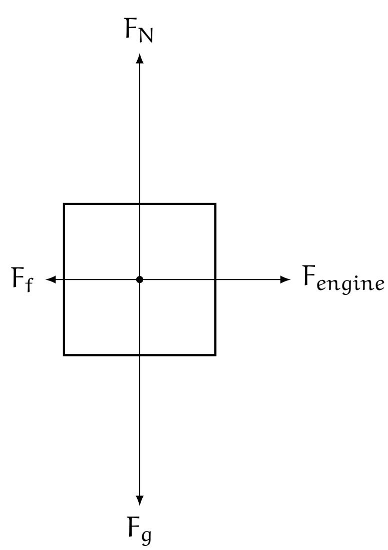
This is a simplification: what Newton's Second Law says in total is "the acceleration of a body is directly proportional to the *net force* acting on the body and inversely proportional to the mass of the body." Mathematically, that would be:

$$a = \frac{\Sigma F}{m}$$

Where ΣF is the vector sum of all the individual forces acting on the body. This sum is also called the *net force*. To visualize the magnitude and direction of the net force, it can be very helpful to draw a free body diagram.

4.1 Interpreting Free Body Diagrams

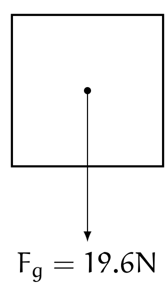
Before you learn to draw your own free body diagrams (FBDs), let's examine and analyze a few. Here is a FBD for an accelerating car:



We can consider the x and y axes separately. On the y-axis, we see that the weight (F_g) is the same magnitude as the normal force (F_N) because the arrows are the same length. This means in the y direction, the forces are balanced, and thus we do not expect to see acceleration in the y direction. This makes intuitive sense: if you are accelerating your car on a flat road, your car does not go up into the air. How about the x direction? Here we see friction (F_f) pulling the car backwards while the engine is pushing the car forwards. In this case, the forces are *not* balanced. See how the F_{engine} arrow is longer than the F_f arrow? This means the forward force from the engine is *greater in magnitude* than the backward force of F_f. Therefore, the *net force* in the x direction is to the right, and the car will accelerate to the right.

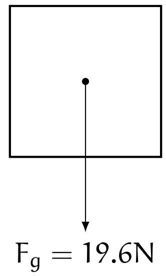
4.2 Drawing Free Body Diagrams

A free body diagram consists of the object, usually represented as a simple square, with various arrows representing the forces acting on the object. The direction and relative lengths of the arrows show the direction and relative strength of the forces. Here is a free body diagram for a 2-kg hammer in free fall through the air on Earth:



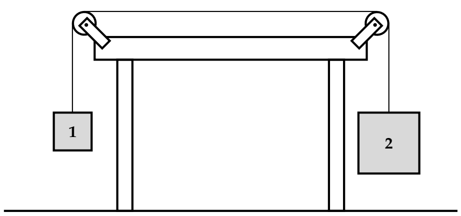
4.2.1 Objects At Rest

If an object is at rest, then its acceleration must be $0\frac{m}{s^2}$. By Newton's Second Law, the *net force* acting on that object must also be zero. Does this mean there are no forces acting on the object? NO! It means all the forces are *balanced*: for every upwards force, there is an equal downwards force, and so on. Consider the same 2-kg hammer, but this time it is sitting on a table. We know gravity must be acting on the hammer, so let's begin with that:


$F_g = 19.6N$

Exercise 3 Blocks on a Table

[This exercise was originally presented as a free-response question on the 2015 AP Physics 1 exam.] Two blocks are connected with a string and two pulleys over a table, as shown in the figure. Block 1 has a mass of 3 kg and Block 2 has a mass of 9 kg. The blocks are released from rest. Treat the string as massless and the pulleys as massless and frictionless when answering the questions below.

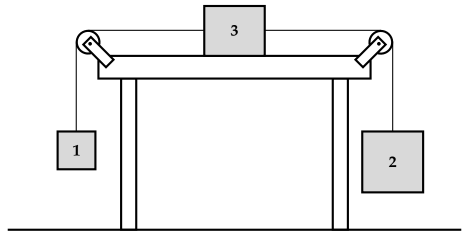


1. Draw free body diagrams for both blocks. Correctly show the relative magnitude of the forces using the relative lengths of the vectors.
2. Before doing any calculations, describe each block's acceleration as (a) positive or negative, (b) having a magnitude greater or less than g. Take the upward direction as positive.
3. Calculate the acceleration of each block. Take the upward direction as positive and express your answer in terms of g.

Working Space

Answer on Page 48

Now here's an interesting question: what happens if we add another mass to the system? Imagine that a third block with a mass of 15 kg is laid on the table from the previous exercise and the string is attached to each side, as shown below.



We could redraw free body diagrams for all three blocks, noting that the tensions on either side of Block 3 will be different. Then, we would have another system of equations to solve and we could find the acceleration. Or, we could think about the entire system. Originally, the net force on the two-block system was 58.8 Newtons (the weight of the larger block minus the weight of the smaller block). And the mass of the entire two-block system was 12 kilograms. We can apply Newton's Second Law:

$$a = \frac{58.8\text{N}}{12\text{kg}} = 4.9\frac{\text{m}}{\text{s}^2} = \frac{1}{2}g$$

(Notice, this is the same answer we found in the previous exercise). What changes when we add a third, 15-kg block? The net force on the system doesn't change: the table pushes back up on the third block with a force equal to the block's weight. However, the total mass of the system has changed: it's increased. Now the total mass is 27 kg, and we can again apply Newton's Second Law:

$$a = \frac{58.8\text{N}}{27\text{kg}} \approx 2.18\frac{\text{m}}{\text{s}^2}$$

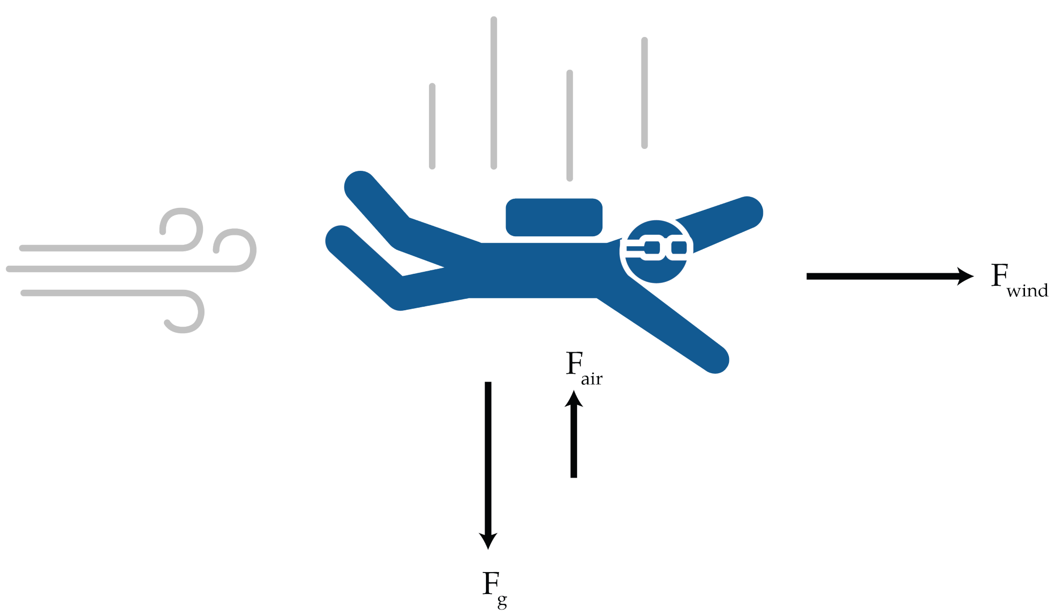
This answer makes sense: the mass of the system has increased while the net force acting on the system has stayed the same, resulting in a slower acceleration.

Chapter 5

Vectors

We have talked a some about forces, but in the calculations that we have done, we have only talked about the magnitude of a force. It is equally important to talk about its direction. To do the math on things with a magnitude and a direction (like forces), we need vectors.

For example, if you jump out of a plane (hopefully with a parachute), several forces with different magnitudes and directions will be acting upon you. Gravity will push you straight down. That force will be proportional to your weight. If there were a wind from the west, it would push you toward the east. That force will be proportional to the square of the speed of the wind and approximately proportional to your size. Once you are falling, there will be resistance from the air that you are pushing through — that force will point in the opposite direction from the direction you are moving and will be proportional to the square of your speed.



To figure out the net force (which will tell us how we will accelerate), we will need to add these forces together. To do this, we need to learn to do math with vectors.

5.1 Adding Vectors

A vector is typically represented as a list of numbers, with each number representing a particular dimension. For example, if you are creating a 3-dimensional vector representing a force, it will have three numbers representing the amount of force in each of the three axes. For example, if a force of one newton is in the direction of the x-axis, you might represent the vector as $v = [1, 0, 0]$. Another vector might be $u = [0.5, 0.9, 0.7]$. You can see examples of 2-dimensional and 3-dimensional vectors in figures 5.1 and 5.2.

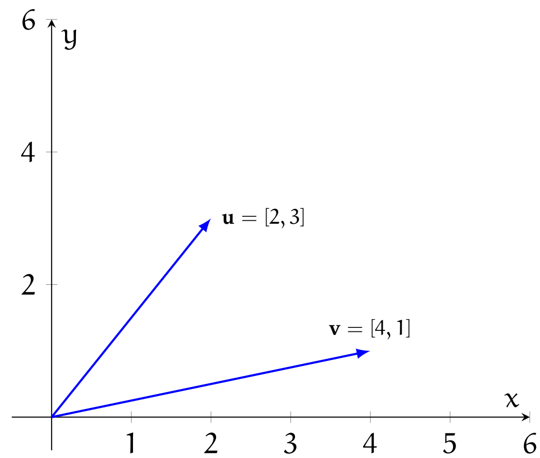


Figure 5.1: 2-dimensional vectors, **u** and **v**

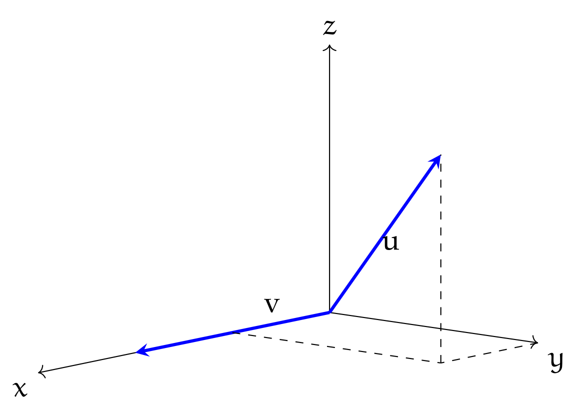


Figure 5.2: 3-dimensional vectors, **u** and **v**

Thinking visually, when we add to vectors, we put the starting point second vector at the ending point of the first vector. This is illustrated for 2-dimensional vectors in figure 5.3 and for 3-dimensional vectors in figure 5.4.

If you know the vectors, you will just add them element-wise:

$$u + v = [0.5, 0.9, 0.7] + [1.0, 0.0, 0.0] = [1.5, 0.9.0.7]$$

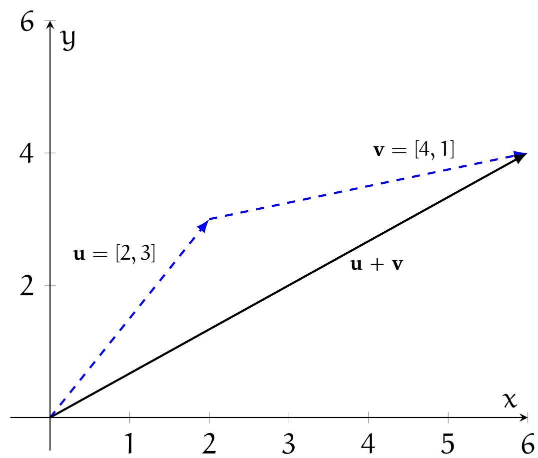


Figure 5.3: A visual representation of adding 2-dimensional vectors, **u** and **v**

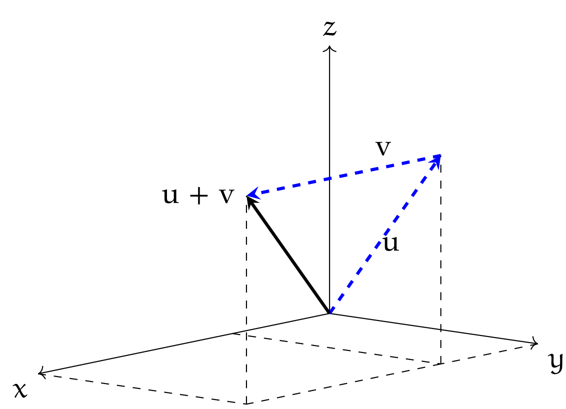


Figure 5.4: A visual representation of adding 3-dimensional vectors, **u** and **v**

These vectors have 3 components, so we say they are *3-dimensional*. Vectors can have any number of components. For example, the vector $[-12.2, 3, \pi, 10000]$ is 4-dimensional.

You can only add two vectors if they have the same dimension.

$$[12, -4] + [-1, 5] = [11, 1]$$

Addition is commutative; if you have two vectors a and b, then $a + b$ is the same as $b + a$.

Addition is also associative: If you have three vectors a, b, and c, it doesn't matter which order you add them in. That is, $a + (b + c) = (a + b) + c$.

A 1-dimensional vector is just a number. We say it is a *scalar*, not a vector.

Exercise 4 Adding vectors

Add the following vectors:

- $[1, 2, 3] + [4, 5, 6]$
- $[-1, -2, -3, -4] + [4, 5, 6, 7]$
- $[\pi, 0, 0] + [0, \pi, 0] + [0, 0, \pi]$

Working Space

Answer on Page 48

Exercise 5 Adding Forces

You are adrift in space, near two different stars. The gravity of one star is pulling you towards it with a force of $[4.2, 5.6, 9.0]$ newtons. The gravity of the other star is pulling you towards it with a force of $[-100.2, 30.2, -9.0]$ newtons. What is the net force?

Working Space

Answer on Page 49

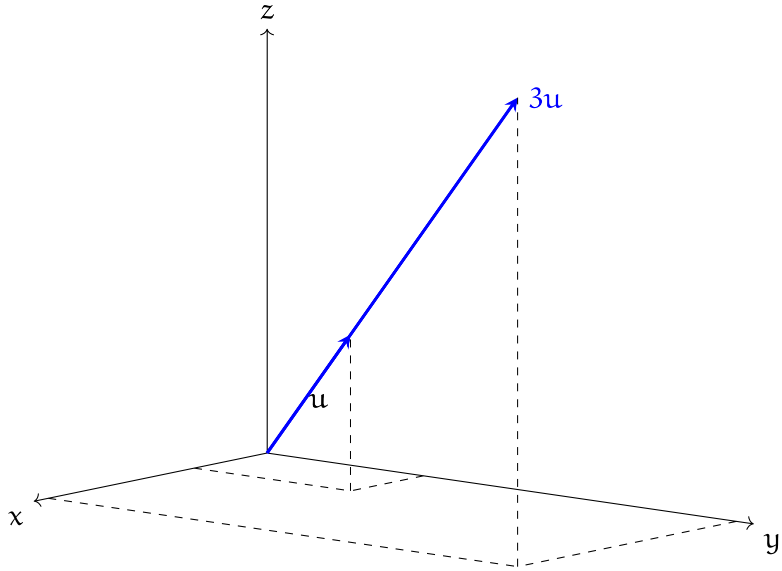


Figure 5.5: To multiply vectors, the vector gets stretched in the same direction a amount

5.2 Multiplying a vector with a scalar

It is not uncommon to multiply a vector by a scalar. For example, a rocket engine might have a force vector v. If you fire 9 engines in the exact same direction, the resulting force vector would be $9v$.

Visually, when we multiply a vector u by a scalar a, we get a new vector that goes in the same direction as u but has a magnitude a times as long as u. A visual is presented in figure 5.5.

When you multiply a vector by a scalar, you simply multiply each of the components by the scalar:

$$3 \times [0.5, 0.9, 0.7] = [1.5, 2.7, 3.6]$$

Exercise 6 Multiplying a vector and a scalar

Working Space

Simplify the following expressions:

- $2 \times [1, 2, 3]$
- $[-1, -2, -3, -4] \times -2$
- $\pi[\pi, 2\pi, 3\pi]$

Answer on Page 49

Note that when you multiply a vector times a negative number, the new vector points in the opposite direction (see figure 5.6).

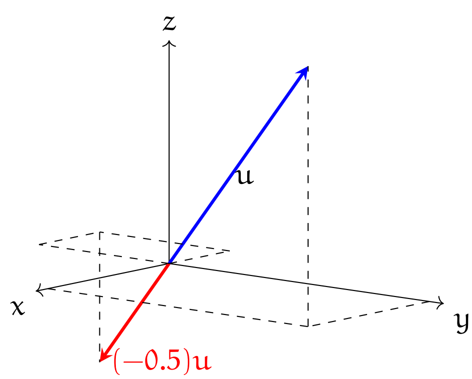


Figure 5.6: Multiplying a vector by a negative number reverses the direction of the vector.

5.3 Vector Subtraction

As you might guess, when you subtract one vector from another, you just do element-wise subtraction:

$$[4, 2, 0] - [3, -2, 9] = [1, 4, -9]$$

So, $u - v = u + (-1v)$.

Visually, you reverse the one that is being subtracted (see figure 5.7):

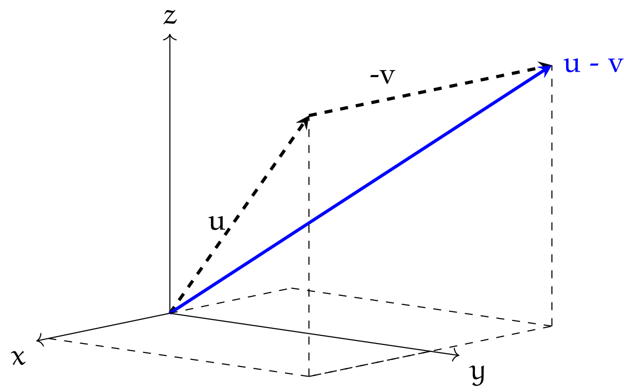


Figure 5.7: To subtract a vector, you reverse it, then add the reversed vector.

5.4 Magnitude of a Vector

The *magnitude* of a vector is just its length. We write the magnitude of a vector v as $|v|$.

We compute the magnitude using the pythagorean theorem. If $v = [3, 4, 5]$, then

$$|v| = \sqrt{3^2 + 4^2 + 5^2} = \sqrt{50} \approx 7.07$$

(You might notice that the notation for the magnitude is exactly like the notation for absolute value. If you think of a scalar as a 1-dimensional vector, the absolute value and the magnitude are the same. For example, the absolute value of -5 is 5. If you take the magnitude of the one-dimensional vector $[-5]$,you get $\sqrt{25} = 5$.)

Where does this equation come from? Consider a 2-dimensional vector, $\mathbf{v} = [3, 4]$. This means the the vector represents 3 units in the x-direction, and 4 units in the y-direction. We can then visualize a right triangle, with the vector being the hypotenuse and the legs being the x- and y-components of the vector (see figure 5.8). As you recall, the length of the hypotenuse of a right triangle is the square root of the sum of the squares of the legs. That is:

$$c = \sqrt{a^2 + b^2}$$

Where c is the length of the hypotenuse and a and b are the lengths of the legs.

We won't prove it here, but this method holds for higher-dimension vectors as well.

Magnitude of Vectors

For an n-dimensional vector, $\mathbf{v} = [x_1, x_2, x_3, \cdots, x_n]$, the magnitude of the vector is

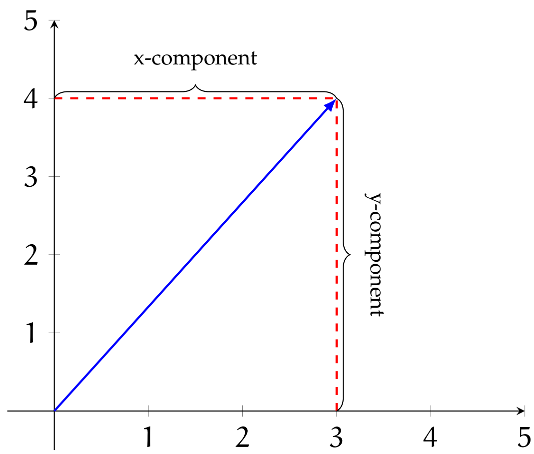


Figure 5.8: The magnitude of a vector can be thought of as the length of a hypotenuse of a right triangle.

given by:

$$|\mathbf{v}| = \sqrt{x_1^2 + x_2^2 + x_3^2 + \cdots + x_n^2}$$

Notice that if you scale up a vector, its magnitude scales by the same amount. For example:

$$|7[3,4,5]| = 7\sqrt{50} \approx 7 \times 7.07$$

Here is why that is true. Suppose we have a vector, $\mathbf{u} = [a, b, c]$. Then the magnitude of $\mathbf{u}$ is given by:

$$|\mathbf{u}| = \sqrt{a^2 + b^2 + c^2}$$

If we scale $\mathbf{u}$ to create $\mathbf{v}$ such that $\mathbf{v} = k\mathbf{u} = [ka, kb, kc]$, where k is some constant. Then the magnitude of $\mathbf{v}$ is given by:

$$|\mathbf{v}| = \sqrt{(ka)^2 + (kb)^2 + (kc)^2}$$

We can expand and simplify this equation:

$$|\mathbf{v}| = \sqrt{k^2a^2 + k^2b^2 + k^2c^2}$$

$$|\mathbf{v}| = \sqrt{k^2\left(a^2 + b^2 + c^2\right)}$$

$$|\mathbf{v}| = \left(\sqrt{k^2}\right)\sqrt{a^2 + b^2 + c^2}$$

$$|\mathbf{v}| = |k|\sqrt{a^2 + b^2 + c^2} = |k|\,|\mathbf{u}|$$

So, if you scale a vector, the magnitude of the resulting vector is the absolute value of the scale factor times the magnitude of the original vector.

The rule then is: If you have any vector v and any scalar k:

$$|kv| = |k||v|$$

5.4.1 Unit Vectors

A *unit vector* is a vector whose magnitude is 1. For any non-zero vector $\mathbf{v}$, the unit vector pointing in the same direction is

$$\vec{\mathbf{u}} = \frac{\mathbf{v}}{|\mathbf{v}|}.$$

For example, if $\mathbf{v} = [3, 4, 5]$ then

$$|\mathbf{v}| = \sqrt{3^2 + 4^2 + 5^2} = \sqrt{50},$$

so

$$\vec{\mathbf{u}} = \frac{1}{\sqrt{50}}[3, 4, 5] = \left[\frac{3}{\sqrt{50}}, \frac{4}{\sqrt{50}}, \frac{5}{\sqrt{50}}\right].$$

$$\boxed{\hat{\mathbf{u}} = \frac{\mathbf{v}}{|\mathbf{v}|}}, \qquad |\vec{\mathbf{u}}| = 1.$$

Exercise 7 Magnitude of a Vector

Find the magnitude of the following vectors:

- $[1, 1, 1]$
- $[-5, -5, -5]$ (that is the same as $-5 \times [1, 1, 1]$)
- $[3, 4, -4] + [-2, -3, 5]$

Working Space

Answer on Page 49

5.5 Vectors in Python

NumPy is a library that allows you to work with vectors in Python. You might need to install it on your computer. This is done with `pip`. `pip3` installs things specifically for Python 3.

```
pip3 install NumPy
```

We can think of a vector as a list of numbers. There are also grids of numbers known as *matrices*. NumPy deals with both in the same way, so it refers to both of them as arrays.

The study of vectors and matrices is known as *Linear Algebra*. Some of the functions we need are in a sublibrary of NumPy called `linalg`.

As a convention, everyone who uses NumPy, imports it as *np*.

Create a file called `first_vectors.py`:

```
import NumPy as np

# Create two vectors
v = np.array([2,3,4])
u = np.array([-1,-2,3])
print(f"u = {u}, v = {v}")

# Add them
w = v + u
print(f"u + v = {w}")

# Multiply by a scalar
w = v * 3
print(f"v * 3 = {w}")

# Get the magnitude
# Get the magnitude
mv = np.linalg.norm(v)
mu = np.linalg.norm(u)
print(f"|v| = {mv}, |u| = {mu}")
```

When you run it, you should see:

```
> python3 first_vectors.py
u = [-1 -2  3], v = [2 3 4]
u + v = [1 1 7]
```

```
v * 3 = [ 6  9 12]
|v| = 5.385164807134504, |u| = 3.7416573867739413
```

5.5.1 Formatting Floats

The numbers 5.385164807134504 and 3.7416573867739413 are pretty long. You probably want them rounded off after a couple of decimal places.

Numbers with decimal places are called *floats*. In the placeholder for your float, you can specify how you want it formatted, including the number of decimal places.

Change the last line to look like this:

```
print(f"|v| = {mv:.2f}, |u| = {mu:.2f}")
```

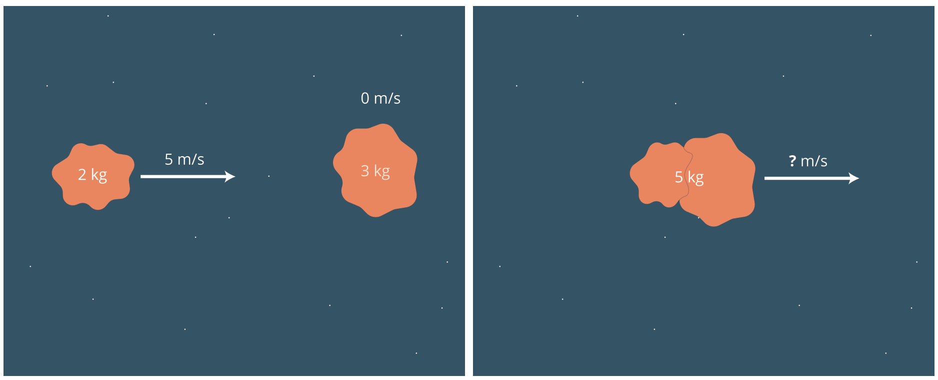
When you run the code, it will be neatly rounded off to two decimal places:

```
|v| = 5.39, |u| = 3.74
```

Chapter 6

Momentum

Let's say a 2 kg block of putty is flying through space at 5 meters per second, and it collides with a larger 3 kg block of putty that is not moving at all. When the two blocks deform and stick to each other, how fast will the resultant big block be moving?



Every object has *momentum*. The momentum is a vector quantity — it points in the direction that the object is moving and has a magnitude equal to its mass times its speed.

$$p = mv$$

Given a set of objects that are interacting, we can sum all their momentum vectors to get the total momentum. In such a set, the total momentum will stay constant.

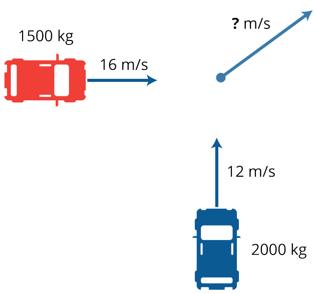
$$p = p_1 + p_2 = m_1 v_1 + m_2 v_2$$

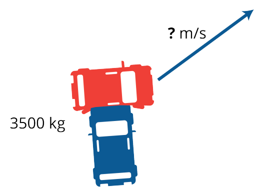
In our example, one object has a momentum vector of magnitude of 10 kg m/s, the other has a momentum of magnitude 0. Once they have merged, they have a combined mass of 5 kg. This means the velocity vector must have magnitude 2 m/s and pointing in the same direction that the first mass was moving.

Exercise 8 Cars on Ice

A car weighing 1000 kg is going north at 12 m/s. Another car weighing 1500 kg is going east at 16 m/s. They both hit a patch of ice (with zero friction) and collide. Steel is bent, and the two objects become one. How what is the velocity vector (direction and magnitude) of the new object sliding across the ice?

Working Space





Answer on Page 49

Note that kinetic energy ($1/2mv^2$) is *not* conserved here. Before the collision, the moving putty block has $(1/2)(2)(5^2) = 25$ joules of kinetic energy. Afterward, the big block has $(1/2)(5)(2^2) = 10$ joules of kinetic energy. What happened to the energy that was lost? It was used up deforming the putty.

What if the blocks were marble instead of putty? Then there would be very little deforming, so kinetic energy *and* momentum would be conserved. The two blocks would end up having different velocity vectors.

Let's assume for a moment that they strike each other straight on, so there is motion in only one direction, both before and after the collision. Can we solve for the speeds of the first block (v_1) and the second block (v_2)?

We end up with two equations. Conservation of momentum says:

$$2v_1 + 3v_2 = 10$$

Conservation of kinetic energy says:

$$(1/2)(2)(v_1^2) + (1/2)(3)(v_2^2) = 25$$

Using the first equation, we can solve for v_1 in terms of v_2:

$$v_1 = \frac{10 - 3v_2}{2}$$

Substituting this into the second equation, we get:

$$\left(\frac{10 - 3v_2}{2}\right)^2 + \frac{3v_2^2}{2} = 25$$

Simplifying, we get:

$$v_2^2 - 4v_2 + 0 = 0$$

This quadratic has two solutions: $v_2 = 0$ and $v_2 = 4$. $v_2 = 0$ represents the situation before the collision. Substituting in $v_2 = 4$:

$$v_1 = \frac{10 - 3(4)}{2} = -1$$

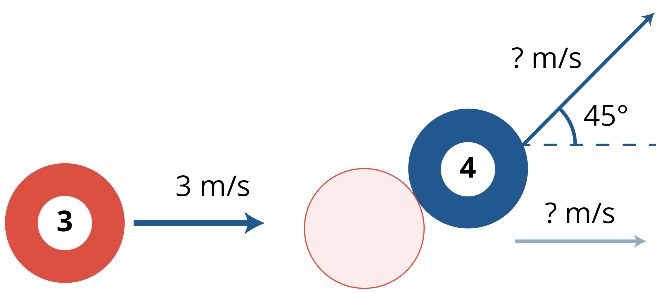
Thus, if the blocks are hard enough that kinetic energy is conserved, after the collision, the smaller block will be heading in the opposite direction at 1 m/s. The larger block will be moving at 4 m/s in the direction of the original motion.

Exercise 9 Billiard Balls

Working Space

A billiard ball weighing 0.4 kg and traveling at 3 m/s hits a billiard ball (same weight) at rest. It strikes obliquely (neither perpendicular nor parallel), so that the ball at rest starts to move at a 45 degree angle from the path of the ball that hit it.

Assuming all kinetic energy is conserved, what is the velocity vector of each ball after the collision?



Answer on Page 50

Appendix A

Answers to Exercises

Answer to Exercise 1 (on page 5)

At it's apex, the hammer's velocity is $0m/s$, so we will find the time where $v = 0m/s$:

$$0\frac{m}{s} = 12\frac{m}{s} - 9.8\frac{m}{s^2} \cdot t$$

$$t = \frac{12\frac{m}{s}}{9.8\frac{m}{s^2}} \approx 1.22s$$

Answer to Exercise 2 (on page 20)

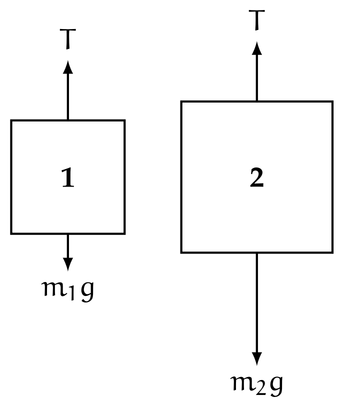
1. D, B/C/F, A, E; displacement is the difference in position between the starting and ending points. Object D moves backwards and therefore has a negative displacement. Objects B, C, and F all end in the same position they started and therefore have zero displacement. Objects A and E both move forward and have positive displacement. Object E moves forward 4 units while Object A only moves forward by 1 unit. Therefore, object E has a greater positive displacement than object A.

2. C, D, B/F, A, E; instantaneous velocity is given by the slope of a position-time graph at the indicated time (in this case, $t = 7.5s$). Since the graphs show 10 seconds of motion and there are 4 tick marks on each x-axis, each unit on the x-axis represents 2.5 seconds of time. Therefore, $t = 7.5s$ is the third tick mark on the x-axis. At $t = 7.5s$, the graphs of objects C and D have negative slopes and therefore negative velocities. Since the slope for object C is steeper than the slope for object D, object C's *speed* is greater than object D's, and object C's *velocity* is more negative than object D's. The graphs of object's B and F are horizontal at $t = 7.5s$ and therefore their velocities are zero. Graphs A and E have positive slopes, and since E is steeper, object E has a greater speed and more positive velocity than object A.

3. B/F, A, D, E, C; distance is a scalar and therefore always positive. B and F do not change position, and therefore travel a distance of 0. A moves forward 1 unit. D moves backwards 3 units (for a *distance* of 3). E moves forward 4 units. C moves forwards 3 units, then backwards 3 additional units, for a total of 6 units of distance.

Answer to Exercise 3 (on page 28)

1. Each block is acted on by gravity and the tension of the string. The force of gravity on block 2 is 3 times that of block 1, and the tension vector should be between the lengths of the gravity vectors:



2. Block 1 will have a positive acceleration with a magnitude less than g. Block 2 will have a negative acceleration with a magnitude greater than g.

3. Since the blocks are connected, they will move as a system and therefore have the same magnitude acceleration. From this, the free body diagrams, and Newton's Second Law, we know that:

$$m_1 a = T - m_1 g$$

$$m_2 (-a) = T - m_2 g$$

We know m_1 and m_2, so the two unknowns are T and a. One way to solve this system of equations would be to subtract equation 2 from equation 1:

$$m_1 a + m_2 a = m_2 g - m_1 g$$

$$a = \frac{m_2 - m_1}{m_1 + m_2} g$$

Substituting for the masses, we see that:

$$a = \frac{9-3}{9+3} g = \frac{6}{12} g = \frac{1}{2} g$$

Therefore, Block 1 is accelerating at $+\frac{1}{2}g$ and Block 2 is accelerating at $-\frac{1}{2}g$.

Answer to Exercise 4 (on page 34)

- $[1, 2, 3] + [4, 5, 6] = [5, 7, 9]$
- $[-1, -2, -3, -4] + [4, 5, 6, 7] = [3, 3, 3, 3]$
- $[\pi, 0, 0] + [0, \pi, 0] + [0, 0, \pi] = [\pi, \pi, \pi]$

Answer to Exercise 5 (on page 34)

To get the net force, you add the two forces:

$$F = [4.2, 5.6, 9.0] + [-100.2, 30.2, -9.0] = [-96, 35.8, 0.0] \text{ newtons}$$

Answer to Exercise 6 (on page 36)

- $2 \times [1, 2, 3] = [2, 4, 6]$
- $[-1, -2, -3, -4] \times -3 = [3, 6, 9, 12]$
- $\pi[\pi, 2\pi, 3\pi] = \pi^2, 2\pi^2, 3\pi^2]$

Answer to Exercise 7 (on page 39)

- $|[1, 1, 1]| = \sqrt{3} \approx 1.73$
- $|[-5, -5, -5]| = |-5 \times [1, 1, 1]| = 5\sqrt{3} \approx 8.66$
- $|[3, 4, 5] + [-2, -3, -4]| = |[1, 1, 1]| = \sqrt{3} \approx 1.73$

Answer to Exercise 8 (on page 44)

The momentum of the first car is 12,000 kg m/s in the north direction.

The momentum of the second car is 24,000 kg m/s in the east direction.

The new object will be moving northeast. What is the angle compared with the east?

$$\theta = \arctan \frac{12,000}{24,000} \approx 0.4636 \text{ radians } \approx 26.565 \text{ degrees north of east}$$

The magnitude of the momentum of the new object is $\sqrt{12,000^2 + 24,000^2} \approx 26,833$ kg m/s

Its new mass is 2,5000 kg. So the speed will be $26,833/2,500 = 10.73$ m/s.

Answer to Exercise 9 (on page 46)

The original forward momentum was 1.2 kg m/s. The original kinetic energy is $(1/2)(0.4)(3^2)$ = 1.8 joules.

Let s be the post-collision speed of the ball that had been at rest. Let x and y be the forward and sideways speeds (post-collision) of the other ball. Conservation of kinetic energy says

$$(1/2)(0.4)(s^2) + (1/2)(0.4)(x^2 + y^2) = 1.8$$

Forward momentum is conserved:

$$0.4\frac{s}{\sqrt{2}} + 0.4x = 1.2$$

Which can be rewritten:

$$x = 3 - \frac{s}{\sqrt{2}}$$

Sideways momentum stays zero:

$$(0.4)\frac{s}{\sqrt{2}} - 0.4y = 0.0$$

Which can be rewritten:

$$y = \frac{s}{\sqrt{2}}$$

Substituting into to the conservation of kinetic energy equation above:

$$(1/2)(0.4)(s^2) + (1/2)(0.4)(\left(3 - \frac{s}{\sqrt{2}}\right)^2 + \left(\frac{s}{\sqrt{2}}\right)^2 = 1.8$$

Which can be rewritten:

$$s^2 - \frac{3}{\sqrt{2}}s + 0 = 0$$

There are two solutions to this quadratic: $s = 0$ (before collision) and $s = \frac{3}{\sqrt{2}}$. Thus,

$$y = \frac{3}{2}$$

and

$$x = 3 - \frac{3}{2} = \frac{3}{2}$$

So, both balls careen off at 45° angles at the exact same speed.

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