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Chapter 1

Solving Quadratics

A quadratic function has three terms: $ax^2 + bx + c$. a, b, and c are known as the *coeffcients*. The coefficients can be any constant, except that a can never be zero. (If a is zero, it is a linear function, not a quadratic.)

When you have an equation with a quadratic function on one side and a zero on the other, you have a quadratic equation. For example:

$$72x^2 - 12x + 1.2 = 0$$

How can you find the values of x that will make this equation true?

You can always reduce a quadratic equation so that the first coefficient is 1, so that your equation looks like this:

$$x^2 + bx + c = 0$$

For example, if you are asked to solve $4x^2 + 8x - 19 = -2x^2 - 7$

$$4x^2 + 8x - 19 = -2x^2 - 7$$

$$6x^2 + 8x - 12 = 0$$

$$x^2 + \frac{4}{3}x - 2 = 0$$

Here, $b = \frac{4}{3}$ and $c = -2$.

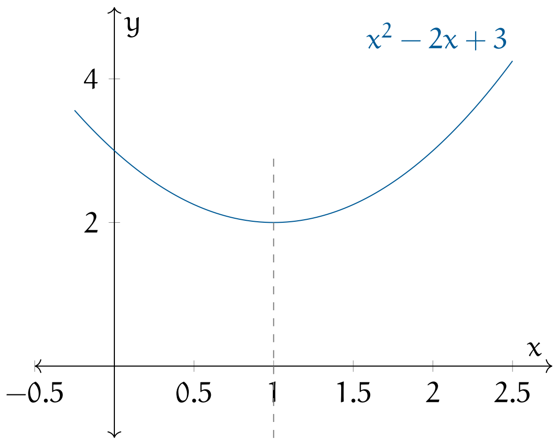
$x^2 + bx + c = 0$ when

$$x = -\frac{b}{2} \pm \frac{\sqrt{b^2 - 4c}}{2}$$

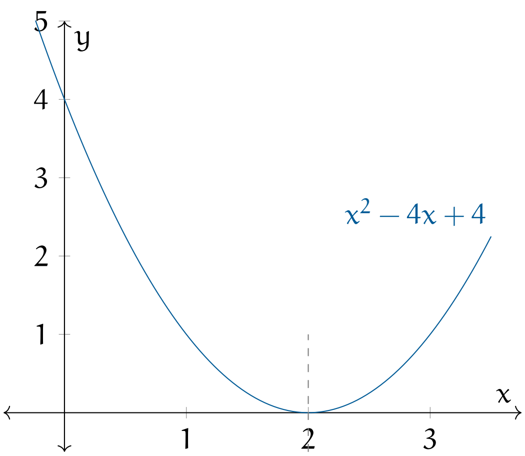
What does this mean?

For any b and c, the graph of $x^2 + bx + c$ is a parabola that goes up on each end. Its low point is at $x = -\frac{b}{2}$.

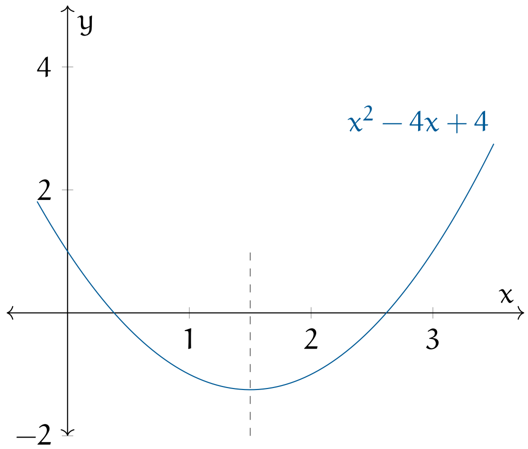
If there are no real roots ($b^2 - 4c < 0$), which means the parabola never gets low enough to cross the x-axis:



If there is one real root $(b^2 - 4c = 0)$, it means that the parabola only touches the x-axis.



If there are two real roots $(b^2 - 4c > 0)$, it means that the parabola crosses the x-axis twice as it dips below and then returns:



Exercise 1 Roots of a Quadratic

Working Space

In the last chapter, you found that the function for the height of your flying hammer is:

$$p = -\frac{1}{2}9.8t^2 + 12t + 2$$

At what time will the hammer hit the ground?

Answer on Page 33

1.1 The Traditional Quadratic Formula

If the last explanation was a little tricky to understand, the quadratic formula is a nifty tool.

The Quadratic Formula

$ax^2 + bx + c = 0$ when

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Chapter 2

Complex Numbers

Complex numbers are an extension of real numbers, which in turn are an extension of rational numbers. In mathematics, the set of complex numbers is a number system that extends the real number line to a full two dimensions, using the imaginary unit, which is denoted by i, with the property that $i^2 = -1$.

2.1 Definition

A complex number is a number of the form $a + bi$, where a and b are real numbers, and i is the imaginary unit, with the property that $i^2 = -1$. The real part of the complex number is a, and the imaginary part is b.

2.2 Why Are Complex Numbers Necessary?

Complex numbers are essential to many fields of science and engineering. Here are a few reasons why:

2.2.1 Roots of Negative Numbers

In the real number system, the square root of a negative number does not exist, because there is no real number that you can square to get a negative number. The introduction of the imaginary unit i, which has the property that $i^2 = -1$, allows us to take square roots of negative numbers and gives rise to complex numbers.

2.2.2 Polynomial Equations

The fundamental theorem of algebra states that every non-constant polynomial equation with complex coefficients has a complex root. This theorem guarantees that polynomial equations of degree n always have n roots in the complex plane.

2.2.3 Physics and Engineering

In physics and engineering, complex numbers are used to represent waveforms in control systems, in quantum mechanics, and many other areas. Their properties make many mathematical manipulations more convenient.

2.3 Adding Complex Numbers

The addition of complex numbers is straightforward. If we have two complex numbers $z_1 = a + bi$ and $z_2 = c + di$, their sum is defined as:

$$z_1 + z_2 = (a + c) + (b + d)i \tag{2.1}$$

In other words, you add the real parts to get the real part of the sum, and add the imaginary parts to get the imaginary part of the sum.

2.4 Multiplying Complex Numbers

The multiplication of complex numbers is a bit more involved. If we have two complex numbers $z_1 = a + bi$ and $z_2 = c + di$, their product is defined as:

$$z_1 \cdot z_2 = (a + bi) \cdot (c + di) = ac + adi + bci - bd = (ac - bd) + (ad + bc)i \tag{2.2}$$

Note the last term comes from $i^2 = -1$. You multiply the real parts and the imaginary parts just as you would in a binomial multiplication, and remember to replace i^2 with -1.

Chapter 3

Introduction to Sequences

A sequence is a list of numbers in a particular order. {1, 3, 5, 7, 9} is a sequence. So is $\{\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\cdots\}$. There are many types of sequences. We will present two of the most common types in this chapter: arithmetic and geometric sequences.

Sequences are generally represented like this:

$$a_1, a_2, a_3, a_4, ..., a_n, ...$$

The first number, a_1, is called the *first term*, a_2 is the *second term*, and a_n is the *nth term*. A sequence can be finite or infinite. If the sequence is infinite, we represent that with ellipses $(\cdots)$ at the end of the list, to indicate that there are more numbers.

We can also write formulas to represent a sequence. Take the first example, the finite sequence {1, 3, 5, 7, 9}. Notice that each term is two more than the previous term. We can define the sequence *recursively* by defining the n^{th} term as a function of the $(n-1)^{th}$ term. In our example, we see that $a_n = a_{n-1} + 2$ with $a_1 = 1$ for $1 \leq n \leq 5$. This is called a recursive formula, because you have to already know the $(n-1)^{th}$ term to find the n^{th} term.

Another way to write a formula for a sequence is to find a rule for the n^{th} term. In our example sequence, the first term is 1 plus 0 times 2, the second term is 1 plus 1 times 2, the third term is 1 plus 2 times 2, and so one. Did you notice the pattern? The n^{th} term is 1 plus (n-1) times 2. We can write this mathematically:

$$a_n = 1 + 2(n-1) \text{ for } 1 \leq n \leq 5$$

This is called the *explicit* formula because each term is explicitly defined. Notice that for the second way of writing a formula, we don't have to state what the first term is — the formula tells us.

3.1 Arithemtic sequences

Our first example sequence, {1, 3, 5, 7, 9} is a *finite*, *arithmetic* sequence. We know it is finite because there is a limited number of terms in the sequence (in this case, 5). How do we know it is arithmetic?

An arithmetic sequence is one where you add the same number every time to get the next term. Our example is an arithmetic sequence because you add 2 to get the next term every time. That number that you add is called the *common difference*, so we can say the sequence {1, 3, 5, 7, 9} has a common difference of 2. The common difference can be positive (in the case of an increasing arithmetic sequence) or negative (in the case of a decreasing arithmetic sequence). Formally, we can find the common difference of an arithmetic sequence by subtracting the $(n-1)^{th}$ term from the n^{th} term:

$$d = a_n - a_{n-1}$$

Exercise 2

Which of the following are arithmetic sequences? For the arithmetic sequences, find the common difference.

1. $\{\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, ...\}$
2. {5, 8, 11, 14, 17, ...}
3. {3, -1, -5, -9, ...}
4. {-1, 2, -3, 4, -5, 6, ...}

Working Space

Answer on Page 33

3.1.1 Formulas for arithmetic sequences

If you are given an arithmetic sequence, you can write an explicit or recursive formula. You can think of the formula as a function where the domain (input) is restricted to integers greater than or equal to one. Let's write explicit and recursive formulas for the sequence {3, -1, -5, -9, ...}.

For either type of formula, we need to identify the common difference. Since each term is 4 less than the previous term, the common difference is -4 (see figure 3.1). This means the n^{th} term is the $(n-1)^{th}$ term minus 4. The general form of a recursive formula is $a_n = a_{n-1} + d$, where d is the common difference. For our example, the common difference is -4, so we can write a recursive formula:

$$a_n = a_{n-1} - 4$$

However, this formula doesn't tell us what a_1 is! For recursive formulas, you have to specify the first term in the sequence. So, the *complete* recursive formula for the sequence is:

$$a_n = a_{n-1} - 4$$
$$a_1 = 3$$

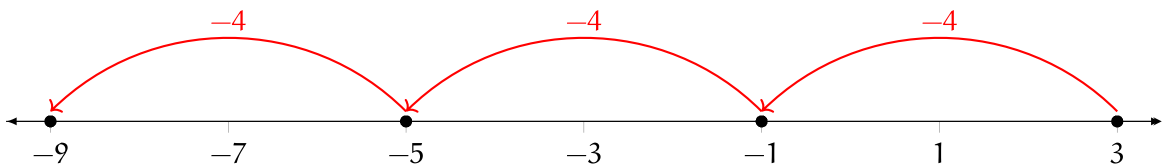


Figure 3.1: The common difference in the sequence {3, -1, -5, -9,...} is -4

Recursive formulas make it easy to see how each term is related to the next term. However, it is difficult to use recursive formulas to find a specific term. Say we wanted to know the 7^{th} term in the sequence. Well, from the formula, we know that:

$$a_7 = a_6 - 4$$

What is a_6? Again, we see that

$$a_6 = a_5 - 4$$

Now we have to find a_5! If we keep going, we see that:

$$a_5 = a_4 - 4$$
$$a_4 = a_3 - 4$$
$$a_3 = a_2 - 4$$
$$a_2 = a_1 - 4$$

Since we were told a_1, we can find a_2 and propagate our terms back up the chain to find a_7:

$$a_2 = 3 - 4 = -1$$
$$a_3 = a_2 - 4 = -1 - 4 = -5$$
$$a_4 = a_3 - 4 = -5 - 4 = -9$$
$$a_5 = a_4 - 4 = -9 - 4 = -13$$
$$a_6 = a_5 - 4 = -13 - 4 = -17$$
$$a_7 = a_6 - 4 = -17 - 4 = -21$$

Ultimately, we see that $a_7 = -21$. That was a lot of work! You can imagine that for higher

n terms, such as the 100^{th} or 1000^{th} term, this method becomes cumbersome. This is where the explicit formula is more useful.

The general form of an explicit formula for an arithmetic sequence is

$$a_n = a_1 + d \times (n - 1)$$

where d is the common difference. For our example sequence, {3, -1, -5, -9, ...}, the common difference is -4. So the explicit formula is

$$a_n = 3 + (-4)(n - 1) = 3 - 4(n - 1)$$

You may be tempted to distribute and simplify, which is fine and yields an equivalent formula:

$$a_n = 7 - 4n$$

Now, to find the 7^{th} term, all we have to do is substitute $n = 7$:

$$a_7 = 3 - 4(7 - 1) = 3 - 4(6) = 3 - 24 = -21$$

We get the same answer with much less effort!

Exercise 3

An arithmetic sequence is defined by the recursive formula $a_n = a_{n-1} + 5$ with $a_1 = -4$. Write the first 5 terms of the sequence and determine an explicit formula for the same sequence.

Working Space

Answer on Page 33

Exercise 4

The first four terms of an arithmetic sequence are $\{\pi, \frac{3\pi}{2}, 2\pi, \frac{5\pi}{2}\}$. What is the common difference? Write explicit and recursive formulas for the infinite sequence.

Working Space

Answer on Page 34

3.2 Geometric sequences

Let's look at the other sequence given as an example at the beginning of the chapter: $\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \cdots\}$. How is each term related to the previous term? Well, $\frac{1}{4}$ is half of $\frac{1}{2}$, and $\frac{1}{8}$ is half of $\frac{1}{4}$, so each term is the previous term multiplied by $\frac{1}{2}$. When each term in a sequence is a multiple of the previous term, this is a *geometric* sequence. The number we multiply by each time (in our example, this is $\frac{1}{2}$), which is called the *common ratio*. The common ratio can be positive or negative, but not zero.

An easy way to determine the common ratio (r) is to divide the n^{th} term by the $(n-1)^{th}$ term. In our example sequence, the first term is $\frac{1}{2}$ and the second is $\frac{1}{4}$.

$$r = \frac{a_2}{a_1} = \frac{1/4}{1/2} = \frac{1}{2}$$

which returns the common ratio we already identified, $r = \frac{1}{2}$.

If the common ratio is negative, then the sequence will "flip" back and forth from positive to negative. For example, suppose there is a geometric sequence such that $a_1 = 1$ and $r = -2$. Then the first 5 terms are {1, -2, 4, -8, 16}. Whenever you see a sequence going back and forth from positive to negative, that means the common ratio is negative.

For positive common ratios, if $r > 1$, then the sequence is increasing. And if $r < 1$, the sequence is decreasing.

3.2.1 Formulas for geometric sequences

Like arithmetic sequences, we can write recursive and explicit formulas. For geometric sequences, the recursive formula has the general form:

$$a_n = r(a_{n-1})$$

where r is the common ratio and a_1 is specified. For our example sequence, $\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \cdots\}$, the recursive formula is:

$$a_n = \frac{1}{2}a_{n-1}$$
$$a_1 = \frac{1}{2}$$

In a geometric sequence, each term is the first term, a_1, multiplied by the common ratio, r, $n - 1$ times. Therefore, the general form of an explicit formula for a geometric function is:

$$a_n = (a_1)r^{n-1}$$

Again, for our example sequence, $a_1 = \frac{1}{2}$ and $r = \frac{1}{2}$, so the explicit formula is:

$$a_n = (\frac{1}{2})(\frac{1}{2})^{(n-1)}$$

Exercise 5

Which of the following are geometric sequences? For each geometric sequence, determine the common ratio.

1. {2, -4, 6, -8,...}
2. {4, 2, 1, $\frac{1}{2}$, ...}
3. {-5, 25, -125, 525, ...}
4. {2, 0, -2, -4, ...}

Working Space

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Exercise 6

A geometric sequence is defined by the recursive formula $a_n = a_{n-1} \times \frac{3}{2}$ with $a_1 = 1$. Write the first five terms of the sequence and determine an explicit formula for the same sequence.

Working Space

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Exercise 7

The first four terms of a geometric sequence are {-4, 2, -1, $\frac{1}{2}$}. What is the common ratio? Write recursive and explicit formulas for the infinite sequence.

Working Space

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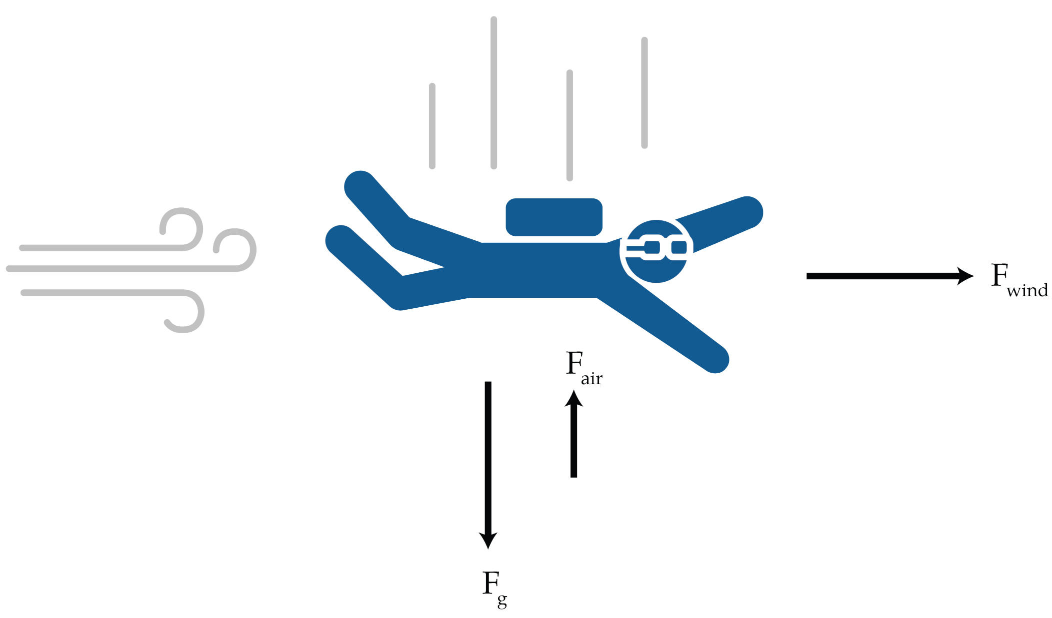
Chapter 4
Vectors

We have talked a some about forces, but in the calculations that we have done, we have only talked about the magnitude of a force. It is equally important to talk about its direction. To do the math on things with a magnitude and a direction (like forces), we need vectors.

For example, if you jump out of a plane (hopefully with a parachute), several forces with different magnitudes and directions will be acting upon you. Gravity will push you straight down. That force will be proportional to your weight. If there were a wind from the west, it would push you toward the east. That force will be proportional to the square of the speed of the wind and approximately proportional to your size. Once you are falling, there will be resistance from the air that you are pushing through — that force will point in the opposite direction from the direction you are moving and will be proportional to the square of your speed.



To figure out the net force (which will tell us how we will accelerate), we will need to add these forces together. To do this, we need to learn to do math with vectors.

4.1 Adding Vectors

A vector is typically represented as a list of numbers, with each number representing a particular dimension. For example, if you are creating a 3-dimensional vector representing a force, it will have three numbers representing the amount of force in each of the three axes. For example, if a force of one newton is in the direction of the x-axis, you might represent the vector as $v = [1, 0, 0]$. Another vector might be $u = [0.5, 0.9, 0.7]$. You can see examples of 2-dimensional and 3-dimensional vectors in figures 4.1 and 4.2.

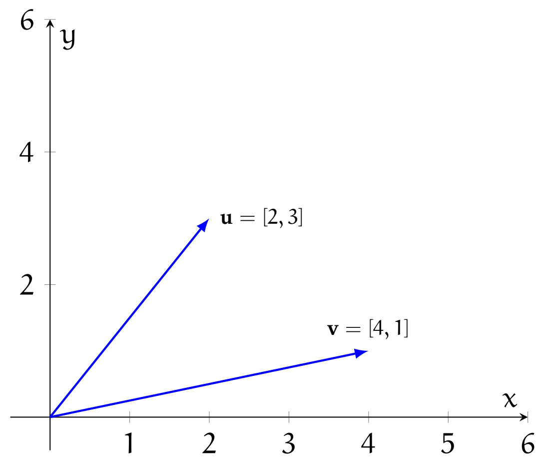


Figure 4.1: 2-dimensional vectors, **u** and **v**

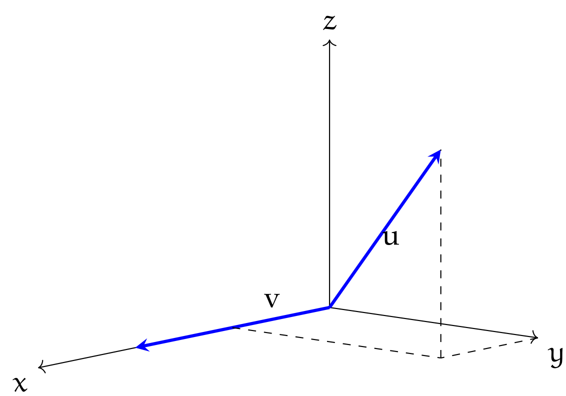


Figure 4.2: 3-dimensional vectors, **u** and **v**

Thinking visually, when we add to vectors, we put the starting point second vector at the ending point of the first vector. This is illustrated for 2-dimensional vectors in figure 4.3 and for 3-dimensional vectors in figure 4.4.

If you know the vectors, you will just add them element-wise:

$$u + v = [0.5, 0.9, 0.7] + [1.0, 0.0, 0.0] = [1.5, 0.9.0.7]$$

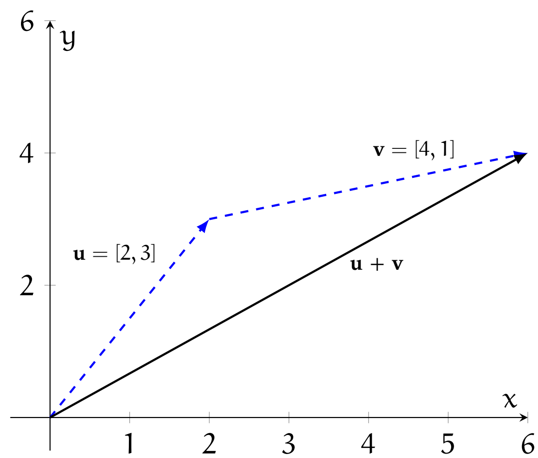


Figure 4.3: A visual representation of adding 2-dimensional vectors, **u** and **v**

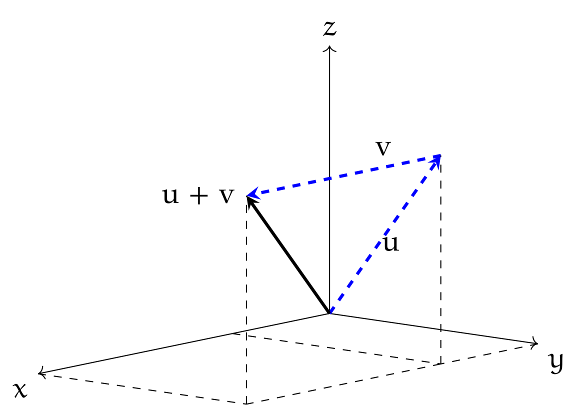


Figure 4.4: A visual representation of adding 3-dimensional vectors, **u** and **v**

These vectors have 3 components, so we say they are *3-dimensional*. Vectors can have any number of components. For example, the vector $[-12.2, 3, \pi, 10000]$ is 4-dimensional.

You can only add two vectors if they have the same dimension.

$$[12, -4] + [-1, 5] = [11, 1]$$

Addition is commutative; if you have two vectors a and b, then $a + b$ is the same as $b + a$.

Addition is also associative: If you have three vectors a, b, and c, it doesn't matter which order you add them in. That is, $a + (b + c) = (a + b) + c$.

A 1-dimensional vector is just a number. We say it is a *scalar*, not a vector.

Exercise 8 Adding vectors

Add the following vectors:

- $[1, 2, 3] + [4, 5, 6]$
- $[-1, -2, -3, -4] + [4, 5, 6, 7]$
- $[\pi, 0, 0] + [0, \pi, 0] + [0, 0, \pi]$

Working Space

Answer on Page 34

Exercise 9 Adding Forces

You are adrift in space, near two different stars. The gravity of one star is pulling you towards it with a force of $[4.2, 5.6, 9.0]$ newtons. The gravity of the other star is pulling you towards it with a force of $[-100.2, 30.2, -9.0]$ newtons. What is the net force?

Working Space

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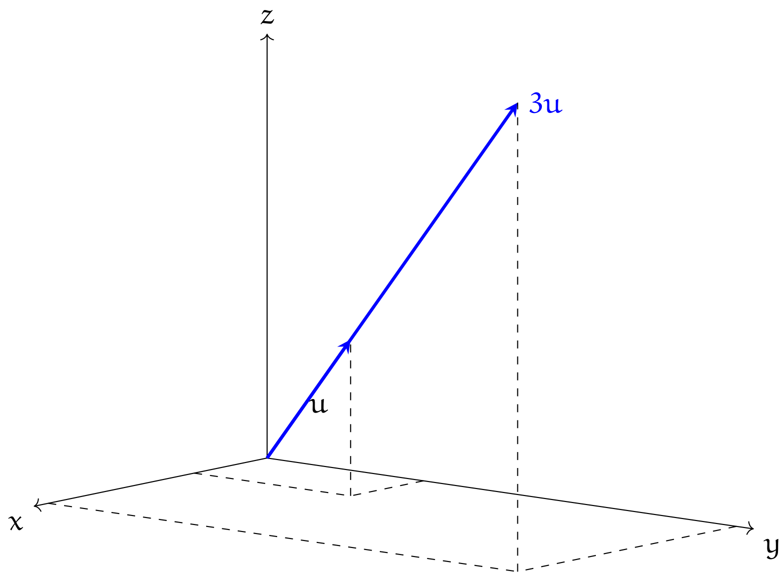


Figure 4.5: To subtract vectors, you reverse the vector that is being subtracted

4.2 Multiplying a vector with a scalar

It is not uncommon to multiply a vector by a scalar. For example, a rocket engine might have a force vector v. If you fire 9 engines in the exact same direction, the resulting force vector would be $9v$.

Visually, when we multiply a vector u by a scalar a, we get a new vector that goes in the same direction as u but has a magnitude a times as long as u. A visual is presented in figure 4.5.

When you multiply a vector by a scalar, you simply multiply each of the components by the scalar:

$$3 \times [0.5, 0.9, 0.7] = [1.5, 2.7, 3.6]$$

Exercise 10 Multiplying a vector and a scalar

Simplify the following expressions:

- $2 \times [1, 2, 3]$
- $[-1, -2, -3, -4] \times -2$
- $\pi[\pi, 2\pi, 3\pi]$

Working Space

Answer on Page 35

Note that when you multiply a vector times a negative number, the new vector points in the opposite direction (see figure 4.6).

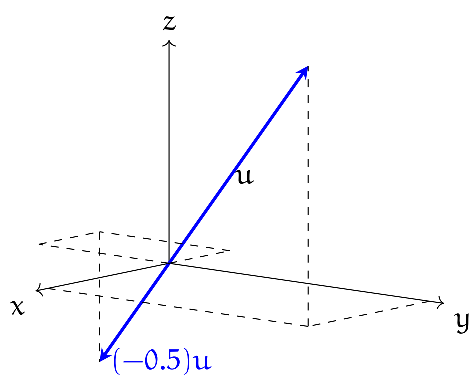


Figure 4.6: Multiplying a vector by a negative number reverses the direction of the vector.

4.3 Vector Subtraction

As you might guess, when you subtract one vector from another, you just do element-wise subtraction:

$$[4, 2, 0] - [3, -2, 9] = [1, 4, -9]$$

So, $u - v = u + (-1v)$.

Visually, you reverse the one that is being subtracted (see figure 4.7):

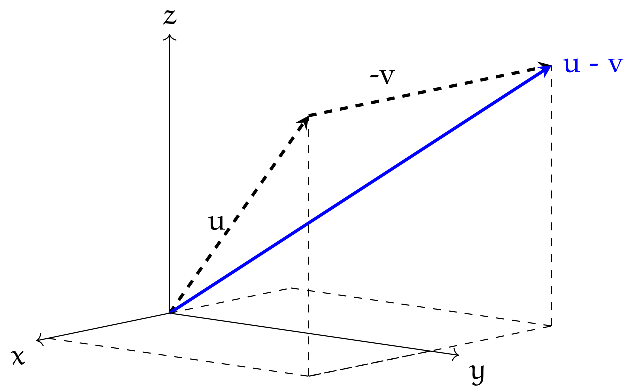


Figure 4.7: To subtract a vector, you reverse it, then add the reversed vector.

4.4 Magnitude of a Vector

The *magnitude* of a vector is just its length. We write the magnitude of a vector v as $|v|$.

We compute the magnitude using the pythagorean theorem. If $v = [3, 4, 5]$, then

$$|v| = \sqrt{3^2 + 4^2 + 5^2} = \sqrt{50} \approx 7.07$$

(You might notice that the notation for the magnitude is exactly like the notation for absolute value. If you think of a scalar as a 1-dimensional vector, the absolute value and the magnitude are the same. For example, the absolute value of -5 is 5. If you take the magnitude of the one-dimensional vector $[-5]$,you get $\sqrt{25} = 5$.)

Where does this equation come from? Consider a 2-dimensional vector, $\mathbf{v} = [3, 4]$. This means the the vector represents 3 units in the x-direction, and 4 units in the y-direction. We can then visualize a right triangle, with the vector being the hypotenuse and the legs being the x- and y-components of the vector (see figure 4.8). As you recall, the length of the hypotenuse of a right triangle is the square root of the sum of the squares of the legs. That is:

$$c = \sqrt{a^2 + b^2}$$

Where c is the length of the hypotenuse and a and b are the lengths of the legs.

We won't prove it here, but this method holds for higher-dimension vectors as well.

Magnitude of Vectors

For an n-dimensional vector, $\mathbf{v} = [x_1, x_2, x_3, \cdots, x_n]$, the magnitude of the vector is

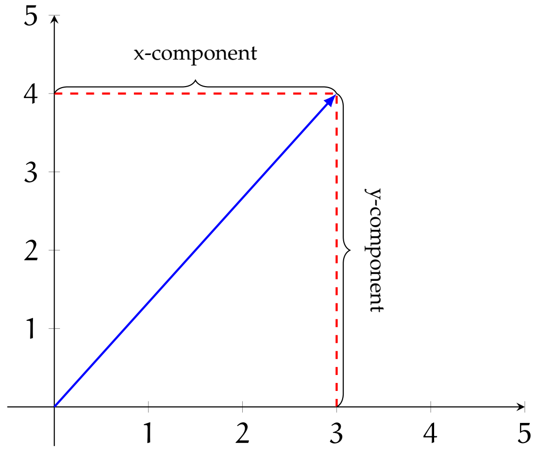


Figure 4.8: The magnitude of a vector can be thought of as the length of a hypotenuse of a right triangle.

given by:

$$|\mathbf{v}| = \sqrt{x_1^2 + x_2^2 + x_3^2 + \cdots + x_n^2}$$

Notice that if you scale up a vector, its magnitude scales by the same amount. For example:

$$|7[3, 4, 5]| = 7\sqrt{50} \approx 7 \times 7.07$$

Here is why that is true. Suppose we have a vector, $\mathbf{u} = [a, b, c]$. Then the magnitude of $\mathbf{u}$ is given by:

$$|\mathbf{u}| = \sqrt{a^2 + b^2 + c^2}$$

If we scale $\mathbf{u}$ to create $\mathbf{v}$ such that $\mathbf{v} = k\mathbf{u} = [ka, kb, kc]$, where k is some constant. Then the magnitude of $\mathbf{v}$ is given by:

$$|\mathbf{v}| = \sqrt{(ka)^2 + (kb)^2 + (kc)^2}$$

We can expand and simplify this equation:

$$|\mathbf{v}| = \sqrt{k^2a^2 + k^2b^2 + k^2c^2}$$

$$|\mathbf{v}| = \sqrt{k^2\left(a^2 + b^2 + c^2\right)}$$

$$|\mathbf{v}| = \left(\sqrt{k^2}\right)\sqrt{a^2 + b^2 + c^2}$$

$$|\mathbf{v}| = |k|\sqrt{a^2 + b^2 + c^2} = |k|\,|\mathbf{u}|$$

So, if you scale a vector, the magnitude of the resulting vector is the absolute value of the scale factor times the magnitude of the original vector.

The rule then is: If you have any vector v and any scalar k:

$$|kv| = |k||v|$$

Exercise 11 Magnitude of a Vector

Find the magnitude of the following vectors:

- $[1, 1, 1]$
- $[-5, -5, -5]$ (that is the same as $-5 \times [1, 1, 1]$)
- $[3, 4, -4] + [-2, -3, 5]$

Working Space

Answer on Page 35

4.5 Vectors in Python

NumPy is a library that allows you to work with vectors in Python. You might need to install it on your computer. This is done with `pip`. `pip3` installs things specifically for Python 3.

```
pip3 install NumPy
```

We can think of a vector as a list of numbers. There are also grids of numbers known as *matrices*. NumPy deals with both in the same way, so it refers to both of them as arrays.

The study of vectors and matrices is known as *Linear Algebra*. Some of the functions we need are in a sublibrary of NumPy called `linalg`.

As a convention, everyone who uses NumPy, imports it as *np*.

Create a file called `first_vectors.py`:

```
import NumPy as np

# Create two vectors
v = np.array([2,3,4])
u = np.array([-1,-2,3])
print(f"u = {u}, v = {v}")

# Add them
w = v + u
print(f"u + v = {w}")

# Multiply by a scalar
w = v * 3
print(f"v * 3 = {w}")

# Get the magnitude
# Get the magnitude
mv = np.linalg.norm(v)
mu = np.linalg.norm(u)
print(f"|v| = {mv}, |u| = {mu}")
```

When you run it, you should see:

```
> python3 first_vectors.py
u = [-1 -2  3], v = [2 3 4]
u + v = [1 1 7]
v * 3 = [ 6  9 12]
|v| = 5.385164807134504, |u| = 3.7416573867739413
```

4.5.1 Formatting Floats

The numbers 5.385164807134504 and 3.7416573867739413 are pretty long. You probably want them rounded off after a couple of decimal places.

Numbers with decimal places are called *floats*. In the placeholder for your float, you can specify how you want it formatted, including the number of decimal places.

Change the last line to look like this:

```
print(f"|v| = {mv:.2f}, |u| = {mu:.2f}")
```

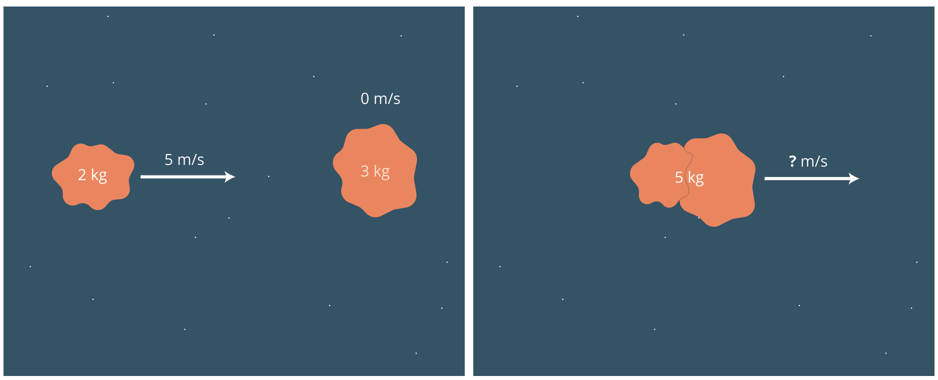
When you run the code, it will be neatly rounded off to two decimal places:

```
|v| = 5.39, |u| = 3.74
```

Chapter 5

Momentum

Let's say a 2 kg block of putty is flying through space at 5 meters per second, and it collides with a larger 3 kg block of putty that is not moving at all. When the two blocks deform and stick to each other, how fast will the resultant big block be moving?



Every object has *momentum*. The momentum is a vector quantity — it points in the direction that the object is moving and has a magnitude equal to its mass times its speed.

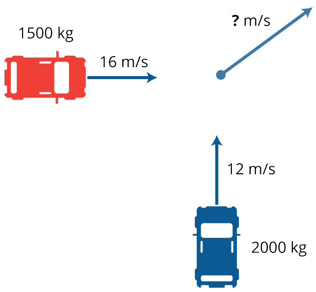
Given a set of objects that are interacting, we can sum all their momentum vectors to get the total momentum. In such a set, the total momentum will stay constant.

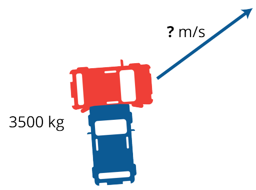
In our example, one object has a momentum vector of magnitude of 10 kg m/s, the other has a momentum of magnitude 0. Once they have merged, they have a combined mass of 5 kg. This means the velocity vector must have magnitude 2 m/s and pointing in the same direction that the first mass was moving.

Exercise 12 Cars on Ice

A car weighing 1000 kg is going north at 12 m/s. Another car weighing 1500 kg is going east at 16 m/s. They both hit a patch of ice (with zero friction) and collide. Steel is bent, and the two objects become one. How what is the velocity vector (direction and magnitude) of the new object sliding across the ice?

Working Space





Answer on Page 35

Note that kinetic energy ($1/2mv^2$) is *not* conserved here. Before the collision, the moving putty block has $(1/2)(2)(5^2) = 25$ joules of kinetic energy. Afterward, the big block has $(1/2)(5)(2^2) = 10$ joules of kinetic energy. What happened to the energy that was lost? It was used up deforming the putty.

What if the blocks were marble instead of putty? Then there would be very little deforming, so kinetic energy *and* momentum would be conserved. The two blocks would end up having different velocity vectors.

Let's assume for a moment that they strike each other straight on, so there is motion in only one direction, both before and after the collision. Can we solve for the speeds of the first block (v_1) and the second block (v_2)?

We end up with two equations. Conservation of momentum says:

$$2v_1 + 3v_2 = 10$$

Conservation of kinetic energy says:

$$(1/2)(2)(v_1^2) + (1/2)(3)(v_2^2) = 25$$

Using the first equation, we can solve for v_1 in terms of v_2:

$$v_1 = \frac{10 - 3v_2}{2}$$

Substituting this into the second equation, we get:

$$\left(\frac{10 - 3v_2}{2}\right)^2 + \frac{3v_2^2}{2} = 25$$

Simplifying, we get:

$$v_2^2 - 4v_2 + 0 = 0$$

This quadratic has two solutions: $v_2 = 0$ and $v_2 = 4$. $v_2 = 0$ represents the situation before the collision. Substituting in $v_2 = 4$:

$$v_1 = \frac{10 - 3(4)}{2} = -1$$

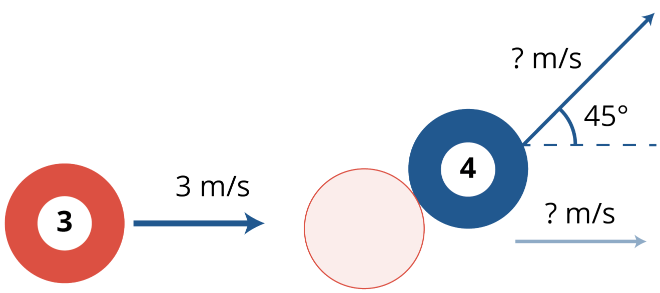
Thus, if the blocks are hard enough that kinetic energy is conserved, after the collision, the smaller block will be heading in the opposite direction at 1 m/s. The larger block will be moving at 4 m/s in the direction of the original motion.

Exercise 13 Billiard Balls

Working Space

A billiard ball weighing 0.4 kg and traveling at 3 m/s hits a billiard ball (same weight) at rest. It strikes obliquely (neither perpendicular nor parallel), so that the ball at rest starts to move at a 45 degree angle from the path of the ball that hit it.

Assuming all kinetic energy is conserved, what is the velocity vector of each ball after the collision?



Answer on Page 35

Appendix A

Answers to Exercises

Answer to Exercise 1 (on page 5)

For what t is $-4.9t^2 + 12t + 2 = 0$? Start by dividing both sides of the equation by -4.9.

$$t^2 - 2.45t - 0.408 = 0$$

The roots of this are at

$$x = -\frac{b}{2} \pm \frac{\sqrt{b^2 - 4c}}{2} = -\frac{-2.45}{2} \pm \frac{\sqrt{(-2.45)^2 - 4(-0.408)}}{2} = 1.22 \pm 1.36$$

We only care about the root after we release the hammer $(t > 0)$.

$1.22 + 1.36 = 2.58$ seconds after releasing the hammer, it will hit the ground.

Answer to Exercise 2 (on page 10)

1. not arithmetic
2. arithmetic, common difference is 3
3. arithmetic, common difference is -4
4. not arithmetic

Answer to Exercise 3 (on page 12)

The first five terms are {-4, 1, 6, 11, 16} and an explicit formula is $a_n = -4 + 5(n - 1)$.

Answer to Exercise 4 (on page 13)

The common difference is $\frac{3\pi}{2} - \pi = \frac{\pi}{2}$. The recursive formula is $a_n = a_{n-1} + \frac{\pi}{2}$ with $a_1 = \pi$. The explicit formula is $a_n = \pi + \frac{\pi}{2}(n-1)$.

Answer to Exercise 5 (on page 14)

1. not geometric
2. geometric sequence with common ratio $r = \frac{1}{2}$
3. geometric sequence with common ration $r = -5$
4. not geometric

Answer to Exercise 6 (on page 15)

The first five terms are $\{1, \frac{3}{2}, \frac{9}{4}, \frac{27}{8}, \frac{81}{16}\}$. An explicit formula for this sequence is $a_n = 1(\frac{3}{2})^{(n-1)}$.

Answer to Exercise 7 (on page 15)

The common ratio is $\frac{a_n}{a_{n-1}} = \frac{2}{-4} = -\frac{1}{2}$. A recursive formula would be $a_n = a_{n-1} \times -\frac{1}{2}$ with $a_1 = -4$. An explicit formula would be $a_n = (-4)(-\frac{1}{2})^{(n-1)}$.

Answer to Exercise 8 (on page 20)

- $[1, 2, 3] + [4, 5, 6] = [5, 7, 9]$
- $[-1, -2, -3, -4] + [4, 5, 6, 7] = [3, 3, 3, 3]$
- $[\pi, 0, 0] + [0, \pi, 0] + [0, 0, \pi] = [\pi, \pi, \pi]$

Answer to Exercise 9 (on page 20)

To get the net force, you add the two forces:

$$F = [4.2, 5.6, 9.0] + [-100.2, 30.2, -9.0] = [-96, 35.8, 0.0] \text{ newtons}$$

Answer to Exercise 10 (on page 22)

- $2 \times [1, 2, 3] = [2, 4, 6]$
- $[-1, -2, -3, -4] \times -3 = [3, 6, 9, 12]$
- $\pi[\pi, 2\pi, 3\pi] = \pi^2, 2\pi^2, 3\pi^2]$

Answer to Exercise 11 (on page 25)

- $|[1, 1, 1]| = \sqrt{3} \approx 1.73$
- $|[-5, -5, -5]| = | - 5 \times [1, 1, 1]| = 5\sqrt{3} \approx 8.66$
- $|[3, 4, 5] + [-2, -3, -4]| = |[1, 1, 1]| = \sqrt{3} \approx 1.73$

Answer to Exercise 12 (on page 30)

The momentum of the first car is 12,000 kg m/s in the north direction.

The momentum of the second car is 24,000 kg m/s in the east direction.

The new object will be moving northeast. What is the angle compared with the east?

$$\theta = \arctan \frac{12,000}{24,000} \approx 0.4636 \text{ radians } \approx 26.565 \text{ degrees north of east}$$

The magnitude of the momentum of the new object is $\sqrt{12,000^2 + 24,000^2} \approx 26,833$ kg m/s

Its new mass is 2,5000 kg. So the speed will be $26,833/2,500 = 10.73$ m/s.

Answer to Exercise 13 (on page 32)

The original forward momentum was 1.2 kg m/s. The original kinetic energy is $(1/2)(0.4)(3^2)$

= 1.8 joules.

Let s be the post-collision speed of the ball that had been at rest. Let x and y be the forward and sideways speeds (post-collision) of the other ball. Conservation of kinetic energy says

$$(1/2)(0.4)(s^2) + (1/2)(0.4)(x^2 + y^2) = 1.8$$

Forward momentum is conserved:

$$0.4\frac{s}{\sqrt{2}} + 0.4x = 1.2$$

Which can be rewritten:

$$x = 3 - \frac{s}{\sqrt{2}}$$

Sideways momentum stays zero:

$$(0.4)\frac{s}{\sqrt{2}} - 0.4y = 0.0$$

Which can be rewritten:

$$y = \frac{s}{\sqrt{2}}$$

Substituting into to the conservation of kinetic energy equation above:

$$(1/2)(0.4)(s^2) + (1/2)(0.4)(\left(3 - \frac{s}{\sqrt{2}}\right)^2 + \left(\frac{s}{\sqrt{2}}\right)^2 = 1.8$$

Which can be rewritten:

$$s^2 - \frac{3}{\sqrt{2}}s + 0 = 0$$

There are two solutions to this quadratic: $s = 0$ (before collision) and $s = \frac{3}{\sqrt{2}}$. Thus,

$$y = \frac{3}{2}$$

and

$$x = 3 - \frac{3}{2} = \frac{3}{2}$$

So, both balls careen off at 45° angles at the exact same speed.

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