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Right-Triangle Trigonometry

How tall is the flagpole in front of your school? You could climb it with a tape measure. Or you could stand 20 meters away, tilt your phone until it points at the top, read the angle, and work out the height in ten seconds. Surveyors map mountains this way, astronomers measure the distance to stars this way, and the GPS in your phone does a fancier version of it. The tools are three ratios called sine, cosine, and tangent. This chapter introduces them for right triangles, then extends them to every triangle with the Law of Sines and the Law of Cosines.

1.1 Why the Ratios Exist

Every right triangle has one right angle and two acute angles. Pick one of the acute angles and call it θ (the Greek letter theta). The sides get names relative to θ (figure 1.1):

- The *hypotenuse* is the longest side, across from the right angle.
- The *opposite* side is across from θ .
- The *adjacent* side is the leg that touches θ (and is not the hypotenuse).

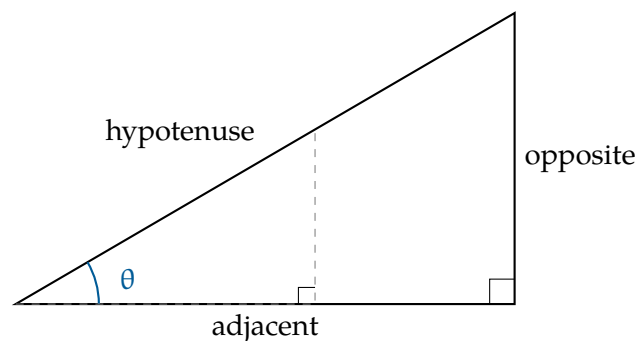


Figure 1.1: The sides of a right triangle, named relative to the angle θ . The smaller dashed triangle has the same angles, so it is similar to the big one.

Here is the key idea. Two right triangles that share an acute angle θ have all three angles equal, so they are *similar*: one is a scaled copy of the other (see the Congruence chapter). Scaling multiplies every side by the same factor, so the *ratios* of the sides do not change. The ratio $\frac{\text{opposite}}{\text{hypotenuse}}$ is the same for a tiny triangle and a huge one; it depends only on θ . That is why we can give these ratios names.

1.2 Sine, Cosine, and Tangent

For an acute angle θ in a right triangle:

$$\sin \theta = \frac{\text{opposite}}{\text{hypotenuse}}, \quad \cos \theta = \frac{\text{adjacent}}{\text{hypotenuse}}, \quad \tan \theta = \frac{\text{opposite}}{\text{adjacent}}$$

A famous memory aid is *SOH-CAH-TOA*: Sine is Opposite over Hypotenuse, Cosine is Adjacent over Hypotenuse, Tangent is Opposite over Adjacent.

Example: A right triangle has legs 3 and 4. Find the sine, cosine, and tangent of the angle θ opposite the side of length 3.

Solution: By the Pythagorean theorem, the hypotenuse is $\sqrt{3^2 + 4^2} = 5$. The opposite side is 3 and the adjacent side is 4, so

$$\sin \theta = \frac{3}{5} = 0.6, \quad \cos \theta = \frac{4}{5} = 0.8, \quad \tan \theta = \frac{3}{4} = 0.75$$

Two facts follow straight from the definitions. Dividing,

$$\frac{\sin \theta}{\cos \theta} = \frac{\text{opp/hyp}}{\text{adj/hyp}} = \frac{\text{opp}}{\text{adj}} = \tan \theta$$

And since $\text{opp}^2 + \text{adj}^2 = \text{hyp}^2$ by the Pythagorean theorem, dividing by hyp^2 gives the Pythagorean identity:

$$\sin^2 \theta + \cos^2 \theta = 1$$

(Here $\sin^2 \theta$ means $(\sin \theta)^2$.) Check it with the 3-4-5 triangle: $0.6^2 + 0.8^2 = 0.36 + 0.64 = 1$.

1.2.1 Using a Calculator

Your calculator (or phone, or Python) knows the sine, cosine, and tangent of every angle. In this chapter, we measure angles in degrees, so put your calculator in **degree mode**. Check: $\sin 30^\circ$ should be exactly 0.5. If you get -0.988 , your calculator is in radians mode, and it has computed the sine of 30 radians instead.

1.2.2 Two Special Triangles

For two shapes of right triangle, you can find the ratios exactly, without a calculator.

The 45-45-90 triangle. Cut a square with side 1 along its diagonal. Each half is a right triangle with two 45° angles, legs of 1, and hypotenuse $\sqrt{1^2 + 1^2} = \sqrt{2}$.

The 30-60-90 triangle. Cut an equilateral triangle with side 2 in half from a corner straight down to the middle of the opposite side. Each half has angles 30° , 60° , and 90° , a hypotenuse of 2, a short leg of 1, and a long leg of $\sqrt{2^2 - 1^2} = \sqrt{3}$.

Reading the ratios off these triangles gives:

θ	$\sin \theta$	$\cos \theta$	$\tan \theta$
30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
45°	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$

Notice that $\sin 30^\circ = \cos 60^\circ$. That is no coincidence: the side opposite the 30° angle is adjacent to the 60° angle.

1.3 Solving Right Triangles

To *solve* a triangle means to find all its sides and angles. In a right triangle, you need only two more facts: two sides, or one side and one acute angle.

1.3.1 Finding a Side

When you look up at something, the angle between your line of sight and the horizontal is the *angle of elevation*. Looking down, it is the *angle of depression*.

Example: You stand 20 m from a flagpole. Holding your phone at eye level, 1.5 m above the ground, you measure the angle of elevation to the top as 35° . How tall is the flagpole?

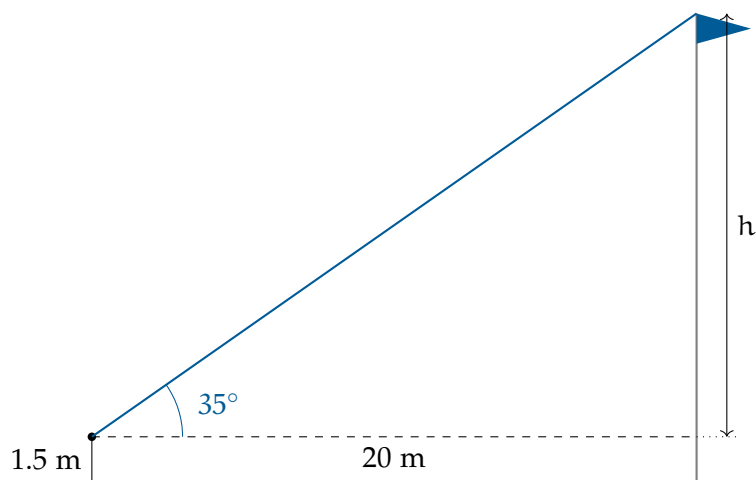


Figure 1.2: Measuring a flagpole with an angle of elevation.

Solution: The part of the pole above your eye, h , is opposite the 35° angle, and the 20 m distance is adjacent. Tangent connects them:

$$\tan 35^\circ = \frac{h}{20} \implies h = 20 \tan 35^\circ \approx 20(0.7002) \approx 14.0 \text{ m}$$

Add your eye height: the flagpole is about $14.0 + 1.5 = 15.5$ m tall.

1.3.2 Finding an Angle

To find an angle from two sides, run a trig function backward. The calculator keys $\sin^{-1}$, $\cos^{-1}$, and $\tan^{-1}$ (often a second function above the trig keys) do exactly that: $\sin^{-1}(0.5) = 30^\circ$ because $\sin 30^\circ = 0.5$. These are the *inverse trigonometric functions*, and a later chapter studies them in detail.

Example: A 6 m ladder leans against a wall, with its top 5.5 m above the ground. What angle does it make with the ground? How far is its foot from the wall?

Solution: The height is opposite the angle θ at the ground, and the ladder is the hypotenuse, so

$$\sin \theta = \frac{5.5}{6} \approx 0.9167 \implies \theta = \sin^{-1}(0.9167) \approx 66.4^\circ$$

The foot of the ladder is adjacent to θ : $6 \cos 66.4^\circ \approx 2.40$ m from the wall. (Ladder makers recommend about 75° , so this ladder is a bit too flat for comfort.)

Exercise 1 Ratios from Sides

A right triangle has legs 5 and 12. Find its hypotenuse, and find $\sin \theta$, $\cos \theta$, and $\tan \theta$ for the angle θ opposite the side of length 5.

*Working Space**Answer on Page 69***Exercise 2 Kite and Cliff**

1. A kite's string is 50 m long and makes a 55° angle with the ground. How high is the kite above the hand holding the string?
2. From the top of a 30 m cliff, the angle of depression to a boat is 12° . How far is the boat from the base of the cliff?

*Working Space**Answer on Page 69***Exercise 3 Designing a Ramp**

A wheelchair ramp must rise 0.75 m, and building codes say its angle with the ground can be at most 4.8° . What is the shortest the ramp (the sloped surface) can be?

*Working Space**Answer on Page 69*

1.4 Beyond Right Triangles

Most triangles do not have a right angle. A surveyor measuring across a pond, a pilot plotting a course with a crosswind, or a phone locating itself from three cell towers all face triangles of any shape. Two laws handle them. Both use the same labels: angles A , B , and C at the corners, and sides a , b , and c opposite them (figure 1.3).

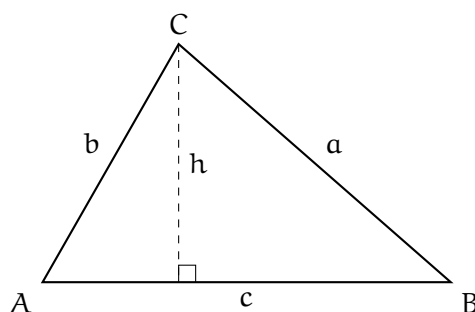


Figure 1.3: Side a is opposite angle A , and so on. The dashed altitude h splits the triangle into two right triangles.

1.5 The Law of Sines

Drop the altitude h from C , as in figure 1.3. It splits the triangle into two right triangles. In the left one, the hypotenuse is b , so $h = b \sin A$. In the right one, the hypotenuse is a , so $h = a \sin B$. Both equal h , so $b \sin A = a \sin B$, which rearranges to $\frac{a}{\sin A} = \frac{b}{\sin B}$. Dropping an altitude from a different corner brings in c the same way.

The Law of Sines

In any triangle,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

In words: each side is proportional to the sine of the angle across from it. Bigger angles sit across from longer sides.

Use the Law of Sines when you know an angle and the side across from it, plus one more piece.

Example: In a triangle, $A = 50^\circ$, $B = 60^\circ$, and $a = 10$. Solve the triangle.

Solution: The angles add up to 180° , so $C = 70^\circ$. Then

$$b = \frac{a \sin B}{\sin A} = \frac{10 \sin 60^\circ}{\sin 50^\circ} \approx 11.31, \quad c = \frac{a \sin C}{\sin A} = \frac{10 \sin 70^\circ}{\sin 50^\circ} \approx 12.27$$

Check: the largest angle, C , is across from the longest side, c .

1.5.1 The Ambiguous Case

In the Congruence chapter, you learned that there is no SSA test: two sides and an angle that is not between them do not always pin down a triangle. Here is that fact in action.

Example: Solve the triangle with $A = 40^\circ$, $a = 7$, and $b = 10$.

Solution: The Law of Sines gives

$$\sin B = \frac{b \sin A}{a} = \frac{10 \sin 40^\circ}{7} \approx 0.918$$

The calculator says $B = \sin^{-1}(0.918) \approx 66.7^\circ$. But there is a second angle with the same sine: $180^\circ - 66.7^\circ = 113.3^\circ$. (The height of a point on a circle is the same at θ and at $180^\circ - \theta$, as you will see in the next chapter.) Both work, because each leaves room for C :

- $B \approx 66.7^\circ$, $C \approx 73.3^\circ$, $c \approx 10.4$
- $B \approx 113.3^\circ$, $C \approx 26.7^\circ$, $c \approx 4.9$

Two different triangles fit the same three facts. Figure 1.4 shows why: a side of length a swinging from C can reach the bottom line in two places.

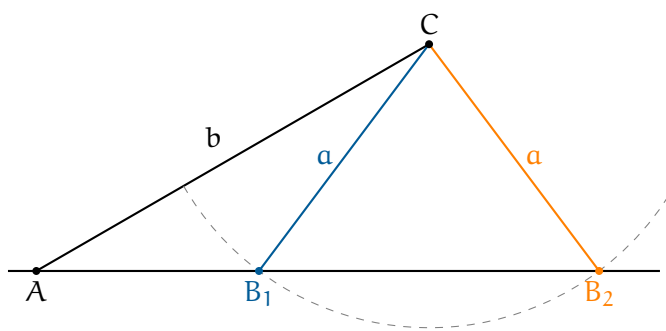


Figure 1.4: The ambiguous case. With angle A and sides b and a fixed, the side a can end at B_1 or at B_2 .

If $\frac{b \sin A}{a}$ comes out bigger than 1, there is no triangle at all: side a is too short to reach. And if the second angle leaves no room ($A + B \geq 180^\circ$), there is only one.

Exercise 4 Law of Sines

In a triangle, $A = 35^\circ$, $C = 105^\circ$, and $c = 20$. Find B , a , and b .

Working Space

Answer on Page 69

Exercise 5 How Many Triangles?

How many triangles have $A = 30^\circ$ and $b = 8$, if

1. $a = 5$?

2. $a = 3$?

Working Space

Answer on Page 69

1.6 The Law of Cosines

The Law of Sines needs an angle and the side across from it. If you know two sides and the angle *between* them, or all three sides, you need the Law of Cosines.

Put angle C at the origin, with side b along the x -axis, so A is at $(b, 0)$. Side a leaves the origin at angle C , so its far end is at $B = (a \cos C, a \sin C)$. (Picture a right triangle with hypotenuse a under that point.) Side c runs from A to B , so by the distance formula,

$$\begin{aligned} c^2 &= (a \cos C - b)^2 + (a \sin C)^2 \\ &= a^2 \cos^2 C - 2ab \cos C + b^2 + a^2 \sin^2 C \\ &= a^2(\cos^2 C + \sin^2 C) + b^2 - 2ab \cos C \\ &= a^2 + b^2 - 2ab \cos C \end{aligned}$$

using the Pythagorean identity in the last step. (This picture assumes C is acute, but the formula holds for every angle once you meet the cosine of obtuse angles in the next chapter.)

The Law of Cosines

In any triangle,

$$c^2 = a^2 + b^2 - 2ab \cos C$$

When $C = 90^\circ$, $\cos C = 0$, and this is the Pythagorean theorem. The $-2ab \cos C$ term is a correction for angles that are not right: a narrower angle pulls the ends closer together, and a wider one pushes them apart.

Example: To measure across a pond, a surveyor stands at a point C on shore. One end of the pond is 120 m away, the other is 150 m away, and the angle between the two sight lines is 70° . How wide is the pond?

Solution:

$$c^2 = 120^2 + 150^2 - 2(120)(150) \cos 70^\circ \approx 14400 + 22500 - 12313 = 24587$$

so $c \approx 156.8$ m.

Example: A triangle has sides 5, 6, and 7. Find its largest angle.

Solution: The largest angle is across from the longest side, so let $c = 7$ and solve the Law of Cosines for $\cos C$:

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab} = \frac{25 + 36 - 49}{60} = 0.2 \implies C = \cos^{-1}(0.2) \approx 78.5^\circ$$

1.6.1 The Area of Any Triangle

The altitude gives one more useful formula. In figure 1.3, the area is $\frac{1}{2}(\text{base})(\text{height}) = \frac{1}{2}ch$, and $h = b \sin A$, so

$$\text{Area} = \frac{1}{2}bc \sin A$$

Half the product of two sides times the sine of the angle between them. For example, a triangle with sides 8 and 11 and a 40° angle between them has area $\frac{1}{2}(8)(11) \sin 40^\circ \approx 28.3$.

1.7 Which Tool Should I Use?

What you know	What to use
A right triangle	SOH-CAH-TOA and the Pythagorean theorem
Two angles and any side (AAS or ASA)	Law of Sines
Two sides and the angle between them (SAS)	Law of Cosines
All three sides (SSS)	Law of Cosines
Two sides and an angle not between them (SSA)	Law of Sines, but check for 0, 1, or 2 triangles

1.8 Triangles in Python

Python's `math` library has `sin`, `cos`, and their inverses, but they work in radians, not degrees. The functions `math.radians` and `math.degrees` convert back and forth:

```

1  from math import sin, cos, acos, sqrt, radians, degrees
2
3  def third_side(a, b, angle_C):
4      """Law of Cosines: the side opposite angle C (in degrees)."""
5      C = radians(angle_C)          # math.cos expects radians
6      return sqrt(a**2 + b**2 - 2*a*b*cos(C))
7
8  def angle_opposite(a, b, c):
9      """Law of Cosines, solved for the angle opposite side c."""
10     return degrees(acos((a**2 + b**2 - c**2) / (2*a*b)))
11
12 print(round(third_side(120, 150, 70), 1))    # the pond
13 print(round(angle_opposite(5, 6, 7), 1))     # the 5-6-7 triangle
14 print(round(third_side(3, 4, 90), 1))        # a right angle gives Pythagoras

```

156.8
78.5
5.0

Exercise 6 Law of Cosines

Working Space

- Two sides of a triangle are 8 and 11, and the angle between them is 40° . Find the third side.
- A triangle has sides 7, 8, and 9. Find

its largest angle.

Answer on Page 70

Exercise 7 **Off Course**

A boat sails 20 km due north, then turns 30° to the right and sails another 15 km. How far is it from where it started?

Working Space

Answer on Page 70

Exercise 8 **A Triangle Solver**

Write a Python function `area(a, b, angle_C)` that returns the area of a triangle from two sides and the angle between them, in degrees. Test it on the triangle with sides 8 and 11 and a 40° angle.

Working Space

Answer on Page 70

Trigonometric Functions

Sound, light, tides, a swinging pendulum, the alternating current in your walls: they all rise and fall in the same smooth pattern. That pattern is the sine wave, and this chapter shows where it comes from.

You already know that in a right triangle with an angle θ , the sine of θ is the opposite side divided by the hypotenuse. That definition only works for angles between 0 and $\frac{\pi}{2}$. Here is one that works for every real number.

Treat the number θ as an angle. Draw a ray from the origin at angle θ , measured counter-clockwise from the positive x -axis, out to the *unit circle* (the circle of radius 1 centered at the origin). The y value of the point where the ray hits the circle is $\sin(\theta)$. For example, $\sin(\frac{4\pi}{3}) = -\frac{\sqrt{3}}{2}$ (see figure 2.1). For angles between 0 and $\frac{\pi}{2}$, this matches the triangle definition: the hypotenuse is 1, so opposite over hypotenuse is just the height.

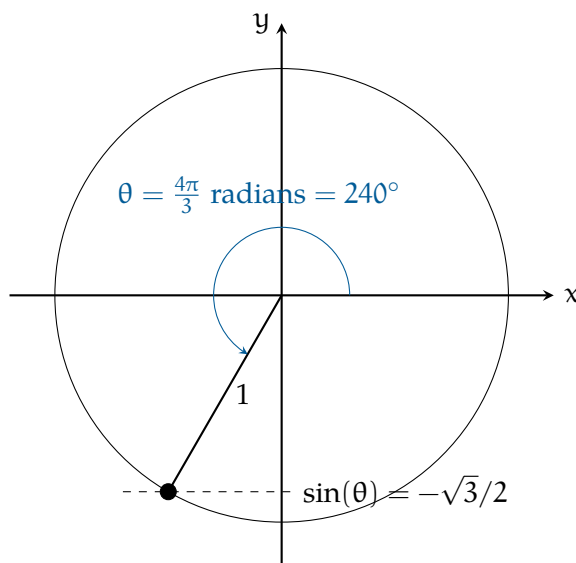
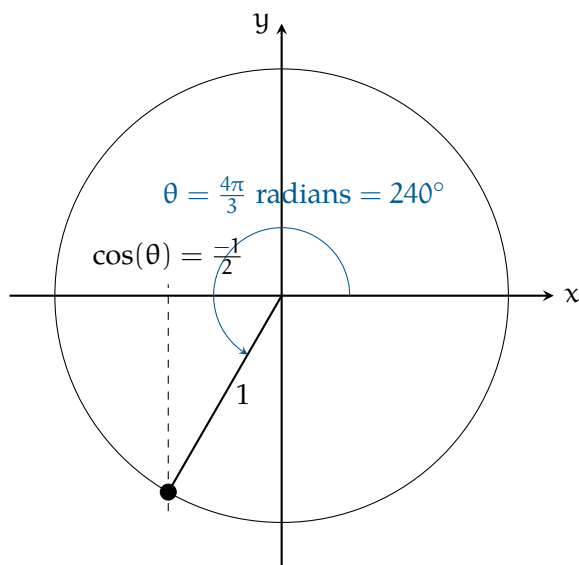


Figure 2.1: $\sin \frac{4\pi}{3} = -\frac{\sqrt{3}}{2}$

From here on, angles are in radians unless we say otherwise. Engineers and mathematicians nearly always use radians. Put your calculator in radians mode now; forgetting is the most common trig mistake there is.

The cosine of θ is the x value of the same point (figure 2.2).

Figure 2.2: $\cos \frac{4\pi}{3} = -\frac{1}{2}$

Three facts fall right out of the picture:

- $\sin^2(\theta) + \cos^2(\theta) = 1$. The cosine and sine are the legs of a right triangle whose hypotenuse is the radius, 1.
- $\sin(\theta + 2\pi) = \sin(\theta)$. Go once around the circle, and you are back where you started.
- $\cos(\theta) = \sin(\theta + \frac{\pi}{2})$. Turn the picture a quarter turn and see.

Worked example. Find $\sin(\frac{\pi}{2})$ and $\cos(\frac{\pi}{2})$.

Solution. An angle of $\frac{\pi}{2}$ points straight up, to $(0, 1)$. So $\cos(\frac{\pi}{2}) = 0$ and $\sin(\frac{\pi}{2}) = 1$.

Exercise 9 Around the circle

Without a calculator, find $\sin(\pi)$, $\cos(\pi)$, $\sin(\frac{3\pi}{2})$, and $\cos(2\pi)$. Then use the Pythagorean identity to find $\cos(\theta)$ if $\sin(\theta) = 0.6$ and θ is between 0 and $\frac{\pi}{2}$.

Working Space

Answer on Page 70

2.1 Graphs of sine and cosine

Here is a graph of $y = \sin(x)$:

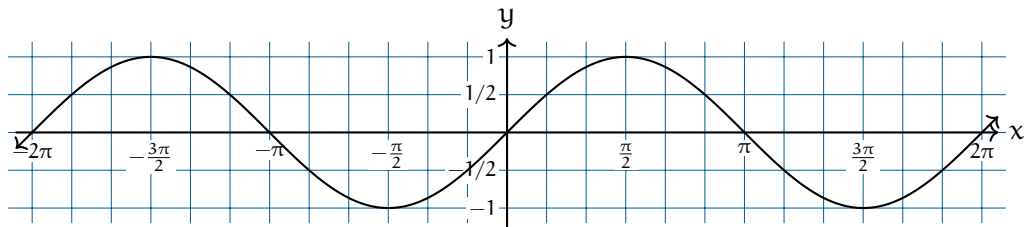


Figure 2.3: $y = \sin(x)$.

Waves! The pattern repeats every 2π , forever in both directions. Since $\cos(\theta) = \sin(\theta + \frac{\pi}{2})$, the cosine graph is the same wave slid $\frac{\pi}{2}$ to the left:

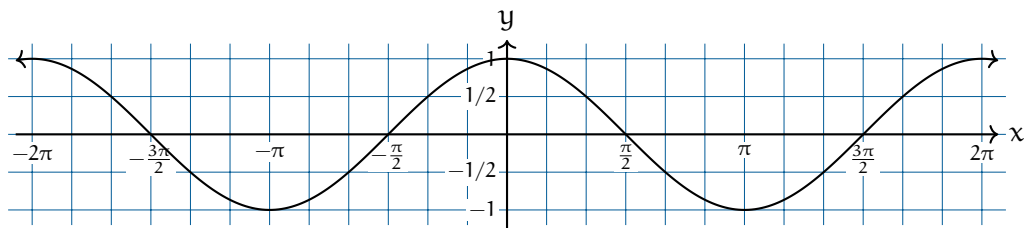


Figure 2.4: $y = \cos(x)$.

2.2 Plot cosine in Python

Create a file called `cos.py`:

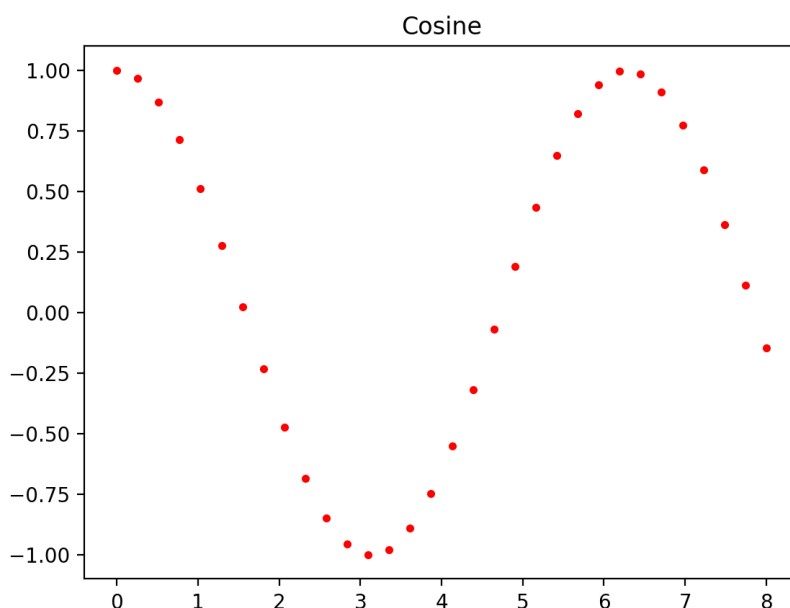
```

1 import numpy as np
2 import matplotlib.pyplot as plt
3
4 until = 8.0
5
6 # Make a plot of cosine
7 thetas = np.linspace(0, until, 32)
8 cosines = []
9 for theta in thetas:
10     cosines.append(np.cos(theta))
11
12 # Plot the data
13 fig, ax = plt.subplots()
14 ax.plot(thetas, cosines, 'r.', label="Cosine")
15 ax.set_title("Cosine")

```

16 `plt.show()`

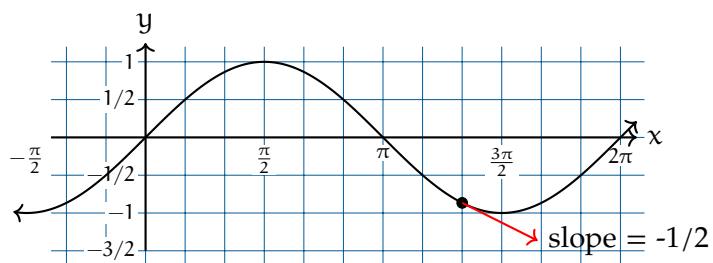
This plots 32 points on the cosine wave between 0 and 8. Run it, and you should see something like this:



2.3 The slope of a sine wave

Here is a wonderful fact. At any θ , the slope of the sine graph equals $\cos(\theta)$.

For example, $\sin(\frac{4\pi}{3}) = -\frac{\sqrt{3}}{2}$ and $\cos(\frac{4\pi}{3}) = -\frac{1}{2}$. A line tangent to the sine curve at that point has slope $-\frac{1}{2}$:



Check it at the top of the wave, $\theta = \frac{\pi}{2}$: the curve is flat there, and sure enough $\cos(\frac{\pi}{2}) = 0$.

In calculus, the slope function is called the *derivative*, so we say “the derivative of sine is cosine.” And the slope of the cosine graph? It is $-\sin(\theta)$. You will prove both in the calculus chapters.

2.4 A weight on a spring

Fill a roller skate with heavy rocks and attach it to a wall with a stiff spring. Push the skate toward the wall and let go. It rolls back and forth; engineers say it *oscillates*.

You can probably guess:

- If the spring is stronger, the skate will oscillate more times per minute.
- If the rocks are lighter, the skate will oscillate more times per minute.

The force from the spring is proportional to how far it is stretched or squeezed from its relaxed length. Springs are sold by their “spring rate,” in pounds per inch or newtons per meter. A spring with a rate of 5 newtons per meter pushes back with 0.5 newtons when you squeeze it 10 cm, and with 1 newton when you squeeze it 20 cm.

Let us simulate the skate. Copy `cos.py` to `spring.py` and add the simulation. Every hundredth of a second, the code nudges the position by the velocity, finds the spring force from the new position, and nudges the velocity by the acceleration:

```

1 import numpy as np
2 import matplotlib.pyplot as plt
3
4 until = 8.0
5
6 # Constants
7 mass = 4.0 # kg of skate and rocks
8 spring_rate = 4.0 # newtons per meter
9 time_step = 0.01 # s
10
11 # Initial state
12 displacement = 1.0 # meters from the relaxed position
13 velocity = 0.0
14 time = 0.0 # seconds
15
16 # Lists to gather data
17 displacements = []
18 times = []
19
20 # Run it for a little while
21 while time <= until:
22     # Record data
23     displacements.append(displacement)

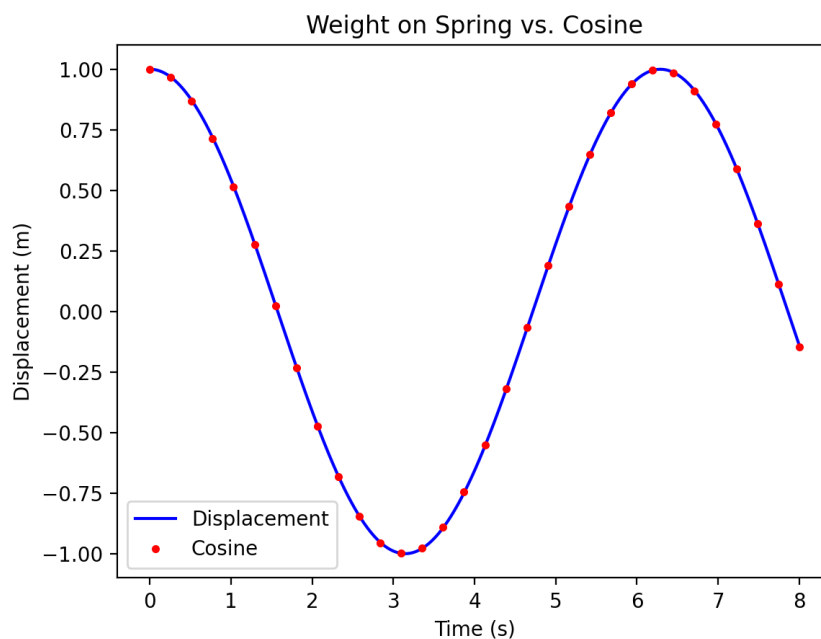
```

```

24     times.append(time)
25
26     # Calculate the next state
27     time += time_step
28     displacement += time_step * velocity
29     force = -spring_rate * displacement
30     acceleration = force / mass
31     velocity += time_step * acceleration
32
33 # Make a plot of cosine
34 thetas = np.linspace(0, until, 32)
35 cosines = []
36 for theta in thetas:
37     cosines.append(np.cos(theta))
38
39 # Plot the data
40 fig, ax = plt.subplots()
41 ax.plot(times, displacements, 'b', label="Displacement")
42 ax.plot(thetas, cosines, 'r.', label="Cosine")
43
44 ax.set_title("Weight on Spring vs. Cosine")
45 ax.set_xlabel("Time (s)")
46 ax.set_ylabel("Displacement (m)")
47 ax.legend()
48 plt.show()

```

Run it. The skate's position and the cosine curve land right on top of each other.



Why does the skate follow a cosine curve? Sine and cosine waves appear whenever an object's acceleration is proportional to -1 times its displacement:

$$a \propto -p$$

where a is acceleration and p is the displacement from equilibrium.

Velocity is the slope of the position graph, and acceleration is the slope of the velocity graph. So the skate's position must follow a function f whose slope-of-the-slope is proportional to $-f$:

$$f(t) \propto -f''(t)$$

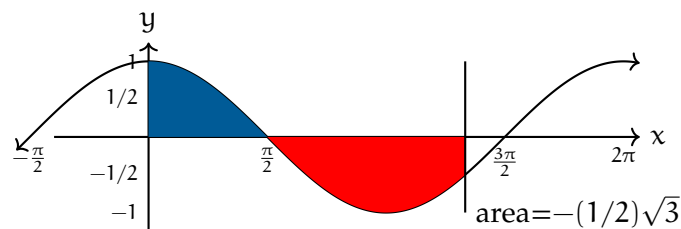
Cosine does exactly that. Its slope is $-\sin$, and the slope of $-\sin$ is $-\cos$. Take the slope twice, and you get the original function times -1 .

That is why sine waves are everywhere. Anything that gets pushed back toward the middle harder the farther it strays (a spring, a guitar string, a swing, the air in a flute) moves in a sine wave.

In our simulation, the spring rate and mass are equal, which makes the wave line up exactly with $\cos(t)$. A stiffer spring or lighter skate gives a faster wave, just as you guessed.

2.5 Integral of sine and cosine

Now look at area. Take the area between the cosine graph and the x axis from 0 to $\frac{4\pi}{3}$, counting area below the axis as negative. The total is $-\frac{\sqrt{3}}{2}$, which is exactly $\sin(\frac{4\pi}{3})$.



That is no coincidence. The area under cosine from 0 to any θ is always $\sin(\theta)$. In calculus, we say “the integral of cosine is sine.”

Trigonometric Identities

3.1 The Unit Circle

A calculator will happily tell you that $\sin(\frac{\pi}{3}) \approx 0.8660254$. But the exact answer is $\frac{\sqrt{3}}{2}$, and knowing that lets you simplify, spot patterns, and check your work. In this chapter you will learn the handful of exact values worth memorizing, and a few identities that let you build every other value from them.

Everything starts with the unit circle (figure 3.1): the circle of radius 1 centered at the origin. The point at angle θ (measured counterclockwise from the positive x -axis) has coordinates $(\cos \theta, \sin \theta)$.

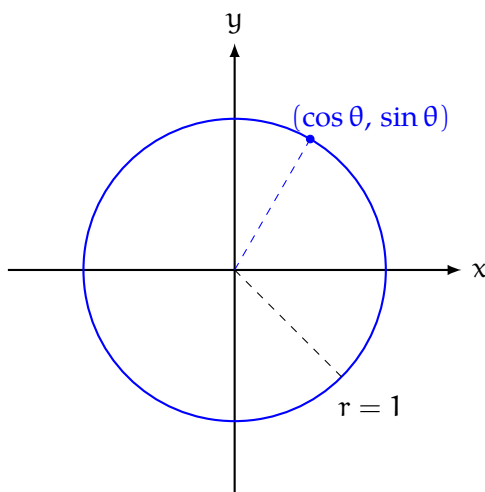


Figure 3.1: The Unit Circle is a circle with radius 1 centered at the origin

Why does this agree with the triangle definitions you already know? Pick a point (x_o, y_o) on the circle in the first quadrant. The segment from the origin to it makes an angle θ with the positive x -axis (figure 3.2). Drop a vertical segment down to $(x_o, 0)$, and you have a right triangle. What are its legs?

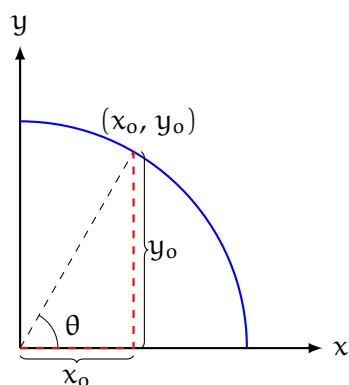


Figure 3.2: Drawing a line from any point on the circle to the origin creates an angle with the horizontal

Recall SOH-CAH-TOA: sine is opposite over hypotenuse. The hypotenuse is a radius, so it has length 1, and the opposite leg has length y_0 . So $\sin \theta = \frac{y_0}{1} = y_0$. The y -coordinate is the sine.

Exercise 10

In the same way, show that the x -coordinate of the point is $\cos \theta$.

Working Space

Answer on Page 70

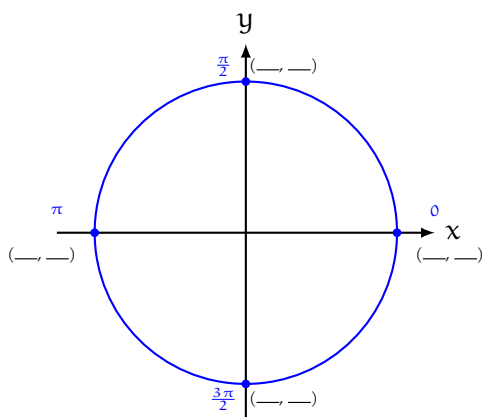
So every point on the unit circle is $(\cos \theta, \sin \theta)$.

Exercise 11

Fill in the unit circle with the coordinates for $\theta = 0, \pi/2, \pi$, and $3\pi/2$. Use this to determine:

Working Space

1. $\sin \frac{\pi}{2}$
2. $\cos \frac{3\pi}{2}$
3. $\sin \pi$
4. $\cos -\pi$



Answer on Page 71

3.1.1 Exact Values of Key Angles

Two triangles give us every exact value we need: the 30-60-90 and the 45-45-90.

Cut an equilateral triangle with sides of length 1 in half (figure 3.3). Each half is a 30-60-90 triangle with hypotenuse 1. Call the horizontal leg A and the vertical leg B.

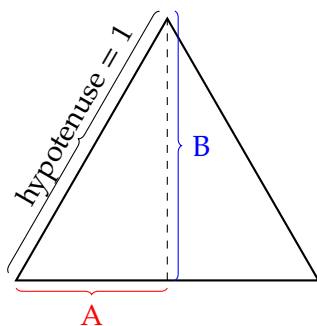


Figure 3.3: A 30-60-90 triangle is made by vertically bisecting an equilateral triangle

The cut split the bottom side in half, so A is $\frac{1}{2}$. That means $\cos 60^\circ = \cos \frac{\pi}{3} = \frac{1}{2}$. To find the length of side B, we can use the Pythagorean theorem:

$$B^2 = C^2 - A^2, \text{ where } C \text{ is the hypotenuse}$$

$$B^2 = 1^2 - \left(\frac{1}{2}\right)^2$$

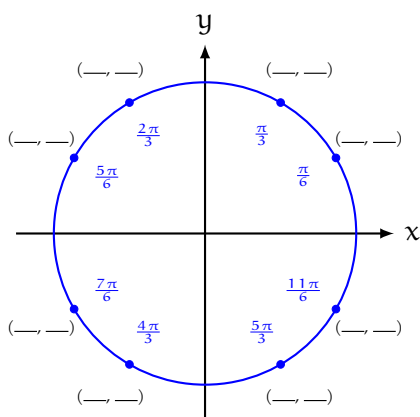
$$B^2 = \frac{3}{4}$$

$$B = \frac{\sqrt{3}}{2}$$

Therefore, $\sin 60^\circ = \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$.

Exercise 12

Use symmetry to complete the blank unit circle below. (Hint: We just showed that the (x, y) coordinate for $\frac{\pi}{3}$ is $(1/2, \sqrt{3}/2)$).



Working Space

Answer on Page 71

Now the 45-45-90 triangle (figure 3.4). It is isosceles, so its legs are equal: $A = B$. Its hypotenuse is a radius, so $C = 1$. The Pythagorean theorem gives us A :

$$A^2 + B^2 = C^2$$

$$A^2 + A^2 = 1^2$$

$$2A^2 = 1$$

$$A^2 = \frac{1}{2}$$

$$A = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

Therefore, each leg has a length of $\sqrt{2}/2$, and the (x, y) coordinates for $\theta = 45^\circ = \pi/4$ are $(\sqrt{2}/2, \sqrt{2}/2)$.

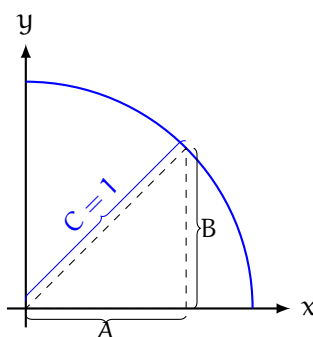
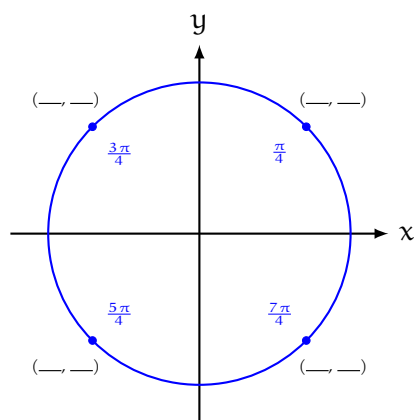


Figure 3.4: The two legs of a 45-45-90 triangle are the same length

Exercise 13

Use symmetry to complete the blank unit circle below.



Working Space

Answer on Page 72

Exercise 14

Without a calculator and using only your completed unit circles, determine the value requested (angles are given in

Working Space

radians unless otherwise indicated).

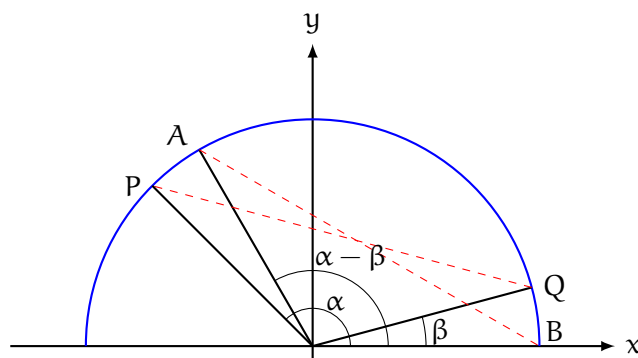
1. $\cos \frac{3\pi}{2}$
2. $\sin \frac{\pi}{4}$
3. $\sin -\frac{\pi}{6}$
4. $\cos \frac{4\pi}{3}$
5. $\sin \frac{3\pi}{4}$
6. $\cos -\frac{\pi}{3}$
7. $\sin 45^\circ$
8. $\sin 270^\circ$
9. $\sin -60^\circ$
10. $\sin 150^\circ$

Answer on Page 72

3.2 Sum and Difference Formulas

What is $\cos(75^\circ)$? It is not on our circle. But $75^\circ = 45^\circ + 30^\circ$, and both of those are. If only we knew how to combine them. We do not get to just add: $\cos(45^\circ) + \cos(30^\circ) \approx 1.57$, which is bigger than 1, so it cannot be a cosine. We need a real formula, and here is a clever way to get one.

Consider 4 points on the unit circle: B at $(1, 0)$, Q at some angle β , P at some angle α , and A at angle $\alpha - \beta$ (see figure 3.5).

Figure 3.5: $\overline{AB} = \overline{PQ}$

The angle between P and Q is $\alpha - \beta$, and so is the angle between A and B. So $\triangle POQ$ is just $\triangle AOB$ rotated, and the distance from P to Q equals the distance from A to B. Because this is a Unit Circle, $P = (\cos \alpha, \sin \alpha)$, $Q = (\cos \beta, \sin \beta)$, and $A = (\cos(\alpha - \beta), \sin(\alpha - \beta))$. Let us use the distance formula to find the length of $\overline{PQ}$:

$$\begin{aligned}\overline{PQ} &= \sqrt{(\cos \alpha - \cos \beta)^2 + (\sin \alpha - \sin \beta)^2} = \\ &= \sqrt{\cos^2 \alpha - 2 \cos \alpha \cos \beta + \cos^2 \beta + \sin^2 \alpha - 2 \sin \alpha \sin \beta + \sin^2 \beta} = \\ &= \sqrt{(\cos^2 \alpha + \sin^2 \alpha) + (\cos^2 \beta + \sin^2 \beta) - 2 \cos \alpha \cos \beta - 2 \sin \alpha \sin \beta}\end{aligned}$$

Recall that for any angle, θ , $\sin^2 \theta + \cos^2 \theta = 1$. Substituting this identity, we see that:

$$\overline{PQ} = \sqrt{1 + 1 - 2 \sin \alpha \sin \beta - 2 \cos \alpha \cos \beta} = \sqrt{2 - 2 \sin \alpha \sin \beta - 2 \cos \alpha \cos \beta}$$

Set that aside and find $\overline{AB}$ the same way:

$$\begin{aligned}\overline{AB} &= \sqrt{[\cos(\alpha - \beta) - 1]^2 + [\sin(\alpha - \beta) - 0]^2} = \\ &= \sqrt{\cos^2(\alpha - \beta) - 2 \cos(\alpha - \beta) + 1 + \sin^2(\alpha - \beta)} = \\ &= \sqrt{\cos^2(\alpha - \beta) + \sin^2(\alpha - \beta) + 1 - 2 \cos(\alpha - \beta)} \\ &= \sqrt{2 - 2 \cos(\alpha - \beta)} = \overline{AB}\end{aligned}$$

Since $\overline{AB} = \overline{PQ}$, set them equal:

$$\sqrt{2 - 2 \sin \alpha \sin \beta - 2 \cos \alpha \cos \beta} = \sqrt{2 - 2 \cos(\alpha - \beta)}$$

Squaring both sides and subtracting 2, we find:

$$-2 \sin \alpha \sin \beta - 2 \cos \alpha \cos \beta = -2 \cos (\alpha - \beta)$$

Finally, we can divide both sides by negative 2 to get the difference of angles formula for cosine:

$$\cos (\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$$

Similar arguments give the other three formulas. (For a head start: replace β with $-\beta$ to get the cosine sum formula, since $\cos(-\beta) = \cos \beta$ and $\sin(-\beta) = -\sin \beta$.)

Now we can answer the opening question:

$$\cos(75^\circ) = \cos 45^\circ \cos 30^\circ - \sin 45^\circ \sin 30^\circ = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2} \cdot \frac{1}{2} = \frac{\sqrt{6} - \sqrt{2}}{4} \approx 0.2588$$

Sum and Difference Formulas

$$\cos (\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$$

$$\cos (\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$$

$$\sin (\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$$

$$\sin (\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta$$

Exercise 15

Without a calculator, find the exact value requested:

Working Space

1. $\sin \frac{\pi}{12}$

2. $\cos \frac{7\pi}{12}$

3. $\tan \frac{13\pi}{12}$ (hint: $\tan \theta = \sin \theta / \cos \theta$)

Answer on Page 72

3.3 Double and Half Angle Formulas

Set $\alpha = \beta$ in a sum formula, and you get a formula for twice an angle.

Example: Derive a formula for $\cos 2\theta$ in terms of trigonometric functions of θ .

Solution: Using the sum formula for cosine, we see that:

$$\begin{aligned}\cos 2\theta &= \cos(\theta + \theta) \\ &= \cos \theta \cos \theta - \sin \theta \sin \theta = \cos^2 \theta - \sin^2 \theta\end{aligned}$$

Noting that $\sin^2 \theta = 1 - \cos^2 \theta$:

$$\cos 2\theta = 2 \cos^2 \theta - 1$$

Alternatively, we could note that $\cos^2 \theta = 1 - \sin^2 \theta$:

$$\cos 2\theta = 1 - 2 \sin^2 \theta$$

All three versions are worth knowing:

$$\cos 2\theta = \cos^2 \theta - \sin^2 \theta$$

Exercise 16

Derive a formula for $\sin 2\theta$ in terms of trigonometric functions of θ .

Working Space

Answer on Page 73

Run a double-angle formula backward, and you get a half-angle formula. Start with

$$\cos 2\theta = 2 \cos^2 \theta - 1$$

Let $\theta = \alpha/2$, then:

$$\cos \alpha = 2 \cos^2 (\alpha/2) - 1$$

Rearranging to solve for $\cos(\alpha/2)$:

$$2\cos^2(\alpha/2) = \cos\alpha + 1$$

$$\cos^2(\alpha/2) = \frac{\cos\alpha + 1}{2}$$

$$\cos(\alpha/2) = \pm\sqrt{\frac{1 + \cos\alpha}{2}}$$

Exercise 17

Derive a formula for $\sin(\alpha/2)$.

Working Space

Answer on Page 73

Two rearrangements will be very useful for integrals in a later chapter:

Squared Trigonometric Identities

$$\sin^2\theta = \frac{1 - \cos 2\theta}{2}$$

$$\cos^2\theta = \frac{1 + \cos 2\theta}{2}$$

These are the half-angle identities, squared and with $\alpha = 2\theta$.

Exercise 18 Half of thirty

Use a half-angle formula to find $\cos(15^\circ)$ exactly. Then check that your answer matches $\cos(15^\circ) = \cos(45^\circ - 30^\circ)$ from the difference formula.

Working Space

Answer on Page 73

Volumes

How much concrete does a new skate park need? How much water fits in a swimming pool? How heavy is that boulder? Each question starts with a volume. This chapter gives you the formulas for the common solids, and one big idea that ties them together: slice the solid and add up the slices.

4.1 Volumes of Common Solids

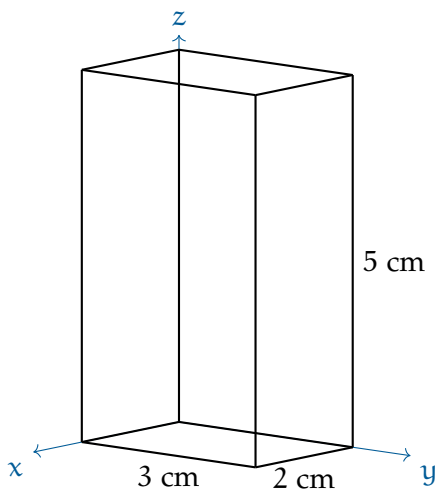
4.1.1 Rectangular Prism

The volume of a rectangular solid is the product of its three dimensions. If a block of ice is 5 cm tall, 3 cm wide, and 2 cm deep, its volume is $5 \times 3 \times 2 = 30$ cubic centimeters.

Volume of a rectangular solid.

A rectangular solid with height h , width w and length/depth l has volume:

$$V = lwh$$



A cubic centimeter is the same as a milliliter, and a milliliter of ice has a mass of about 0.92 grams. So the block of ice has a mass of $30 \times 0.92 = 27.6$ grams.

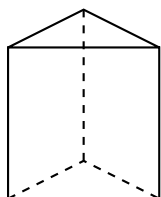
4.1.2 Triangular Prism

Stretch a triangle straight out of the page, and you get a triangular prism: 2 triangular faces and 3 rectangular ones. Think of a slice of cheese or a camping tent.

Volume of a triangular prism.

The area of a triangle with base w and height h is $\frac{1}{2}wh$. A triangular prism with that triangle as its end and length l has volume:

$$V = \frac{1}{2}lwh$$



4.1.3 Spheres

Volume of a Sphere

A sphere with a radius of r has a volume of

$$v = \frac{4}{3}\pi r^3$$

(For completeness, the surface area of that sphere would be

$$a = 4\pi r^2$$

That is exactly four times the area of a circle with the same radius.)

Exercise 19 Flying Sphere

Working Space

An iron sphere is traveling at 5 m/s (and

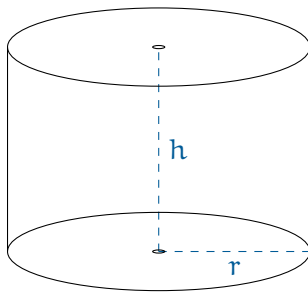
is not spinning). The sphere has a radius of 1.5 m. Iron has a density of 7,800 kg per cubic meter. How much kinetic energy does the sphere have?

Answer on Page 74

4.1.4 Cylinders

A soup can is a right cylinder: the top and bottom are identical circles on parallel planes, and the side stands perpendicular to them.

Worked example. A cylinder has radius 2 and height 5. Its volume is $\pi(2)^2(5) = 20\pi \approx 62.8$ cubic units.



Volume of a cylinder

The volume of the right cylinder of radius r and height h is given by:

$$v = \pi r^2 h$$

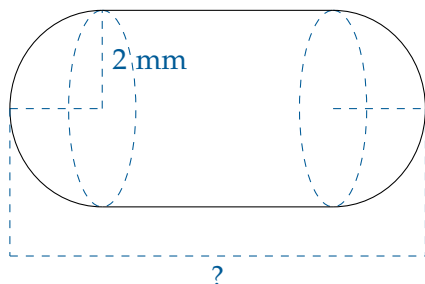
In other words, it is the area of the base times the height.

Exercise 20 Tablet

Working Space

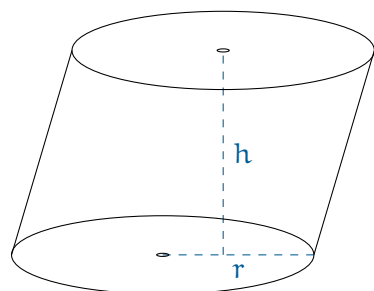
A drug company needs to create a tablet with volume of 90 cubic millimeters.

The tablet will be a cylinder with a half sphere on each end. The radius will be 2 mm. How long should the tablet be?



Answer on Page 74

What if the base and top are identical, but the side leans? That is an *oblique cylinder*.

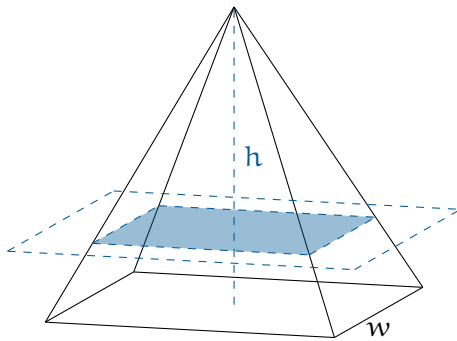


The volume is still the area of the base times the height, as long as you measure the height perpendicular to the top and bottom.

Why? Picture a stack of coins. Push the stack so it leans. Every coin is the same coin it was before, so the volume has not changed, and the stack is just as tall as before. Any two solids whose slices at every height have the same area have the same volume. This idea is called *Cavalieri's principle*, and the rest of this chapter runs on it.

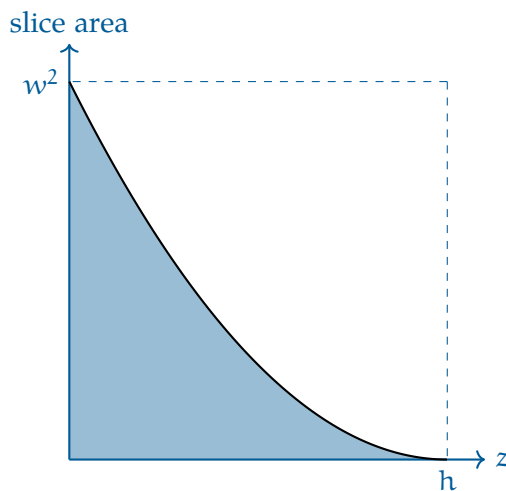
4.1.5 Pyramids

Slice a pyramid with a square base parallel to its base, and every slice is a square: big near the bottom, tiny near the top. (A pyramid with a triangular base is called a *tetrahedron*.)



We can figure out the area of the slice at every height z . For example, at $z = 0$, the slice would have area w^2 . At $z = h$, the slice would have zero area. What about an arbitrary z in between? The edge of the square would be $w(1 - \frac{z}{h})$. The area of the slice would be $w^2(1 - \frac{z}{h})^2$.

The graph of this would look like this:



Each slice is a thin plate, so the volume is the sum of all the slice areas: the area under this curve. In calculus you will learn to find areas like this exactly. For now, take our word that the colored region is one third of the rectangle around it.

So the volume of a square-based pyramid is $\frac{1}{3}hw^2$.

In fact:

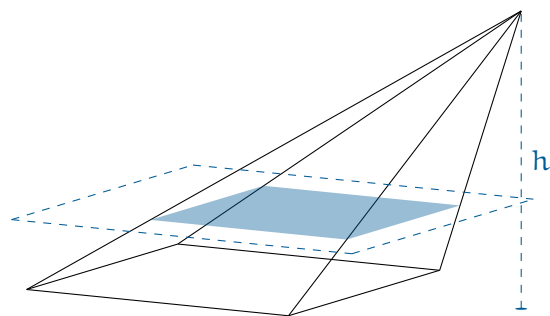
Volume of a pyramid

The volume of pyramid whose base has an area of b and height h is given by:

$$V = \frac{1}{3}hb$$

regardless of the shape of the base. A cone is a pyramid with a circular base, so its volume is $\frac{1}{3}\pi r^2 h$.

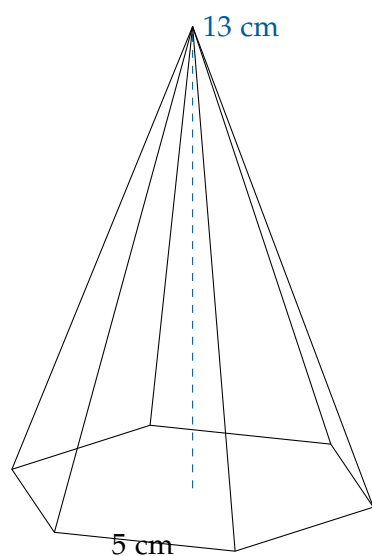
Thanks to Cavalieri, this is true even for oblique pyramids:



Exercise 21 Hexagon-based Pyramid

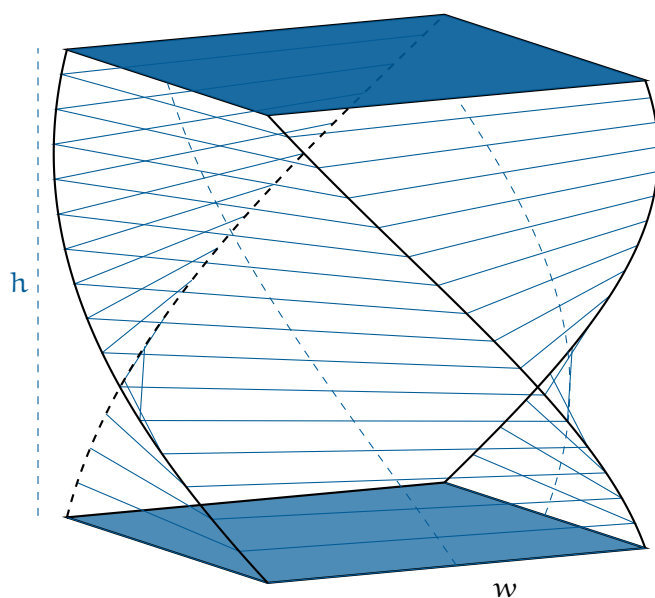
Working Space

There is a pyramid with a regular hexagon for a base. Each edge is 5 cm long. The pyramid is 13 cm tall. What is its volume?

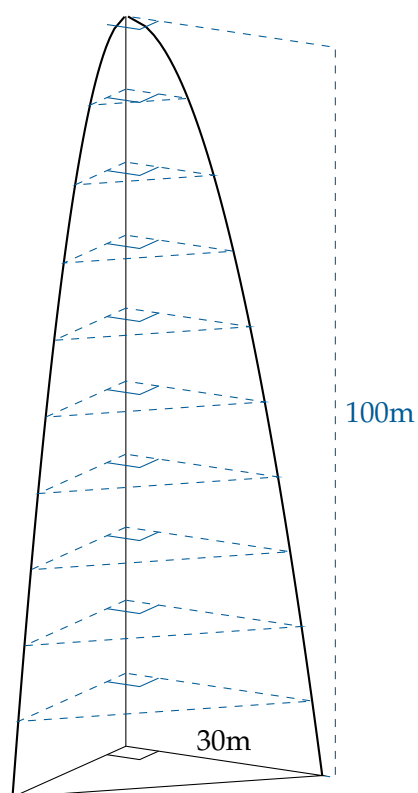


Answer on Page 75

Graphing the area of each slice and finding the area under the curve works for all sorts of strange shapes. Here is a twisted tower whose slices are all squares of width w :



Every slice has area w^2 , just like a straight box, so the twisted tower has volume hw^2 .



Exercise 22 Volume of a building

Working Space

An architect is designing a hotel with a right triangular base; the base is 30 meters on each leg. The building gets narrower as you get closer to the top, and finally shrinks to a point. The spine of the building is where the right angle is. That spine is straight and perpendicular to the ground.

Each floor has a right isosceles triangle as its floor plan. The length of each leg is given by this formula:

$$w = 30\sqrt{1 - \frac{z}{100}}$$

So, the width of the building is 30 me-

ters at height $z = 0$. At 100 meters, the building comes to a point. It will look like this:

What is the volume of the building in cubic meters?

Answer on Page 76

4.2 Looking Ahead

Every volume in this chapter came from the same idea: slice the solid, find the area of each slice, and add the slices up. When the slices change smoothly, “adding them up” means finding the area under a curve. In the calculus volumes, you will learn to do that exactly with an integral, and you will prove the sphere and cone formulas yourself.

Exercise 23 Another Cylinder

A cylinder has radius 3 and height 4. Find its exact volume.

Working Space

Answer on Page 77

Conic Sections

Shine a flashlight straight at a wall, and the spot of light is a circle. Tilt the flashlight, and the circle stretches into an ellipse. Tilt it more, and the edge of the light opens up into a parabola and then a hyperbola. You are slicing a cone of light with the flat wall.

Those four curves are the *conic sections*: the curves you get where a plane cuts through a cone. They show up everywhere. Planets move in ellipses, satellite dishes are parabolas, and the path of a comet that visits the sun once and never returns is a hyperbola.

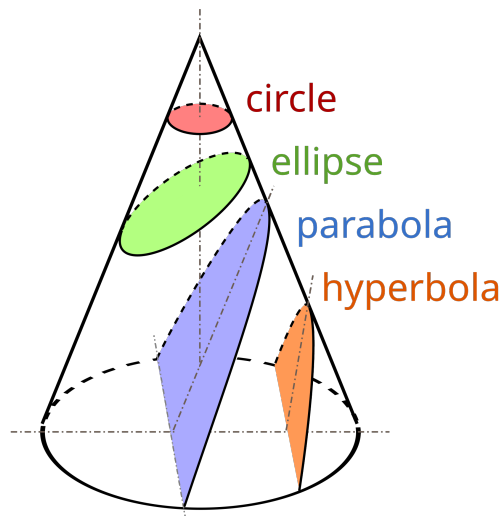


Figure 5.1: A plane slicing a cone. A level cut gives a circle; a tilted cut gives an ellipse; a cut parallel to the side of the cone gives a parabola; a steeper cut gives (one branch of) a hyperbola.

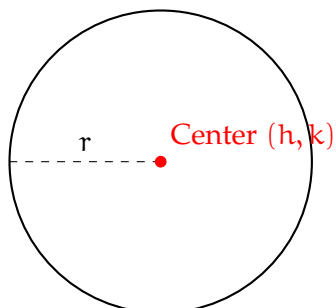
Source: Wikimedia Commons, Public Domain: https://upload.wikimedia.org/wikipedia/commons/thumb/1/11/Conic_Sections.svg/1920px-Conic_Sections.svg.png

Every equation in this chapter is fun to type into Desmos. Change the numbers and watch the curve respond.

5.1 Circle

A circle is the set of all points in a plane at a fixed distance (the radius r) from a fixed point (the center (h, k)). The distance formula turns that sentence straight into an equation:

$$(x - h)^2 + (y - k)^2 = r^2 \quad (5.1)$$



Worked example. Find the center and radius of the circle $x^2 - 6x + y^2 + 4y = 12$.

Solution. Complete the square in x and in y . Add 9 and 4 to both sides:

$$(x^2 - 6x + 9) + (y^2 + 4y + 4) = 12 + 9 + 4$$

$$(x - 3)^2 + (y + 2)^2 = 25$$

The center is $(3, -2)$ and the radius is 5.

Exercise 24 Radius of a Circle

What are the center and radius of each circle?

1. $x^2 + y^2 = 4$
2. $(x + 1)^2 + (y - 5)^2 = 49$
3. $x^2 + y^2 + 10y = 0$

Working Space

Answer on Page 77

5.2 Ellipse

An ellipse is the set of all points whose distances to two fixed points, the *foci*, add up to the same total. You can draw one: push two thumbtacks into a board, drop a loop of string over them, pull the string tight with a pencil, and trace all the way around.

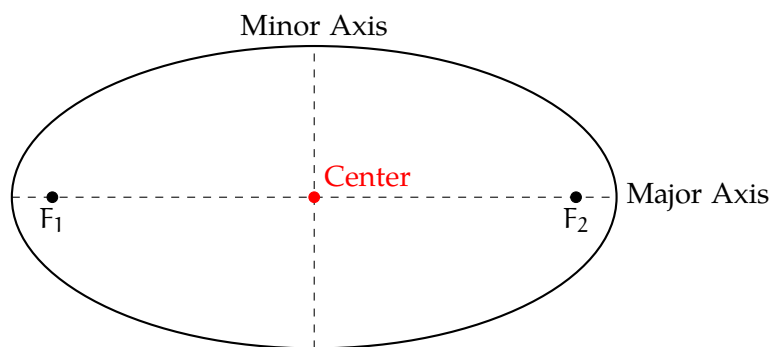
For an ellipse centered at the origin with semi-major axis a (half the long width) and semi-minor axis b (half the short width), with $a > b$, the equation is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad (5.2)$$

The foci sit on the major axis, a distance c from the center, where

$$c^2 = a^2 - b^2$$

When $a = b$, the foci merge at the center and the ellipse is a circle.



Exercise 25 Where are the thumbtacks?

An ellipse has equation $\frac{x^2}{25} + \frac{y^2}{9} = 1$. How wide and how tall is it? Where are its foci? How long is the string loop you would need to draw it?

Working Space

Answer on Page 77

5.3 Hyperbola

A hyperbola is the set of all points whose distances to two foci *differ* by the same amount. It comes from slicing a *double cone* (two cones tip to tip) with a plane steep enough to cut both cones, so a hyperbola has two separate branches.

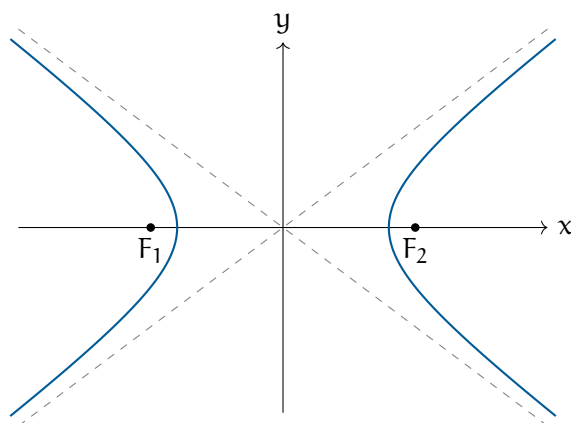
A hyperbola centered at the origin that opens left and right has the equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad (5.3)$$

Swap the roles of x and y , and it opens up and down:

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1 \quad (5.4)$$

Far from the center, the branches hug two straight lines called asymptotes. For the left-right hyperbola, they are $y = \pm \frac{b}{a}x$. The foci are a distance c from the center, where $c^2 = a^2 + b^2$.



The hyperbola $\frac{x^2}{4} - \frac{y^2}{2.25} = 1$ with its asymptotes $y = \pm 0.75x$ (dashed) and foci $(\pm 2.5, 0)$.

5.4 Parabola

A parabola is the set of all points that are the same distance from a fixed point (the *focus*) and a fixed line (the *directrix*). A parabola that opens up or down has the equation

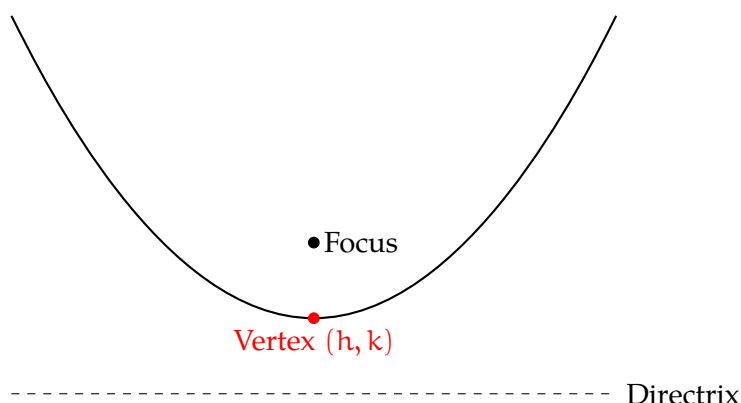
$$y = a(x - h)^2 + k \quad (5.5)$$

and one that opens left or right has the equation

$$x = a(y - k)^2 + h \quad (5.6)$$

where (h, k) is the *vertex*, the turning point. The focus is inside the curve, a distance

$p = \frac{1}{4|a|}$ from the vertex, and the directrix is the same distance away on the other side.



Here is the parabola's superpower. Every ray of light that comes in parallel to the axis bounces off the curve straight to the focus. That is why a satellite dish is a parabola, with the receiver at the focus. Run it backward, and a bulb at the focus of a flashlight's mirror sends out a straight beam.

Worked example. Where is the focus of $y = x^2$?

Solution. Here $a = 1$ and the vertex is $(0, 0)$, so $p = \frac{1}{4}$. The focus is at $(0, \frac{1}{4})$ and the directrix is $y = -\frac{1}{4}$. Check with the point $(1, 1)$: its distance to the directrix is $1 + \frac{1}{4} = \frac{5}{4}$, and its distance to the focus is $\sqrt{1^2 + (\frac{3}{4})^2} = \sqrt{\frac{25}{16}} = \frac{5}{4}$. They match.

Exercise 26 Where does the receiver go?

A satellite dish has a cross section shaped like $y = \frac{1}{8}x^2$, measured in decimeters. How far above the bottom of the dish should the receiver be mounted?

Working Space

Answer on Page 77

Only the parabolas that open up or down are graphs of functions. Circles, ellipses, hyperbolas, and sideways parabolas all fail the vertical line test, so we describe them with equations instead.

Exercise 27 **Name that conic**

Name the conic each equation describes:

Working Space

1. $4x^2 + 4y^2 = 36$

2. $4x^2 + 9y^2 = 36$

3. $4x^2 - 9y^2 = 36$

4. $y = 4x^2 - 9$

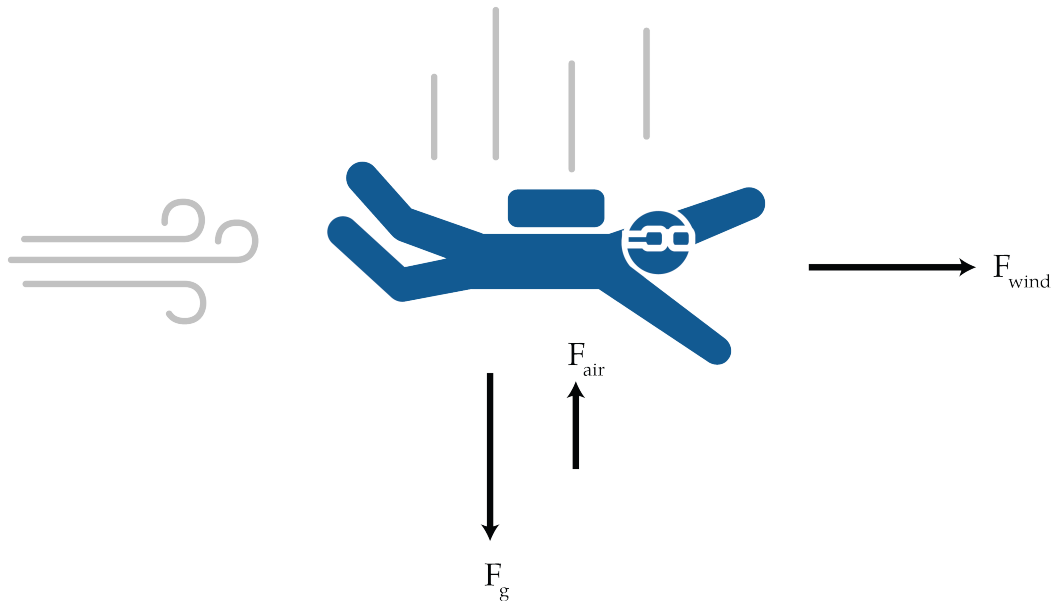
Answer on Page 77

Vectors

6.1 Vectors

So far, we have described forces by their size alone: 10 newtons, 500 newtons. But a push to the left and a push to the right are very different, even if they are the same size. To do math with things that have both a size and a direction, like forces and velocities, we need *vectors*.

Imagine jumping out of a plane (with a parachute, please). Several forces act on you at once, each with its own size and direction. Gravity pulls you straight down with a force proportional to your mass. If there were a wind from the west, it would push you toward the east. That force will be proportional to the square of the speed of the wind and approximately proportional to your size. Once you are falling, there will be resistance from the air that you are pushing through — that force will point in the opposite direction from the direction you are moving and will be proportional to the square of your speed.



To find the net force, which tells you how you will accelerate, you must add these forces together, directions and all. That takes vector math.

6.1.1 Adding Vectors

A vector is typically represented as a list of numbers, with each number representing a particular dimension. For example, if you are creating a 3-dimensional vector representing a force, it will have three numbers representing the amount of force in each of the three axes. For example, if a force of one newton is in the direction of the x -axis, you might represent the vector as $v = [1, 0, 0]$. Another vector might be $u = [0.5, 0.9, 0.7]$. You can see examples of 2-dimensional and 3-dimensional vectors in figures 6.1 and 6.2.

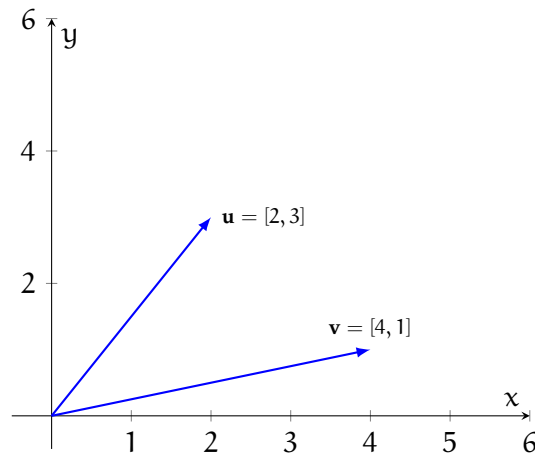


Figure 6.1: 2-dimensional vectors, u and v

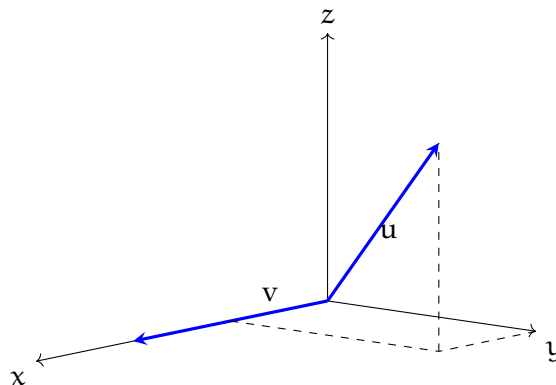


Figure 6.2: 3-dimensional vectors, u and v

To add two vectors visually, put the tail of the second vector at the tip of the first. The sum runs from the tail of the first to the tip of the second. Think of walking: three blocks east, then four blocks north, gets you to the same place as walking straight along the sum. Figure 6.3 shows this in 2 dimensions and figure 6.4 in 3.

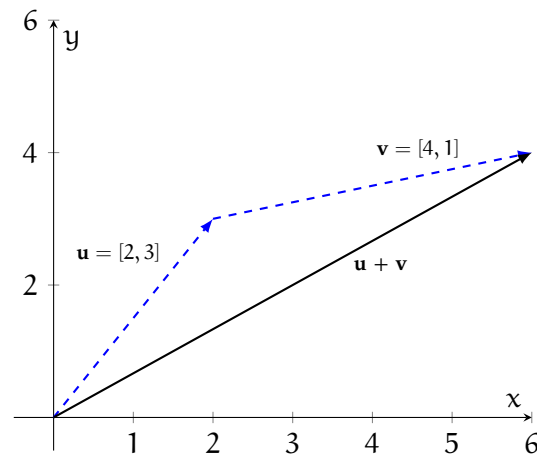


Figure 6.3: A visual representation of adding 2-dimensional vectors, $\mathbf{u}$ and $\mathbf{v}$

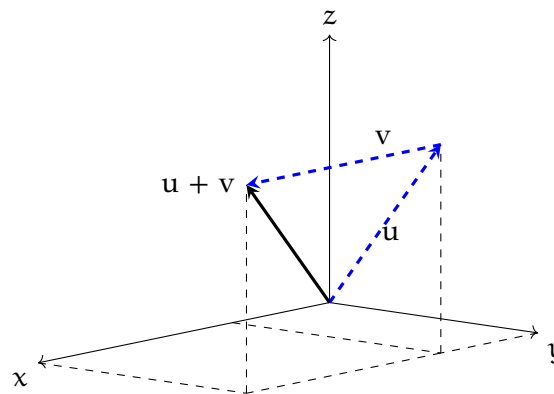


Figure 6.4: A visual representation of adding 3-dimensional vectors, $\mathbf{u}$ and $\mathbf{v}$

If you know the vectors, you will just add them element-wise:

$$\mathbf{u} + \mathbf{v} = [0.5, 0.9, 0.7] + [1.0, 0.0, 0.0] = [1.5, 0.9, 0.7]$$

These vectors have 3 components, so we say they are *3-dimensional*. Vectors can have any number of components. For example, the vector $[-12.2, 3, \pi, 10000]$ is 4-dimensional.

You can only add two vectors if they have the same dimension.

$$[12, -4] + [-1, 5] = [11, 1]$$

Addition is commutative; if you have two vectors $\mathbf{a}$ and $\mathbf{b}$, then $\mathbf{a} + \mathbf{b}$ is the same as $\mathbf{b} + \mathbf{a}$.

Addition is also associative: If you have three vectors a , b , and c , it does not matter which order you add them in. That is, $a + (b + c) = (a + b) + c$.

A 1-dimensional vector is just a number. We say it is a *scalar*, not a vector.

Exercise 28 Adding vectors

Add the following vectors:

- $[1, 2, 3] + [4, 5, 6]$
- $[-1, -2, -3, -4] + [4, 5, 6, 7]$
- $[\pi, 0, 0] + [0, \pi, 0] + [0, 0, \pi]$

Working Space

Answer on Page 77

Exercise 29 Adding Forces

You are adrift in space, near two different stars. The gravity of one star is pulling you towards it with a force of $[4.2, 5.6, 9.0]$ newtons. The gravity of the other star is pulling you towards it with a force of $[-100.2, 30.2, -9.0]$ newtons. What is the net force?

Working Space

Answer on Page 78

6.1.2 Multiplying a vector with a scalar

A rocket engine pushes with some force vector v . Fire 9 identical engines pointed the same way, and the total force is $9v$. That is multiplying a vector by a scalar.

Visually, when we multiply a vector u by a scalar a , we get a new vector that goes in the same direction as u but has a magnitude a times as long as u . A visual is presented in

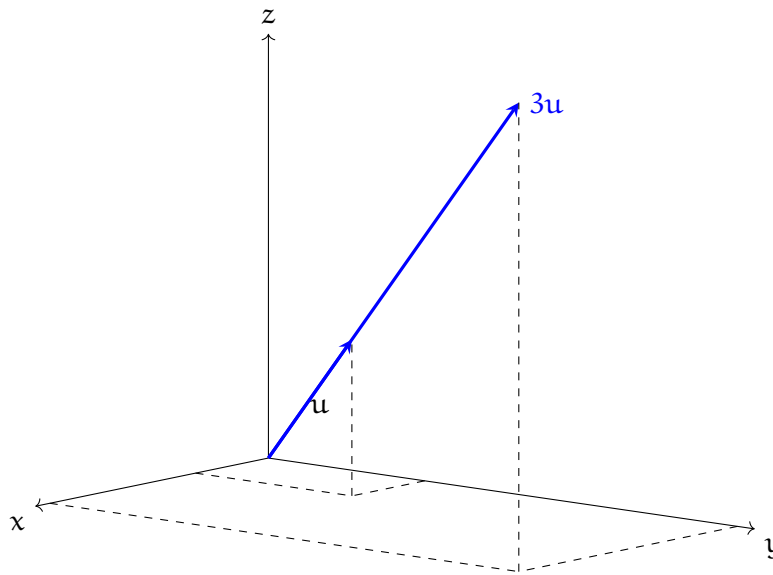


Figure 6.5: Multiplying a vector by a scalar a stretches it by a factor of a without changing its direction.

figure 6.5.

When you multiply a vector by a scalar, you simply multiply each of the components by the scalar:

$$3 \times [0.5, 0.9, 0.7] = [1.5, 2.7, 2.1]$$

Exercise 30 Multiplying a vector and a scalar

Simplify the following expressions:

Working Space

- $2 \times [1, 2, 3]$
- $[-1, -2, -3, -4] \times -2$
- $\pi[\pi, 2\pi, 3\pi]$

Answer on Page 78

Note that when you multiply a vector times a negative number, the new vector points in the opposite direction (see figure 6.6).

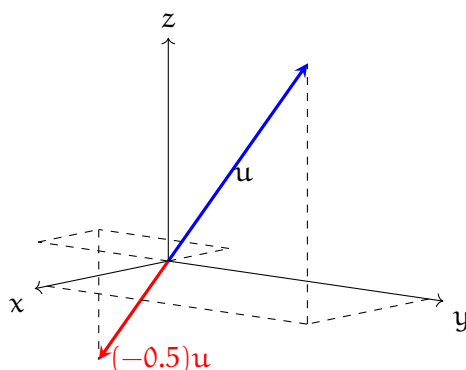


Figure 6.6: Multiplying a vector by a negative number reverses the direction of the vector.

6.1.3 Vector Subtraction

As you might guess, when you subtract one vector from another, you just do element-wise subtraction:

$$[4, 2, 0] - [3, -2, 9] = [1, 4, -9]$$

So, $u - v = u + (-1v)$.

Visually, you reverse the one that is being subtracted (see figure 6.7):

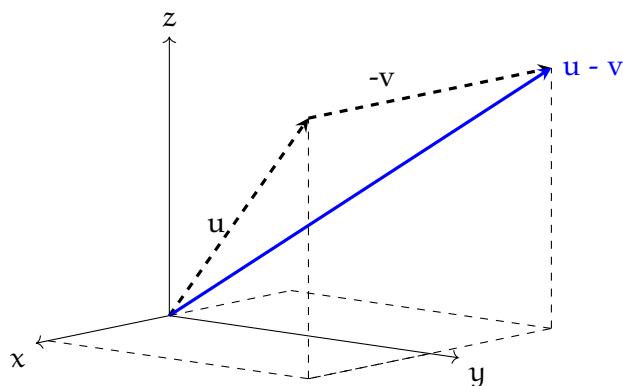


Figure 6.7: To subtract a vector, you reverse it, then add the reversed vector.

6.1.4 Magnitude of a Vector

The *magnitude* of a vector is just its length. We write the magnitude of a vector v as $|v|$.

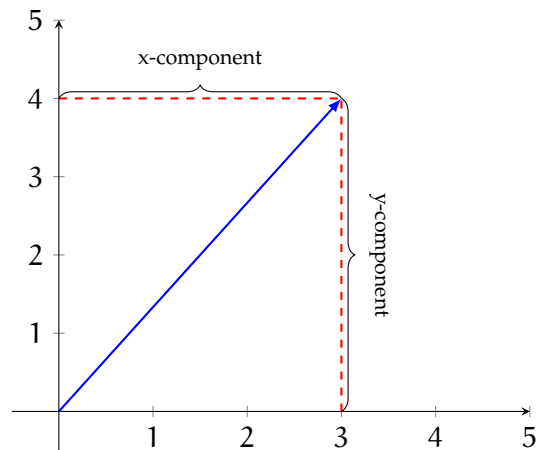


Figure 6.8: The magnitude of a vector can be thought of as the length of a hypotenuse of a right triangle.

We compute the magnitude using the Pythagorean theorem. If $\mathbf{v} = [3, 4, 5]$, then

$$|\mathbf{v}| = \sqrt{3^2 + 4^2 + 5^2} = \sqrt{50} \approx 7.07$$

(You might notice that the notation for the magnitude is exactly like the notation for absolute value. If you think of a scalar as a 1-dimensional vector, the absolute value and the magnitude are the same. For example, the absolute value of -5 is 5 . If you take the magnitude of the one-dimensional vector $[-5]$, you get $\sqrt{25} = 5$.)

Where does this equation come from? Consider a 2-dimensional vector, $\mathbf{v} = [3, 4]$. This means the vector represents 3 units in the x -direction, and 4 units in the y -direction. We can then visualize a right triangle, with the vector being the hypotenuse and the legs being the x - and y -components of the vector (see figure 6.8). As you recall, the length of the hypotenuse of a right triangle is the square root of the sum of the squares of the legs. That is:

$$c = \sqrt{a^2 + b^2}$$

where c is the length of the hypotenuse and a and b are the lengths of the legs. For $[3, 4]$, that is $\sqrt{9 + 16} = 5$.

In 3 dimensions, use the theorem twice: $[3, 4]$ along the floor has length 5, and climbing 5 more straight up makes a new right triangle with hypotenuse $\sqrt{5^2 + 5^2} = \sqrt{50}$. The same trick works in any number of dimensions.

Magnitude of Vectors

For an n -dimensional vector, $\mathbf{v} = [x_1, x_2, x_3, \dots, x_n]$, the magnitude of the vector is given by:

$$|\mathbf{v}| = \sqrt{x_1^2 + x_2^2 + x_3^2 + \dots + x_n^2}$$

Notice that if you scale up a vector, its magnitude scales by the same amount. For example:

$$|7[3, 4, 5]| = 7\sqrt{50} \approx 7 \times 7.07$$

Here is why that is true. Suppose we have a vector, $\mathbf{u} = [a, b, c]$. Then the magnitude of $\mathbf{u}$ is given by:

$$|\mathbf{u}| = \sqrt{a^2 + b^2 + c^2}$$

If we scale $\mathbf{u}$ to create $\mathbf{v}$ such that $\mathbf{v} = k\mathbf{u} = [ka, kb, kc]$, where k is some constant. Then the magnitude of $\mathbf{v}$ is given by:

$$|\mathbf{v}| = \sqrt{(ka)^2 + (kb)^2 + (kc)^2}$$

We can expand and simplify this equation:

$$|\mathbf{v}| = \sqrt{k^2a^2 + k^2b^2 + k^2c^2}$$

$$|\mathbf{v}| = \sqrt{k^2(a^2 + b^2 + c^2)}$$

$$|\mathbf{v}| = \left(\sqrt{k^2}\right) \sqrt{a^2 + b^2 + c^2}$$

$$|\mathbf{v}| = |k| \sqrt{a^2 + b^2 + c^2} = |k| |\mathbf{u}|$$

So, if you scale a vector, the magnitude of the resulting vector is the absolute value of the scale factor times the magnitude of the original vector.

The rule then is: If you have any vector $\mathbf{v}$ and any scalar k :

$$|k\mathbf{v}| = |k||\mathbf{v}|$$

Unit Vectors

A *unit vector* is a vector whose magnitude is 1. For any non-zero vector $\vec{v}$, the unit vector $\vec{u}$ pointing in the same direction is

$$\vec{u} = \frac{\vec{v}}{|\vec{v}|}.$$

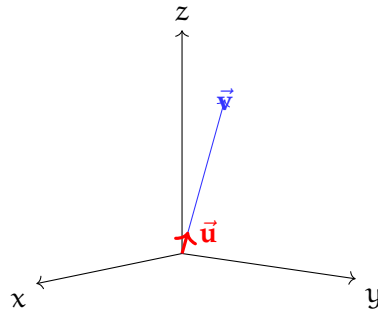
For example, if $\vec{v} = [3, 4, 5]$ then

$$|\vec{v}| = \sqrt{3^2 + 4^2 + 5^2} = \sqrt{50},$$

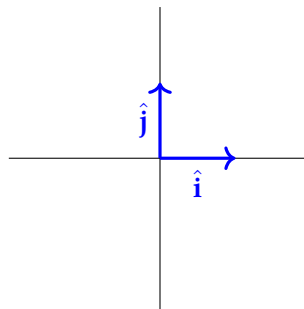
so

$$\vec{u} = \frac{1}{\sqrt{50}} [3, 4, 5] = \left[\frac{3}{\sqrt{50}}, \frac{4}{\sqrt{50}}, \frac{5}{\sqrt{50}} \right].$$

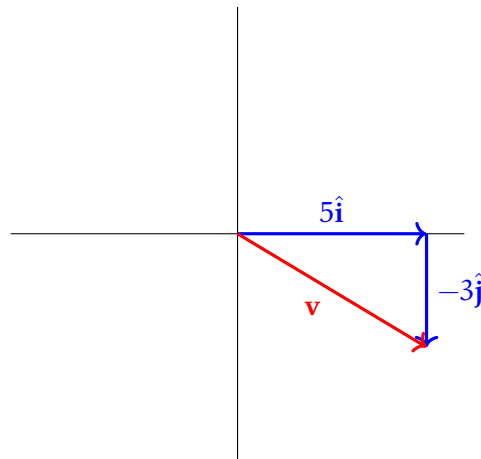
$$\boxed{\hat{\mathbf{u}} = \frac{\mathbf{v}}{|\mathbf{v}|}}, \quad |\vec{u}| = 1.$$



Unit vectors are often written with a hat. Three of them get special names: $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ point one unit along the x , y , and z axes.



So $\vec{v} = 5\hat{\mathbf{i}} - 3\hat{\mathbf{j}}$ means “5 units in the x direction and -3 units in the y direction,” which is just $[5, -3]$. In 3D, add a $\hat{\mathbf{k}}$ term.



Exercise 31 Magnitude of a Vector

Find the magnitude of the following vectors:

- $[1, 1, 1]$
- $[-5, -5, -5]$ (that is the same as $-5 \times [1, 1, 1]$)
- $[3, 4, -4] + [-2, -3, 5]$

Working Space

Answer on Page 78

6.1.5 Vectors in Python

NumPy is a Python library for working with vectors. If it is not installed on your computer yet, install it with pip3:

```
1 pip3 install numpy
```

We can think of a vector as a list of numbers. There are also grids of numbers known as *matrices*. NumPy deals with both in the same way, so it refers to both of them as arrays.

The study of vectors and matrices is known as *Linear Algebra*. Some of the functions we

need are in a sublibrary of NumPy called `linalg`.

As a convention, everyone who uses NumPy, imports it as *np*.

Create a file called `first_vectors.py`:

```

1  import numpy as np
2
3  # Create two vectors
4  v = np.array([2,3,4])
5  u = np.array([-1,-2,3])
6  print(f"u = {u}, v = {v}")
7
8  # Add them
9  w = v + u
10 print(f"u + v = {w}")
11
12 # Multiply by a scalar
13 w = v * 3
14 print(f"v * 3 = {w}")
15
16 # Get the magnitude
17 mv = np.linalg.norm(v)
18 mu = np.linalg.norm(u)
19 print(f"|v| = {mv}, |u| = {mu}")

```

When you run it, you should see:

```

1  > python3 first_vectors.py
2  u = [-1 -2  3], v = [2  3  4]
3  u + v = [1  1  7]
4  v * 3 = [ 6  9 12]
5  |v| = 5.385164807134504, |u| = 3.7416573867739413

```

Formatting Floats

The numbers 5.385164807134504 and 3.7416573867739413 are pretty long. You probably want them rounded off after a couple of decimal places.

Numbers with decimal places are called *floats*. In the placeholder for your float, you can specify how you want it formatted, including the number of decimal places.

Change the last line to look like this:

```

1  print(f"|v| = {mv:.2f}, |u| = {mu:.2f}")

```

When you run the code, it will be neatly rounded off to two decimal places:

```
1 |v| = 5.39, |u| = 3.74
```

6.2 The Dot Product

If you have two vectors $\mathbf{u} = [u_1, u_2, \dots, u_n]$ and $\mathbf{v} = [v_1, v_2, \dots, v_n]$, we define the *dot product* $\mathbf{u} \cdot \mathbf{v}$ as

$$\mathbf{u} \cdot \mathbf{v} = (u_1 \times v_1) + (u_2 \times v_2) + \dots + (u_n \times v_n)$$

The output of the dot product is a *scalar* quantity.

For example,

$$[2, 4, -3] \cdot [5, -1, 1] = 2 \times 5 + 4 \times -1 + -3 \times 1 = 3$$

Multiply matching components and add. That hardly sounds exciting, but dot products are *incredibly* useful. The graphics chip that renders your favorite video game spends much of its time computing dot products, billions of them per second, to figure out how brightly each surface is lit.

Exercise 32 Basic dot products

Compute the dot product of each pair of vectors:

Working Space

- $[1, 2, 3], [4, 5, -6]$
- $[\pi, 2\pi], [2, -1]$
- $[0, 0, 0, 0], [10, 10, 10, 10]$

Answer on Page 78

6.2.1 Properties of the dot product

Sometimes we need an easy way to say “The vector of appropriate length is filled with zeros.” We use the notation $\vec{0}$ to represent this. Then, for any vector $\mathbf{v}$, this is true:

$$\mathbf{v} \cdot \vec{0} = 0$$

The dot product is commutative:

$$\mathbf{v} \cdot \mathbf{u} = \mathbf{u} \cdot \mathbf{v}$$

The dot product of a vector with itself is its magnitude squared:

$$\mathbf{v} \cdot \mathbf{v} = |\mathbf{v}|^2$$

If you have a scalar a , then:

$$(\mathbf{v}) \cdot (a\mathbf{u}) = a(\mathbf{v} \cdot \mathbf{u})$$

So, if $\mathbf{v}$ and $\mathbf{w}$ are vectors that go in the same direction,

$$\mathbf{v} \cdot \mathbf{w} = |\mathbf{v}||\mathbf{w}|$$

If $\mathbf{v}$ and $\mathbf{w}$ are vectors that go in opposite directions,

$$\mathbf{v} \cdot \mathbf{w} = -|\mathbf{v}||\mathbf{w}|$$

If $\mathbf{v}$ and $\mathbf{w}$ are vectors that are perpendicular to each other, their dot product is zero:

$$\mathbf{v} \cdot \mathbf{w} = 0$$

6.2.2 Cosines and dot products

Here is what makes the dot product so useful. For any two vectors $\mathbf{v}$ and $\mathbf{u}$,

$$\mathbf{v} \cdot \mathbf{u} = |\mathbf{v}||\mathbf{u}| \cos \theta$$

where θ is the angle between them.

So, for example, if two vectors $\mathbf{v}$ and $\mathbf{u}$ are perpendicular, the angle between them is $\pi/2$. The cosine of $\pi/2$ is 0. The dot product of any two perpendicular vectors is always 0. In fact, if the dot product of two non-zero vectors is 0, the vectors *must be* perpendicular (see figure 6.9 for an example of perpendicular 2-dimensional vectors).

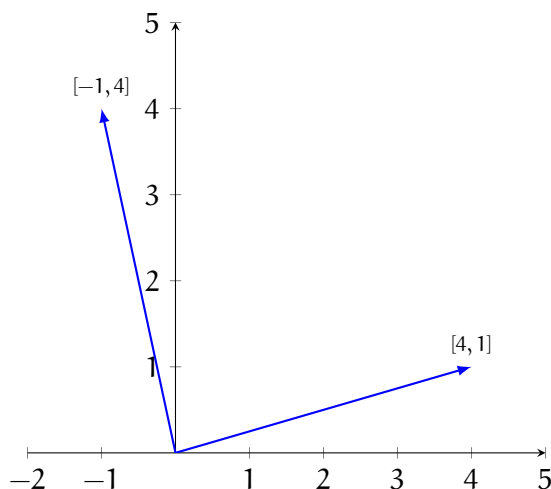


Figure 6.9: The dot product of any two perpendicular vectors is zero.

If you have two non-zero vectors $\mathbf{v}$ and $\mathbf{u}$, you can always compute the angle between them:

$$\theta = \arccos \left(\frac{\mathbf{v} \cdot \mathbf{u}}{|\mathbf{v}||\mathbf{u}|} \right)$$

Arccos, short for arccosine and also written $\cos^{-1}$, undoes cosine. Cosine takes an angle and gives the x-coordinate of that point on the unit circle; arccos takes the x-coordinate and gives back an angle. But which angle? Many angles share the same cosine (figures 6.10 and 6.11).

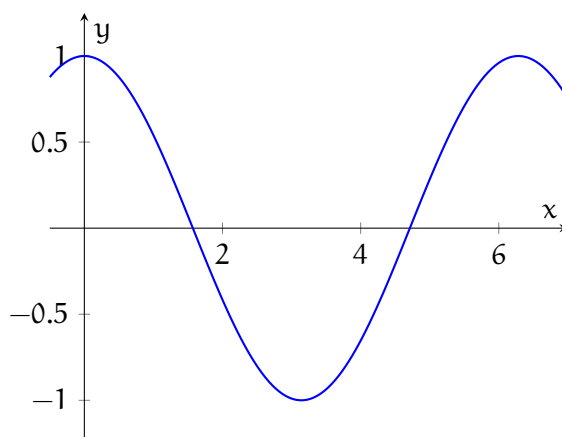


Figure 6.10: Cosine is a function: there is exactly one output for every input.

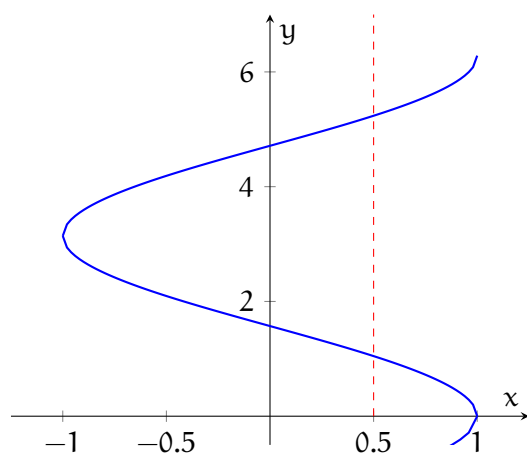


Figure 6.11: Flipping the cosine graph does not give a function: many angles have the same cosine, so one input has many outputs (red dashed line). Arccos keeps only the outputs between 0 and π .

To make arccos a function, we keep only answers between 0 and π . That is exactly the range of possible angles between two vectors, so it works out perfectly.

Example: What is the angle between $\mathbf{u} = [\sqrt{3}, 1]$ and $\mathbf{v} = [0, -1]$?

Solution: We know that $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}| |\mathbf{v}| \cos \theta$. Therefore, we also know that:

$$\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}| |\mathbf{v}|}$$

First, let us compute the dot product:

$$\mathbf{u} \cdot \mathbf{v} = \sqrt{3} \cdot 0 + 1 \cdot -1 = -1$$

And therefore:

$$\cos \theta = \frac{-1}{|\mathbf{u}| |\mathbf{v}|}$$

Now, let us find the magnitudes of both vectors:

$$|\mathbf{u}| = \sqrt{(\sqrt{3})^2 + (1)^2} = 2$$

$$|\mathbf{v}| = \sqrt{(0)^2 + (-1)^2} = 1$$

Substituting for the magnitudes, we find that:

$$\cos \theta = \frac{-1}{2 \cdot 1} = \frac{-1}{2}$$

To solve for θ , we take the arccos of both sides:

$$\arccos(\cos \theta) = \theta = \arccos \frac{-1}{2}$$

What angles have a cosine of $-1/2$? We know that $2\pi/3, 4\pi/3, 8\pi/3$, etc., all have a cosine of $-1/2$. Because the range of arccos is restricted to between 0 and π , our result is:

$$\theta = \arccos \frac{-1}{2} = \frac{2\pi}{3}$$

.

Therefore, the angle between $\mathbf{u}$ and $\mathbf{v}$ is $2\pi/3$ (or 120°).

Exercise 33 Using dot products

What is the angle between these each pair of vectors:

Working Space

- $[1, 0], [0, 1]$
- $[3, 4], [4, 3]$
- $[2, -1, 2], [-1, 2, -2]$

- $[-5, 0, -1]$, $[2, 3, -4]$

[Answer on Page 79](#)

6.2.3 Dot products in Python

NumPy will let you do dot products using the symbol `@`. Open `first_vectors.py` and add the following to the end of the script:

```

1 # Take the dot product
2 d = v @ u
3 print("v @ u =", d)
4
5 # Get the angle between the vectors
6 a = np.arccos(d / (mv * mu))
7 print(f"The angle between u and v is {a * 180 / np.pi:.2f} degrees")

```

When you run it you should get:

```

1 v @ u = 4
2 The angle between u and v is 78.55 degrees

```

6.2.4 Work and Power

In an earlier chapter, work was force times distance. Push a train with 10 newtons as it rolls 5 meters down the track, and you do 50 joules of work.

Now push the train sideways, straight at the side of the car, as it rolls those same 5 meters. You do no work at all: the train did not move one centimeter in the direction you pushed.

Dot products handle both cases. The *work* you do is the dot product of your force vector and the displacement vector (the vector from where the train started to where it ended up):

$$W = \vec{F} \cdot \vec{d} = |\vec{F}||\vec{d}| \cos \theta$$

$$P = \vec{F} \cdot \vec{v} = |\vec{F}||\vec{v}| \cos \theta$$

Likewise, *power* is the dot product of the force vector and the velocity vector. If you push

a sled with 10 newtons while it moves at 2 meters per second, but your push is 20 degrees off, you are not delivering 20 watts. You are delivering $10 \times 2 \times \cos(20 \text{ degrees}) \approx 18.8$ watts of power.

Exercise 34 Power at an angle

A person pushes a skateboard with a force of 40 N. The skateboard is moving forward at 3 m/s, but the person's push is angled 30° above the direction of motion. How much power is being transferred to the skateboard?

Working Space

Answer on Page 79

The same idea explains potential energy. If a cart moves a distance L up a ramp at angle θ , its displacement is $\vec{d} = [L \cos \theta, L \sin \theta]$. Gravity is $\vec{F}_g = [0, -mg]$.

Therefore,

$$\vec{F}_g \cdot \vec{d} = [0, -mg] \cdot [L \cos \theta, L \sin \theta] = -mgL \sin \theta.$$

Since $L \sin \theta$ is the height h the cart climbs, gravity does $-mgh$ joules of work. That is why the cart gains mgh of potential energy: someone else had to do mgh of work against gravity. Notice that the angle of the ramp dropped out. Only the height matters.

Exercise 35 Work done on a Box

A force $\vec{F} = [12, 5]$ N pushes a box whose displacement is $\vec{d} = [3, 0]$ m. How much work is done?

Working Space

Answer on Page 79

6.3 Cross Product

6.3.1 The Cross Product

The dot product of two vectors is a number that measures how much they *line up*. The *cross product* of two 3D vectors is a *vector*: it points perpendicular to both of them, and its length measures how much they *do not* line up. Physicists use it for anything that twists: turning a wrench, a spinning top, the force a magnet exerts on a moving charge.

The formula is often written as a grid with the unit vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ in the top row. (In linear algebra, you will recognize the grid as a *determinant*.) For now, treat it as a memory aid: to get each component, cover that component's column and cross-multiply the four numbers that remain, then flip the sign of the middle one.

Given two vectors in $\mathbb{R}^3$,

$$\vec{\mathbf{a}} = \langle a_x, a_y, a_z \rangle, \quad \vec{\mathbf{b}} = \langle b_x, b_y, b_z \rangle,$$

their *cross product* is defined as

$$\vec{\mathbf{a}} \times \vec{\mathbf{b}} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

Expanding this yields:

$$\vec{\mathbf{a}} \times \vec{\mathbf{b}} = (a_y b_z - a_z b_y) \mathbf{i} - (a_x b_z - a_z b_x) \mathbf{j} + (a_x b_y - a_y b_x) \mathbf{k}.$$

where $\mathbf{i}, \mathbf{j}$, and $\mathbf{k}$ are unit vectors.

Geometric Interpretation

The cross product $\vec{\mathbf{a}} \times \vec{\mathbf{b}}$ has the following properties:

1. It is perpendicular to both $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$, making it *orthogonal* as well (orthogonality can be thought of as 3D perpendicularity).
2. Orthogonality implies $\vec{\mathbf{a}} \times \vec{\mathbf{b}} = \vec{\mathbf{c}}$, $\vec{\mathbf{a}} \cdot \vec{\mathbf{c}} = 0$ and $\vec{\mathbf{b}} \cdot \vec{\mathbf{c}} = 0$.
3. Its magnitude is

$$\|\vec{\mathbf{a}} \times \vec{\mathbf{b}}\| = \|\vec{\mathbf{a}}\| \|\vec{\mathbf{b}}\| \sin \theta,$$

where θ is the angle between $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$.

4. The magnitude equals the area of the parallelogram spanned by $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$.

5. Two parallel vectors $\vec{a}$ and $\vec{b}$ result in the zero vector, $\vec{0}$.

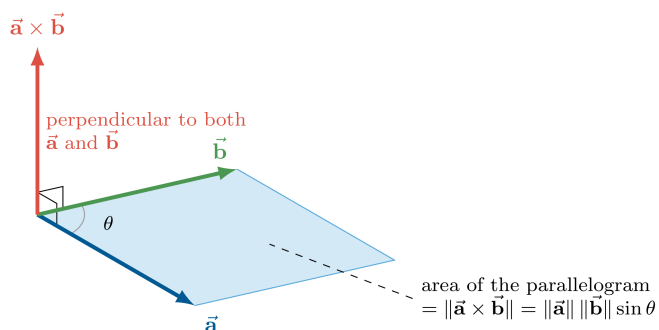


Figure 6.12: The cross product $\vec{a} \times \vec{b}$ is perpendicular to both $\vec{a}$ and $\vec{b}$, and its magnitude equals the area of the parallelogram they span.

Exercise 36 Using the cross product

Find the cross product of the vectors

$$\vec{v}_1 = \langle 4, 5, 6 \rangle \text{ and } \vec{v}_2 = \langle 3, 7, -8 \rangle$$

Working Space

Answer on Page 80

Answers to Exercises

Answer to Exercise 1 (on page 7)

The hypotenuse is $\sqrt{25 + 144} = 13$. Then $\sin \theta = \frac{5}{13}$, $\cos \theta = \frac{12}{13}$, and $\tan \theta = \frac{5}{12}$.

Answer to Exercise 2 (on page 7)

1. The height is opposite the angle, and the string is the hypotenuse: $50 \sin 55^\circ \approx 41.0$ m.
2. The angle of depression from the cliff equals the angle of elevation from the boat (they are alternate angles between parallel horizontal lines). So $\tan 12^\circ = \frac{30}{d}$ and $d = \frac{30}{\tan 12^\circ} \approx 141$ m.

Answer to Exercise 3 (on page 7)

The rise is opposite the angle and the ramp is the hypotenuse: $\sin 4.8^\circ = \frac{0.75}{L}$, so $L = \frac{0.75}{\sin 4.8^\circ} \approx 8.96$ m. A steeper angle would allow a shorter ramp, so this is the minimum.

Answer to Exercise 4 (on page 10)

$B = 180^\circ - 35^\circ - 105^\circ = 40^\circ$. Then $a = \frac{20 \sin 35^\circ}{\sin 105^\circ} \approx 11.88$ and $b = \frac{20 \sin 40^\circ}{\sin 105^\circ} \approx 13.31$.

Answer to Exercise 5 (on page 10)

1. $\sin B = \frac{8 \sin 30^\circ}{5} = 0.8$, so $B \approx 53.1^\circ$ or $B \approx 126.9^\circ$. Both leave room for C ($30^\circ + 126.9^\circ < 180^\circ$), so there are two triangles.
2. $\sin B = \frac{8 \sin 30^\circ}{3} \approx 1.33$. No angle has a sine greater than 1, so there is no triangle: side a is too short to reach.

Answer to Exercise 6 (on page 12)

1. $c^2 = 64 + 121 - 176 \cos 40^\circ \approx 50.2$, so $c \approx 7.08$.
2. The largest angle is across from 9: $\cos C = \frac{49+64-81}{2(7)(8)} = \frac{32}{112} \approx 0.286$, so $C \approx 73.4^\circ$.

Answer to Exercise 7 (on page 13)

Turning 30° off a straight line makes the angle between the two legs, at the turning point, $180^\circ - 30^\circ = 150^\circ$. Then $d^2 = 20^2 + 15^2 - 2(20)(15) \cos 150^\circ \approx 625 + 519.6 = 1144.6$, so $d \approx 33.8$ km. (Without the turn, it would be 35 km away.)

Answer to Exercise 8 (on page 13)

```
1 from math import sin, radians
2
3 def area(a, b, angle_C):
4     return 0.5 * a * b * sin(radians(angle_C))
5
6 print(round(area(8, 11, 40), 1))    # 28.3
```

Answer to Exercise 9 (on page 16)

An angle of π points to $(-1, 0)$, so $\sin(\pi) = 0$ and $\cos(\pi) = -1$. An angle of $\frac{3\pi}{2}$ points to $(0, -1)$, so $\sin(\frac{3\pi}{2}) = -1$. An angle of 2π is a full turn, back to $(1, 0)$, so $\cos(2\pi) = 1$.

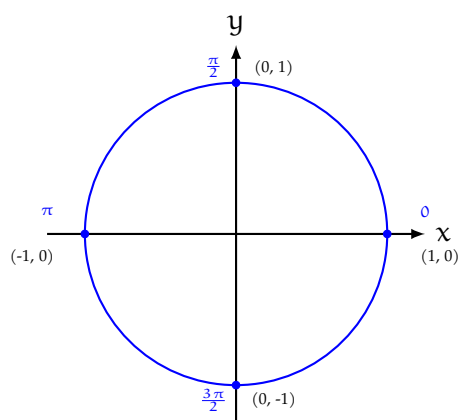
$\cos^2(\theta) = 1 - 0.6^2 = 0.64$, so $\cos(\theta) = 0.8$ (positive, because the point is in the first quadrant).

Answer to Exercise 10 (on page 24)

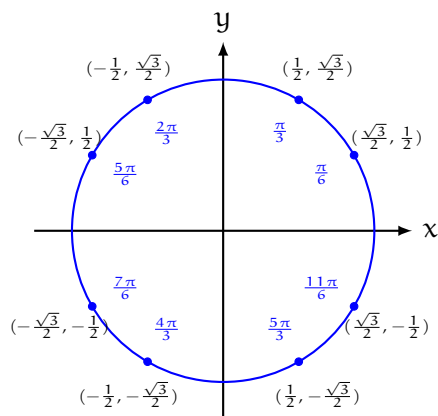
We know that for a right triangle, $\cos \theta = \frac{\text{adjacent leg}}{\text{hypotenuse}}$. For a right triangle inscribed in the Unit Circle, the adjacent leg is parallel to the x-axis and has the same length as the x-value of the coordinate point on the circle. Additionally, the length of the hypotenuse is 1. Therefore, $\cos \theta = \frac{x_0}{1} = x_0$.

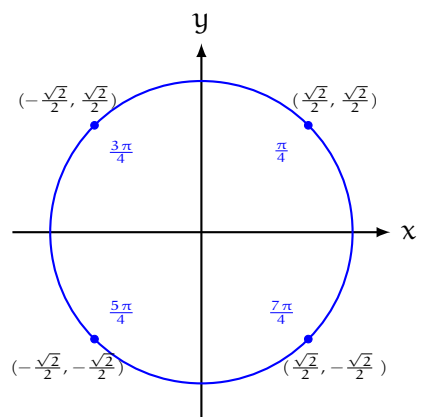
Answer to Exercise 11 (on page 24)

1. $\sin \frac{\pi}{2} = 1$
2. $\cos \frac{3\pi}{2} = 0$
3. $\sin \pi = 0$
4. $\cos -\pi = -1$ (Negative angles are measured clockwise from the x -axis, so $\theta = -\pi$ is at the same angle as $\theta = \pi$.)



Answer to Exercise 12 (on page 26)



Answer to Exercise 13 (on page 27)**Answer to Exercise 14 (on page 27)**

1. 0
2. $\sqrt{2}/2$
3. $-1/2$
4. $-1/2$
5. $\sqrt{2}/2$
6. $1/2$
7. $\sqrt{2}/2$
8. -1
9. $-\sqrt{3}/2$
10. $1/2$

Answer to Exercise 15 (on page 30)

1. $\sin(\pi/12) = \sin(\pi/3 - \pi/4) = \sin \frac{\pi}{3} \cos \frac{\pi}{4} - \cos \frac{\pi}{3} \sin \frac{\pi}{4} = \left(\frac{\sqrt{3}}{2}\right) \cdot \left(\frac{\sqrt{2}}{2}\right) - \left(\frac{1}{2}\right) \cdot \left(\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4} = \frac{\sqrt{6}-\sqrt{2}}{4}$
2. $\cos(7\pi/12) = \cos(4\pi/12 + 3\pi/12) = \cos(\pi/3 + \pi/4) = \cos \frac{\pi}{3} \cos \frac{\pi}{4} - \sin \frac{\pi}{3} \sin \frac{\pi}{4} = \left(\frac{1}{2}\right) \cdot$

$$\left(\frac{\sqrt{2}}{2}\right) - \left(\frac{\sqrt{3}}{2}\right) \cdot \left(\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{4} - \frac{\sqrt{6}}{4} = \frac{\sqrt{2}-\sqrt{6}}{4}$$

3. $\tan(13\pi/12) = \frac{\sin(13\pi/12)}{\cos(13\pi/12)}$ First, we will find $\sin(13\pi/12)$: $\sin(13\pi/12) = \sin(3\pi/12 + 10\pi/12) = \sin(\pi/4 + 5\pi/6) = \sin \frac{\pi}{4} \cos \frac{5\pi}{6} + \cos \frac{\pi}{4} \sin \frac{5\pi}{6} = \left(\frac{\sqrt{2}}{2}\right) \cdot \left(-\frac{\sqrt{3}}{2}\right) + \left(\frac{\sqrt{2}}{2}\right) \cdot \left(\frac{1}{2}\right) = \frac{-\sqrt{6}}{4} + \frac{\sqrt{2}}{4} = \frac{\sqrt{2}-\sqrt{6}}{4}$. Next we find $\cos(13\pi/12) = \cos(\pi/4 + 5\pi/6) = \cos \frac{\pi}{4} \cos \frac{5\pi}{6} - \sin \frac{\pi}{4} \sin \frac{5\pi}{6} = \left(\frac{\sqrt{2}}{2}\right) \cdot \left(-\frac{\sqrt{3}}{2}\right) - \left(\frac{\sqrt{2}}{2}\right) \cdot \left(\frac{1}{2}\right) = \frac{-\sqrt{6}}{4} - \frac{\sqrt{2}}{4} = \frac{-\sqrt{6}-\sqrt{2}}{4}$. And therefore $\tan(13\pi/12) = \frac{\sin 13\pi/12}{\cos 13\pi/12} = \frac{\sqrt{2}-\sqrt{6}}{4} \cdot \frac{4}{-\sqrt{6}-\sqrt{2}} = \frac{\sqrt{6}-\sqrt{2}}{\sqrt{6}+\sqrt{2}} = \frac{\sqrt{6}-\sqrt{2}}{\sqrt{6}+\sqrt{2}} \cdot \frac{\sqrt{6}-\sqrt{2}}{\sqrt{6}-\sqrt{2}} = \frac{6-2\sqrt{12}+2}{6-2} = \frac{8-4\sqrt{3}}{4} = 2 - \sqrt{3}$

Answer to Exercise 16 (on page 31)

$$\sin 2\theta = \sin(\theta + \theta) = \sin \theta \cos \theta + \cos \theta \sin \theta = 2 \sin \theta \cos \theta$$

Answer to Exercise 17 (on page 32)

Similar to $\cos(\alpha/2)$, we begin with the double angle formula for cosine, but another version:

$$\cos 2\theta = 1 - 2 \sin^2 \theta$$

Substituting $\theta = \alpha/2$:

$$\cos \alpha = 1 - 2 \sin^2(\alpha/2)$$

And rearranging to solve for $\sin(\alpha/2)$:

$$\sin(\alpha/2) = \pm \sqrt{\frac{1 - \cos \alpha}{2}}$$

Answer to Exercise 18 (on page 32)

15° is in the first quadrant, so take the positive root:

$$\cos(15^\circ) = \sqrt{\frac{1 + \cos 30^\circ}{2}} = \sqrt{\frac{2 + \sqrt{3}}{4}} = \frac{\sqrt{2 + \sqrt{3}}}{2} \approx 0.9659$$

The difference formula gives $\cos 45^\circ \cos 30^\circ + \sin 45^\circ \sin 30^\circ = \frac{\sqrt{6}+\sqrt{2}}{4} \approx 0.9659$. They agree: squaring $\frac{\sqrt{6}+\sqrt{2}}{4}$ gives $\frac{8+2\sqrt{12}}{16} = \frac{2+\sqrt{3}}{4}$.

Answer to Exercise 19 (on page 34)

The volume of the sphere (in cubic meters) is

$$\frac{4}{3}\pi(1.5)^3 = 4.5\pi \approx 14.14$$

The mass (in kg) is $14.14 \times 7800 \approx 110,269$

The kinetic energy (in joules) is

$$k = \frac{110269 \times 5^2}{2} = 1,378,373$$

About 1.4 million joules.

Answer to Exercise 20 (on page 35)

In your mind, you can disassemble the tablet into a sphere (made up of the two ends) and a cylinder (between the two ends).

The volume of the sphere (in cubic millimeters) is

$$\frac{4}{3}\pi(2)^3 = \frac{32}{3}\pi \approx 33.5$$

Thus, the cylinder part has to be $90 - 33.5 = 56.5$ cubic mm. The cylinder part has a radius of 2 mm. If the length of the cylinder part is x , then

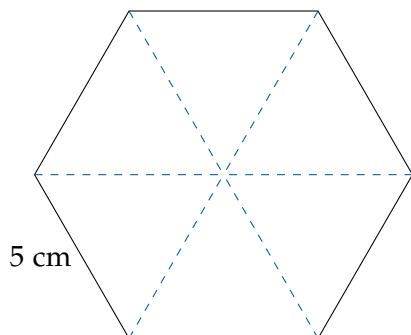
$$\pi 2^2 x = 56.5$$

Thus $x = \frac{56.5}{4\pi} \approx 4.5$ mm.

The cylinder part is 4.5 mm long. Add 2 mm for each rounded end, and the whole tablet is 8.5 mm long.

Answer to Exercise 21 (on page 38)

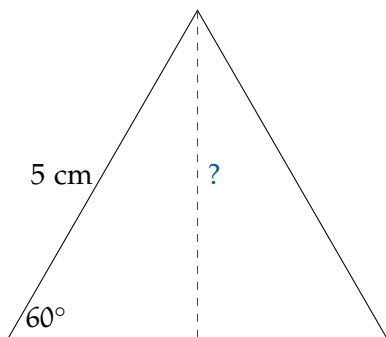
First, you need to find the area of the base, which is a regular hexagon:



All the angles in this picture are 60° or $\frac{\pi}{3}$ radians. This means each line is 5 cm long.

This tells us that we need to find the area of one of these triangles and multiply that by six.

Every triangle has a base of 5cm. How tall are they?



$$5 \sin 60^\circ = 5 \frac{\sqrt{3}}{2}$$

Which is about 4.33 cm.

So, the area of single triangle is

$$\frac{1}{2}(5) \left(5 \frac{\sqrt{3}}{2} \right) = 25 \frac{\sqrt{3}}{4}$$

And the area of the whole hexagon is six times that:

$$75\frac{\sqrt{3}}{2}$$

Thus, the volume of the pyramid is:

$$\frac{1}{3}hb = \frac{1}{3}13\left(75\frac{\sqrt{3}}{2}\right)$$

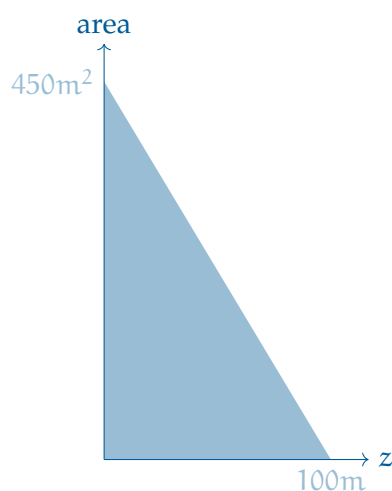
About 281.46 cubic centimeters.

Answer to Exercise 22 (on page 40)

The area at height z is given by:

$$a = \frac{1}{2}w^2 = \frac{1}{2}\left(30\sqrt{1 - \frac{z}{100}}\right)^2 = \frac{1}{2}900\left(1 - \frac{z}{100}\right)$$

If we plot that, it looks like this:



The blue region is a triangle: $\frac{1}{2}(450)(100) = 22,500$.

The building has a volume of 22,500 cubic meters.

Answer to Exercise 23 (on page 41)

$V = \pi(3)^2(4) = 36\pi$ cubic units, or about 113.1 cubic units.

Answer to Exercise 24 (on page 44)

1. Center $(0, 0)$; $r^2 = 4$, so $r = 2$.
2. Center $(-1, 5)$, radius 7.
3. Add 25 to both sides: $x^2 + (y + 5)^2 = 25$. Center $(0, -5)$, radius 5.

Answer to Exercise 25 (on page 45)

$a = 5$ and $b = 3$, so it is 10 wide and 6 tall. $c^2 = 25 - 9 = 16$, so the foci are at $(\pm 4, 0)$.

At the right end of the ellipse, $(5, 0)$, the distances to the foci are 1 and 9, which add to 10. So the string from tack to pencil to tack is 10 long. The loop also runs straight between the tacks, another 8, so the loop is 18 long.

Answer to Exercise 26 (on page 47)

$a = \frac{1}{8}$, so $p = \frac{1}{4 \cdot \frac{1}{8}} = 2$. Mount the receiver at the focus, 2 decimeters (20 cm) above the bottom of the dish.

Answer to Exercise 27 (on page 48)

1. Circle: dividing by 4 gives $x^2 + y^2 = 9$, radius 3.
2. Ellipse: dividing by 36 gives $\frac{x^2}{9} + \frac{y^2}{4} = 1$.
3. Hyperbola: dividing by 36 gives $\frac{x^2}{9} - \frac{y^2}{4} = 1$.
4. Parabola: only x is squared.

Answer to Exercise 28 (on page 52)

- $[1, 2, 3] + [4, 5, 6] = [5, 7, 9]$
- $[-1, -2, -3, -4] + [4, 5, 6, 7] = [3, 3, 3, 3]$
- $[\pi, 0, 0] + [0, \pi, 0] + [0, 0, \pi] = [\pi, \pi, \pi]$

Answer to Exercise 29 (on page 52)

To get the net force, you add the two forces:

$$\mathbf{F} = [4.2, 5.6, 9.0] + [-100.2, 30.2, -9.0] = [-96, 35.8, 0.0] \text{ newtons}$$

Answer to Exercise 30 (on page 53)

- $2 \times [1, 2, 3] = [2, 4, 6]$
- $[-1, -2, -3, -4] \times -2 = [2, 4, 6, 8]$
- $\pi[\pi, 2\pi, 3\pi] = [\pi^2, 2\pi^2, 3\pi^2]$

Answer to Exercise 31 (on page 58)

- $||[1, 1, 1]|| = \sqrt{3} \approx 1.73$
- $||[-5, -5, -5]|| = |-5 \times [1, 1, 1]| = 5\sqrt{3} \approx 8.66$
- $||[3, 4, -4] + [-2, -3, 5]|| = ||[1, 1, 1]|| = \sqrt{3} \approx 1.73$

Answer to Exercise 32 (on page 60)

- $[1, 2, 3] \cdot [4, 5, -6] = 4 + 10 - 18 = -4$
- $[\pi, 2\pi] \cdot [2, -1] = 2\pi - 2\pi = 0$
- $[0, 0, 0, 0] \cdot [10, 10, 10, 10] = 0 + 0 + 0 + 0 = 0$

Answer to Exercise 33 (on page 64)

- $[1, 0] \cdot [0, 1] = 0$. The angle must be $\pi/2$.
- $[3, 4] \cdot [4, 3] = 24$. $\|[3, 4]\| \|[4, 3]\| \cos(\theta) = 24$. $\cos(\theta) = \frac{24}{(5)(5)}$. $\theta = \arccos(\frac{24}{25}) \approx 0.284$ radians.
- $[2, -1, 2] \cdot [-1, 2, -2] = 4 - 2 - 4 = -2$. $\|[2, -1, 2]\| = \sqrt{4 + 1 + 4} = \sqrt{9} = 3$. $\|[-1, 2, -2]\| = \sqrt{1 + 4 + 4} = \sqrt{9} = 3$. $3(3) \cos \theta = -2$. $\theta = \arccos(-2/9) \approx 1.795$ radians.
- $[-5, 0, -1] \cdot [2, 3, -4] = -10 + 0 + 4 = -6$. $\|[-5, 0, -1]\| = \sqrt{25 + 0 + 1} = \sqrt{26}$. $\|[2, 3, -4]\| = \sqrt{4 + 9 + 16} = \sqrt{29}$. $\sqrt{26}(\sqrt{29}) \cos \theta = -6$. $\theta = \arccos(\frac{-6}{\sqrt{26}\sqrt{29}}) \approx 1.791$ radians.

Answer to Exercise 34 (on page 66)

Power is the dot product of the force vector and the velocity vector:

$$P = \vec{F} \cdot \vec{v}$$

Since the force and velocity are at an angle, we can use:

$$P = |\vec{F}| |\vec{v}| \cos \theta$$

Here, $|\vec{F}| = 40$ N, $|\vec{v}| = 3$ m/s, and $\theta = 30^\circ$:

$$P = 40(3) \cos(30^\circ)$$

Since $\cos(30^\circ) \approx 0.866$:

$$P \approx 120(0.866) \approx 104 \text{ W}$$

Only the part of the force pointing in the direction of motion transfers power to the skateboard. The angled part still matters physically, but it does not all help move the skateboard forward. So the power transferred to the skateboard is about 104 watts.

Answer to Exercise 35 (on page 66)

The work done is the dot product of the force vector and the displacement vector:

$$W = \vec{F} \cdot \vec{d}$$

Here, the force is $\vec{F} = [12, 5]$ N and the displacement is $\vec{d} = [3, 0]$ m. Take the dot product:

$$W = [12, 5] \cdot [3, 0] = 12(3) + 5(0) = 36 \text{ J}$$

The horizontal part of the force does work because the box moves horizontally. The vertical part of the force does no work because the box does not move vertically. So the total work done is 36 joules.

Answer to Exercise 36 (on page 68)

Setting up the matrix, we get:

$$\begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ 4 & 5 & 6 \\ 3 & 7 & -8 \end{bmatrix}$$

$$\begin{aligned} & (5(-8) - (7)(6))\vec{i} - ((4)(-8) - (6)(3))\vec{j} + ((4)(7) - (5)(3))\vec{k} \\ & (-40 - 42)\vec{i} - (-32 - 18)\vec{j} + (28 - 15)\vec{k} \\ & -82\vec{i} + 50\vec{j} + 13\vec{k} \implies \langle -82, 50, 13 \rangle \end{aligned}$$

Check: the answer should be perpendicular to both vectors. $\langle 4, 5, 6 \rangle \cdot \langle -82, 50, 13 \rangle = -328 + 250 + 78 = 0$ and $\langle 3, 7, -8 \rangle \cdot \langle -82, 50, 13 \rangle = -246 + 350 - 104 = 0$.



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