Contents

Chapter 1

Trigometric Functions

As mentioned in an earlier chapter, in a right triangle where one angle is θ, the sine of θ is the length of the side opposite θ divided by the length of the hypotenuse.

The sine function is defined for any real number. We treat that real number θ as an angle, we draw a ray from the origin out to the unit circle. The y value of that point is the sine. For example, the $\sin(\frac{4\pi}{3})$ is $-\sqrt{3}/2$ (see figure 1.1).

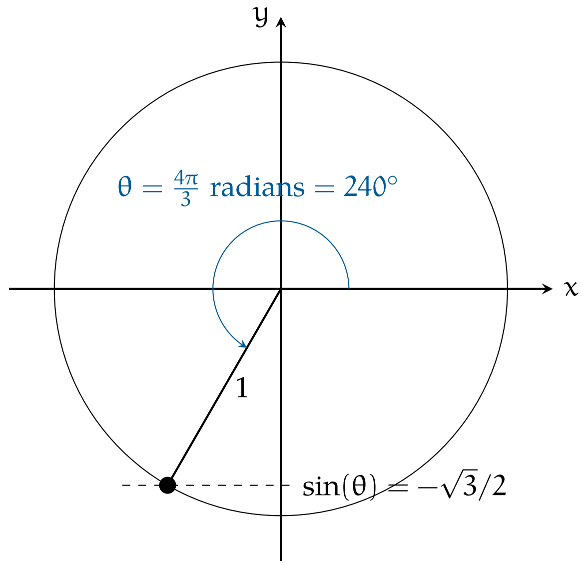


Figure 1.1: $\sin \frac{4\pi}{3} = \frac{-\sqrt{3}}{2}$

(Note that in this section, we will be using radians instead of degrees unless otherwise noted. While degrees are more familiar to most people, engineers and mathematicians nearly always use radians when solving problems. Your calculator should have a radians mode and a degrees mode; you want to be in radians mode.)

Similarly, we define cosine using the unit circle. To find the cosine of θ, we draw a ray from the origin at the angle θ. The x component of the point where the ray intersects the unit circle is the cosine of θ (shown in figure 1.2).

From this description, it is easy to see why $\sin(\theta)^2 + \cos(\theta)^2 = 1$. They are the legs of a right triangle with a hypotenuse of length 1.

It should also be easy to see why $\sin(\theta) = \sin(\theta + 2\pi)$: Each time you go around the circle, you come back to where you started.

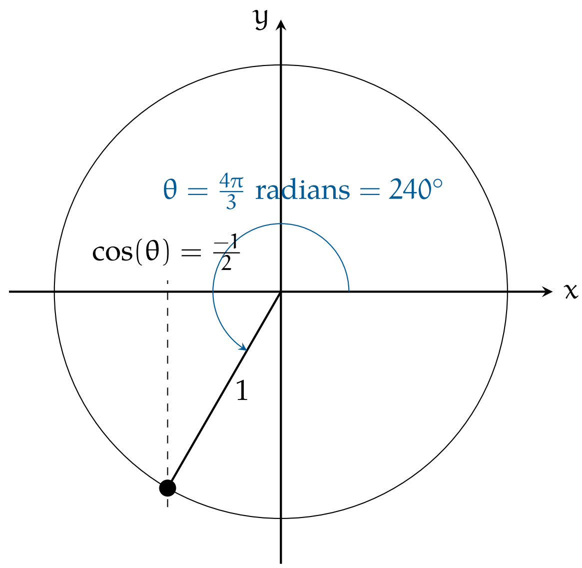


Figure 1.2: $\cos \frac{4\pi}{3} = \frac{-1}{2}$

Can you see why $\cos(\theta) = \sin(\theta + \pi/2)$? Turn the picture sideways.

1.1 Graphs of sine and cosine

Here is a graph of $y = \sin(x)$: It looks like waves, right? It goes forever to the left and

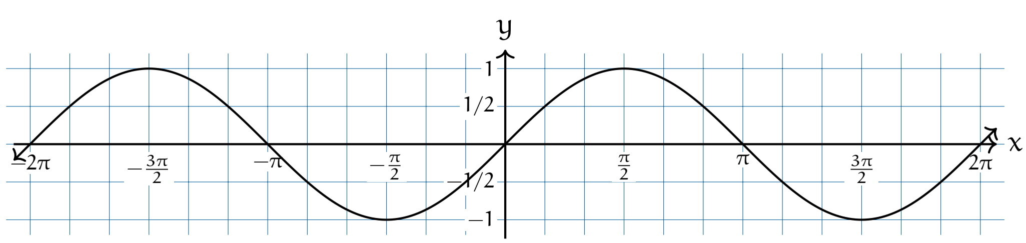


Figure 1.3: $y = \sin(x)$.

right. Remembering that $\cos(\theta) = \sin(\theta+\pi/2)$, we can guess what the graph of $y = \cos(x)$ looks like:

1.2 Plot cosine in Python

Create a file called `cos.py`:

```
import numpy as np
import matplotlib.pyplot as plt
```

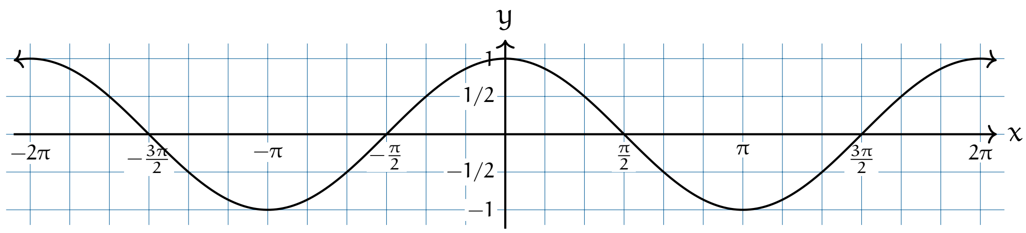


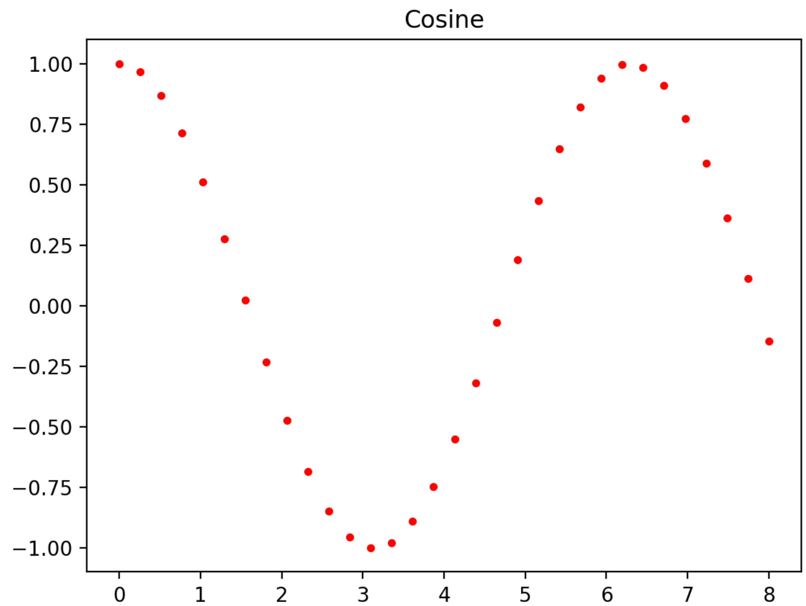
Figure 1.4: $y = \cos(x)$.

```
until = 8.0

# Make a plot of cosine
thetas = np.linspace(0, until, 32)
cosines = []
for theta in thetas:
    cosines.append(np.cos(theta))

# Plot the data
fig, ax = plt.subplots()
ax.plot(thetas, cosines, 'r.', label="Cosine")
ax.set_title("Cosine")
plt.show()
```

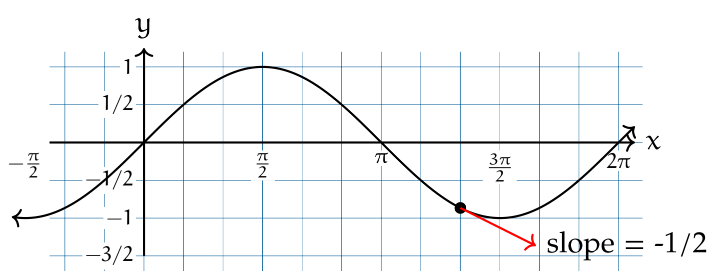
This will plot 32 points on the cosine wave between 0 and 8. When you run it, you should see something like this:



1.3 Derivatives of trigonometic functions

Here is a wonderful property of sine and cosine functions: At any point θ, the slope of the sine graph at θ equals $\cos(\theta)$.

For example, we know that $\sin(4\pi/3) = -(1/2)\sqrt{3}$ and $\cos(4\pi/3) = -1/2$. If we drew a line tangent to the sine curve at this point, it would have a slope of -1/2:



We say "The derivative of the sine function is the cosine function."

Can you guess the derivative of the cosine function? For any θ, the slope of the graph of the $\cos(\theta)$ is $-\sin(\theta)$.

Exercise 1 Derivatives of Trig Functions Practice 1

Working Space

Use the limit definition of a derivative to show that $\frac{d}{dx}\cos x = -\sin x$

Answer on Page 39

The derivatives of all the trigonometric functions are presented below:

$\frac{d}{dx}\sin x = \cos x$	$\frac{d}{dx}\csc x = -\csc x \cdot \cot x$
$\frac{d}{dx}\cos x = -\sin x$	$\frac{d}{dx}\sec x = \sec x \cdot \tan x$
$\frac{d}{dx}\tan x = \sec^2 x$	$\frac{d}{dx}\cot x = -\csc^2 x$

Example: Find the derivative of $f(x)$ if $f(x) = x^2 \sin x$ **Solution**: Using the product rule, we find that:

$$\frac{d}{dx}f(x) = (x^2)\frac{d}{dx}(\sin x) + (\sin x)\frac{d}{dx}(x^2)$$

Taking the derivatives:

$$= x^2(\cos x) + 2x(\sin x)$$

Exercise 2 Derivatives of Trig Functions 2

Working Space

Find the derivative of the following functions:

1. $f(x) = \frac{\sec x}{1+\tan x}$
2. $y = \sec t \tan t$
3. $f(\theta) = \frac{\theta}{4-\tan\theta}$
4. $f(t) = 2\sec t - \csc t$
5. $f(\theta) = \frac{\sin\theta}{1+\cos\theta}$
6. $f(x) = \sin x \cos x$

Answer on Page 40

1.4 A weight on a spring

Let's say you fill a rollerskate with heavy rocks and attach it to the wall with a stiff spring. If you push the skate toward the wall and release it, it will roll back and forth. Engineers would say "The skate will oscillate." Intuitively, you can probably guess:

- If the spring is stronger, the skate will oscillate more times per minute.
- If the rocks are lighter, the skate will oscillate more times per minute.

The force that the spring exerts on the skate is proportional to how far its length is from its relaxed length. When you buy a spring, the manufacturer advertises its "spring rate", which is in pounds per inch or newtons per meter. If a spring has a rate of 5 newtons per meter, that means that if you stretch or compress it 10 cm, it will push back with a force of 0.5 newtons. If you stretch or compress it 20 cm, it will push back with a force of 1 newton.

Let's write a simulation of the skate-on-a-spring. Duplicate `cos.py`, and name the new copy `spring.py`. Add code to implement the simulation:

```
import numpy as np
import matplotlib.pyplot as plt

until = 8.0

# Constants
mass = 100 # kg
spring_constant = -1 # newtons per meter displacement
time_step = 0.01 # s

# Initial state
displacement = 1.0 # height above equilibrium in meters
velocity = 0.0
time = 0.0 # seconds

# Lists to gather data
displacements = []
times = []

# Run it for a little while
while time <= until:
    # Record data
    displacements.append(displacement)
    times.append(time)
```

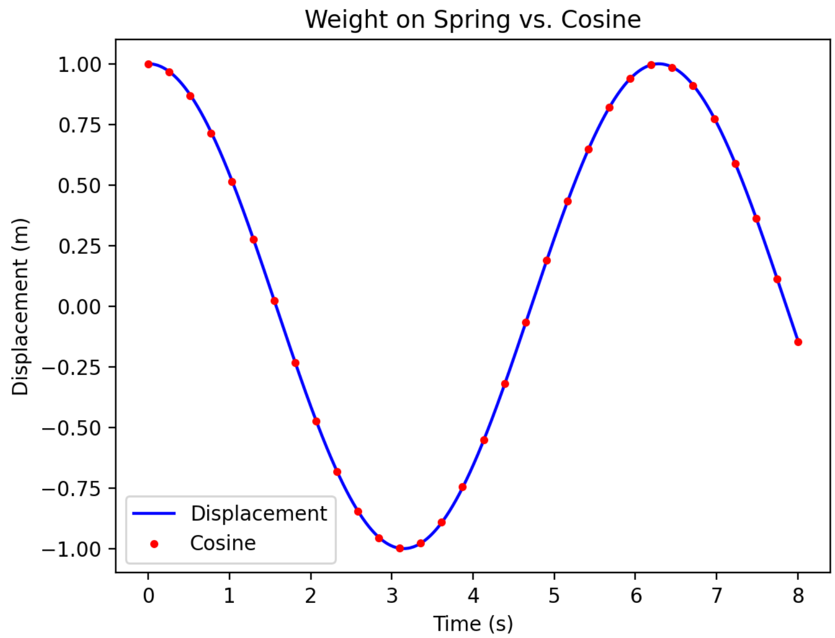
```
    # Calculate the next state
    time += time_step
    displacement += time_step * velocity
    force = spring_constant * displacement
    acceleration = force / mass
    velocity += acceleration

# Make a plot of cosine
thetas = np.linspace(0, until, 32)
cosines = []
for theta in thetas:
    cosines.append(np.cos(theta))

# Plot the data
fig, ax = plt.subplots()
ax.plot(times, displacements, 'b', label="Displacement")
ax.plot(thetas, cosines, 'r.', label="Cosine")

ax.set_title("Weight on Spring vs. Cosine")
ax.set_xlabel("Time (s)")
ax.set_ylabel("Displacement (m)")
ax.legend()
plt.show()
```

When you run it, you should get a plot of your spring and the cosine graph on the same plot.



The position of the skate is following a cosine curve. Why?

Because a sine or cosine waves happen whenever the acceleration of an object is proportional to -1 times its displacement. Or in symbols:

$$a \propto -p$$

where a is acceleration and p is the displacement from equilibrum.

Remember that if you take the derivative of the displacement, you get the velocity. And if you take the derivative of that, you get acceleration. So, the weight on the spring must follow a function f such that

$$f(t) \propto -f''(t)$$

Remember that the derivative of the $\sin(\theta)$ is $\cos(\theta)$.

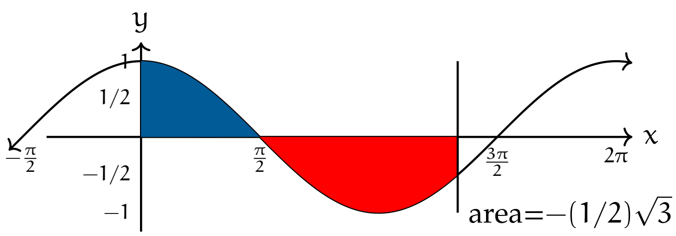
And the derivative of the $\cos(\theta)$ is $-\sin(\theta)$

These sorts of waves have an almost-magical power: Their acceleration is proportional to -1 times their displacement.

Thus, sine waves of various magnitudes and frequencies are ubiquitous in nature and technology.

1.5 Integral of sine and cosine

If we take the area between the graph and the x axis of the cosine function (and if the function is below the x axis, it counts as negative area), from 0 to $4\pi/3$, we find that it is equal to $-(1/2)\sqrt{3}$.



We say "The integral of the cosine function is the sine function."

1.5.1 Integrals of Trig Functions Practice

Exercise 3

Evaluate the following integrals:

1. $\int \sec x \tan x \, dx$

Working Space

Answer on Page 40

Chapter 2

Trigonometric Identities

2.1 The Unit Circle

There are some values of $\sin\theta$ and $\cos\theta$ that will be useful to know off the top of your head. The Unit Circle will help you in this memorization process (see figure 2.1). When a circle of radius 1 is centered at the origin, the Cartesian coordinates of any point on the circle correspond to the values of cosine and sine of the angle above the horizontal (how far you've rotated from the positive x-axis).

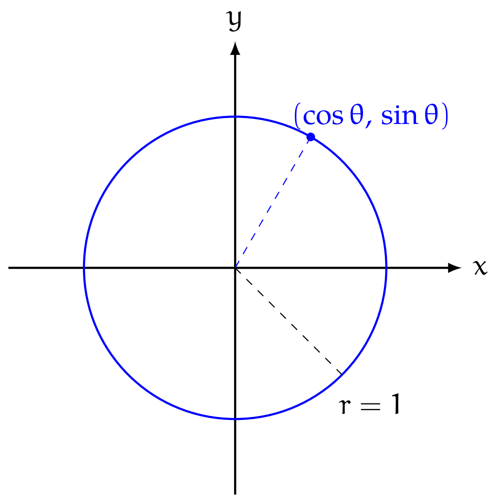


Figure 2.1: The Unit Circle is a circle with radius 1 centered at the origin

Let's take a closer look at a triangle in the first quadrant to see why this is true. Imagine some point on the circle, (x_o, y_o). Drawing a line from that point back to the origin creates an angle θ between the imaginary line and the positive x-axis (see figure 2.2). Extending an imaginary vertical down to $(x_o, 0)$, then an imaginary horizontal from $(x_o, 0)$ to the origin, creates a right triangle. What can we say about the legs of the triangle?

Recall SOH-CAH-TOA from a previous chapter. This acronym tells us that, for a right triangle, the sine of an angle is given by the ratio of the length of the leg opposite the angle to the hypotenuse. In our case, then, $\sin\theta = \frac{y_o}{1} = y_o$. [Remember: We are dealing with the Unit Circle, which has a radius of one. Examining figure 2.2 shows you that the hypotenuse of the imaginary triangle is the same as the circle's radius.] This means that the y-coordinate of any point on the Unit Circle is the sine of the angle of rotation from the horizontal.

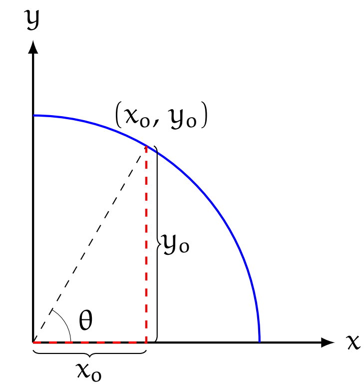


Figure 2.2: Drawing a line from any point on the circle to the origin creates an angle with the horizontal

Exercise 4

In a similar manner as we did with $\sin\theta$ above, prove the x-coordinate of any point on the unit circle is equal to $\cos\theta$, where θ is the angle of rotation from the horizontal.

Working Space

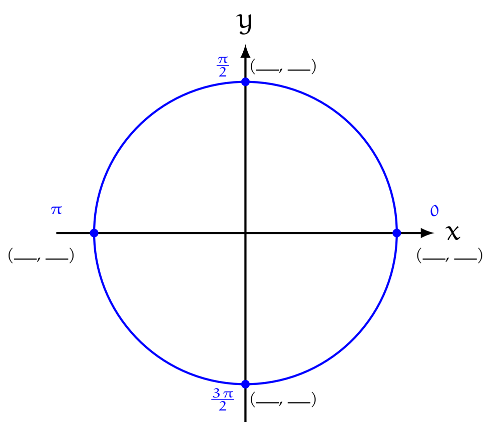
Answer on Page 40

From these exercises, we can see that each (x, y) coordinate on the circle is equal to $(\cos(\theta), \sin(\theta))$.

Exercise 5

Fill in the unit circle with the coordinates for $\theta = 0$, $\pi/2$, π, and $3\pi/2$. Use this to determine:

1. $\sin \frac{\pi}{2}$
2. $\cos \frac{3\pi}{2}$
3. $\sin \pi$
4. $\cos -\pi$



Working Space

Answer on Page 40

2.1.1 Exact Values of Key Angles

We will examine two triangles. First, a 30-60-90 triangle, then a 45-45-90 triangle. As shown in figure 2.3, you can get a 30-60-90 triangle with hypotenuse 1 by dividing an equilateral triangle in half. We will label the horizontal leg of the 30-60-90 triangle A and the vertical leg B.

From the figure, we see that the length of A is half that of the hypotenuse, which in this case is $\frac{1}{2}$. This means the $\cos 60° = \cos \frac{\pi}{3} = \frac{1}{2}$. To find the length of side B, we can use the Pythagorean theorem:

$$B^2 = C^2 - A^2, \text{ where C is the hypotenuse}$$

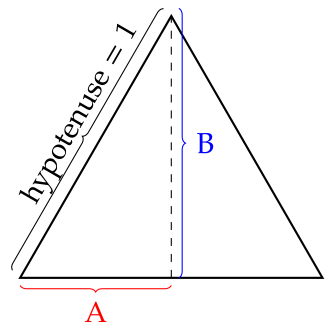


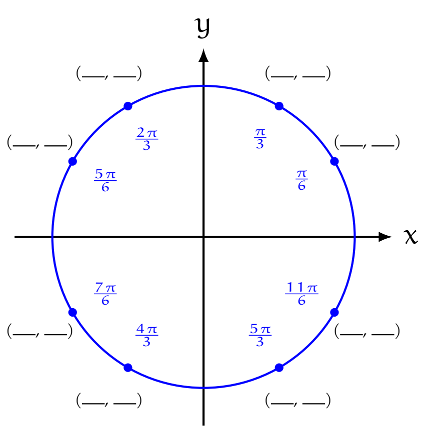
Figure 2.3: A 30-60-90 triangle is made by vertically bisecting an equilateral triangle

$$B^2 = 1^2 - \left(\frac{1}{2}\right)^2$$
$$B^2 = \frac{3}{4}$$
$$B = \frac{\sqrt{3}}{2}$$

Therefore, $\sin 60^\circ = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$.

Exercise 6

Use symmetry to complete the blank unit circle below. (Hint: We just showed that the (x, y) coordinate for $\frac{\pi}{3}$ is $(1/2, \sqrt{3}/2)$).



Working Space

Answer on Page 41

Now we will look at a 45-45-90 triangle (see figure 2.4), which will allow us to complete our Unit Circle. Recall that a 45-45-90 triangle is an isosceles triangle in addition to being a right triangle. This means both the legs are the same length. Using the Pythagorean theorem, we would say $A = B$. We also know that $C = 1$, since our triangle is inscribed in the unit circle. Let's find A:

$$A^2 + B^2 = C^2$$
$$A^2 + A^2 = 1^2$$
$$2A^2 = 1$$
$$A^2 = \frac{1}{2}$$
$$A = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

Therefore, each leg has a length of $\sqrt{2}/2$, and the (x, y) coordinates for $\theta = 45° = \pi/4$ are $(\sqrt{2}/2, \sqrt{2}/2)$.

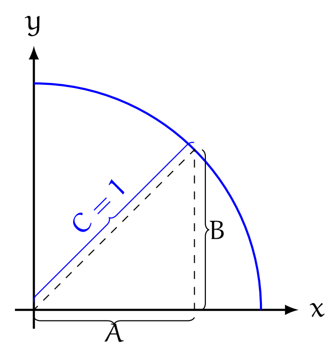
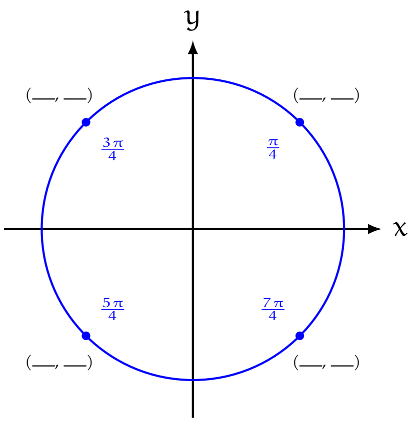


Figure 2.4: The two legs of a 45-45-90 triangle are the same length

Exercise 7

Use symmetry to complete the blank unit circle below.



Working Space

Answer on Page 41

Exercise 8

Without a calculator and using only your completed unit circles, determine the value requested (angles are given in radians unless otherwise indicated).

1. $\cos\frac{3\pi}{2}$
2. $\sin\frac{\pi}{4}$
3. $\sin -\frac{\pi}{6}$
4. $\cos\frac{4\pi}{3}$
5. $\sin\frac{3\pi}{4}$
6. $\cos -\frac{\pi}{3}$
7. $\sin 45°$
8. $\sin 270°$
9. $\sin -60°$
10. $\sin 150°$

Working Space

Answer on Page 42

2.2 Sum and Difference Formulas

Consider 4 points on the unit circle: B at $(1, 0)$, Q at some angle β, P at some angle α, and A at angle $\alpha - \beta$ (see figure 2.5).

The distance from P to Q is the same as the distance from A to B, since ΔPOQ is a rotation of ΔAOB. Because this is a Unit Circle, $P = (\cos\alpha, \sin\alpha)$, $Q = (\cos\beta, \sin\beta)$, and $A = (\cos\alpha - \beta, \sin\alpha - \beta)$. Let's use the distance formula to find the length of $\overline{PQ}$:

$$\overline{PQ} = \sqrt{(\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2} =$$

$$\sqrt{\cos^2\alpha - 2\cos\alpha\cos\beta + \cos^2\beta + \sin^2\alpha - 2\sin\alpha\sin\beta + \sin^2\beta} =$$

$$\sqrt{\left(\cos^2\alpha + \sin^2\alpha\right) + \left(\cos^2\beta + \sin^2\beta\right) - 2\cos\alpha\cos\beta - 2\sin\alpha\sin\beta}$$

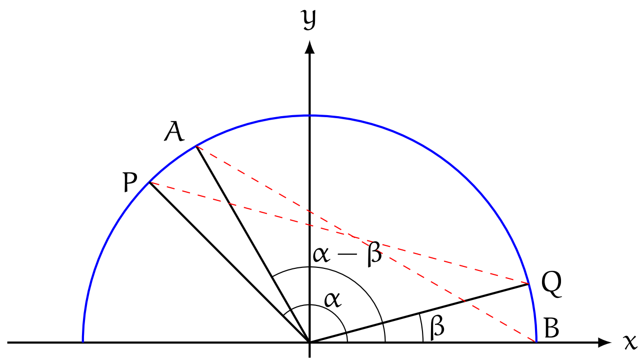


Figure 2.5: $\overline{AB} = \overline{PQ}$

Recall that for any angle, θ, $\sin^2\theta + \cos^2\theta = 1$. Substituting this identity, we see that:

$$\overline{PQ} = \sqrt{1 + 1 - 2\sin\alpha\sin\beta - 2\cos\alpha\cos\beta} = \sqrt{2 - 2\sin\alpha\sin\beta - 2\cos\alpha\cos\beta}$$

Let's leave this simplified equation for $\overline{PQ}$ alone for the moment and similarly find $\overline{AB}$:

$$\overline{AB} = \sqrt{[\cos(\alpha-\beta) - 1]^2 + [\sin(\alpha-\beta) - 0]^2} =$$

$$\sqrt{\cos^2(\alpha-\beta) - 2\cos(\alpha-\beta) + 1 + \sin^2(\alpha-\beta)} =$$

$$\sqrt{\cos^2(\alpha-\beta) + \sin^2(\alpha-\beta) + 1 - 2\cos(\alpha-\beta)}$$

$$= \sqrt{2 - 2\cos(\alpha-\beta)} = \overline{AB}$$

Recall that we've established $\overline{AB} = \overline{PQ}$. We can set the statements equal to each other:

$$\sqrt{2 - 2\sin\alpha\sin\beta - 2\cos\alpha\cos\beta} = \sqrt{2 - 2\cos(\alpha-\beta)}$$

Squaring both sides and subtracting 2, we find:

$$-2\sin\alpha\sin\beta - 2\cos\alpha\cos\beta = -2\cos(\alpha-\beta)$$

Finally, we can divide both sides by negative 2 to get the difference of angles formula for cosine:

$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

There are similar formulas for the sine and cosine of the sum of two angles, and for the sine of the difference of two angles, which we won't derive here.

Sum and Difference Formulas

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$
$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$
$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$
$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

Exercise 9

Without a calculator, find the exact value requested:

1. $\sin\frac{\pi}{12}$
2. $\cos\frac{7\pi}{12}$
3. $\tan\frac{13\pi}{12}$ (hint: $\tan\theta = \sin\theta/\cos\theta$)

Working Space

Answer on Page 42

2.3 Double and Half Angle Formulas

We can easily derive a formula for twice an angle by letting $\alpha = \beta$ for a sum formula.

Example: Derive a formula for $\cos 2\theta$ in terms of trigonometric functions of θ.

Solution: Using the sum formula for cosine, we see that:

$$\cos 2\theta = \cos(\theta+\theta)$$

$$= \cos\theta\cos\theta - \sin\theta\sin\theta = \cos^2\theta - \sin^2\theta$$

Noting that $\sin^2\theta = 1 - \cos^2\theta$: $\boxed{\cos 2\theta = 2\cos^2\theta - 1}$ Alternatively, we could note that $\cos^2\theta = 1 - \sin^2\theta$: $\boxed{\cos 2\theta = 1 - 2\sin^2\theta}$

Or additionally, $\boxed{\cos 2\theta = \cos^2\theta - \sin^2\theta}$

Exercise 10

Derive a formula for $\sin 2\theta$ in terms of trigonometric functions of θ.

Working Space

Answer on Page 42

We can use these double-angle formulas for find half-angle formulas. Consider the double-angle formula for cosine:

$$\cos 2\theta = 2\cos^2\theta - 1$$

Let $\theta = \alpha/2$, then:

$$\cos\alpha = 2\cos^2(\alpha/2) - 1$$

Rearranging to solve for $\cos(\alpha/2)$:

$$2\cos^2(\alpha/2) = \cos\alpha + 1$$

$$\cos^2(\alpha/2) = \frac{\cos\alpha + 1}{2}$$

$$\cos(\alpha/2) = \pm\sqrt{\frac{1 + \cos\alpha}{2}}$$

Exercise 11

Derive a formula for $\sin(\alpha/2)$.

Working Space

Answer on Page 43

There are two identities that will be very useful for integrals in a future chapter:

Squared Trigonometric Identities

$$\sin^2\theta = \frac{1-\cos 2\theta}{2}$$
$$\cos^2\theta = \frac{1+\cos 2\theta}{2}$$

These are just specific re-writings of the half-angle identities.

Chapter 3

Volumes of Common Solids

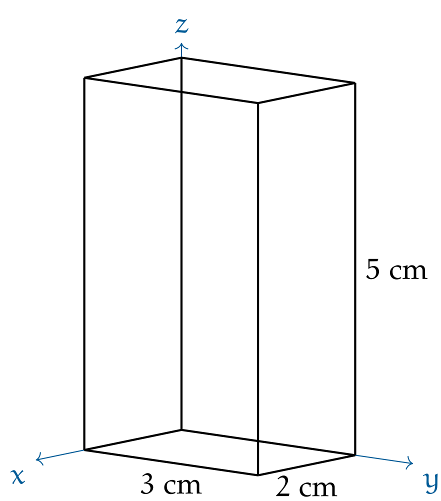
3.1 Rectangular Prism

The volume of a rectangular solid is the product of its three dimensions. If a block of ice is 5 cm tall, 3 cm wide, and 2 cm deep, its volume is $5 \times 3 \times 2 = 30$ cubic centimeters.

Volume of a rectangular solid.

A rectangular solid with height h, width w and length/depth l has volume:

$$V = lwh$$



A cubic centimeter is the same as a milliliter. A milliliter of ice weighs about 0.92 grams. This means the block of ice would have a mass of $30 \times 0.92 = 27.6$ grams.

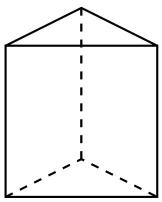
3.2 Triangular Prism

Triangular prisms are 3D versions of triangles (imagine stretching a triangle out of the page). It has 2 triangular faces and 3 rectangular faces.

Volume of a triangular prism.

Recall the area of a triangle is $V = \frac{1}{2}wh$ where w is the width or base and h is the height of the triangle. A triangular prism with height h, width w and length/depth l has volume:

$$V = \frac{1}{2}lwh$$



3.3 Spheres

Volume of a Sphere

A sphere with a radius of r has a volume of

$$v = \frac{4}{3}\pi r^3$$

(For completeness, the surface area of that sphere would be

$$a = 4\pi r^2$$

Note that a circle of radius r is one quarter of this: πr^2.)

Exercise 12 Flying Sphere

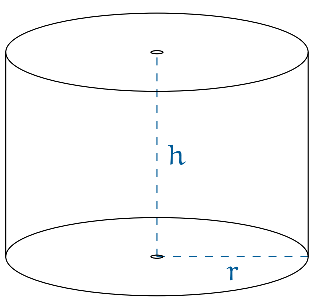
Working Space

An iron sphere is traveling at 5 m/s (and is not spinning). The sphere has a radius of 1.5 m. Iron has a density of 7,800 kg per cubic meter. How much kinetic energy does the sphere have?

Answer on Page 43

3.4 Cylinders

The base and the top of a right cylinder are identical circles. The circles are on parallel planes. The sides are perpendicular to those planes.



Volume of a cylinder

The volume of the right cylinder of radius r and height h is given by:

$$v = \pi r^2 h$$

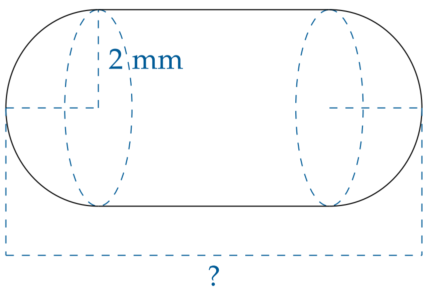
In other words, it is the area of the base times the height.

Exercise 13 Tablet

A drug company needs to create a tablet with volume of 90 cubic millimeters.

The tablet will be a cylinder with half spheres on each end. The radius will be 2mm.

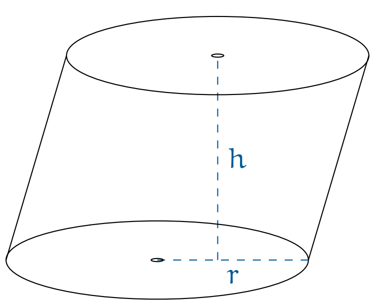
How long do they need to make the tablet to be?



Working Space

Answer on Page 43

What if the base and top are identical, but the sides aren't perpendicular to the base? This is called *oblique cylinder*.

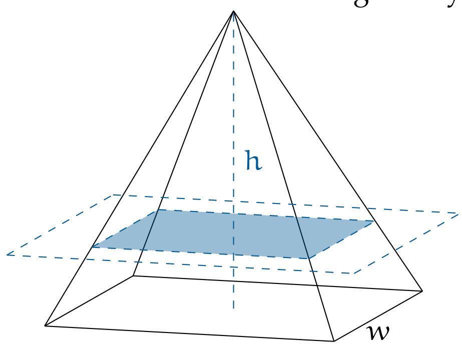


The volume is still the height times the area of the base. Note, however, that the height is measured perpendicular to the bottom and top.

Why is this the case?

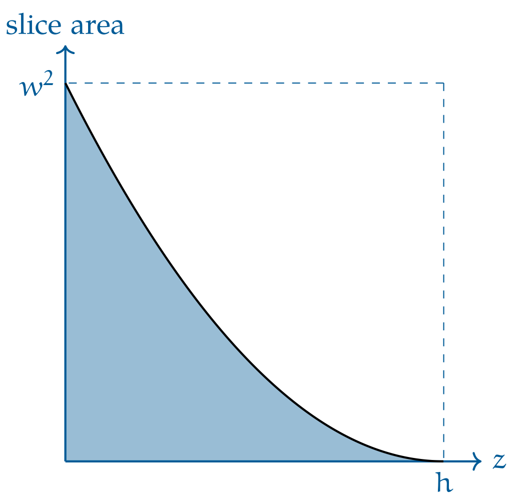
3.5 Pyramids

On a solid with a flat base, the line that we use to measure height is always perpendicular to the plane of the base. We can take slices through the solid that are parallel to that base plane. For example, if we have a pyramid with a square base, each slice will be a square — small squares near the top, larger squares near the bottom. The sides of the pyramids are all triangles, so these are referred to as Triangular Pyramids, just pyramids, or sometimes, Tetrahedrons.



We can figure out the area of the slice at every height z. For example, at $z = 0$, the slice would have area w^2. At $z = h$, the slice would have zero area. What about an arbitrary z in between? The edge of the square would be $w(1 - \frac{z}{h})$. The area of the slice would be $w^2(1 - \frac{z}{h})^2$

The graph of this would look like this:



The volume is given by the area under the curve and above the axis. Once you learn integration, you will be extra good at finding the area under the curve. In this case, we will just tell you that the colored region in the picture is one third of the rectangle.

Thus, the area of a square-based pyramid is $\frac{1}{3}hw^2$.

In fact:

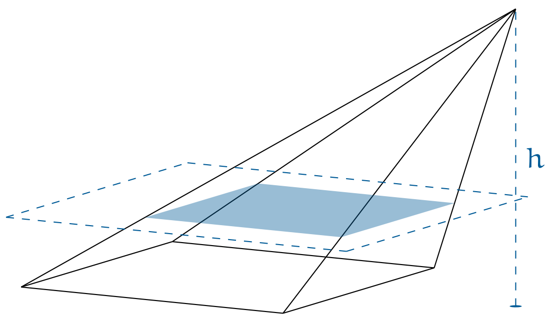
Volume of a pyramid

The volume of pyramid whose base has an area of b and height h is given by:

$$V = \frac{1}{3}hb$$

Regardless of the shape of the base.

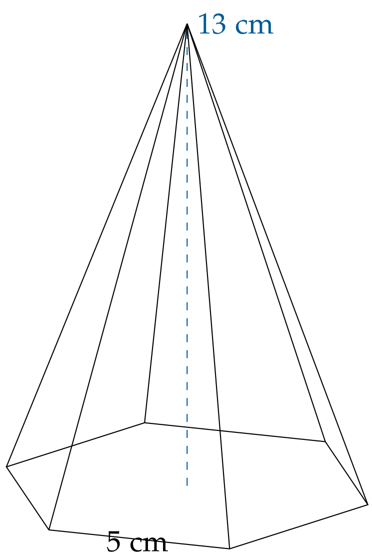
Note that this is true even for oblique pyramids:



Exercise 14 Hexagon-based Pyramid

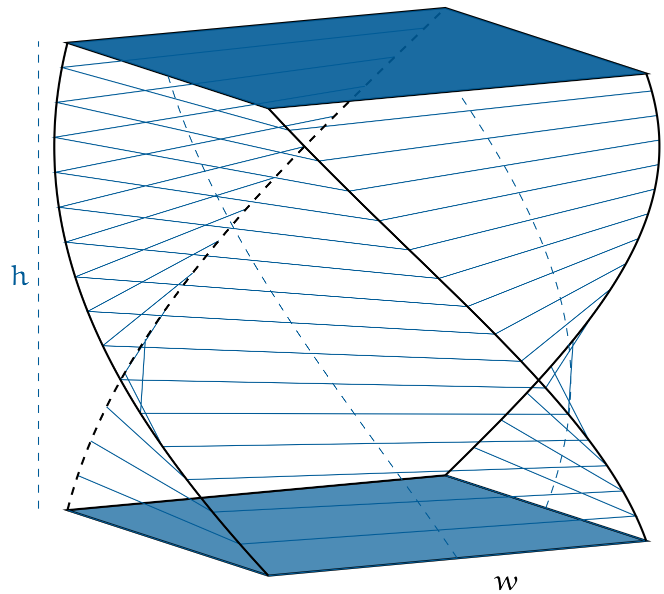
Working Space

There is a pyramid with a regular hexagon for a base. Each edge is 5 cm long. The pyramid is 13 cm tall. What is its volume?



Answer on Page 44

Note that plotting the area of each slice and finding the area under the curve will let you find the area of many things. For example, let's say that you have a four-sided spiral, where each face has the same width w:



Every slice still has an area of w^2, which means this figure has a volume of hw^2.

Exercise 15 Volume of a building

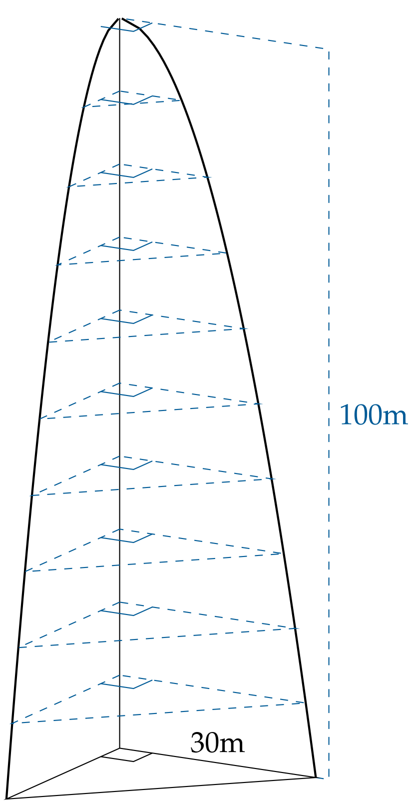
Working Space

An architect is designing a hotel with a right triangular base; the base is 30 meters on each leg. The building gets narrower as you get closer to the top, and finally shrinks to a point. The spine of the building is where the right angle is. That spine is straight and perpendicular to the ground.

Each floor has a right isosceles triangle as its floor plan. The length of each leg is given by this formula:

$$w = 30\sqrt{1 - \frac{z}{100}}$$

So, the width of the building is 30 meters at height $z = 0$. At 100 meters, the building comes to a point. It will look like this:



What is the volume of the building in cubic meters?

Chapter 4

Conic Sections

In mathematics, conic sections (or simply conics) are curves obtained as the intersection of the surface of a cone with a plane. The three types of conic section are the hyperbola, the parabola, and the ellipse. The circle is a special case of the ellipse, though historically it was sometimes called a fourth type. All of the equations below can be graphed on programs like Desmos.

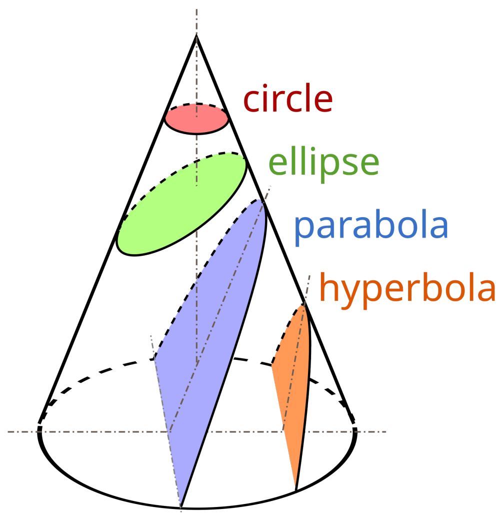


Figure 4.1: Visualization of conic sections.

Source: Wikimedia Commons, Public Domain: `https://upload.wikimedia.org/wikipedia/commons/thumb/1/11/Conic_Sections.svg/1920px-Conic_Sections.svg.png`

4.1 Definitions

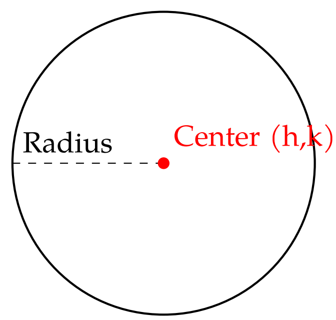
Each type of conic sections can be defined as follows:

4.1.1 Circle

A circle is the set of all points in a plane that are at a given distance (the radius) from a given point (the center). The standard equation for a circle with center (h, k) and radius

r is:

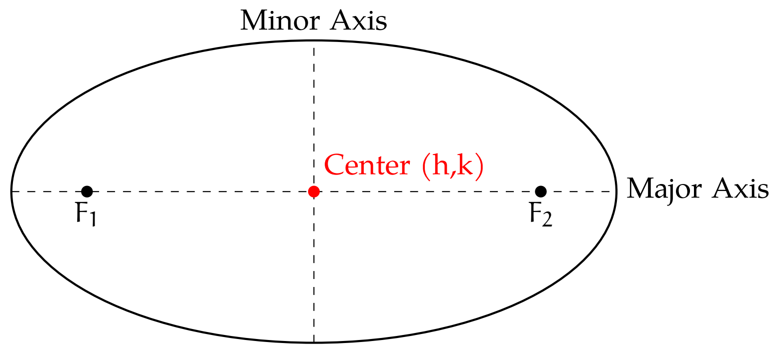
$$(x-h)^2 + (y-k)^2 = r^2 \tag{4.1}$$



4.1.2 Ellipse

An ellipse is the set of all points such that the sum of the distances from two fixed points (the foci) is constant. The standard equation for an ellipse centered at the origin with semi-major axis a and semi-minor axis b is:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{4.2}$$



4.1.3 Hyperbola

A hyperbola is the set of all points such that the absolute difference of the distances from two fixed points (the foci) is constant. A hyperbola is formed from slicing a *double-cone* — two cones placed tip-to-tip — parallel to or angled off of the central axes. The standard equation for a hyperbola centered at the origin is:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \tag{4.3}$$

or

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1 \tag{4.4}$$

depending on the orientation of the hyperbola.

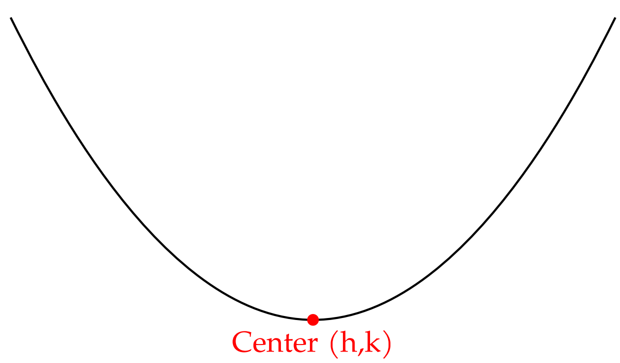
4.1.4 Parabola

A parabola is the set of all points that are equidistant from a fixed point (the focus) and a fixed line (the directrix). The standard equation for a parabola that opens upwards or downwards is:

$$y = a(x - h)^2 + k \tag{4.5}$$

and that opens leftwards or rightwards is:

$$x = a(y - k)^2 + h \tag{4.6}$$



where (h, k) is the vertex of the parabola, and a is a scalar.

Note that only the parabola out of these four is a function, as passes vertical line test. The other three cannot be expressed as functions, only equations.

Appendix A

Answers to Exercises

Answer to Exercise 1 (on page 7)

We start by writing out the limit:

$$\frac{d}{dx}\cos x = \lim_{h\to 0}\frac{\cos x + h - \cos x}{h}$$

Applying the sum formula for $\cos(x+h)$, we get:

$$= \lim_{h\to 0}\frac{\cos x \cos h - \sin x \sin h - \cos x}{h}$$

Rearranging to group the $\cos x$ and applying the Difference Rule:

$$= \lim_{h\to 0}\frac{\cos x \cos h - \cos x}{h} - \lim_{h\to 0}\frac{\sin x \sin h}{h}$$

Applying the Constant Multiple Rule:

$$= \cos x \lim_{h\to 0}\frac{\cos h - 1}{h} - \sin x \lim_{h\to 0}\frac{\sin h}{h}$$

Recalling that $\lim_{h\to 0}\frac{\sin h}{h} = 1$,

$$= \cos x \lim_{h\to a}\frac{\cos h - 1}{h} - \sin x \cdot 1$$

Recalling that $\lim_{h\to 0}\frac{\cos h - 1}{h} = 0$:

$$= \cos x \cdot 0 - \sin x = -\sin x$$

Therefore, $\frac{d}{dx}\cos x = -\sin x$

Answer to Exercise 2 (on page 7)

1. $\frac{\sec x(\tan x - 1)}{(1+\tan x)^2}$
2. $\sec t[\sec^2 t + \tan^2 t]$
3. $\frac{4 - \tan\theta + \theta\sec^2\theta}{(4-\tan\theta)^2}$
4. $2\sec t\tan t + \csc t\cot t$
5. $\frac{2}{(1+\cos\theta)^2}$
6. $\cos^2 x - \sin^2 x$

Answer to Exercise 3 (on page 11)

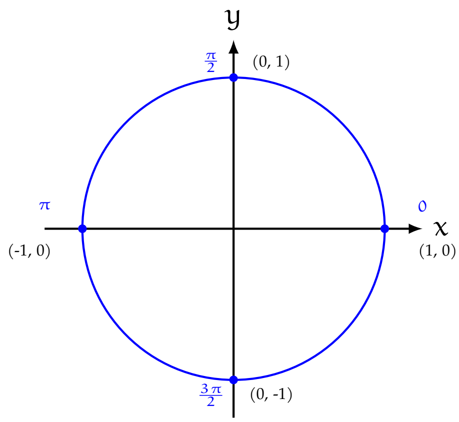
1. $\sec x + C$

Answer to Exercise 4 (on page 14)

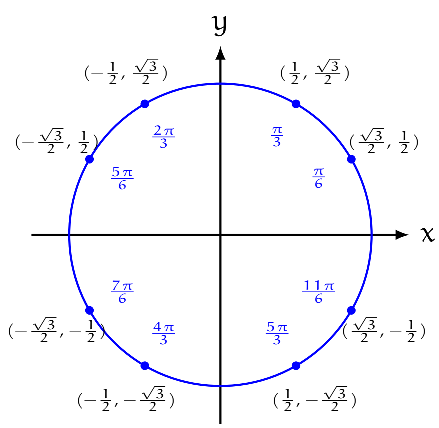
We know that for a right triangle, $\cos\theta = \frac{\text{adjacent leg}}{\text{hypotenuse}}$. For a right triangle inscribed in the Unit Circle, the adjacent leg is parallel to the x-axis and has the same length as the x-value of the coordinate point on the circle. Additionally, the length of the hypotenuse is 1. Therefore, $\cos\theta = \frac{x_o}{1} = x_o$.

Answer to Exercise 5 (on page 15)

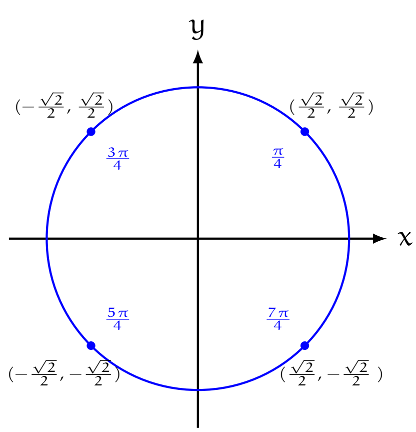
1. $\sin\frac{\pi}{2} = 1$
2. $\cos\frac{3\pi}{2} = 0$
3. $\sin\pi = 0$
4. $\cos -\pi = -1$ (Negative angles are measured clockwise from the x-axis, so $\theta = -\pi$ is at the same angle as $\theta = \pi$.)



Answer to Exercise 6 (on page 16)



Answer to Exercise 7 (on page 18)



Answer to Exercise 8 (on page 19)

1. 0
2. $\sqrt{2}/2$
3. $-1/2$
4. $-1/2$
5. $\sqrt{2}/2$
6. $1/2$
7. $\sqrt{2}/2$
8. -1
9. $-\sqrt{3}/2$
10. $1/2$

Answer to Exercise 9 (on page 21)

1. $\sin(\pi/12) = \sin(\pi/3 - \pi/4) = \sin\frac{\pi}{3}\cos\frac{\pi}{4} - \cos\frac{\pi}{3}\sin\frac{\pi}{4} = \left(\frac{\sqrt{3}}{2}\right)\cdot\left(\frac{\sqrt{2}}{2}\right) - \left(\frac{1}{2}\right)\cdot\left(\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4} = \frac{\sqrt{6}-\sqrt{2}}{4}$
2. $\cos(7\pi/12) = \cos(4\pi/12 + 3\pi/12) = \cos(\pi/3 + \pi/4) = \cos\frac{\pi}{3}\cos\frac{\pi}{4} - \sin\frac{\pi}{3}\sin\frac{\pi}{4} = \left(\frac{1}{2}\right)\cdot\left(\frac{\sqrt{2}}{2}\right) - \left(\frac{\sqrt{3}}{2}\right)\cdot\left(\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{4} - \frac{\sqrt{6}}{4} = \frac{\sqrt{2}-\sqrt{6}}{4}$
3. $\tan(13\pi/12) = \frac{\sin(13\pi/12)}{\cos(13\pi/12)}$ First, we will find $\sin(13\pi/12)$: $\sin(13\pi/12) = \sin(3\pi/12 + 10\pi/12) = \sin(\pi/4 + 5\pi/6) = \sin\frac{\pi}{4}\cos\frac{5\pi}{6} + \cos\frac{\pi}{4}\sin\frac{5\pi}{6} = \left(\frac{\sqrt{2}}{2}\right)\cdot\left(\frac{-\sqrt{3}}{2}\right) + \left(\frac{\sqrt{2}}{2}\right)\cdot\left(\frac{1}{2}\right) = \frac{-\sqrt{6}}{4} = \frac{\sqrt{2}}{4} = \frac{\sqrt{2}-\sqrt{6}}{4}$. Next we find $\cos(13\pi/12) = \cos(\pi/4 + 5\pi/6) = \cos\frac{\pi}{4}\cos\frac{5\pi}{6} - \sin\frac{\pi}{4}\sin\frac{5\pi}{6} = \left(\frac{\sqrt{2}}{2}\right)\cdot\left(\frac{-\sqrt{3}}{2}\right) - \left(\frac{\sqrt{2}}{2}\right)\cdot\left(\frac{1}{2}\right) = \frac{-\sqrt{6}}{4} - \frac{\sqrt{2}}{4} = \frac{-\sqrt{6}-\sqrt{2}}{4}$. And therefore $\tan(13\pi/12) = \frac{\sin 13\pi/12}{\cos 13\pi/12} = \frac{\sqrt{2}-\sqrt{6}}{4}\cdot\frac{4}{-\sqrt{6}-\sqrt{2}} = \frac{\sqrt{6}-\sqrt{2}}{\sqrt{6}+\sqrt{2}} = \frac{\sqrt{6}-\sqrt{2}}{\sqrt{6}+\sqrt{2}}\cdot\frac{\sqrt{6}-\sqrt{2}}{\sqrt{6}-\sqrt{2}} = \frac{6-2\sqrt{12}+2}{6-2} = \frac{8-4\sqrt{3}}{4} = 2-\sqrt{3}$

Answer to Exercise 10 (on page 22)

$\sin 2\theta = \sin\theta + \theta = \sin\theta\cos\theta + \cos\theta\sin\theta = 2\sin\theta\cos\theta$

Answer to Exercise 11 (on page 23)

Similar to $\cos(\alpha/2)$, we begin with the double angle formula for cosine, but another version:

$$\cos 2\theta = 1 - 2\sin^2\theta$$

Substituting $\theta = \alpha/2$:

$$\cos\alpha = 1 - 2\sin^2(\alpha/2)$$

And rearranging to solve for $\sin(\alpha/2)$:

$$\sin(\alpha/2) = \pm\sqrt{\frac{1-\cos\alpha}{2}}$$

Answer to Exercise 12 (on page 27)

The volume of the sphere (in cubic meters) is

$$\frac{4}{3}\pi(1.5)^3 = 4.5\pi \approx 14.14$$

The mass (in kg) is $14.14 \times 7800 \approx 110,269$

The kinetic energy (in joules) is

$$k = \frac{110269 \times 5^2}{2} = 1,378,373$$

About 1.4 million joules.

Answer to Exercise 13 (on page 28)

In your mind, you can disassemble the tablet into a sphere (made up of the two ends) and a cylinder (between the two ends).

The volume of the sphere (in cubic millimeters) is

$$\frac{4}{3}\pi(2)^3 = \frac{32}{3}\pi \approx 33.5$$

Thus, the cylinder part has to be $90 - 33.5 = 56.5$ cubic mm. The cylinder part has a radius of 2 mm. If the length of the cylinder part is x, then

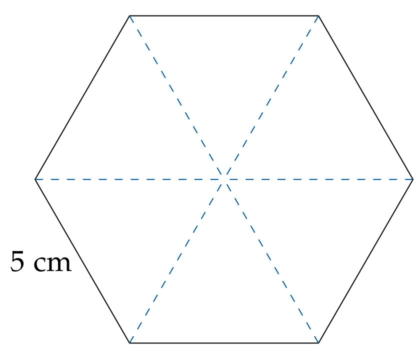
$$\pi 2^2 x = 56.5$$

Thus $x = \frac{56.5}{4\pi} \approx 4.5$ mm.

The cylinder part of the table needs to be 4.5mm. Thus the entire tablet is 8.5mm long.

Answer to Exercise 14 (on page 31)

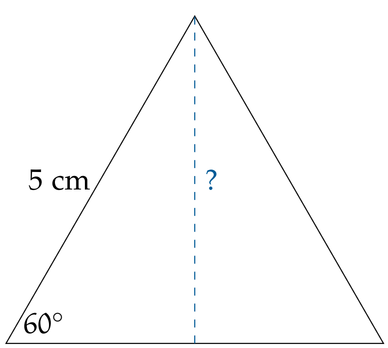
First, you need to find the area of the base, which is a regular hexagon:



All the angles in this picture are 60° or $\frac{\pi}{3}$ radians. This means each line is 5 cm long.

This tells us that we need to find the area of one of these triangles and multiply that by six.

Every triangle has a base of 5cm. How tall are they?



$$5 \sin 60^\circ = 5\frac{\sqrt{3}}{2}$$

Which is about 4.33 cm.

So, the area of single triangle is

$$\frac{1}{2}(5)\left(5\frac{\sqrt{3}}{2}\right) = 25\frac{\sqrt{3}}{4}$$

And the area of the whole hexagon is six times that:

$$75\frac{\sqrt{3}}{2}$$

Thus, the volume of the pyramid is:

$$\frac{1}{3}hb = \frac{1}{3}13\left(75\frac{\sqrt{3}}{2}\right)$$

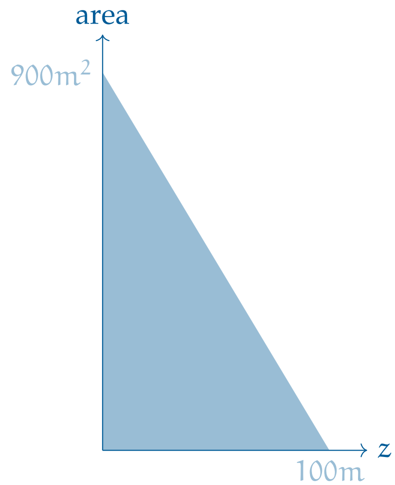
About 281.46 cubic centimeters.

Answer to Exercise 15 (on page 33)

The area at height z is given by:

$$a = \frac{1}{2}w^2 = \frac{1}{2}\left(30\sqrt{1 - \frac{z}{100}}\right)^2 = \frac{1}{2}900\left(1 - \frac{z}{100}\right)$$

If we plot that, it looks like this:



What is the area of the blue region? $\frac{1}{2}(900)(100) = 45,000$

The building will be 45 thousand cubic meters.

Index