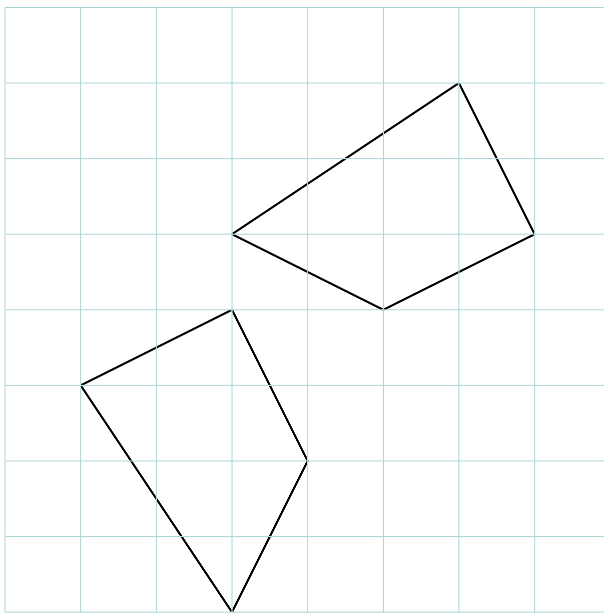
Contents

Chapter 1

Congruence

1.1 Rigid Transformations

Look at this picture of two geometric figures.



They are the same shape, right? If you cut one out with scissors, it would lay perfectly on top of the other. In geometry, we say they are *congruent*.

What is the official definition of “congruent”? Two geometric figures are congruent if you can transform one into the other using only rigid transformations. You might be wondering now, what are rigid transformations? A transformation is *Rigid* if it doesn’t change the distances between the points or the measure of the angles between the lines they form. The following are all rigid transformations:

- Translations
- Rotations
- Reflections

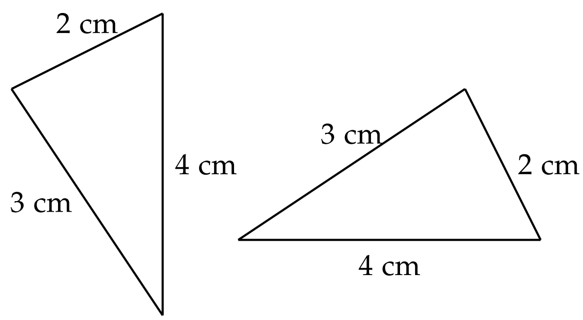
Once again, imagine cutting out one figure with scissors and trying to match it with the second figure; your actions are rigid transformations:

- Translations - Sliding the cutout left and right and up and down
- Rotations - Rotating the cutout clockwise and counterclockwise
- Reflection - Flipping the piece of paper over

A transformation is rigid if it is some combination of translations, rotations, or reflections.

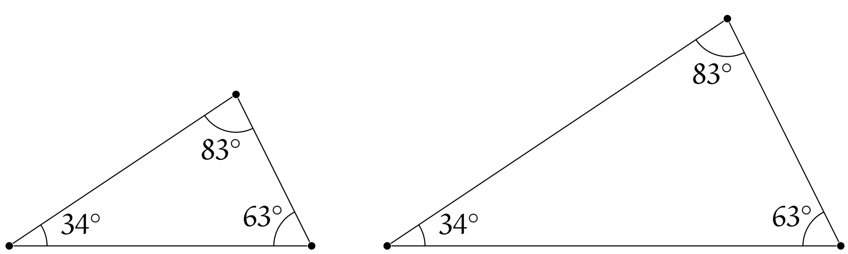
1.2 Triangle Congruency

If the sides of two triangles have the same length, the triangles must be congruent:



To be precise, the Side-Side-Side Congruency Test says that two triangles are congruent if three sides in one triangle are the same length as the corresponding sides in the other. We usually refer to this as the SSS test.

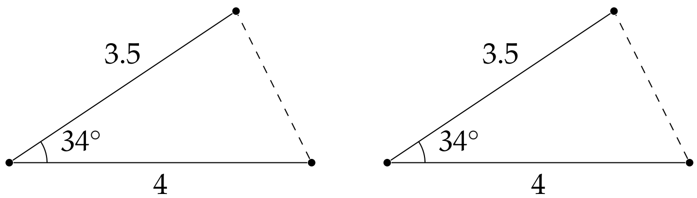
Note that two triangles with all three angles equal are not necessarily congruent. For example, here are two triangles with the same interior angles, but they are different sizes:



These triangles are not congruent, but they are *similar*. Meaning they have the same shape, but are not necessarily the same size.

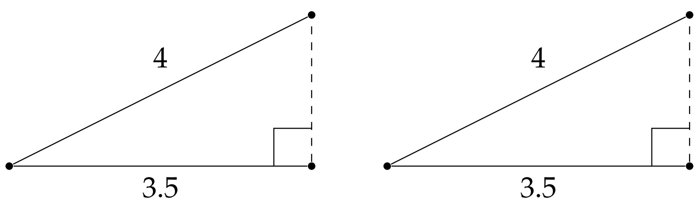
Therefore, if you know two angles of a triangle, you can calculate the third. This is why it makes sense to say "If two triangles have two angles that are equal, they are similar triangles." And if two similar triangles have one side that is equal in length, they must be the same size — so they are congruent. Thus, the Side-Angle-Angle Congruency Test says that two triangles are congruent if two angles and one side match.

What if you know that two triangles have two sides that are the same length, and that the angle between them is also equal?



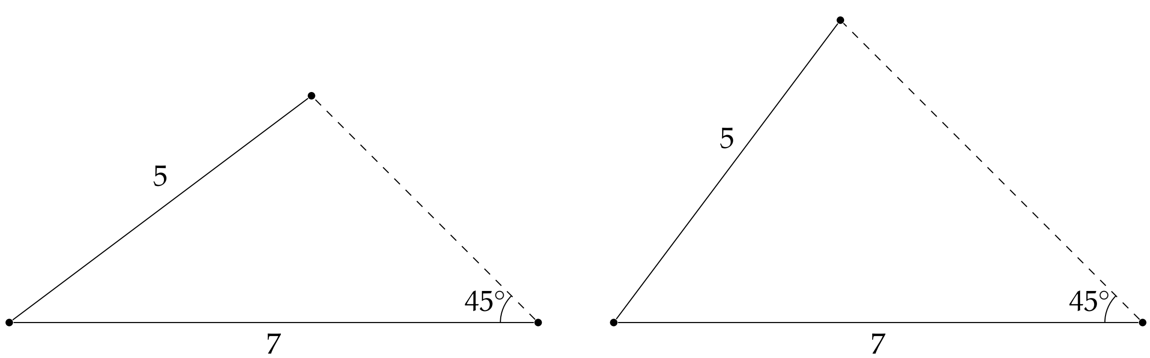
Yes, they must be congruent. This is the Side-Angle-Side Congruency Test.

What if the angle isn't the one between the two known sides? If it is a right angle, you can be certain the two triangles are congruent. (How do we know? Because the Pythagorean Theorem tells us that we can calculate the length of the third side. There is only one possibility, so all three sides must be the same length.)

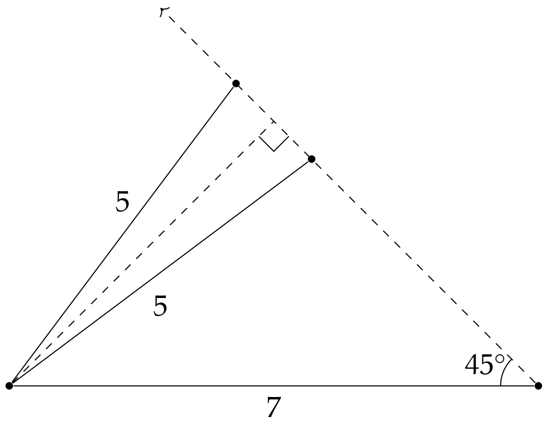


In this case, the third side of each triangle must be $\sqrt{4^2 - 3.5^2} \approx 1.9$.

What if the know angle is less than 90°? *The triangles are not necessarily congruent.* For example, let's say that there are two triangles with sides of length 5 and 7 and that the corresponding angle (at the end of the side of length 7) on each is 45°. Two different triangles satisfy this:



Let's look at this another way by laying one triangle on top of the other:



This is why there is *not* a general Side-Side-Angle Congruency Test.

Here, then, is the list of common congruency tests:

- Side-Side-Side: All three sides have the same measure
- Side-Angle-Angle: Two angles and one side have the same measure
- Side-Angle-Side: Two sides and the angle between them have the same measure
- Side-Side-Right: They are right triangles and have two sides have the same measure

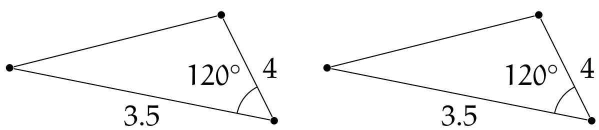
Once proven congruent, we can use the Congruent Parts of Congruent Triangles Theorem, or CPCTC. This concludes that their corresponding parts are also congruent. All three corresponding angles and all three corresponding sides must be the same (defined by congruence).

FIXME diagram here

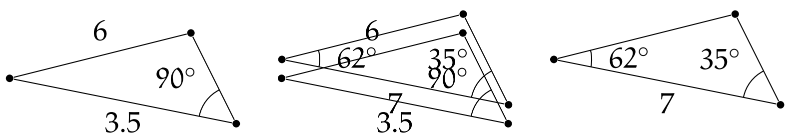
Exercise 1 Congruent Triangles

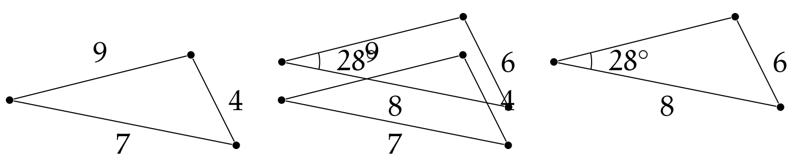
Working Space

Ted is terrible at drawing triangles; he always draws them exactly the same. Fortunately, he has marked these diagrams with the sides and angles that he measured. For each pair of triangles, write whether you know them to be congruent and which congruency test proves it. For example:



(These drawings are clearly not accurate, but you are told the measurements are.) The answer is "Congruent by the Side-Angle-Side test."





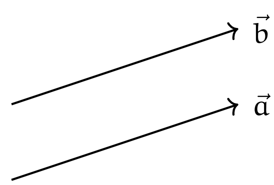
Answer on Page 39

Chapter 2

Parallel and Perpendicular

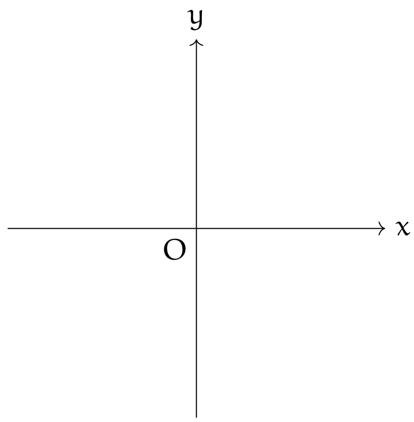
A vector is a line or ray with defined length (referred to as magnitude) and direction (given in degrees or radians). Velocity, force, and displacement are all examples of quantities that can be represented as vectors. Vectors are commonly drawn as arrows, where the length of the arrow corresponds to the magnitude and the direction of the arrow shows the direction of the vector.

Understanding how vectors relate to each other—such as being parallel or perpendicular—is fundamental in mathematics and physics. Two vectors are said to be parallel if they have the same or opposite direction. In simpler terms, if two vectors are pointing in the same direction (even if their magnitudes differ), they are considered parallel. For example, imagine you have a vector representing the direction and speed of a car moving north. If you have another vector representing the direction and speed of a different car also moving north, these vectors are parallel. We have already seen parallel lines when talking about parallel circuits, meaning they offer multiple paths of flow but don't intersect.



On the other hand, if two vectors point are in completely opposite directions, they are still considered parallel. For example, if one vector represents a car moving north and the other represents a car moving south, these vectors are parallel, but in opposite directions.

Perpendicular vectors, as the name suggests, are vectors that intersect each other at a right angle, forming a 90-degree angle. If we imagine a sheet of paper, drawing a horizontal vector and a vertical vector on that paper would create perpendicular vectors. In this case, the horizontal vector represents left-right direction, while the vertical vector represents up-down direction. Perpendicular vectors are often seen in geometric shapes, such as squares and rectangles, where their sides intersect at right angles. The coordinate planes are also perpendicular.



A fundamental property of perpendicular vectors is that their dot product is zero. The dot product is a mathematical operation that measures the extent to which two vectors align with each other. When two vectors are perpendicular, their dot product is always zero. This property provides a useful tool for determining whether two given vectors are perpendicular.

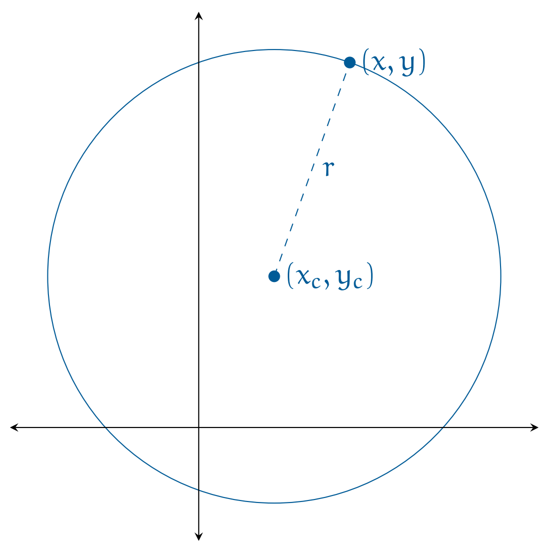
Understanding parallel and perpendicular vectors is essential in various areas of mathematics and physics. For example, in geometry, knowledge of perpendicular vectors helps us determine whether lines are perpendicular or parallel. In physics, vectors can represent forces, velocities, or displacements, and identifying parallel or perpendicular vectors aids in analyzing motion and forces acting on objects.

In summary, parallel vectors have the same or opposite direction, while perpendicular vectors intersect at a right angle. Recognizing these relationships between vectors enables us to solve problems involving geometry, physics, and many other fields. As you delve deeper into the exciting world of vectors, keep an eye out for parallel and perpendicular relationships, as they often hold valuable insights and solutions. We are going to get into graphing these two lines and their equations in the next chapter!

Chapter 3

Circles

A circle is the set of points (x, y) that are a particular distance r from a particular point (x_c, y_c). We say that r is the *radius* and (x_c, y_c) is the *center*.



Area and Radius

If the radius of a circle is r, the area of its interior (a) is given by

$$a = \pi r^2$$

Exercise 2 Area of a Circle

The paint you have says "One liter covers 6 square meters."

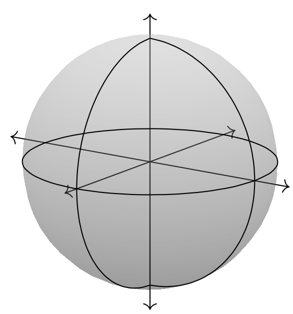
You are painting the top of a circular table with a radius of 3 meters.

How much paint will you need?

Working Space

Answer on Page 39

Note that a circle lives in a particular plane. In 3D, the points (x, y, z) that are a particular distance r from a particular point (x_c, y_c, z_c) are a sphere:



The distance all the way across the middle of a circle (or a sphere) is its *diameter*. The diameter is always twice the radius.

For the rest of the chapter, we will be talking about circles, points, and lines *in a plane*.

Circumference and Diameter

The circumference (c) of a circle is the distance around the circle. If the diameter is d,

$$c = \pi d$$

Exercise 3 Circumference

Working Space

Using a tape measure, you figure out that the circumference of a tree in your yard is 64 cm.

Assuming the trunk is basically circular, what is its diameter?

Answer on Page 40

Exercise 4 Splitting a Pie

Working Space

A pie has a radius of 13 cm. 7 friends all want equal sized sectors. You have a tape measure to assist you.

How many centimeters will each outer crust be?

Answer on Page 40

3.1 Arc Length

Previously, you learned that angles can be measured in degrees and radians. A circle is 360^o (see figure 3.1).

This means a circle is also 2π radians:

$$360^\circ \cdot \frac{\pi}{180^\circ} = 2\pi$$

You may be wondering: why is it that there are π radians in a 180° angle? A radian is

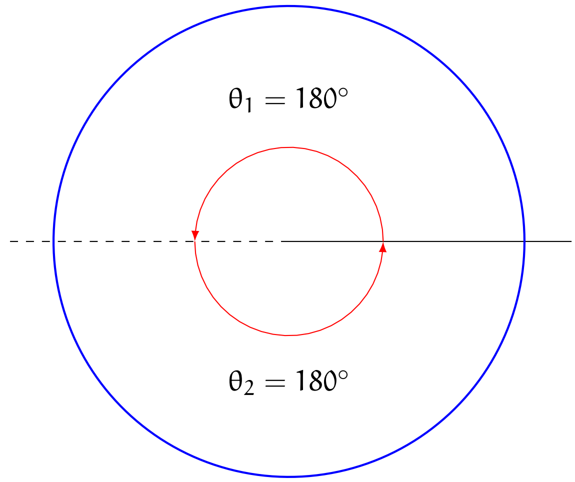


Figure 3.1: The total internal angle of a circle is $\theta_1 + \theta_2 = 360°$

defined such that one radian is the angle at the center of a circle which defines an arc of the circumference equal to the radius of the circle (see figure 3.2).

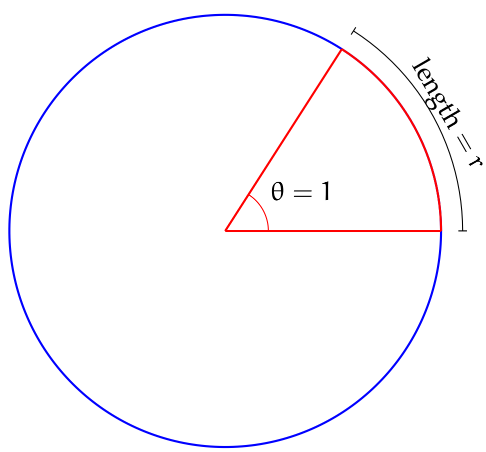
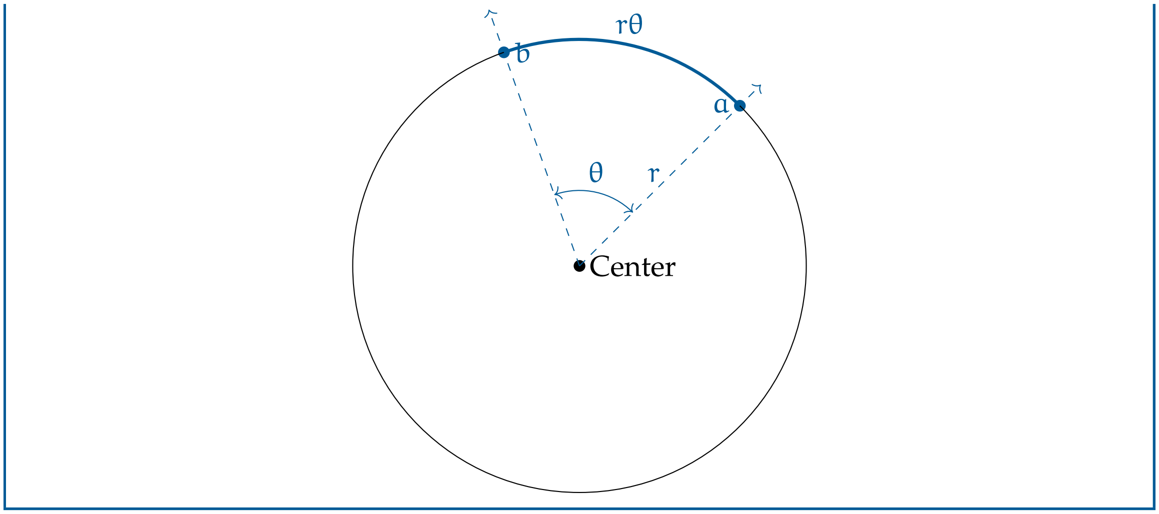


Figure 3.2: When the center angle is 1 radian, the length of the arc is equal to the radius of the circle

This makes it very straightforward to find the lengths of arcs if we know the center angle in radians. The arc length is just $r\theta$, where θ is the center angle in radians.

Length of an Arc

If you have two points a and b on a circle, the ray from the center through a and the ray from the center through b form an angle. If θ is the angle in radians and r is the radius of the circle, the distance from a to b on the circle is $r\theta$.



This shows us why π radians $= 180°$. Recall the formula for circumference: $c = \pi d$, where d is the diameter of the circle. Since the diameter is twice the radius, we can also say that $c = 2\pi r$, where r is the radius of the circle. The circumference of the circle is just an arc where the central angle is the entirety of the circle. Since we know that the length of an arc is $r\theta$, we can find the total internal angle of a circle in radians:

$$2\pi r = r\theta$$

$$\theta = 2\pi$$

This is how we know $360° = 2\pi$ radians.

Exercise 5 Angle of Rotation

A car tire has a radius of approximately 25 centimeters. If you roll your car forward 10 cm, by how many radians has your tire rotated?

Working Space

Answer on Page 40

Exercise 6 Arc Length Ranking

Rank the following arc lengths from longest to shortest (the central angle that defines the arc and the radius of the circle are provided:

1. central angle of $\frac{\pi}{4}$ and a radius of 2 cm
2. central angle of π and a radius of 1 cm
3. central angle of $\frac{\pi}{10}$ and a radius of 5 cm
4. central angle of $\frac{3\pi}{4}$ and a radius of 3 cm

Working Space

Answer on Page 40

Exercise 7 Arc Length

Working Space

You have been asked to find the radius of a very large cylindrical tank. You have a tape measure, but it is only 15 meters long and doesn't reach all the way around the tank.

However, you have a compass. So you stick one end of the tape measure to the side of the tank and measure the orientation of the wall at that point. You then walk the 15 meters and measure the orientation of the wall there.

You find that 15 meters represents 72 degrees of arc.

What is the radius of the tank in meters?

Answer on Page 41

3.2 Sector Area

We already know the area of a circle is given by $A = \pi r^2$. What about a piece of a circle? Let's start with a straightforward example:

Example: A pizza with a radius of 15 cm is divided into 6 equal pieces. What is the area of each piece?

Solution: First, we find the area of the entire pizza:

$$A = \pi r^2$$

$$A = \pi (18 \text{ cm})^2$$

$$A = 324\pi \text{ cm}^2 \approx 1018 \text{ cm}^2$$

Then, we divide by 6, since the pieces of equal sizes:

$$A_{piece} = \frac{A}{6} = \frac{324\pi\ \text{cm}^2}{6} = 54\pi\ \text{cm}^2$$

Let's use this to write a general formula for the area of a sector defined by a central angle θ (see figure 3.3). We know that when a circle is divided into 6 equal sectors, the central angle of each sector is $\theta = \frac{2\pi}{6} = \frac{\pi}{3}$. Additionally, we know the area of each sector is the total area divided by 6: $A_{sector} = \frac{\pi r^2}{6} = \frac{\pi}{6} r^2 = \frac{\theta}{2} r^2$.

Area of a sector

For a sector whose corner is at the center of a circle, the area is given by $A_{sector} = \frac{\theta}{2} r^2$, where θ is the central angle and r is the radius.

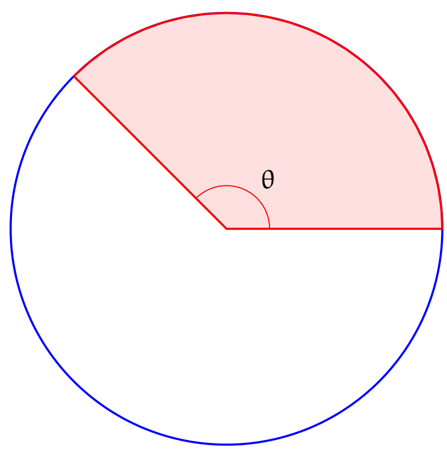


Figure 3.3: The area of a sector with central angle θ is $\frac{\theta}{2} r^2$

Exercise 8 Area of a sector

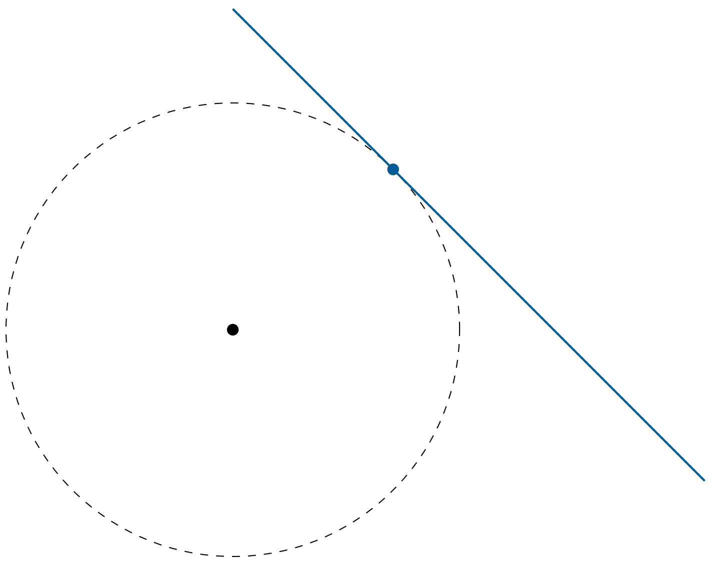
You are tasked with painting a large, circular logo on the side of a building. If a liter of paint covers covers 6 square meters and the logo is 5 meters wide, how many liters of red paint will you need to paint a sector whose central angle is $\frac{3\pi}{4}$ radians?

Working Space

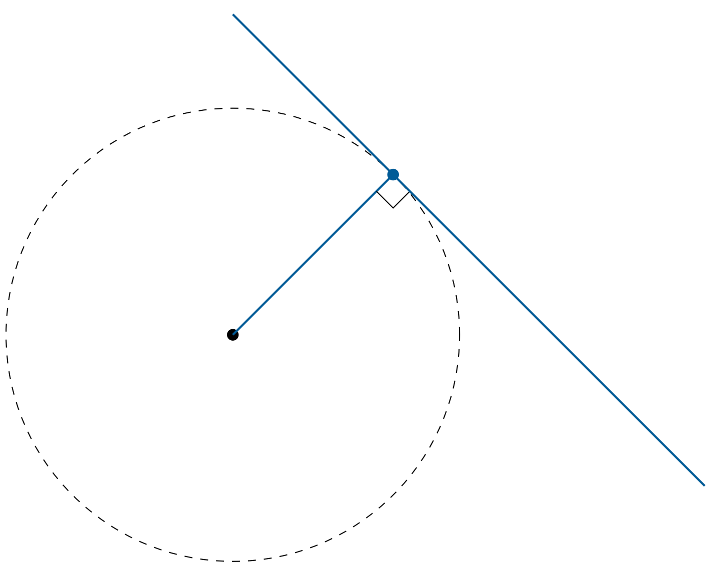
Answer on Page 41

3.3 Tangents

A line that is *tangent* to a circle touches it at exactly one point:



The tangent line is always perpendicular to the radius to the point of tangency:



Exercise 9 Painting a Comet

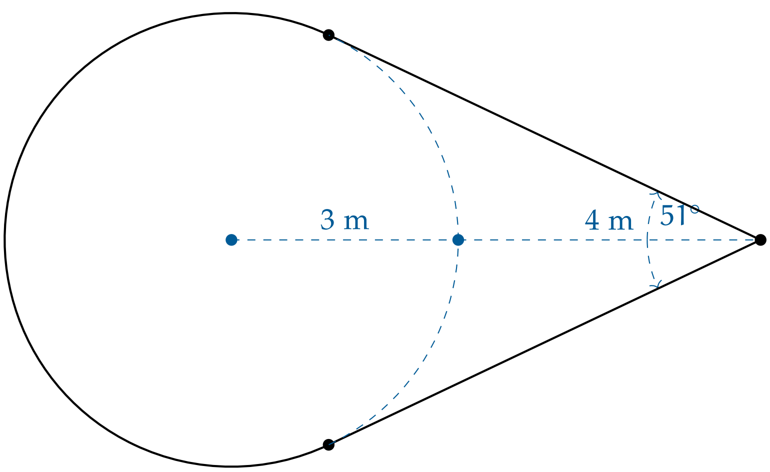
Working Space

You have been asked to paint a comet and its tail in yellow on the floor of a gymnasium.

A liter of yellow paint covers 6 square meters.

First you draw a circle with a radius of 3 meters. You then mark a point D on the floor 7 meters from the center of the circle. Then you draw two tangent lines that pass through D.

You use a protractor to measure the angle at which the tangent lines meet: about 51°



Before you paint the area contained by the circle and the two tangent lines, how much paint will you need?

Answer on Page 41

Chapter 4

Functions and Their Graphs

Functions are a major part of science, engineering, and math. You can think of a function as a machine: you put something into the machine, it processes it, and out comes something else: a product. Just as we often use the variable x to stand in for a number, we often use the variable f to stand in for a function. $f(x)$ is said as "f of x".

For example, we might ask, "Let the function f be defined like this:

$$f(x) = -5x^2 + 12x + 2$$

What is the value of $f(3)$, said as "f of 3"?

You would run the number 3 through "the machine": $-5(3^2)+12(3)+2 = -7$. The answer would be "$f(3)$ is 7".

However, some functions are not defined for every possible input. For example:

$$f(x) = \frac{1}{x}$$

This is defined for any x except 0, because you can't divide 1 by 0. The set of values that a function can process is called its *domain*, and resulting output values are called the range. It is important to note that functions have only one *output* for each *input*. However, multiple inputs can have the same output. A relationship where one input can result in the more than one output is not a function, but a relation. This can be proven by the vertical line test.

Exercise 10 Domain of a function

Working Space

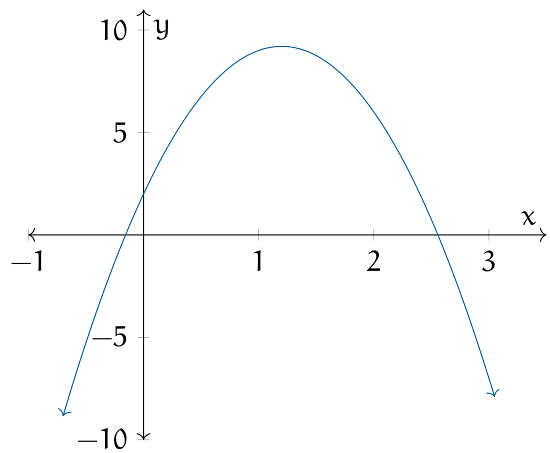
Let the function f be given by $f(x) = \sqrt{x-3}$. What is its domain?

Answer on Page 43

4.1 Graphs of Functions

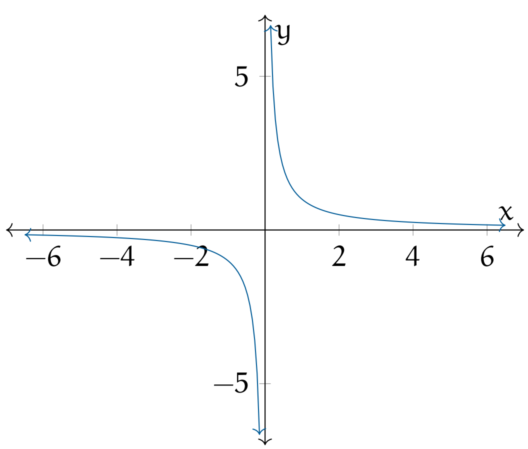
If you have a function, f, its graph is the set of pairs (x, y) such that $y = f(x)$. We usually draw a picture of this set, called a *graph*. The graph not only includes the picture, but also the values of x and y used to create it.

Here is the graph of the function $f(x) = -5x^2 + 12x + 2$:



(Note that this is just part of the graph; it goes infinitely in both directions. Remember your vectors!)

Here is the graph of the function $f(x) = \frac{1}{x}$:



To draw a graph, take each x value (usually in increments of 1) and determine its corresponding y value. Then, plot those points on a graph, using the origin as the starting point.

Exercise 11 Draw a graph

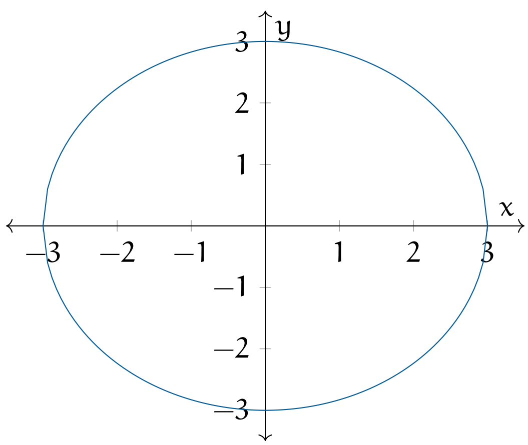
Let the function f be given by $f(x) = -3x + 3$. Sketch its graph.

Working Space

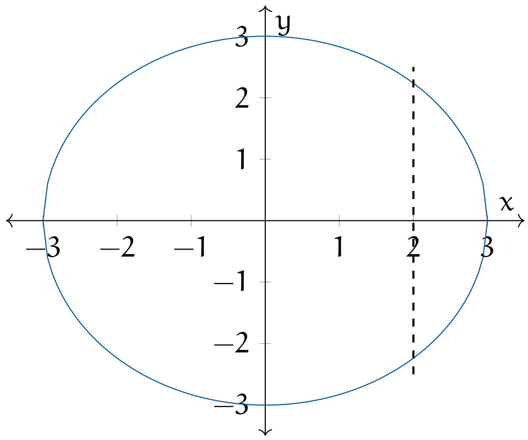
Answer on Page 43

4.2 Can this be expressed as a function?

Note that not all sets can be expressed as graphs of functions. For example, here is the set of points (x, y) such that $x^2 + y^2 = 9$:



This cannot be the graph of a function, because what would $f(0)$ be? 3 or -3? This set fails what we call "the vertical line test": If any vertical line contains more than one point from the set, it isn't the graph of a function. For example, the vertical line $x = 2$ would cross the graph twice:

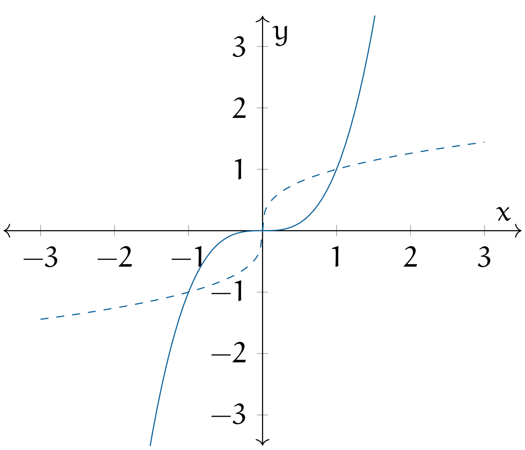


4.3 Inverses

Some functions have inverse functions. If a function f is a machine that turns number x into y, the inverse (usually denoted f^{-1}) is the machine that turns y back into x.

For example, let $f(x) = 5x + 1$. Its inverse is $f^{-1}(x) = (x - 1)/5$. (Spot check it: $f(3) = 16$ and $f^{-1}(16) = 3$)

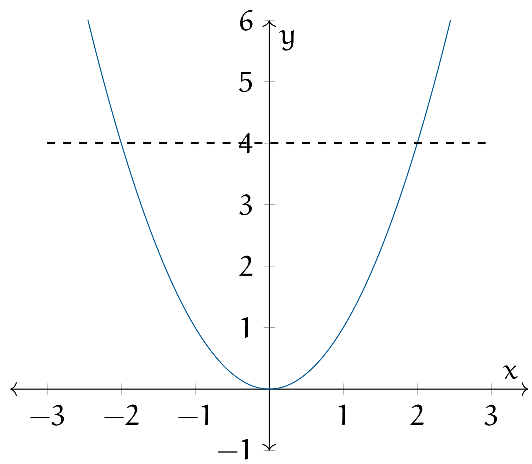
Does the function $f(x) = x^3$ have an inverse? Yes, $f^{-1}(x) = \sqrt[3]{x}$. Let's plot the function (solid line) and its inverse (dashed):



The inverse is the same as the function, just with its axes swapped. The inverse function is *reflected* across the line $y = x$. This tells us how to solve for an inverse: We swap x and y and solve for y.

For example, if you are given the function $f(x) = 5x + 1$, its graph is all (x, y) such that $y = 5x + 1$. The graph of its inverse is all (x, y), such that $x = 5y + 1$. Solving for y gives you $y = (x - 1)/5$. So we can say that $f^{-1}(x) = (x - 1)/5$, QED.

Not every function has an inverse. For example, $f(x) = x^2$. Note that $f(2) = f(-2) = 4$. What would $f^{-1}(4)$ be? 2 or -2? This implies the "horizontal line test": If any horizontal line contains more than one point of a function's graph, that function has no inverse. If a function passes the horizontal line test, it is called "one-to-one", meaning there is exactly one x that gives each y.



In some problems, you need an inverse, but you don't need the whole domain, so you trim the domain to a set you can define an inverse on. This allow you to make claims such as "If we restrict the domain to the nonnegative numbers, the function $f(x) = x^2 - 5$ has an inverse: $f^{-1}(x) = \sqrt{x+5}$.

This raises the question: What is the domain of the inverse function f^{-1}?

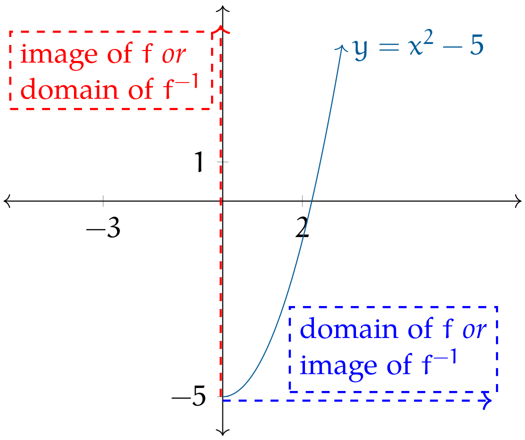
If we let X be the domain of f, we can run every member of X through "the machine" and gather them in a set on the other side. This set would be the *image* of the f "machine". (This is the *range* of f.)

What is the image of $f(x) = x^2 - 5$? It is the set of all real numbers greater than or equal to -5. We write this:

$$\{x \in \mathbb{R} | x \geq -5\}$$

Now we can say: **The range (or image) of the function is the domain of the inverse function.**

In inverse functions, the domain and range get swapped: the domain of the function is the range of the inverse function, and visa versa. In our example, we can use any number greater than or equal to -5 as input into the inverse function.



Exercise 12 Find the inverse

Working Space

Let $f(x) = (x-3)^2 + 2$. Sketch the graph.

Using all the real numbers as a domain, does this function have an inverse?

How would you restrict the domain to make the function invertible?

What is the inverse of that restricted function?

What is the domain of the inverse?

Answer on Page 43

Exercise 13

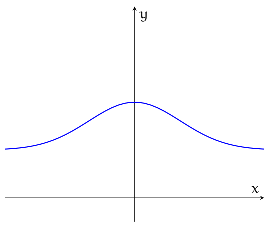
A function is given by a table of values, a graph, or a written description. Determine whether it is one-to-one.

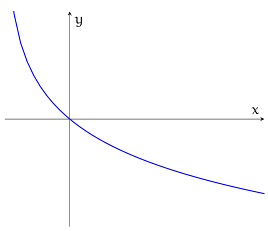
1.

x	1	2	3	4	5	6
$f(x)$	1.5	2.0	3.6	5.3	2.8	2.0

2.

x	1	2	3	4	5	6
$f(x)$	1.0	1.9	2.8	3.5	3.1	2.9

3. 

4. 

5. $f(t)$ is the height of a football t seconds after kickoff

6. $v(t)$ is the velocity of a dropped object

Working Space

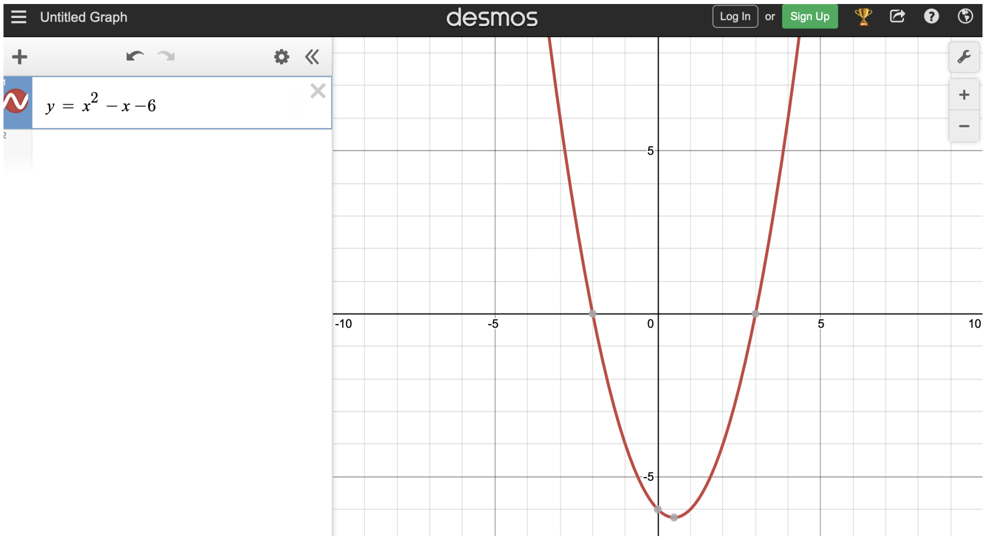
Answer on Page 44

4.4 Graphing Calculators

One really easy way to understand your function better is to use a graphing calculator. Desmos is a great, free online graphing calculator.

In a web browser, go to Desmos: `https://www.desmos.com/calculator`

In the field on the left, enter the function $y = x^2 - x - 6$. (For the exponent, just prefix it with a caret symbol: “x^2”.)



4.5 Even and Odd Functions

4.5.1 Even Functions

An even function is symmetric about the y-axis. That means if you fold the graph along the y-axis, both sides will match perfectly. Note that the input value yields the same output regardless of whether the input value is positive or negative.

Even Functions

A function $f(x)$ is even if

$$f(-x) = f(x)$$

for all x in its domain.

Examples:

- $f(x) = x^2$
- $f(x) = \cos(x)$
- any $f(x) = x^n$ where n is an even number
- $f(x) = |x|$

On a graph, a function will be a mirror image across the y-axis.

4.5.2 Odd Functions

An odd function is symmetric about the origin. That means if you rotate the graph 180° about the origin, it lands on itself. Algebraically, when you input a value k, you get some output n; if you negate that input as $-k$, the output is also negated resulting in $-n$.

Odd Functions

A function $f(x)$ is odd if

$$f(-x) = -f(x)$$

for all x in its domain.

Equivallently,

$$f(x) + f(-x) = 0$$

for all x in its domain.

- $f(x) = x^3$
- $f(x) = \sin(x)$
- $f(x) = \tan(x)$
- any $f(x) = x^n$ where n is an odd number

On a graph, rotating the graph 180° about the origin will yield the same graph.

4.5.3 Neither Even Nor Odd

Some functions are neither even nor odd.

For example, $f(x) = x^3 + 1$ does not satisfy either condition.

Chapter 5

Transforming Functions

Recall how we could translate, mirror, or rotate shapes and they would still be congruent shapes? We can do the same with functions, but the functions are not always equal. Let's say we gave you the graph of a function f, like this:

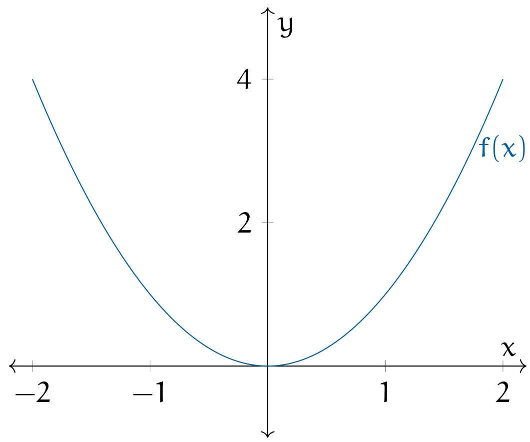


Figure 5.1: A graph of x^2.

We then tell you that the function is $g(x) = f(x) + 1.5$. Can you guess what the graph of g would look like? It is the same graph, just translated up 1.5:

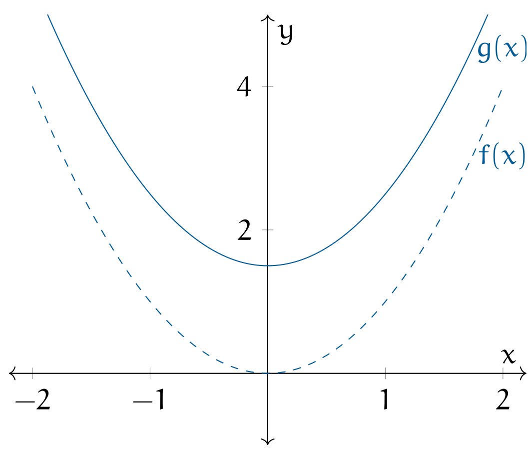


Figure 5.2: Shifted graph of $f(x) + 1.2$.

There are four kinds of transformations that we do all the time:

- Translation up and down in the direction of y axis (the one you just saw)
- Translation left and right in the direction of the x axis
- Scaling up and down along the y axis
- Scaling up and down along the x axis

Next, we will demonstrate each of the four using the graph of $\sin(x)$.

5.1 Translation up and down

When you add a positive constant to a function, you translate the whole graph up that much. A negative constant translates it down.

Here is the graph of $\sin(x) - 0.5$:

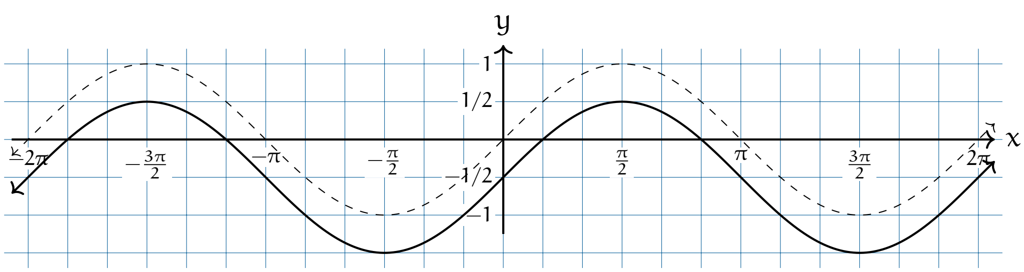


Figure 5.3: A graph of $\sin(x)$ dashed and $\sin(x) - 0.5$ in solid

5.2 Translation left and right

When you add a positive number to x before running it through f, you translate the graph to the left by that amount. Adding a negative number translates the graph to the right.

Here is the graph of $\sin(x - \pi/6)$:

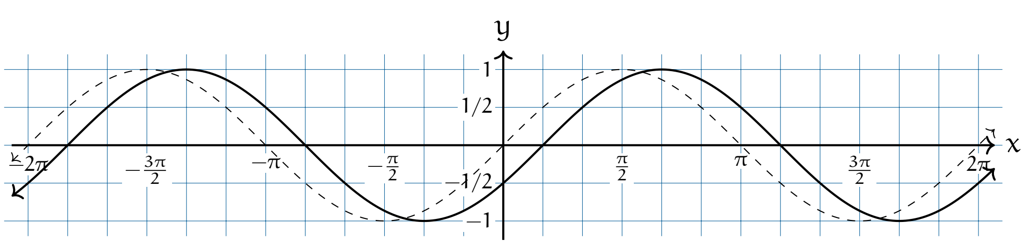


Figure 5.4: A graph of $\sin(x)$ in dashed lines and $\sin(x - \pi/6)$ in solid lines.

Notice the sign:

- Adding to x before processing with the function translates the graph to the *left*.
- Subtracting from x before processing with the function translates the graph to the *right*

5.3 Scaling up and down in the y direction

To scale the function up and down, you multiply the result of the function by a constant. If the constant is larger than 1, it stretches the function up and down.

Here is $y = 2\sin(x)$:

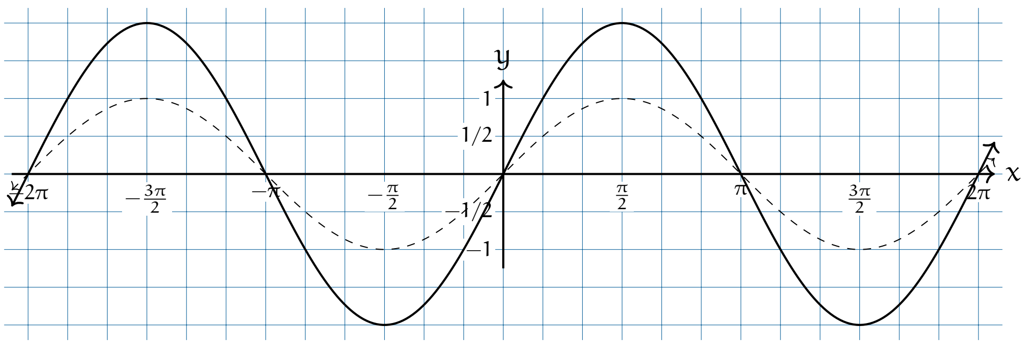


Figure 5.5: Scaling up and down in the y-direction requires a scalar outside of the function.

With a wave like this, we speak of its *amplitude*, which you can think of as its height. The baseline that this wave oscillates around is zero. The maximum distance that it gets from that baseline is its amplitude. Thus, the amplitude here has been increased from 1 to 2.

If you multiply by a negative number, the function gets flipped. Here is $y = -0.5\sin(x)$:

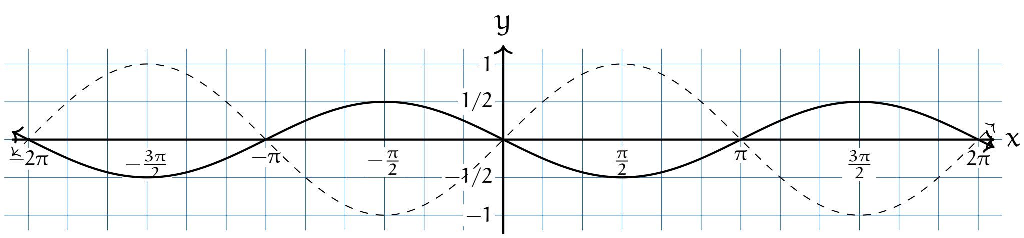


Figure 5.6: Scaling down and flipping or mirroring the function using $y = -0.5\sin(x)$.

Amplitude is never negative. Thus, the amplitude of this wave is 0.5.

5.4 Scaling up and down in the x direction

If you multiply x by a number larger than 1 before running it through the function, the graph gets compressed toward zero.

Here is $y = \sin(3x)$:

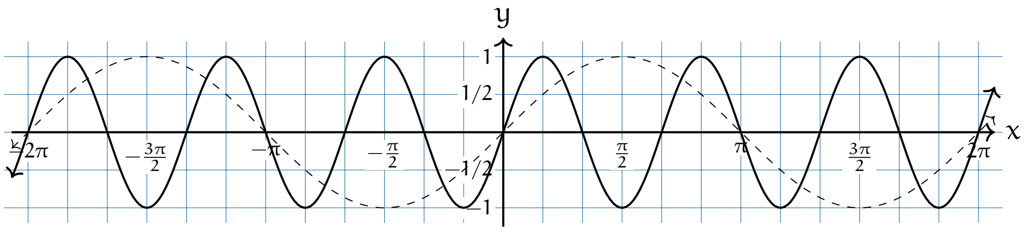


Figure 5.7: A solid $\sin(3x)$ compared to a solid $\sin(x)$ graph.

The distance between two peaks of a wave is known as its *wavelength*. The original wave had a wavelength of 2π. The compressed wave has a wavelength of $2\pi/3$.

If you multiply x by a number smaller than 1, it will stretch the function out, away from the y axis.

If you multiply x by a negative number, it will flip the function around the y axis.

Here is $y = 2^{(-0.5x)}$. Notice that it has flipped around the y axis and is stretched out along the x axis.

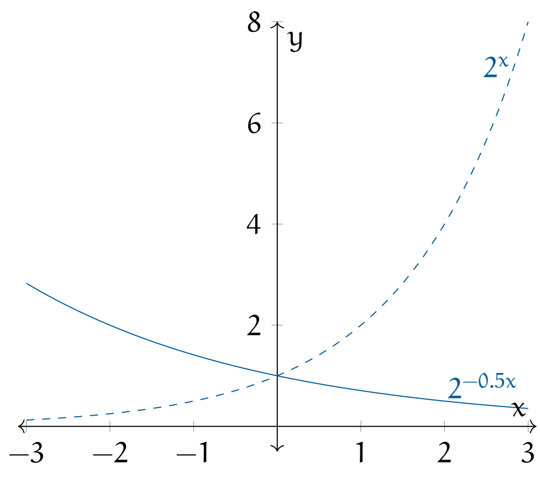


Figure 5.8: A base function 2^x in dashes compared to $2^{-0.5x}$ in solid.

Reflection over *x*-axis	$(x, y) \to (x, -y)$
Reflection over *y*-axis	$(x, y) \to (-x, y)$
Translation	$(x, y) \to (x + a, y + b)$
Dilation	$(x, y) \to (kx, ky)$
Rotation 90° counterclockwise	$(x, y) \to (-y, x)$
Rotation 180°	$(x, y) \to (-x, -y)$

Figure 5.9: A table of different transforms on a function

5.5 Order is important!

We can combine these transformations. This allows us, for example, to translate a function up 2, then scale along the y axis by 3.

Here is $y = 2.0(\sin(x) + 1)$:

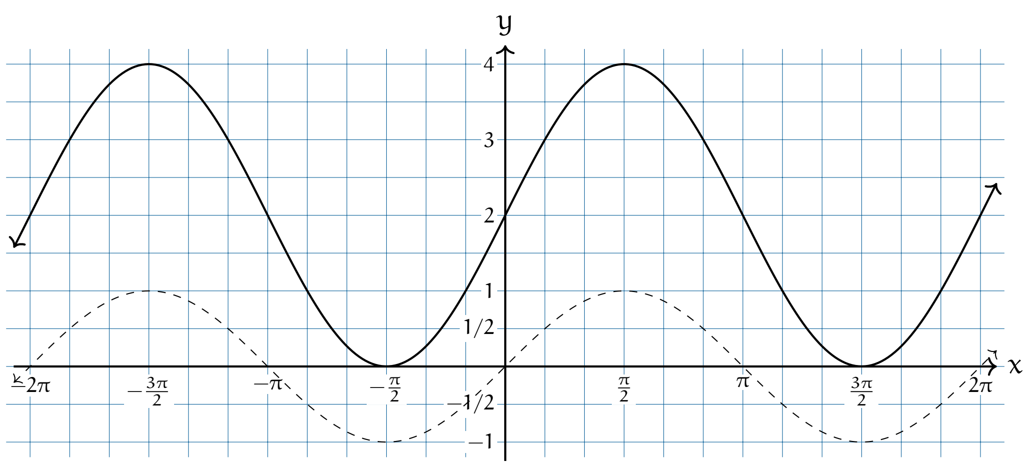


Figure 5.10: A graph of the base function $\sin(x)$ and the transformed function $2(\sin(x)+1)$

A function is often a series of steps. Here are the steps in $f(x) = 2(\sin(x) + 1)$:

1. Take the sine of x
2. Add 1 to that
3. Multiply that by 2

Note that this function can be distributed into $f(x) = 2\sin(x) + 2$ by bringing the 2 in both of the steps.

What if we change the order? Here are the steps in $g(x) = 2\sin(x) + 1$:

1. Take the sine of x
2. Multiply that by 2
3. Add 1 to that

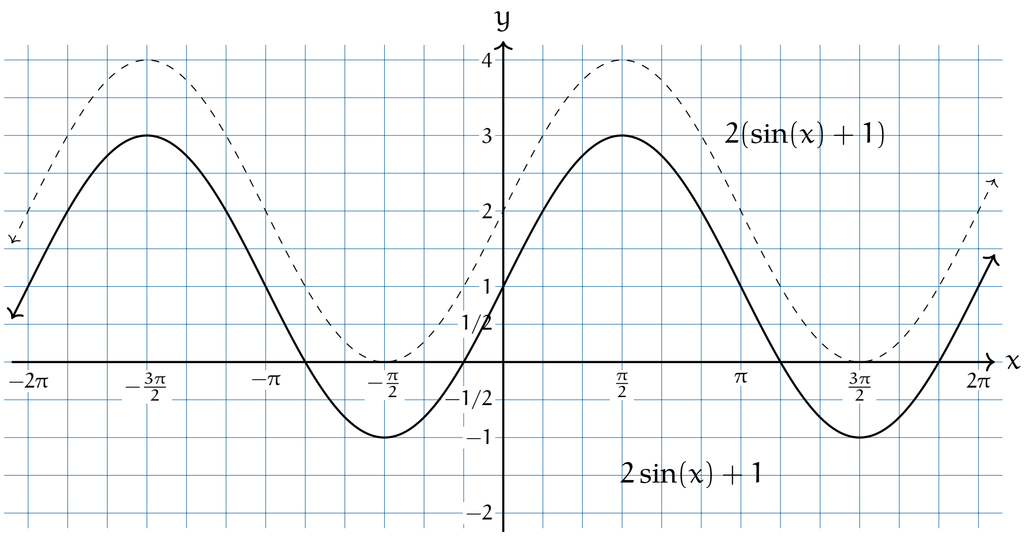
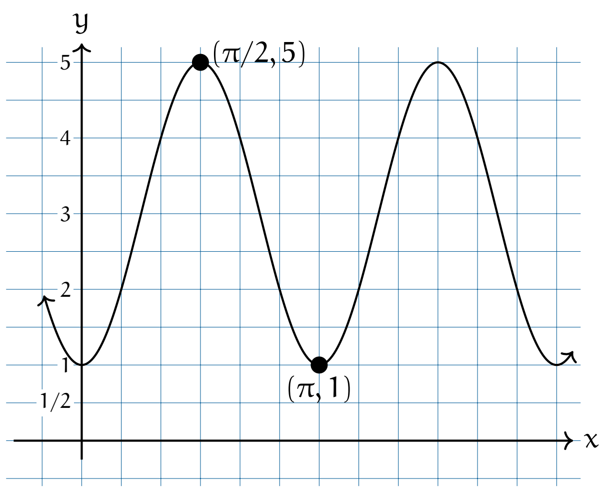


Figure 5.11: The functions $2\sin(x) + 1$ in solid compared to $2(\sin(x) + 1)$ in dashes.

The moral: You can do multiple transformations of your function, but the order in which you do them is important.

Exercise 14 Transforms

Find a function that creates a sine wave such that the top of the first crest is at the point $(\frac{\pi}{2}, 5)$ and the bottom of the trough that follows is at $(\pi, 1)$.



Working Space

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5.6 Effects on even and odd Functions

Now that you've worked with transformations like shifts and scalings, you can use symmetry to quickly assess how those changes affect a function.

1. Vertical shifts (like $f(x)+c$) do not change whether a function is even or odd — but they break symmetry about the origin or y-axis. Example: $x^2 + 1$ is still even, but it's not symmetric about the origin.
2. Horizontal shifts (like $f(x-c)$) also typically destroy even/odd symmetry. Example: $f(x) = \sin(x)$ is odd. But $f(x) = \sin(x - \pi/2)$ is neither even nor odd.
3. Vertical scalings (like $a \cdot f(x)$) preserve symmetry type. Example: x^2 versus $3x^2$.
4. Horizontal scalings (like $f(bx)$) preserve symmetry as long as the center of the transformation stays aligned. Example: $f(x) = x^3$ is odd. $f(x) = (2x)^3$ is still odd.

Remember these key ideas:

1. Reflecting an even function over the y-axis? It remains unchanged.
2. Reflecting an odd function over the origin? Still unchanged.
3. But any shift left or right often breaks the symmetry, so check $f(-x)$ again if you've applied one.

Appendix A

Answers to Exercises

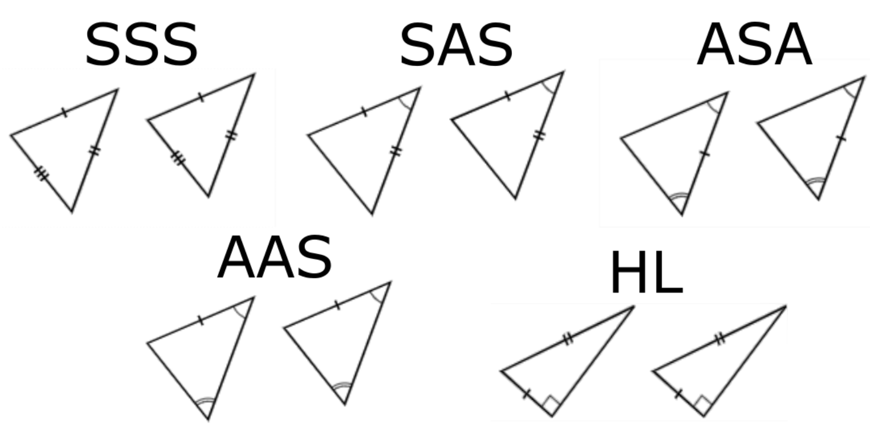
Answer to Exercise 1 (on page 7)

Congruent by the Side-Side-Right Congruency Test.

Congruent by the Side-Side-Side Congruency Test.

Congruent by the Side-Angle-Angle Congruency Test.

We don't know if they are congruent. The measured angle is not between the measured sides.



Answer to Exercise 2 (on page 12)

The table has a radius of 3 meters.

So the area of its top is $3^2\pi \approx 28.27$.

$$28.27 \text{ square meters} \left(\frac{1 \text{ liter}}{6 \text{ square meters}}\right) = 4.72 \text{ liters}$$

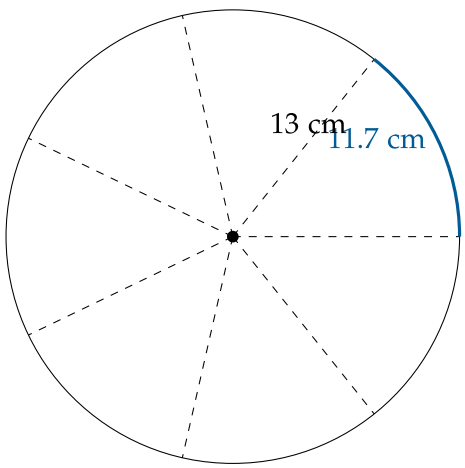
Answer to Exercise 3 (on page 13)

The diameter is
$$\frac{c}{\pi} = \frac{64}{\pi} \approx 20.37 \text{ centimeters}$$

Answer to Exercise 4 (on page 13)

The circumference of the pie is $26\pi \approx 81.7$ centimenters.

The length of the crust for each piece would be about $\frac{81.7}{7} = 11.7$ cm.



Answer to Exercise 5 (on page 15)

If you roll forward by 10 cm, that means you move along the edge of your tire such that the arc length is 10 cm. So, we are looking for a central angle such that $r\theta = 10$ cm. Substituting $r = 25$ cm and solving for θ: $\theta = \frac{10 \text{ cm}}{25 \text{ cm}} = 0.4$ radians.

Answer to Exercise 6 (on page 16)

1. $\frac{\pi}{4} \cdot 2 \text{ cm} = \frac{\pi}{2} \text{ cm}$
2. $\pi \cdot 1 \text{ cm} = \pi \text{ cm}$
3. $\frac{\pi}{10} \cdot 5 \text{ cm} = \frac{\pi}{2} \text{ cm}$
4. $\frac{3\pi}{4} \cdot 3 \text{ cm} = \frac{9\pi}{4} \text{ cm}$

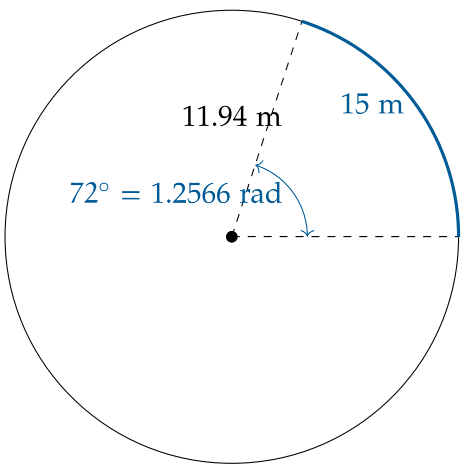
Therefore, from longest to shortest are 4, (1,3), 2 (1 and 3 are the same length).

Answer to Exercise 7 (on page 17)

$$72 \text{ degrees} \left(\frac{2\pi \text{ radians}}{360 \text{ degrees}} \right) \approx 1.2566 \text{ radians}$$

$$15 = 1.2566r$$

$$r = 11.94 \text{ meters}$$

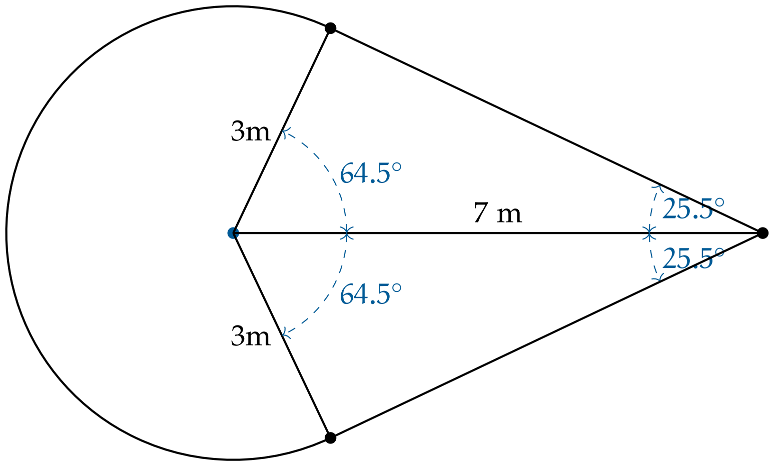


Answer to Exercise 8 (on page 18)

If the logo is 5 meters wide, the diameter is 5 meters and the radius is 2.5 meters. Using the formula for the area of a sector: $A_{sector} = \frac{1}{2}\frac{3\pi}{4}(21.5 \text{ m})^2 \approx 7.363 \text{ m}^2$. Since a liter covers 6 m^2, you will need $\frac{7.363}{6} \approx 1.227$ L of paint.

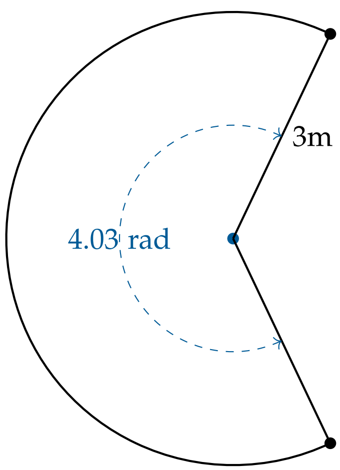
Answer to Exercise 9 (on page 20)

The trick here is to take advantage of the fact that the tangent is perpendicular to the radius to make right triangles:



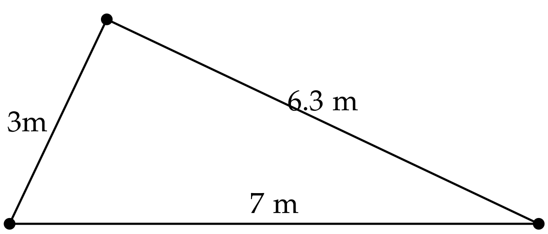
The sector has radius 3 and represents $360 - 2(64.5) = 231° \approx 4.03$ radians.

We are finding the area of this piece:



The area of this piece is $(4.03)(3^2) = 36.27$ square meters.

If a right triangle has a hypotenuse of 7m and one leg is 3m, the other leg is $\sqrt{7^2 - 3^2} = 2\sqrt{10} \approx 6.3$ m.



A right triangle with legs of 3m and 6.3m has an area of 9.45 square meters.

There are two of them, so the total area is $36.27 + 2(18.9) = 74.07$ square meters.

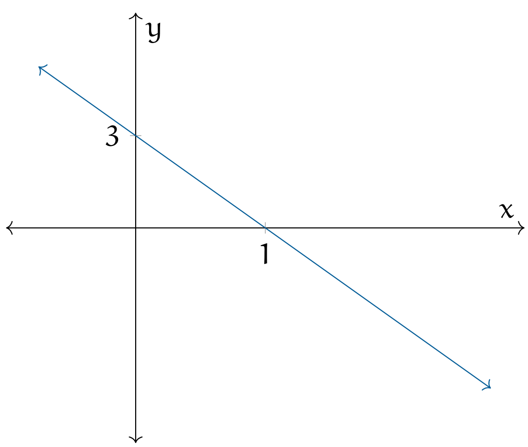
Six square meters per liter, so you need $\frac{74.07}{6} = 12.35$ liters of paint.

Answer to Exercise 10 (on page 21)

You can only take the square root of nonnegative numbers, so the function is only defined when $x - 3 \geq 0$. Thus, the domain is all real numbers greater than or equal to 3.

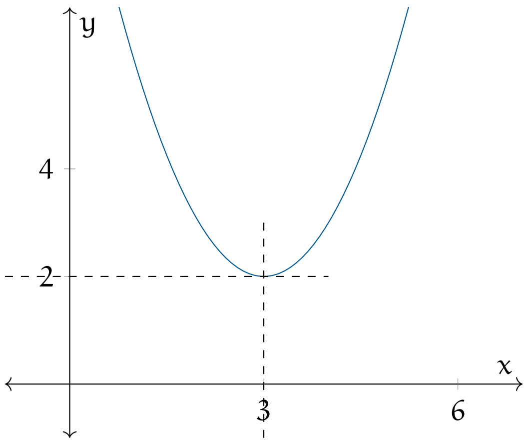
Answer to Exercise 11 (on page 23)

The graph of this function is a line, its slope is -3, and it intersects the y axis at $(0, 3)$.



Answer to Exercise 12 (on page 26)

This graph is the graph of $y = x^2$ that has been moved to the right by three units and up two units:



To prevent any horizontal line from containing more than one point of the graph, you would need to use the left or the right side — either $\{x \in \mathbb{R} | x \leq 3\}$ or $\{x\mathbb{R} | x \geq 3\}$. Most people will choose the right side; the rest of the solution will assume that you did too.

To find the inverse we swap x and y: $x = (y-3)^2 + 2$

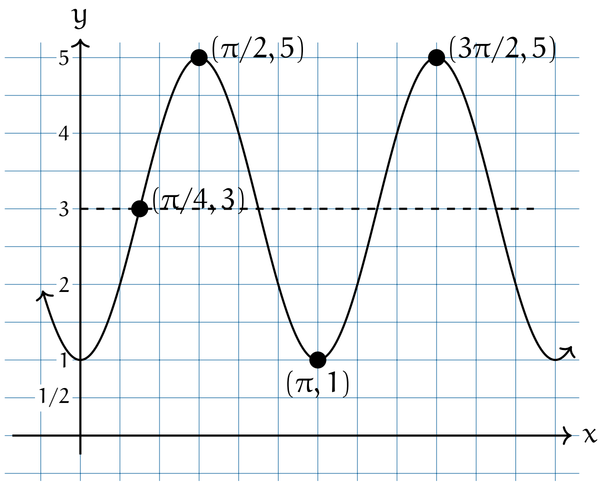
Next, we solve for y to get the inverse: $y = \sqrt{x-2} + 3$

You can take the square root of nonnegative numbers. So the function $f^{-1}(x) = \sqrt{x-2} + 3$ is defined whenever x is greater than or equal to 2.

Answer to Exercise 13 (on page 27)

1. This function is not one-to-one. From $x = 3$ to $x = 4$, the function increases from 3.6 to 5.3, which means it must pass through $f(x_1) = 4.0$. From $x = 4$ to $x = 5$, the function decreases from 5.3 to 2.8, which means it must pass through $f(x_2) = 4.0$ again.
2. This function is not one-to-one by a similar argument in the above solution
3. This function is not one-to-one, because it fails the horizontal line test
4. This function is one-to-one, because it passes the horizontal line test
5. $f(t)$ would not be one-to-one because the football must pass through each height (except the peak height) both on the way up and on the way back down
6. $v(t)$ would be one-to-one because a falling object only speeds up. Therefore, every time has a unique speed.

Answer to Exercise 14 (on page 37)



This wave has an amplitude of 2; its baseline has been translated up to 3.

This wave has wavelength of π. A sine wave usually has a wavelength of 2π, so we need to compress the x axis by a factor of 2.

The wave first crosses its baseline at $pi/4$. The sine wave starts by crossing its baseline, so we need to translate the curve right by $\pi/4$.

$$f(x) = 2\sin(2x - \frac{\pi}{4}) + 3$$

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